Exponential type of hypercyclic entire functions

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Abstract

In this paper the exponential type of hypercyclic entire functions with respect to a sequence $(\Phi_n(D))$ of differential operators is considered, where every Φ_n is an entire function of exponential type. We prove that under suitable conditions certain rates of growth are possible for hypercyclicity while others are not. In particular, our statements extend the negative part of a sharp result on growth of D–hypercyclic entire functions due to Grosse-Erdmann, and are related to a result by Chan and Shapiro about the existence of $\Phi(D)$–hypercyclic functions in certain Hilbert spaces of entire functions.

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1 Introduction and notation

Throughout this paper $\mathbf{C}$ will stand for the complex plane. $\mathbf{N}$ is the set of positive integers, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and $B(a,r)$ $(\overline{B}(a,r))$ is the euclidean open (closed, respectively) disk with center a and radius r $(a \in \mathbf{C},\ r > 0)$. $\mathbf{D}$ is the open unit disk. $H(\mathbf{C})$ denotes, as usual, the linear space of holomorphic functions on G, endowed with the compact-open topology. $H(\mathbf{C})$ becomes a Fréchet space with this topology, so it is a Baire space; it is also separable. An operator on a topological vector space is a continuous linear selfmapping. The differentiation operator D on $H(\mathbf{C})$ is defined as $Df = f'$.

The *exponential type* of an entire function $\Phi(z) = \sum_{j=0}^{\infty} a_j z^j$ on $\mathbf{C}$ is $\tau(\Phi) = \inf\{\mu > 0 :$ there exists $r_0 = r_0(\mu) > 0$ such that $M(\Phi, r) := \max\{|\Phi(z)| : z \in \overline{B}(0,r)\} < \exp(\mu r)\ \forall r > r_0\}$. In other words, $\tau(\Phi) = \limsup_{r\to\infty} \dfrac{\log M(\Phi,r)}{r}$. To every entire function Φ we can associate a "formal" infinite order differential operator with constant coefficients $T = \Phi(D)$, that is, $T = \sum_{j=0}^{\infty} a_j D^j$ with $D^0 = I =$ the identity operator. It is easy to prove (see, for instance, [11, Section 5]) that if Φ has finite exponential type then $\Phi(D)$ defines an operator

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on $H(\mathbf{C})$. In fact, an operator T on $H(\mathbf{C})$ has the latter form if and only if T commutes with D if and only if T commutes with every translation operator τ_a, defined as $\tau_a(f)(z) := f(z+a)$ for all $f \in H(\mathbf{C})$ and all $z, a \in \mathbf{C}$ (see [11, Proposition 5.2]). Note that $\tau_a = \exp(aD)$ for every $a \in \mathbf{C}$.

Let X and Y be topological vector spaces and $T_n : X \to Y$ $(n \in \mathbf{N})$ continuous linear mappings. Then a vector $x \in X$ is said to be *hypercyclic* (or *universal*) for the sequence (T_n) whenever the orbit $\{T_n x : n \in \mathbf{N}\}$ is dense in Y. (T_n) is *hypercyclic* whenever there is at least one hypercyclic vector. If $X = Y$ and T is an operator on X, then T is said to be *hypercyclic* whenever the sequence of iterates (T^n) is hypercyclic, and a vector $x \in X$ is *hypercyclic* for T whenever it is hypercyclic for (T^n). We will employ the following version of the so-called Hypercyclicity Criterion, which can be found in Theorem 2 (and Remark 2 after it) of the survey [13].

Theorem 1.1 *Let X, Y be topological vector spaces, in such a way that X is a Baire space and Y is separable metrizable. Assume that $T_n : X \to Y$ $(n \in \mathbf{N})$ are continuous linear mappings. Suppose that there are dense subsets X_0 of X and Y_0 of Y and (possibly non-linear and discontinuous) mappings $S_n : Y_0 \to Y$ such that*

(i) for every $x \in X_0$, $T_n x \to 0$ $(n \to \infty)$,

(ii) for every $y \in Y_0$, there is an increasing sequence $(n_k) \subset \mathbf{N}$ such that $(S_{n_k} y)$ converges,

(iii) for every $y \in Y_0$, $T_n S_n y \to y$ $(n \to \infty)$.

Then the set of (T_n)–hypercyclic vectors of X is residual, that is, its complement is of first category.

We will also use the next result which guarantees the existence of dense hypercyclic linear submanifolds for densely hereditarily hypercyclic sequences of linear operators, see [5, Theorem 2].

Theorem 1.2 *Let X and Y be two separable metrizable topological vector spaces. If $T_n : X \to Y$ $(n \in \mathbf{N})$ is a sequence of continuous linear mappings such that for each sequence $n_1 < n_2 < n_3 < \dots$ of positive integers there is a dense subset of hypercyclic vectors for the subsequence (T_{n_k}), then there is a dense linear submanifold $M \subset X$ such that every vector $x \in M \setminus \{0\}$ is hypercyclic for (T_n).*

G. D. Birkhoff showed in 1929 that every τ_a $(a \in \mathbf{C} \setminus \{0\})$ is hypercyclic on $H(\mathbf{C})$ [6], and G. R. MacLane [16] obtained in 1952 the same conclusion for D. As a simultaneous generalization of these statements, G. Godefroy and J. H. Shapiro obtained in 1991 the following result, see [11, Theorem 5.1]: If Φ is a nonconstant entire function with finite exponential type then the subset of hypercyclic entire functions for $\Phi(D)$ is residual; the reader is referred to [4] for corresponding statements about sequences of differential operators.

On the other hand, K. G. Grosse–Erdmann proved in 1990 the next sharp statement about the growth of hypercyclic functions, see [12].

Theorem 1.3 *There is no hypercyclic entire function f for D such that $|f(z)| = O(\frac{\exp r}{\sqrt{r}})$ as $|z| = r \to \infty$. However, given any function $\varphi : (0, \infty) \to (0, \infty)$ with $\varphi(r) \to \infty$ as $r \to \infty$, the set of D–hypercyclic functions f with $|f(z)| = O(\varphi(r)\frac{\exp r}{\sqrt{r}})$ as $|z| = r \to \infty$ is dense in $H(\mathbf{C})$.*

The same result was independently obtained by Shkarin [17]. We point out here that MacLane [16] had already shown that there are D–hypercyclic functions of exponential type 1, while S. M. Duyos-Ruiz [10] noted that no D–hypercyclic function can be of exponential type less than 1. G. Herzog [15] proved the existence of a D–hypercyclic function growing no faster than $r \exp r$ as $r \to \infty$. The reader is referred to [1–3] and [12, Section 6] for corresponding results about harmonic functions on $\mathbf{R}^N$ and about shift operators on $H(\mathbf{C})$, respectively. (In [3] and [14], even dense linear manifolds consisting, except for zero, of hypercyclic functions, are obtained). Grosse–Erdmann's theorem cannot be extended in the same way to different operators $\Phi(D)$, that is, there are operators $\Phi(D)$ with a trivial least-possible rate of growth of hypercyclic functions. In fact, Duyos Ruiz [9] proved in 1983 that there are entire functions with arbitrarily slow non-polynomial growth which are hypercyclic for every fixed translation operator τ_a (recall that $\tau_a = \exp{(aD)}$). Nevertheless, if $a \in \mathbf{C} \setminus \{0\}$, it is easy to see –by following step by step the proof of [12]– that the same conclusion of Theorem 1.3 holds for aD–hypercyclic functions just by changing $\exp r$ to $\exp{(cr)}$, where $c = 1/|a|$, that is, c is the common modulus of all solutions of the equation $|\Phi(z)| = 1$, Φ being the function az. Note also that $\Phi(0) = 0$ this time.

In 1991, Chan and Shapiro [8, Theorem 2.1] strengthened Duyos-Ruiz's theorem in the following way: τ_a is hypercyclic on $E^2(\gamma)$ for every $a \in \mathbf{C} \setminus \{0\}$ and every entire function $\gamma(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ satisfying that $\gamma_n > 0$ for each n, the sequence of ratios γ_{n+1}/γ_n decreases to zero as n increases to ∞ and the sequence $n\gamma_n/\gamma_{n-1}$ is monotically decreasing. Here $E^2(\gamma)$ is the Hilbert space of entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which $||f|| \equiv (\sum_{n=0}^{\infty} \gamma_n^{-2}|a_n|^2)^{1/2} < \infty$. In [8, p. 1447] they point out that the proof of their Theorem 2.1 can be modified to give the following result: If $\Phi(z)$ is holomorphic in a neighborhood of the closed unit disk and $\Phi(\mathbf{D})$ intersects the unit circle, then the operator

$\Phi(D)$ is hypercyclic on $E^2(e^z)$. Note that, by [8, Proposition 1.4(a)], every $\Phi(D)$-hypercyclic function for the space $E^2(e^z)$ is a $\Phi(D)$-hypercyclic entire function of exponential type ≤ 1 for the space $H(\mathbf{C})$.

Inspired by the facts about growth described so far, we investigate in this note the growth of hypercyclic entire functions with respect to certain sequences of infinite order linear differential operators and, in particular, with respect to a single operator $\Phi(D)$ satisfying suitable conditions. In a certain important particular case, the critical exponential growth type for hypercyclic entire functions is derived as a consequence.

2 Exponential type of hypercyclic functions

Before establishing our results, we need some notation and several elementary facts. The multiplicity of an entire function Φ for the zero at the origin will be denoted by $m(\Phi)$ ($\in \mathbf{N}_0$). We can associate to each entire function $\Phi(z) = \sum_{j=0}^{\infty} a_j z^j$ the new entire function $\Phi^*(z) = \sum_{j=0}^{\infty} |a_j| z^j$. If $|\Phi(0)| < 1$, let us denote

$$c(\Phi) = \min\{|z| : |\Phi(z)| = 1\},$$

that is, $c(\Phi)$ is the least distance from the origin up to the "level curve" $|\Phi(z)| = 1$. Note that $c(\Phi) \in (0, \infty)$. By continuity, $|\Phi(z)| < 1$ for all $z \in B(0, c(\Phi))$. It is obvious that $c(\Phi^*) \leq c(\Phi)$. Observe that $c(\Phi)$ is the unique $c > 0$ with $\Phi(c) = 1$ whenever the Taylor coefficients of Φ at the origin are ≥ 0.

We are now ready to state our theorems about sequences of operators. The first one is negative and the second one is positive. Corresponding corollaries can be extracted regarding the hypercyclicity of a single operator $\Phi(D)$.

Theorem 2.1 *Assume that (Φ_n) is a sequence of entire functions, each of them with finite exponential type. Let $c > 0$ such that the sequence $(\Phi_n^*(c))$ is bounded. Then there is no hypercyclic entire function f for $(\Phi_n(D))$ such that $|f(z)| = O(\frac{\exp(cr)}{\sqrt{r}})$ as $|z| = r \to \infty$. In particular, there is no hypercyclic function f with $\tau(f) < c$.*

Proof. It is evident that it suffices to show that, for every entire function f with the growth property of the statement, the sequence $((\Phi_n(D)f)(0))$ is not dense in $\mathbf{C}$. In turn, it suffices to show that the latter sequence is bounded. For each $n \in \mathbf{N}$ we have

$$\Phi_n(z) = \sum_{j=0}^{\infty} a_{j,n} z^j,$$

for suitable $a_{j,n} \in \mathbf{C}$ $(j \in \mathbf{N})$. Then

$$\Phi_n^*(z) = \sum_{j=0}^{\infty} |a_{j,n}| z^j.$$

Assume that $(\Phi_n^*(c))$ is bounded. Suppose that f is an entire function such that there exists a positive constant K_1 with $M(f,r) < K_1 \cdot \exp(cr)/\sqrt{r}$ $(r > 0)$. By Cauchy's inequalities,

$$|f^{(j)}(0)| \leq \frac{j! K_1 \cdot \exp(cr)}{r^{j+(1/2)}} \quad (j \in \mathbf{N}_0;\ r > 0).$$

By Stirling's formula, there exists a positive constant K_2 with $j! \leq K_2 \cdot j^{j+(1/2)} \cdot e^{-j}$ $(j \in \mathbf{N})$. By choosing $r = j/c$, one obtains that

$$|f^{(j)}(0)| \leq \frac{K \cdot \exp j \exp(-j) \cdot j^{j+(1/2)} \cdot c^j}{j^{j+(1/2)}},$$

where $K = c^{1/2} K_1 K_2$, that is,

$$|f^{(j)}(0)| \leq K \cdot c^j \quad (j \in \mathbf{N}),$$

so

$$|(\Phi_n(D)f)(0)| = |\sum_{j=0}^{\infty} a_{j,n} f^{(j)}(0)| \leq |\Phi_n(0)| \cdot |f(0)| + K \cdot \sum_{j=1}^{\infty} |a_{j,n}| c^j$$
$$= \Phi_n^*(0)|f(0)| + K \cdot (\Phi_n^*(c) - \Phi_n^*(0)) \leq \Phi_n^*(c) \cdot (|f(0)| + K).$$

Thus, the sequence $((\Phi_n(D)f)(0))$ is bounded, as required.

Corollary 2.2 *Assume that Φ is an entire function with finite exponential type and $|\Phi(0)| < 1$. Then there is no $\Phi(D)$–hypercyclic entire function f for which $|f(z)| = O(\frac{\exp(c(\Phi^*)r)}{\sqrt{r}})$ as $|z| = r \to \infty$. In particular, there is no $\Phi(D)$–hypercyclic entire function f with $\tau(f) < c(\Phi^*)$.*

Proof. We are going to apply Theorem 2.1. Set $c = c(\Phi^*)$. It suffices to see that $(\Phi_n^*(c))$ is bounded, where $\Phi_n := \Phi \cdots \Phi$ (n times). We have from the triangle inequality and the Cauchy product rule for series that $\Phi_n^*(c) \leq (\Phi^*(c))^n = 1$ for every n. The conclusion follows and the proof is finished.

Theorem 2.3 *Assume that (Φ_n) is a sequence of entire functions such that $m(\Phi_n) \to \infty$ $(n \to \infty)$, in such a way that every Φ_n has finite exponential type. Let us set*

$$A = \{z \in \mathbf{C} : \text{the sequence } (\frac{1}{\Phi_n(z)}) \text{ is bounded}\}.$$

(1) If $c \geq 0$ and A has at least one accumulation point in $\overline{B}(0,c)$, then for every $d > c$ there exists a dense linear submanifold of $H(\mathbf{C})$ consisting, except for zero, of $(\Phi_n(D))$–hypercyclic functions f with $\tau(f) < d$.

(2) If $c > 0$ and $A \cap \overline{B}(0,c)$ is infinite, then there exists a dense linear submanifold of $H(\mathbf{C})$ consisting, except for zero, of $(\Phi_n(D))$–hypercyclic functions f with $\tau(f) \leq c$.

Proof. Let us try to apply Theorem 1.1. Fix $d > c$ and pick any $t \in (c,d)$. Take $X = X_t$ and $Y = H(\mathbf{C})$, where

$$X_t := \{f \in H(\mathbf{C}) : \, ||f||_t < \infty \text{ and } ||\sum_{j=k}^{\infty} a_j z^j||_t \to 0 \, (k \to \infty)\}$$

and

$$||f||_t := \sup_{k \in \mathbf{N}_0} \sup_{r>0} \sup_{|z|=r} (|\sum_{j=k}^{\infty} a_j z^j| \, e^{-tr})$$

whenever $f(z) = \sum_{j=0}^{\infty} a_j z^j$. Define $T_n : X \to Y$ by $T_n = \Phi_n(D)|_X$ $(n \in \mathbf{N})$. Note that all functions in X have exponential type less than d. Note also that every function $\exp(az)$ with $|a| < t$ is in X, because if a_j $(j \in \mathbf{N}_0)$ are its Taylor coefficients then $|\sum_{j=k}^{\infty} a_j z^j| \leq e^{|a|r}$ for all k and all z with $|z| = r$. As in [12], it is easy to see that $|| \cdot ||_t$ is a norm on X which makes X a Banach space with a topology which is stronger than that of uniform convergence on compacta, and that the set $X_0 := \{\text{polynomials}\}$ is dense in X. Hence each T_n is a continuous linear mapping from X into Y. If we fix a polynomial P then we get that the sequence $(T_n P)$ is eventually zero due to the condition that $m(\Phi_n) \to \infty$ $(n \to \infty)$. Therefore, trivially, $T_n P \to 0$ for every $P \in X_0$.

On the other hand, under the hypotheses of (1), there exists a set $A_1 \subset A$ with at least one finite accumulation point such that $|a| < t$ for all $a \in A_1$. From the existence of an accumulation point, a combination of Hahn-Banach Theorem, Riesz Theorem and Analytic Continuation Principle (like in, for instance, [11, Section 5]) yields that the set $Y_0 := \text{span}\,\{e^{az} : \, a \in A_1\}$ is dense in $H(\mathbf{C})$. By linearity, it is enough to show that, given $a \in A_1$, there is a sequence $(f_n) \subset X$ with $T_n f_n \to e^{az}$ $(n \to \infty)$ in $H(\mathbf{C})$ such that (f_{n_k}) converges in X for some strictly increasing sequence $(n_k) \subset \mathbf{N}$. Note that, with the notation of Theorem 1.1, $S_n e^{az}$ would be f_n. For this, we define

$$f_n(z) = \frac{e^{az}}{\Phi_n(a)} \quad (n \in \mathbf{N}).$$

Observe that each f_n belongs to X. Then $T_n f_n = \Phi_n(D)(\frac{e^{az}}{\Phi_n(a)}) = e^{az} \to e^{az}$ $(n \to \infty)$. Since $a \in A_1$, there exists $(n_k) \subset \mathbf{N}$ and $\alpha \in \mathbf{C}$ with $1/\Phi_{n_k}(a) \to \alpha$, hence $f_{n_k}(z) \to \alpha e^{az}$ as $k \to \infty$ in X. Consequently, Theorem 1.1 applies and we obtain that there is a residual (so dense) set in X consisting of $(\Phi_n(D))$–hypercyclic functions.

Under the hypotheses of (2), we would take this time

$$X = \bigcap_{t>c} X_t \quad \text{and} \quad Y = H(\mathbf{C}).$$

It is easy to see that X is a Fréchet space when it is endowed with the translation-invariant distance

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \frac{||f-g||_{c_n}}{1 + ||f-g||_{c_n}},$$

where (c_n) is any sequence strictly decreasing to c. We define again $T_n : X \to Y$ $(n \in \mathbf{N})$ by $T_n = \Phi_n(D)|_X$. Observe that the topology of X is stronger than the compact-open topology, and that the set $X_0 := \{\text{polynomials}\}$ is dense in X. This time $\exp(az) \in X$ whenever $|a| \le c$, and all functions in X have exponential type $\le c$. As before, the set $Y_0 := \{e^{az} : a \in A_1\}$ is dense in $H(\mathbf{C})$, where $A_1 := A \cap \overline{B}(0,c)$ this time. From here on, the proof runs through the same steps as part (1) and Theorem 1.1 can be again applied to produce a dense set of $(\Phi_n(D))$-hypercyclic functions in X.

Finally, observe that we have in fact obtained for each increasing sequence $(n_k) \subset \mathbf{N}$ the existence of a dense set in X of $(\Phi_{n_k}(D))$–hypercyclic functions, because the above argument can be applied to every subsequence (Φ_{n_k}). Since X and Y are metrizable separable topological vector spaces (the separability of X is a consequence of the fact that the set of polynomials is dense in X) we have that the hypotheses of Theorem 1.2 are fulfilled for (T_n). Hence there is a linear submanifold $M \subset X$ satisfying that every function $f \in M \setminus \{0\}$ is hypercyclic for $(\Phi_n(D))$; in addition, M is dense *in* X. But X is dense in $H(\mathbf{C})$ for the topology of this last space, because all polynomials are in X and the topology of X is stronger than that of local uniform convergence. Thus, M is also dense in $H(\mathbf{C})$, and the proof is finished.

Corollary 2.4 *Assume that Φ is a nonconstant entire function of finite exponential type with $\Phi(0) = 0$. We have:*

(1) Given $d > c(\Phi)$, there is a dense linear submanifold in $H(\mathbf{C})$ consisting, except for zero, of $\Phi(D)$–hypercyclic functions f with $\tau(f) < d$.

(2) If $|\Phi(z)| = 1$ on an infinite set of points of the circle of center at the origin and radius $c(\Phi)$, then there exists a dense linear submanifold in $H(\mathbf{C})$ consisting, except for zero, of $\Phi(D)$–hypercyclic functions f with $\tau(f) \le c(\Phi)$.

Proof. Note that $m(\Phi_n) = mn \to \infty$ $(n \to \infty)$, where $m > 0$ is the multiplicity of Φ for the zero at the origin and $\Phi_n = \Phi \cdots \Phi$ (n-fold). A simple application of the Maximum Modulus Principle shows that the set $\{z \in \mathbf{C} : |\Phi(z)| > 1\}$ $(\subset \{z \in \mathbf{C} : (1/\Phi_n(z))$ is bounded$\})$ has an accumulation point in $\overline{B}(0, c(\Phi))$. Hence part (1) of Theorem 2.3 applies and one obtains part (1) of this corollary. Part (2) is in turn derived from part (2) of Theorem 2.3, by considering the same sequence (Φ_n).

Nevertheless, it should be pointed out that the conclusion of part (1) of the latter corollary can be deduced from the result of Chan and Shapiro mentioned in Section 1, even *without* assuming $\Phi(0) = 0$ (hence the translations $\tau_a = \exp(aD)$ are included). Indeed, the simple substitution $f(z) \mapsto f(dz)$ $(d > 0)$ yields that if $\Phi(d\mathbf{D})$ intersects the unit disk (which is the same as $d > c(\Phi)$) then $\Phi(D)$ is hypercyclic on $E^2(e^{dz})$, so $\Phi(D)$ is hypercyclic on $H(\mathbf{C})$ and has a hypercyclic entire function with $\tau(f) \le d$. The existence of the dense linear submanifold is a consequence of the fact that $E^2(e^{dz})$ is dense in $H(\mathbf{C})$ by using a general property of hypercyclic operators, namely, if T is hypercyclic on a locally convex space X, then there is a dense T-invariant linear submanifold M of X such that each vector in $M \setminus \{0\}$ is hypercyclic, see [7]. As for the special case $\Phi(D) = D$, part (2) of Corollary 2.4 can be used to get a result containing MacLane's one in Section 1: There is a dense linear manifold in $H(\mathbf{C})$ consisting, except for zero, of D–hypercyclic functions f with $\tau(f) \le 1$ (so $\tau(f) = 1$ by Corollary 2.2 or [12]). Indeed, $A = \{|z| \ge 1\}$ in this case. Note that here $\Phi(\overline{B}(0,1))$ intersects the unit circle. With this hypothesis, Chan and Shapiro [8, p. 1447] proposed the question that whether $\Phi(D)$ is hypercyclic on $E^2(e^z)$. An affirmative answer to this would yield a corresponding affirmative answer for the problem that whether $\Phi(D)$ has a hypercyclic entire function of exponential type ≤ 1. For instance, as far as we know, even the case $\Phi(z) = \frac{1+z}{2}$ (see [8, p. 1447]) remains unsolved.

The next corollary is a joint consequence of Corollary 2.2 and of either part (1) of Corollary 2.4 or Chan-Shapiro's result.

Corollary 2.5 *If Φ is a nonconstant entire function with finite exponential type such that $|\Phi(0)| < 1$ and its Taylor coefficients at the origin are ≥ 0 then*

$$\inf\{\tau(f) : f \text{ is hypercyclic for } \Phi(D)\} = c(\Phi).$$

For instance, $\inf\{\tau(f) : \{\sum_{j=0}^n \binom{n}{j}(-1)^{n-j} f(z+j) : n \in \mathbf{N}\}$ is dense in $H(\mathbf{C})\} = \log 2$. Indeed, just apply the corollary on $\Phi(z) = \exp z - 1$ and take into account that $\exp(D)$ is the 1–translation operator.

To finish, we propose the following open question: For Φ as in Corollary

2.5, give the exact critical rate of growth for $\Phi(D)$-hypercyclic functions, as in [12].

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References

[1] M. P. Aldred and D. H. Armitage, *Harmonic analogues of G.R. MacLane's universal functions*, J. London Math. Soc. (2) **57** (1998), 148–156.

[2] M. P. Aldred and D. H. Armitage, *Harmonic analogues of G.R. MacLane's universal functions II*, J. Math. Anal. Appl. **220** (1998), 382–395.

[3] D. H. Armitage, *Dense vector spaces of universal harmonic functions*, Advances in multivariate approximation (Witten-Bommerholz, 1998), 33–42, Math. Res., 107, Wiley-VCH, Berlin, 1999.

[4] L. Bernal–González, *Hypercyclic sequences of differential and antidifferential operators*, J. Approx. Theory **96** (1999), 323–337.

[5] L. Bernal–González, *Densely hereditarily hypercyclic sequences and large hypercyclic manifolds*, Proc. Amer. Math. Soc. **127** (1999), 3279–3285.

[6] G. D. Birkhoff, *Démonstration d'un théorème élémentaire sur les fonctions entières*, C. R. Acad. Sci. Paris **189** (1929), 473-475.

[7] P. S. Bourdon, *Invariant manifolds of hypercyclic operators*, Proc. Amer. Math. Soc. **118** (1993), 845–847.

[8] K.C. Chan and J.H. Shapiro, *The cyclic behaviour of translation operators on Hilbert spaces of entire functions*, Indiana Univ. Math. J **40** (1991), 1421–1449.

[9] S. M. Duyos Ruiz, *On the existence of universal functions*, Soviet Math. Dokl. **27** (1983), 9–13.

[10] S. M. Duyos Ruiz, *Universal functions and the structure of the space of the entire functions*, Soviet Math. Dokl. **30** (1984), 713–716.

[11] G. Godefroy and J. H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. **98** (1991), 229–269.

[12] K.–G. Grosse–Erdmann, *On the universal functions of G. R. MacLane*, Complex Variables Theory Appl. **15** (1990), 193–196.

[13] K.–G. Grosse–Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. **36** (1999), 345–381.

[14] K.–G. Grosse–Erdmann, *Rate of growth of hypercyclic entire functions*, Indag. Math. **11** (2000), 561–571.

[15] G. Herzog, *Universelle Funktionen*, Diplomarbeit, Universität Karlsruhe, 1988.

[16] G. R. MacLane, *Sequences of derivatives and normal families*, J. Analyse Math. **2** (1952/53), 72-87.

[17] S. A. Shkarin, *On the growth of D–universal functions*, Moscow Univ. Math. Bull. **48** (1993), no. 6, 49-51.

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