On universal entire functions with zero-free derivatives

By

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Abstract. We prove in this note a generalization of a theorem due to G. Herzog on zero-free universal entire functions. Specifically, it is shown that, if a nonnegative integer q and a nonconstant entire function Φ of subexponential type are given, then there is a residual set in the class of entire functions with zero-free derivatives of orders q and $q+1$, such that every member of that set is universal with respect to $\Phi(D)$, where D is the differentiation operator.

1. Introduction and notation. We denote by $\mathbf{C}$ the complex plane, by $\mathbf{N}$ the set of positive integers and by $\mathbf{N_0}$ the set $\mathbf{N} \cup \{0\}$. If $r > 0$, $B(r)$ ($\overline{B}(r)$) is the euclidean open (closed, respectively) disk with center 0 and radius r. We agree that $B(+\infty) = \mathbf{C}$. $H(B(r))$ will stand, as usual, for the space of holomorphic functions in $B(r)$, endowed with the topology of uniform convergence on compact subsets. In $H(\mathbf{C})$, this topology is induced by the metric

$$d(f,g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{||f-g||_j}{1+||f-g||_j}, \tag{1}$$

where $||h||_r = \max_{\overline{B}(r)} |h|$ ($\forall r > 0$). It is well known that $H(B(r))$ is a separable Fréchet space, so it is a Polish space and also a Baire space (see, e.g., [14, pp. 213-214 and 238]). In a Baire space X, a subset is *residual* when it contains a dense G_δ-subset of X or, equivalently, when its complement is of first category. Such a subset is "very large" in X.

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We use a very general notion of universality, which can be found in [10], namely: Let X and Y be nonempty topological spaces and $\mathcal{L}$ be a family of continuous mappings from X into Y. Then an element $x \in X$ is called *universal* with respect to $\mathcal{L}$ if the set $\{Lx : L \in \mathcal{L}\}$ is dense in Y. As in [13], we denote the set of all universal elements by $U(\mathcal{L})$. Universal elements are usually called *hypercyclic* in the case that $X = Y$ is a topological vector space and $\mathcal{L}$ is the sequence of iterates $\{L^n\}_1^\infty$ of a single linear continuous operator L on X (see, for instance, [9]). S. Rolewicz [17] was the first to give an example of a hypercyclic element in the Banach/Hilbert setting.

In 1952, G. R. MacLane [15] stated that there exist entire functions f such that the set of derivatives $\{f^{(n)} : n \in \mathbf{N}\}$ is dense in $H(\mathbf{C})$ or, equivalently, $f \in U(\mathcal{L})$ for $\mathcal{L} = \{D^n : n \in \mathbf{N}\}$, where D is the differentiation operator on $H(\mathbf{C})$, that is, $Df = f'$. The result is also proved in [3] (see also [4]). S. M. Duyos Ruiz [7] has shown that, in fact, there is a residual set of such functions. Furthermore, R. M. Gethner and J. H. Shapiro [8] and K. G. Große-Erdmann [10, Satz 2.2.8] have derived the same result for every simply connected domain. For additional results about the topic, the reader is referred to [2], [11] and [12] and many others in their references. For instance, Große-Erdmann [11] provides a sharp result on growth of D-universal functions.

Returning now to the general case of a family $\mathcal{L}$ of continuous mappings from X into Y, G. Herzog [13] proposed recently the following interesting question: Which additional properties of elements of X are compatible with universality? If $U(\mathcal{L})$ is residual and $A \subset X$ is a G_δ-subset, he proves that under certain conditions on A and $\mathcal{L}$ (see Theorem 1 below) the set $A \cap U(\mathcal{L})$ is residual in A. Then, by using this theorem, he derives the existence of zero-free universal entire functions (for D) having even a zero-free first derivative. The basic tools employed by Herzog are the theory of universality developed by Große-Erdmann [10, specially Satz 1.2.2], Alexandroff's theorem on completeness of G_δ-subsets (see, e.g., [16, pp. 47-48]) and elementary results from Complex Analysis. Herzog [13, Section 3] himself points out that there is no universal entire function f such that $f \cdot f' \cdot f''$ is zero-free, since $\{f \in H(\mathbf{C}) : f \cdot f' \cdot f'' \text{ is zero-free}\} = \{e^{\alpha z+\beta} : \alpha, \beta \in \mathbf{C}, \ \alpha \neq 0\}$ (see [6, p. 433] and [19]).

If $q \in \mathbf{N_0}$, let us denote $A(q) = \{f \in H(\mathbf{C}) : f^{(q)}(z) f^{(q+1)}(z) \neq 0 \ \forall z \in \mathbf{C}\}$. Since $\exp \in \bigcap_{q \in \mathbf{N_0}} A(q)$, every $A(q)$ is nonempty. Our aim in this note is to furnish a strong generalization of Herzog's theorem on zero-free derivatives. Specifically, we show in Theorem 5 that, if a nonnegative integer q and a nonconstant entire function Φ of subexponential type are given, then there is a residual subset in $A(q)$ satisfying that every member of such a subset is universal with respect to the operator $\Phi(D)$, where D is the differentiation operator. In terms of growth orden and type, our result is best possible.

2. Preliminary results. The technique for proving Theorem 5 will be very similar to that in [13], but we need an additional elementary result on antiderivatives together with the "good behaviour" of certain related non-linear operators relatively to convergence (this is Theorem 2; its proof is easy and left to the reader), a strong assertion due to Godefroy and Shapiro [9] (Theorem 3) and, finally, a result which asserts the continuity of subexponential differential operators on every space $H(B(r))$ $(r > 0)$ (Theorem 4). Moreover, it is also employed the above mentioned Theorem 1, which is exactly Theorem 2.1 of [13].

Theorem 1. *Assume that X is a Polish space and Y is a separable metrizable space. Let d_X, d_Y be metrics inducing the topologies of X, Y, respectively. Let $\{A_k : k \in \mathbf{N}\}$ be a sequence of open subsets of X with $A \equiv \bigcap_{k=1}^{\infty} A_k \neq \emptyset$. Let $\mathcal{L} = \{L_n : n \in \mathbf{N}\}$ be a sequence of continuous mappings from X into Y, with $U(\mathcal{L})$ residual in X. Denote $\mathcal{L}|_A = \{L_n|_A : n \in \mathbf{N}\}$, where $L_n|_A$ is the restriction of L_n to A. If*

$$\lim_{k\to\infty} \sup_{n\in\mathbf{N}} \inf_{z\in A} (d_X(a_k, z) + d_Y(L_n a_k, L_n z)) = 0$$

for every sequence $\{a_k\}_1^\infty$ $(a_k \in A_k,\ k \in \mathbf{N})$, *then* $U(\mathcal{L}|_A)$ *is residual in* A.

Theorem 2. *a) If* $q \in \mathbf{N_0}$ *and* $f \in H(B(r))$ *then* $f^{(q)} f^{(q+1)}$ *is zero-free if and only if there exists* $g \in H(B(r))$ *such that*

$$f(z) = \begin{cases} f(0)\exp(\int_0^z \exp(g(t))\,dt) & \text{if } q = 0 \\ \sum_{\nu=0}^{q-1} \frac{f^{(\nu)}(0)}{\nu!} z^\nu + \frac{f^{(q)}(0)}{(q-1)!} \int_0^z (z-t)^{q-1} \exp(\int_0^t \exp(g(u))\,du)\,dt & \text{if } q \geq 1 \end{cases}$$

for all $z \in B(r)$.
b) Assume that $q \in \mathbf{N_0}$ *and* $T : f \in H(B(r)) \mapsto Tf \in H(B(r))$ *is the mapping given by*

$$Tf(z) = \begin{cases} \exp(\int_0^z \exp(f(t))\,dt) & \text{if } q = 0 \\ \int_0^z (z-t)^{q-1} \exp(\int_0^t \exp(f(u))\,du)\,dt & \text{if } q \geq 1. \end{cases}$$

Then T *is a well-defined continuous operator on* $H(B(r))$.

Before stating the next two theorems, we recall that an entire function $\Phi(z) = \sum_{j=0}^\infty a_j z^j$ is said to be *of exponential type* whenever there exist positive constants A and B such that $|\Phi(z)| \leq Ae^{B|z|}$ for all $z \in \mathbf{C}$. Cauchy's inequalities show that this happens if and only if $\limsup_{j\to\infty} (j!|a_j|)^{1/j}$ is finite (cf. [18, Chap. VII]). It is shown in [9, Section 5] that if Φ is of exponential type and $L = \Phi(D)$ (that is, $L = \sum_{j=0}^\infty a_j D^j$, where $D^0 = I =$ the identity operator), then L is a well-defined continuous linear operator on $H(\mathbf{C})$.

By analogy, we adopt the next terminology. We say that an entire function $\Phi(z) = \sum_{j=0}^\infty a_j z^j$ is *of subexponential type* whenever the following property holds: Given $\varepsilon > 0$, there is a positive constant $A = A(\varepsilon)$ such that

$$|\Phi(z)| \leq Ae^{\varepsilon |z|} \quad \forall z \in \mathbf{C},$$

that is, Φ is either of growth order less than one or of growth order one and minimal type. Every entire function of subexponential type is trivially of exponential type. As before, Cauchy's inequalities show that Φ is of subexponential type if and only if $\lim_{j\to\infty} (j!|a_j|)^{1/j} = 0$ (cf., e.g., [5, 2.2.9-11]).

Theorem 3. *Suppose that L is the continuous linear operator on $H(\mathbf{C})$ given by $L = \Phi(D)$, where Φ is a nonconstant entire function of exponential type. Then there is a dense, invariant submanifold of $H(\mathbf{C})$ each of whose non-zero elements is hypercyclic for L.*

It should be pointed out here that a continuous linear operator L on $H(\mathbf{C})$ is of the form $L = \Phi(D)$, where Φ is an entire function of exponential type, if and only if L commutes with each of the translation operators τ_a $(a \in \mathbf{C})$, where $\tau_a f(z) = f(z+a)$ $(f \in H(\mathbf{C}),\ z \in \mathbf{C})$ (see [9, Theorem 5.1, Proposition 5.2] for the proof of Theorem 3 and this note; we just use de case $\mathbf{C}^N = \mathbf{C}$ of [9, Section 5]). Thus the operators L on $H(\mathbf{C})$ commuting with translations are a special class of "infinite order" linear differential operators with constant coefficients.

Under the hypothesis of Theorem 3, $U(\mathcal{L})$ is not empty for $\mathcal{L} = \{L^n : n \in \mathbf{N}\}$. Then $U(\mathcal{L})$ is residual in $H(\mathbf{C})$ (see [8, Proposition 2.1]).

Theorem 4. *Let $\Phi(z) = \sum_{j=0}^{\infty} a_j z^j$ be an entire function of subexponential type and $L = \Phi(D)$. Then L is a well-defined continuous linear operator on $H(B(r))$.*

P r o o f. Fix $t \in (0, r)$ and choose any $s \in (t, r)$. Let $f \in H(B(r))$. Cauchy's inequalities guarantee that $||D^j f||_t \leq \frac{j!\,||f||_s}{(s-t)^j}$ for every $j \geq 0$. Let $\varepsilon = \frac{s-t}{2}$. By hypothesis, there is a positive constant A such that $|a_j| \leq A \cdot \frac{\varepsilon^j}{j!}$ for every $j \geq 0$. Then we have that $\sum_{j=0}^{\infty} ||a_j D^j f||_t = \sum_{j=0}^{\infty} |a_j| \cdot ||D^j f||_t \leq \sum_{j=0}^{\infty} A \cdot \frac{\varepsilon^j}{j!} \cdot \frac{j!\,||f||_s}{(s-t)^j} = A||f||_s \sum_{j=0}^{\infty} (1/2)^j = 2A||f||_s < +\infty$. Therefore $\sum_{j=0}^{\infty} a_j D^j f$ converges uniformly on every closed disk $\overline{B}(t)$ $(0 < t < r)$ and L defines a mapping from $H(B(r))$ into itself. The linearity is trivial and, since $||Lf||_t \leq 2A||f||_s$, we have also obtained that L is continuous on $H(B(r))$. ////

3. The main result. We are now ready to state our theorem on universality. Herzog's result is the special case $q = 0$, $L = D$.

Theorem 5. *Assume that* L *is the continuous linear operator on* $H(\mathbf{C})$ *given by* $L = \Phi(D)$, *where* Φ *is a nonconstant entire function of subexponential type. Fix* $q \in \mathbf{N_0}$ *and set* $A = A(q)$, $\mathcal{L} = \{L^n : n \in \mathbf{N}\}$. *Then the set* $U(\mathcal{L}|_A)$ *is residual in* A.

P r o o f. Firstly, note that $A = \bigcap_{k=1}^{\infty} A_k$ where $A_k = \{f \in H(\mathbf{C}) : \min_{\overline{B}(k)} |f^{(q)} \cdot f^{(q+1)}| > 0\}$. Put $X = Y = H(\mathbf{C})$. It is evident that every A_k is open in $H(\mathbf{C})$, so A is a nonempty G_δ-subset of X. From Theorem 3, $U(\mathcal{L})$ is residual in X. In order to apply Theorem 1, we should demonstrate that, for every fixed sequence $\{f_k\}_1^{\infty}$ ($f_k \in A_k$, $k \in \mathbf{N}$), it holds that

$$\lim_{k\to\infty} \sup_{n\in\mathbf{N}} \inf_{h\in A} (d(f_k, h) + d(L^n f_k, L^n h)) = 0, \tag{2}$$

d being defined by (1). Fix $k \in \mathbf{N}$ and a function $f \in A_k$. There is an $\varepsilon > 0$ such that $f^{(q)}(z)f^{(q+1)}(z) \neq 0$ for all $z \in B(k+2\varepsilon)$. From Theorem 2, there is a function $g \in H(B(k+2\varepsilon))$ such that f is given on $B(k+2\varepsilon)$ by the formula given in that theorem. There exists a sequence of polynomials $\{P_m\}_1^{\infty}$ satisfying $||P_m - g||_{k+\varepsilon} \to 0$ ($m \to \infty$). With the notation of Theorem 2, we have $f(z) = f(0) \cdot Tg(z)$ if $q = 0$ and $f(z) = \sum_{\nu=0}^{q-1} \frac{f^{(\nu)}(0)}{\nu!} z^\nu + \frac{f^{(q)}(0)}{(q-1)!} \cdot Tg(z)$ if $q \geq 1$ for all $z \in B(k+2\varepsilon)$. Let us define

$$h_m(z) = \begin{cases} f(0) \cdot TP_m(z) & \text{if } q = 0 \\ \sum_{\nu=0}^{q-1} \frac{f^{(\nu)}(0)}{\nu!} z^\nu + \frac{f^{(q)}(0)}{(q-1)!} \cdot TP_m(z) & \text{if } q \geq 1 \end{cases}$$

for all $m \in \mathbf{N}$ and for all $z \in \mathbf{C}$. Note that each $h_m \in A$. From Theorem 2, $TP_m \to Tg$ ($m \to \infty$) uniformly on compact subsets of $B(k+\varepsilon)$, so $h_m \to f$ ($m \to \infty$) in the topology of $H(B(k+\varepsilon))$. By Theorem 4, $L^n h_m \to L^n f$ ($m \to \infty$) uniformly on compact subsets of $B(k+\varepsilon)$, for every $n \in \mathbf{N}$. In particular, we obtain that $\lim_{m\to\infty} ||L^n h_m - L^n f||_k = 0$ $\forall n \in \mathbf{N}$. Hence $\lim_{n\to\infty}(||h_m - f||_j + ||L^n h_m - L^n f||_j) = 0$ for every $n \in \mathbf{N}$ and every $j \in \{1, 2, ..., k\}$. Given $\delta > 0$, a positive integer $m = m(\delta, n, k)$ can be found in such a way that $||h_m - f||_j + ||L^n h_m - L^n f||_j < \delta$ for all $j \in \{1, ..., k\}$, so $d(f, h_m) + d(L^n f, L^n h_m) < \sum_{j=1}^{k} \frac{\delta}{2^j} + \sum_{j=k+1}^{\infty} \frac{1}{2^j} + \sum_{j=k+1}^{\infty} \frac{1}{2^j} = \delta + 2^{1-k}$. Then $\inf_{h\in A}(d(f,h) + d(L^n f, L^n h)) < \delta + 2^{1-k}$ $\forall \delta > 0$ and

$\forall n \in \mathbf{N}$. Therefore we get

$$\sup_{n\in\mathbf{N}} \inf_{h\in A} (d(f_k, h) + d(L^n f_k, L^n h)) \leq 2^{1-k} \to 0 \quad (k \to \infty)$$

if $f_k \in A_k$ $(k \in \mathbf{N})$. Consequently, (2) is fulfilled and the proof is complete. ////

Next, we show the optimality of our theorem. The result cannot be extended to all entire functions Φ of exponential type. In fact, to every type $\tau > 0$ there exists an entire function Φ of order 1 and type τ such that the result fails for Φ. One need only consider $\Phi(\zeta) = e^{\tau\zeta}$. Then $\Phi(D)f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}\tau^n = f(z+\tau)$, so that any $\Phi(D)$-hypercyclic element is universal with respect to translates. But it follows from a classical theorem of Hurwitz (see, for instance, [1, p. 178]) that the translates of a zero-free entire function cannot approximate a non-constant entire function with zeros. Hence there cannot exist a $\Phi(D)$-hypercyclic function.

To finish, we point out that if $q \in \mathbf{N_0}$, $\{\Phi_k : k \in \mathbf{N}\}$ is a sequence of non-constant entire functions of subexponential type and $L_k = \Phi_k(D)$ $(k \in \mathbf{N})$, then there is an entire function f with zero-free derivatives of orders q and $q+1$ which is universal with respect to *every* family $\{L_k^n : n \in \mathbf{N}\}$ $(k \in \mathbf{N})$. This is a trivial consequence of the fact that in every Baire space the countable intersection of residual sets is residual.

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