On $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices

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ABSTRACT

A characterization of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices is described, depending on the notions of *distributions*, *ingredients* and *recipes*. In particular, these notions lead to the establishment of some bounds on the number and distribution of 2-coboundaries over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ to use and the way in which they have to be combined in order to obtain a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrix. Exhaustive searches have been performed, so that the table in p. 132 in [4] is corrected and completed. Furthermore, we identify four different operations on the set of coboundaries defining $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices, which preserve orthogonality. We split the set of Hadamard matrices into disjoint orbits, define representatives for them and take advantage of this fact to compute them in an easier way than the usual purely exhaustive way, in terms of *diagrams*. Let $\mathcal{H}$ be the set of cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ having a symmetric diagram. We also prove that the set of Williamson type matrices is a subset of $\mathcal{H}$ of size $\frac{|\mathcal{H}|}{t}$.

1. INTRODUCTION

Hadamard matrices are $n \times n$ square matrices H with entries in $\{1,-1\}$ such that every pair of rows (respectively, columns) are orthogonal, that is, $HH^T = nI_n$.

Due to this nice combinatorial property, Hadamard matrices have many applications in a wide variety of fields, such as Signal Processing, Coding Theory and Cryptography (see [6] for details). Consequently, there is a real interest in knowing enough Hadamard matrices for practical use.

It is a straightforward exercise to prove that the order of a Hadamard matrix has to be 1, 2 or a multiple of 4 (as soon as three or more rows have to be simultaneously orthogonal one to each other). Unfortunately, the Hadamard Conjecture about the existence of these matrices for every order $4t$ remains unproved since the XIXth Century.

Nowadays, there are three orders less than 1000 for which no Hadamard matrix is known: $668 = 4 \cdot 167$, $716 = 4 \cdot 179$, and $892 = 4 \cdot 223$. Furthermore, there are 9 orders in the range $[1000, 2000]$ for which no Hadamard matrix is known (see [6, 5] for details).

One of the most promising techniques for constructing Hadamard matrices is the cocyclic approach (see [7, 8, 6]). A cocyclic matrix M_f over a group G is a matrix $M_f = (f(g,h))$, for f a 2-cocycle over G, that is, a function $f: G \times G \to \{1,-1\}$ such that for every $a,b,c \in G$, $f(a,b) \cdot f(ab,c) \cdot f(a,bc) \cdot f(b,c) = 1$.

Actually, many well known families of Hadamard matrices, such as Sylvester's, Paley's, Williamson's and Ito's, have shown to be cocyclic over appropriate groups (see [6] for details). This has provided inspiration for the Cocyclic Hadamard Conjecture, which states that cocyclic Hadamard matrices exist for every order $4t$.

In this paper we are interested in characterizing cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, which include the family of symmetric Williamson type Hadamard matrices.

Following the indications of [1], we will describe bounds on the number of 2-coboundaries over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ to be combined, as well as their distribution (in terms of what we call *ingredients* and *recipes*), in order to construct a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrix.

This information will allow us to design an exhaustive search for $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices for $3 \leq t \leq 13$, so that the table in p. 132 in [4] is corrected and completed.

Next, we will introduce what we call *diagrams*, a visual representation of the coboundaries which define a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix and we will study four different operations on the set of coboundary matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$: complements, rotations, swappings and dilatations. In particular, these operations extend to operations over $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices, which will be proved to preserve orthogonality. These operations partition the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices into disjoint orbits, which can be easily computed once one element is known. Among all the elements of an orbit, a representative can be chosen, in a standard way that will be made precise.

Finally, by applying these ideas to the task of searching for cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, we have been able to extend the table in p. 132 in [4] for values of t in the range $3 \leq t \leq 23$, and to explain the fact that the set of symmetric Williamson type Hadamard matrices obtained from symmetric diagrams is in proportion $\frac{1}{t}$ with respect to the full set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices.

We organize the paper as follows. Section 2 is dedicated to describe all about $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices. In Section 3 we introduce the notions of *distribution*, *ingredients* and *recipes*, in terms of which we find some upper and lower bounds on the number of 2-coboundaries over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ which have to be combined in order to get $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices, as well as the way in which they have to be distributed, and the results obtained. Section 4 is dedicated to the description of diagrams, after a discussion about the convenience of using the whole set of coboundaries instead of the basis, in order to represent $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices. Section 5 defines four operations on $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices which preserve orthogonality, and which have a nice interpretation in terms of diagrams. These operations split the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices into disjoint orbits, which can be generated from any of their elements (for instance by its representative). In Section 6 we focus on the case of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices obtained from symmetric diagrams, which in turn permit counting the number of Williamson type Hadamard matrices. We also include some final remarks and further work.

2. $\mathbb{Z}_t \times \mathbb{Z}_2^2$-COCYCLIC HADAMARD MATRICES

Consider the group $G = \mathbb{Z}_t \times \mathbb{Z}_2^2$, a basis $\mathcal{B} = \{\partial_2, \ldots, \partial_{4t-2}, \beta_1, \beta_2, \gamma\}$ for 2-cocycles over G is described in [2], and consists of $4t-3$ coboundaries ∂_k, two cocycles β_i coming from inflation and one cocycle γ coming from transgression.

It has been observed that cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ mostly use all the three representative cocycles β_1, β_2 and γ simultaneously (see [4] for details). We will assume that every cocyclic matrix M is obtained as a product $M = M_{\partial_{i_1}} \ldots M_{\partial_{i_w}} \cdot R$, for $2 \le i_1 < \ldots < i_w \le 4t-2$, where $R = M_{\beta_1} \cdot M_{\beta_2} \cdot M_\gamma$.

Here M_{∂_i} refers to the *generalized* coboundary matrix associated to the i^{th}-element in G, with the i^{th}-row and the i^{th}-column negated, as introduced in [2].

In particular, there are three coboundary matrices which are not in $\mathcal{B}$: M_{∂_1}, $M_{\partial_{4t-1}}$ and $M_{\partial_{4t}}$. Consequently, every $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix M_f using R may be expressed as a pointwise product of matrices in $\{M_{\partial_1}, \ldots, M_{\partial_{4t}}\}$ in 8 different ways, just one of which does not use any of M_{∂_1}, $M_{\partial_{4t-1}}$ and $M_{\partial_{4t}}$ (and gives precisely the expression of M_f as a linear combination of elements in $\mathcal{B}$). Actually, suppose that $M_f = R \cdot \prod_{k=1}^{4} \prod_{i_j \in J_k} M_{\partial_{i_j}}$, where $J_k \subset \{1, \ldots, 4t\}$ is a subset of indexes which are congruent to k modulo 4. Then M_f may be expressed as the pointwise product of the coboundary matrices of indexes belonging to any of the following 8 subsets: (J_1, J_2, J_3, J_4), $(\bar{J}_1, \bar{J}_2, J_3, J_4)$, $(J_1, \bar{J}_2, \bar{J}_3, J_4)$, $(J_1, \bar{J}_2, J_3, \bar{J}_4)$, $(\bar{J}_1, J_2, \bar{J}_3, J_4)$, $(\bar{J}_1, J_2, J_3, \bar{J}_4)$, $(J_1, J_2, \bar{J}_3, \bar{J}_4)$, $(\bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{J}_4)$, where $\bar{J}_k = \{k+4i : 0 \le i \le t-1\} \setminus J_k$ denotes the complementary subset of J_k.

It is known that a cocyclic matrix is Hadamard if and only if the summation of each row but the first is zero (this is the cocyclic Hadamard test, see [8, 4] for instance). Furthermore, as proved in [1], a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix is Hadamard if and only if the summation of each of the rows from 5 to $2t+2$ is 0, and an equivalent characterization of the cocyclic Hadamard test may be described in terms of *n-paths* (c_n) and *n-intersections* (I_n) (the interested reader is referred to [1] for a precise definition about paths and intersections). In particular, the following result derives straightforwardly from the work in [1].

Proposition 2.1. *For $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices:*

1. *the summation of a row $n \equiv 1$ mod 4 is zero if and only if*

$$c_n = t, \tag{1}$$

2. *the summation of a row $n \equiv 0, 2, 3$ mod 4 is zero if and only if*

$$c_n = I_n. \tag{2}$$

In the following section we analyze the number c_n of n-paths of a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix. Focusing in rows $n \equiv 1$ *mod* 4, we will obtain upper and lower bounds on the number of coboundaries to combine in order to get a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrix. Furthermore, we will characterize the distribution of these coboundaries in terms of *ingredients* and *recipes*.

3. DISTRIBUTIONS, INGREDIENTS AND RECIPES

Firstly, we analyze the way in which n-paths are generated on $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices, depending on the value of n modulo 4.

Lemma 3.1. *Characterization of n-paths of coboundaries on $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices:*

1. *If $n \equiv 1$ mod 4, $M_{\partial_{i-n+1}}$ forms an n-path with M_{∂_i}.*
2. *If $n \equiv 2$ mod 4, $M_{\partial_{i-n+2-(-1)^i}}$ forms an n-path with M_{∂_i}.*
3. *If $n \equiv 3$ mod 4, $M_{\partial_{i-n+3-2(-1)^{\lceil \frac{i \ mod \ 4}{2} \rceil}}}$ forms an n-path with M_{∂_i}.*
4. *If $n \equiv 0$ mod 4, $M_{\partial_{i-n+4+(-1)^i(1-4(1-\lfloor \frac{i \ mod \ 4}{2} \rfloor))}}$ forms an n-path with M_{∂_i}.*

Proof.

This may be checked by direct inspection.

□

Now we focus our attention on rows $n \equiv 1$ *mod* 4. From Lemma 3.1, it is clear that n-paths consists of groups of coboundaries in the same coset modulo 4.

Lemma 3.2. *Given $1 \leq i \neq j \leq 4t$, $i \equiv j$ mod 4, there exists one and only one row n, $5 \leq n \leq 2t+2$, $n \equiv 1$ mod 4, such that M_{∂_i} and M_{∂_j} form an n-path.*

Corollary 3.3. *Along the $\frac{t-1}{2}$ rows $n \equiv 1$ mod 4, $5 \leq n \leq 2t+2$, any k coboundaries $\partial_{i_1}, \ldots, \partial_{i_k}$ in the same coset modulo 4 give rise to a total amount of $\frac{k(t-k)}{2}$ paths.*

Proof.

Along the $\frac{t-1}{2}$ rows $n \equiv 1$ *mod* 4, $5 \leq n \leq 2t+2$, any k coboundaries $\partial_{i_1}, \ldots, \partial_{i_k}$ in the same coset modulo 4 might give rise to $k\frac{t-1}{2}$ paths. Actually, this is not the case, since we know from Lemma 3.2 that every pair of such coboundaries forms a path at one and only one of these rows n. Thus the total amount of paths has to be reduced in the number of pairs in which the k coboundaries may be grouped. This gives $k\frac{t-1}{2} - \frac{k(k-1)}{2} = k\frac{t-k}{2}$, as claimed.

□

Example: For $t = 5$, the set of $k = 2$ coboundaries $\{\partial_{14}, \partial_{18}\}$ defines exactly $2\frac{5-2}{2} = 3$ paths in rows congruent to 1, namely, 1 path at row 5, $\{(\partial_{18}, \partial_{14})\}$ and 2 paths at row 9, $\{(\partial_{14}), (\partial_{18})\}$.

□

Table A in Appendix [3], shows the total amount $\frac{k(t-k)}{2}$ of paths produced by k coboundaries in the same coset modulo 4, for odd values of t.

Proposition 3.4. *This table has many valuable combinatorial properties:*

1. *The table is symmetric.*
2. *The numbers in the central columns are triangular numbers, of the type* $\frac{n(n+1)}{2}$.
3. *Subtracting from a number in the central columns any of the numbers of the same row, gives as result a triangular number as well.*
4. *Reciprocally, subtracting from a number in the central columns any triangular number gives as result a number of the same row.*

Proof.

1. The table is symmetric, since k coboundaries give rise to $\frac{k(t-k)}{2}$ paths, exactly the same amount of paths produced by $t-k$ coboundaries, $\frac{(t-k)k}{2}$.
2. The numbers in the central columns are triangular numbers. Actually, $\frac{t-1}{2}$ coboundaries give rise to $\frac{n(n+1)}{2}$ paths, where $t = 2n+1$.
3. Subtracting from a number in the central columns any of the numbers of the same row, gives as result a triangular number as well. Indeed, subtracting $\frac{k(t-k)}{2}$ paths from $\frac{t^2-1}{8}$ gives $\frac{t^2-1-4kt+4k^2}{8} = \frac{(t-2k)^2-1}{8} = \frac{\frac{t-2k-1}{2}\cdot\frac{t-2k+1}{2}}{2}$.
4. The argument above fits here as well.

□

Attending to the condition (1), in order to get a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrix, a necessary (but not sufficient!) condition is to select k_i coboundaries in the coset i *mod* 4, such that there is a total amount of $t\frac{t-1}{2}$ paths along the $\frac{t-1}{2}$ rows $n \equiv 1$ *mod* 4, $5 \leq n \leq 2t+2$. This motivates the following definition.

Definition 3.5. *A* distribution *is a tuple* $(\frac{k_0(t-k_0)}{2}, \frac{k_1(t-k_1)}{2}, \frac{k_2(t-k_2)}{2}, \frac{k_3(t-k_3)}{2})$, $0 \leq k_j \leq k_i \leq \frac{t-1}{2}$ *for* $j \geq i$, *such that*

$$\sum_{i=0}^{3} \frac{k_i(t-k_i)}{2} = \frac{t(t-1)}{2}. \tag{3}$$

Proposition 3.6. *For any odd t, there always exists at least one distribution $(\frac{k_0(t-k_0)}{2}, \frac{k_1(t-k_1)}{2}, \frac{k_2(t-k_2)}{2}, \frac{k_3(t-k_3)}{2})$. Furthermore, there are as many different distributions as decompositions of $\frac{t-1}{2}$ as the summation of four triangular numbers.*

Proof.
The maximum possible number of paths is $4\frac{t^2-1}{8} = \frac{t^2-1}{2}$, when $k_i = \frac{t-1}{2}, \forall i$, so that the relation (3) fails to hold by a difference $m = \frac{t^2-1}{2} - \frac{t(t-1)}{2}$.

In 1796 Gauss proved that any positive integer can be decomposed as the summation of three (not necessarily different) triangular numbers, some of which may be eventually zero. Consequently, there exist three triangular numbers $0 \leq t_1, t_2, t_3 \leq \frac{t^2-1}{8}$ such that $m = t_1 + t_2 + t_3$.

Thus $\frac{t^2-1}{2} = 4\frac{t^2-1}{8} - m = \frac{t^2-1}{8} + (\frac{t^2-1}{8} - t_1) + (\frac{t^2-1}{8} - t_2) + (\frac{t^2-1}{8} - t_3)$. Taking into account Proposition 3.4, there exist integers $0 \leq k_3 \leq k_2 \leq k_1 \leq \frac{t-1}{2}$ such that $(\frac{t^2-1}{8} - t_i) = \frac{k_i(t-k_i)}{2}$, and therefore $(\frac{t^2-1}{8}, \frac{k_1(t-k_1)}{2}, \frac{k_2(t-k_2)}{2}, \frac{k_3(t-k_3)}{2})$ is a distribution, in the sense of Definition 3.5.

The second part is a straightforward consequence.

□

This proposition provides a method for finding the set of distributions for a given t, in terms of decompositions of $\frac{t-1}{2}$ as the summation of four triangular numbers $0 \leq t_0 \leq t_1 \leq t_2 \leq t_3 \leq \frac{t^2-1}{8}$.

Proposition 3.7. *Let k be a positive integer. Then:*

1. *k is a triangular number if and only if $\frac{-1+\sqrt{1+8k}}{2}$ is an integer.*
2. *The greatest triangular number less or equal to k is t_n, for $n = \lfloor \frac{-1+\sqrt{1+8k}}{2} \rfloor$.*
3. *If k is decomposed as the summation of m triangular numbers t_{i_j}, $1 \leq j \leq m$, then $\max_j\{t_{i_j}\} \geq t_n$, for $n = \lceil \frac{-1+\sqrt{1+8\frac{k}{m}}}{2} \rceil$.*

Proof.
It suffices to notice that k is a triangular number if and only if there exists an integer n such that $t_n = n\frac{n+1}{2} = k$. Equivalently, if and only if the equation $\frac{n^2}{2} + \frac{n}{2} - k$ has a positive integer solution (which, a fortiori, is $\frac{-1+\sqrt{1+8k}}{2}$).

□

Proposition 3.7 leads straightforwardly to an algorithm for constructing the full set of distributions for a given t (see Algorithm 1 in Appendix [3]).

Table B in Appendix [3], shows the complete set of distributions obtained from Algorithm 1, for $3 \leq t \leq 25$.

Notice that the knowledge of the full set of distributions implies the knowledge about the number of coboundaries which have to be used in order to get a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrix, since each summand $\frac{k_i(t-k_i)}{2}$ is in one to one correspondence to the values k_i and $t - k_i$ (see Table A). In spite of this fact, we may bound the number of coboundaries to be combined a bit further.

Lemma 3.8. *For any decomposition of n into k summands, say $n = n_1 + \ldots + n_k$ We have $\sum_{i=1}^k n_i^2 \geq \frac{1}{k}\left(\sum_{i=1}^k n_i\right)^2$. In fact, one could check that*

$$\sum_{i=1}^{k} n_i^2 - \frac{1}{k}\left(\sum_{i=1}^{k} n_i\right)^2 = \frac{1}{k}\sum_{1\leq j<i\leq k}(n_i - n_j)^2 \geq 0. \quad (4)$$

Proposition 3.9. *Let $(\frac{k_0(t-k_0)}{2}, \frac{k_1(t-k_1)}{2}, \frac{k_2(t-k_2)}{2}, \frac{k_3(t-k_3)}{2})$ be a distribution. Call $n = k_0 + k_1 + k_2 + k_3$. Then*

1.

$$\lceil \frac{t-\sqrt{4t-3}}{2} \rceil \leq k_3 \leq \lfloor \frac{t+\sqrt{4t-3}}{2} \rfloor. \quad (5)$$

2.

$$\lceil 2(t-\sqrt{t}) \rceil \leq n \leq \lfloor 2(t+\sqrt{t}) \rfloor. \quad (6)$$

Proof.
Let (k_0, k_1, k_2, k_3) generate a distribution $(\frac{k_0(t-k_0)}{2}, \frac{k_1(t-k_1)}{2}, \frac{k_2(t-k_2)}{2}, \frac{k_3(t-k_3)}{2})$.
On one hand, condition (3), gives $4k_3^2 - 4k_3t + t^2 - 4t + 3 \leq 0$ which proves (5).

On the other hand, simplifying (3), we get $t\sum_{i=0}^{3} k_i - t^2 + t = \sum_{i=0}^{3} k_i^2$. Now, by Lemma 3.8

$$\sum_{i=0}^{3} k_i^2 \geq \sum_{i=0}^{3} (\frac{n}{4})^2, \quad (7)$$

and so $tn - t^2 + t \geq \dfrac{n^2}{4}$, obtaining (6).

□

Remark 1. Condition (6) may be tightened, depending on the coset of $n = k_0 + k_1 + k_2 + k_3$ modulo 4, substituting the lower bound in (7) by the most homogeneously distributed partition of n into four parts.

Remark 2. The bounds in Proposition 3.9 are very tight, as has been checked experimentally. The first gap occurs for $t = 71$, and consists in just one coboundary.

Once we know that a distribution is available for a given value of t, the next step is looking for appropriate subsets of n_i coboundaries in the cosets $i \ mod \ 4$ in $\mathcal{B}$ such that the amount of n-paths along rows $n \equiv 1 \ mod \ 4$, $5 \leq n \leq 2t-1$ fits that distribution.

Example: For $t = 5$, the distribution $(3, 3, 2, 2)$ corresponds to a repartition of $(2, 2, 1, 1)$ coboundaries in each of the cosets modulo 4 (although we can consider 3 instead of 2 and 4 instead of 1 coboundaries in each case). Now, for each subset of k_i coboundaries, we can compute the number of n-paths defined along the rows congruent to 1 (this can be computed in only one coset, they are disjoint).

□

Definition 3.10. *An* ingredient *produced by a subset of k coboundaries in $\mathcal{B}$ in the same coset modulo 4 is the column vector whose entries are the number of n-paths produced by these k coboundaries along rows $n \equiv 1$ mod 4, $5 \leq n \leq 2t-1$. A* recipe *is a collection of 4 ingredients (one for each different coset i modulo 4), arranged as a matrix of 4 columns, such that the sum of each of the rows is t.*

Example: For $t = 5$, the set of $k = 2$ coboundaries $\{\partial_{14}, \partial_{18}\}$ (and also any set $\{\partial_{14+i}, \partial_{18+i}\}$ *mod* $4t$ for any i) define the ingredient $[1,2]^t$. One recipe for $t = 5$, will be $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

□

Consequently, if a subset of $\{n_1, n_2, n_3, n_4\}$ coboundaries in $\mathcal{B}$ defines a recipe, this subset of coboundaries satisfies the condition (1), and therefore the summation of each of the rows $n \equiv 1$ *mod* 4, $5 \leq n \leq 2t-1$ is zero.

Proposition 3.11. *The notion of recipe does not depend on the order of its ingredients.*

Proof.
Attending to Lemma 3.1, ∂_i forms an n-path with ∂_{i-n+1}, independently on the coset i *mod* 4, for $n \equiv 1$ *mod* 4, $5 \leq n \leq 2t-1$. In particular, n-paths are constructed from those coboundaries in $\mathcal{B}$ in the same coset *mod* 4, which differ in $\frac{n-1}{4}$ positions in the 5-cycles.

This way, if a subset of coboundaries of the coset i *mod* 4 produces an ingredient, the same is produced by the translation of this subset to any other coset *mod* 4.

Eventually, this translation could produce a coboundary ∂_i which is not in $\mathcal{B}$. This is not a source of difficulties, since such prohibited subsets of coboundaries may be substituted by their complements in the 5-cycles. Since the substitution of any amount of paths by their complementary in a cycle does not change the total amount of paths, this operation preserves the ingredient.

□

Finding a recipe is the first step in the process of constructing a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrix, since any subset of coboundaries in $\mathcal{B}$ defining a recipe satisfies condition (1) and conversely.

Turning our attention to rows not congruent to 1, where the number of paths must be equal to the number of intersections in order to fullfill the Hadamard test, the following proposition gives a condition about when the relation (2) is also satisfied (and hence a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrix has been found).

Proposition 3.12. *A subset of coboundaries in $\mathcal{B}$ satisfies condition (2) (i.e. the summation of the n^{th}-row is zero, for $n \equiv 0, 2, 3$ mod 4, $6 \leq n \leq 2t+2$), if and only if the number of n-paths of even length is itself even, half of them starting and ending with coboundaries in cosets i_1, i_2 mod 4, the other half starting and ending in coboundaries in cosets i_3, i_4 mod 4, $i_j \neq i_k$ for $j \neq k$.*

Proof.

As we commented in Section 2., we are using $R = 1_t \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$ as the matrix coming from representative cocycles.

Consequently, intersections in rows $n \equiv 2 \ mod\ 4$ can occur in positions (n, i), for $i \equiv 2, 0 \ mod\ 4$. Similarly, intersections in rows $n \equiv 3 \ mod\ 4$ can occur in positions (n, i), for $i \equiv 2, 3 \ mod\ 4$. Finally, intersections in rows $n \equiv 0 \ mod\ 4$ can occur in positions (n, i), for $i \equiv 3, 0 \ mod\ 4$. Taking into account Lemma 3.1, it follows that n-paths consists in properly alternating coboundaries in:

- Either cosets $(1, 2) \ mod\ 4$, either cosets $(3, 0) \ mod\ 4$, for $n \equiv 2 \ mod\ 4$.
- Either cosets $(2, 0) \ mod\ 4$, either cosets $(1, 3) \ mod\ 4$, for $n \equiv 3 \ mod\ 4$.
- Either cosets $(2, 3) \ mod\ 4$, either cosets $(1, 0) \ mod\ 4$, for $n \equiv 0 \ mod\ 4$.

Hence any n-path of odd length produces exactly one intersection (i.e. shares exactly one negative entry) with R at the n^{th}-row. On the other hand, n-paths of even length produces either 2 or 0 intersections, depending on the cosets modulo 4 of n and the initial coboundary of the n-path. More precisely:

- If $n \equiv 2 \ mod\ 4$, then an n-path of even length will produce two intersections at the n^{th}-row if and only if the coset i of the initial coboundary is $i \equiv 2, 0 \ mod\ 4$.
- If $n \equiv 3 \ mod\ 4$, then an n-path of even length will produce two intersections at the n^{th}-row if and only if the coset i of the initial coboundary is $i \equiv 2, 3 \ mod\ 4$.
- If $n \equiv 0 \ mod\ 4$, then an n-path of even length will produce two intersections at the n^{th}-row if and only if the coset i of the initial coboundary is $i \equiv 3, 0 \ mod\ 4$.

Summing up, each n-path of odd length produces 1 intersection, and each n-path of even length produces either 2 or 0 intersections. Hence, the only circumstance in which the amounts of intersections and n-paths both coincide is precisely when half the n-paths of even length give rise to 2 intersections, whereas the remaining half of n-paths of even length do not produce any intersections at all.

□

Example: For $t = 5$, the set $\{\{\partial_{14}, \partial_{18}\}, \{\partial_3, \partial_{11}\}, \{\partial_8\}, \{\partial_5\}\}$, of $6 = 2 + 2 + 1 + 1$ coboundaries corresponding to the recipe showed after Definition 3.10 defines a Hadamard matrix, because paths of even length at rows congruent to $2, 3, 0$ are balanced. On the other hand, the set $\{\{\partial_{10}, \partial_{14}\}, \{\partial_3, \partial_{11}\}, \{\partial_8\}, \{\partial_5\}\}$ (which corresponds to the same recipe) does not define a Hadamard matrix, because it fails the Hadamard test at row 8, being its 8-paths $(\partial_{14}, \partial_{11}), (\partial_{10}), (\partial_3), (\partial_5), (\partial_8)$, there is only one path of even length, so the number of intersections at row 8 can not be equal to the number of 8-paths.

□

Now it is straightforward to design an algorithm (see algorithm 2 in Appendix [3]), searching exhaustively for $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices for odd t.

Table I shows an exhaustive calculation of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices (last column) for odd t, $3 \leq t \leq 13$, in terms of distributions (second column), number of different ingredients produced by the coboundaries needed (third column) and recipes found (fourth column). The fifth column shows how many of the recipes are "productive", in the sense that they actually give rise to $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices.

t	*distribution*	*#ingredients*	*#recipes*	*#Had.recipes*	*#Had.matrices*
3	$(1,1,1,0)$	$(1,1,1,1)$	4	4	24
5	$(3,3,2,2)$	$(2,2,1,1)$	12	12	120
7	$(6,6,6,3)$	$(4,4,4,1)$	28	24	336
	$(6,5,5,5)$	$(4,3,3,3)$	60	36	504
9	$(10,10,9,7)$	$(10,10,7,4)$	756	108	1944
9	$(9,9,9,9)$	$(7,7,7,7)$	60	24	432
11	$(15,14,14,12)$	$(26,20,20,10)$	5580	120	2640
13	$(21,21,21,15)$	$(74,74,74,14)$	19320	144	3744
	$(21,21,18,18)$	$(74,74,34,34)$	29208	72	1872
	$(20,20,20,18)$	$(57,57,57,34)$	21612	108	2808

TABLE I. $\mathbb{Z}_t \times \mathbb{Z}_2^2$**-cocyclic Hadamard matrices from Algorithm 2**

4. ON BASIS, GENERATORS AND DIAGRAMS

In this section we introduce some elementary notations, and give the notion of *diagram*, a very useful presentation of a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix. In particular, the notion of symmetry in a diagram (see Definition 4.2) will lead to a fast Hadamard test for $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices, which we state in Theorem 4.3.

Cocyclic matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ can be visualized by diagrams which represent the coboundaries which appear in the expression of the matrix.

Definition 4.1. *Given a pointwise product of coboundaries* $M_{\partial_{d_1}} \dots M_{\partial_{d_k}} \cdot R$ *defining a cocyclic matrix over* $\mathbb{Z}_t \times \mathbb{Z}_2^2$*, its diagram is a* $4 \times t$ *matrix, such that* $\{a_{ij}\}_{1\leq i\leq 4,\ 1\leq j\leq t}$ *is* $\times$*,* $(a_{ij} = \times)$ *if* $4t - 4(j-1) - 3 + i \bmod 4t \in \{d_1, \dots, d_k\}$ *and empty elsewhere.*

Remark 3. The definition of diagram has to do with the expression of the matrix in terms of the coboundaries, so every cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ has eight different diagrams.

Example: For instance, one Hadamard matrix for $t = 5$ is given by the following coboundaries $\{\{14, 18\}, \{3, 11\}, \{8\}, \{5\}\}$. A presentation of this subset of coboundaries as a $4 \times t$ matrix, such that coboundaries *mod* i are placed at row $i - 1$ is:

$$\begin{vmatrix} \mathbf{18} & \mathbf{14} & 10 & 6 & 2 \\ 19 & 15 & \mathbf{11} & 7 & \mathbf{3} \\ 20 & 16 & 12 & \mathbf{8} & 4 \\ 17 & 13 & 9 & \mathbf{5} & 1 \end{vmatrix}, \text{or in short,} \begin{vmatrix} \times & \times & - & - & - \\ - & - & \times & - & \times \\ - & - & - & \times & - \\ - & - & - & \times & - \end{vmatrix}$$

□

Diagrams are a very useful tool. This presentation of the coboundaries allows us to read easily the adjacency conditions, the number of paths, their length and the number of intersections they produce at each row n.

Every row of a diagram represents one 5-cycle. Actually, the diagram is not a matrix, but a cylinder, the first and last columns being adjacent. The paths in rows not congruent to 1 can be read by alternating pair of rows in the diagram. In the previous example, the set of coboundaries selected defines the following 8-paths: $(14,11),(3,18),(8),(5)$.

Among the two paths of even length, $(3,18)$ produces 2 intersections (at positions 3 and 15), and $(14,11)$ produces no intersection (it has -1 at positions 14 and 6). So whether they give 0 or 2 intersections comes from the congruency class module 4 of its first (or last) coboundary. This can be easily checked in any diagram.

Definition 4.2. *A diagram associated to a pointwise product of coboundaries $M = M_{\partial_1} \dots M_{\partial_k} R$ is called symmetric if the $\times$ are symmetric with respect to a column. If one of the diagrams representing a cocyclic matrix is symmetric, so are all other diagrams (they are obtained by complementing some of the rows, and symmetry is preserved). We will say, by extension, that the cocyclic matrix presents symmetry (the matrix is not symmetric, the diagram is).*

Theorem 4.3. *If a cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ is represented by a symmetric diagram and has exactly t paths in rows congruent to 1 mod 4, then it is Hadamard.*

Proof.

Taking into account Proposition 2.1, we have to prove that the number of paths coincide with the number of intersections for rows 6, 7, 8 and congruent. Each path of odd length give one intersection, and the only thing to prove is that paths of even length are equally distributed between those giving 0 intersections and those giving 2 intersections. As these paths of even length come in pairs (they do not use any of the coboundaries on the symmetry axis) the character of each path is the opposite of its symmetric, so its number is balanced and the result holds.

□

5. OPERATIONS

In this section we will study four different operations on the set of coboundary matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$: complements, rotations, swappings and dilatations. In particular, these operations extend to operations over $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices, which will be proved to preserve orthogonality. These operations partition the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices into disjoint orbits, which can be easily computed once one element is known. Among all the elements of an orbit, a representative can be chosen, in a standard way that will be made precise.

5.1 Complements. The group $\mathbb{Z}_2$

Given a set of coboundaries, t, and one of the four congruency classes *mod* 4, for instance i-th, $1 \leq i \leq 4$, we can consider the cocyclic matrix defined by the set of coboundaries obtained by substituting the subset of coboundaries belonging to the i-th congruency by its complement.

Definition 5.1. *Let* $\{\{c_{2j_2}\}, \{c_{3j_3}\}, \{c_{4j_4}\}, \{c_{1j_1}\}\}$ *with* $j_k \equiv k$ *mod* 4*, be a set of coboundaries. The complement in the i-th component ,* $1 \leq i \leq 4$*, denoted* $\mathrm{C}_i(\{\{c_{2j_2}\}, \{c_{3j_3}\}, \{c_{4j_4}\}, \{c_{1j_1}\}\})$*, is the union of the complement of* $\{c_{ij_i}\}$ *in the i-th component and the rest of the initial coboundaries.*

For instance , if we choose $i = 2$:

$$\mathrm{C}_2(\{\{14, 18\}, \{3, 11\}, \{8\}, \{5\}\}) = \{\{2, 6, 10\}, \{3, 11\}, \{8\}, \{5\}\}$$

Lemma 5.2. *The only complement to compute is the complement with respect to the component congruent to* 2*.*

Proof.
If we consider any other complement, there will appear one of the coboundaries 1, $4t - 1$ or $4t$. Substitution of the expression of this coboundary in terms of the basis of coboundaries gives us the complement with respect to the congruency 2.
□

If two different sets of coboundaries give the same cocyclic matrix their complements define the same matrix. It suffices to compute the image of the different expressions for a cocyclic matrix and check that we get the different expressions for its image. So, there is no imprecision when we say the *complement matrix* of a cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$.

Theorem 5.3. *If a set a coboundaries define a Hadamard matrix, so does its complement.*

Proof.
It suffices to check that the hypotheses of Theorem 4.3 are satisfied.

In rows congruent to 1, the complement of a set of coboundaries in a cycle is another set which determines the same number of paths. For the other rows, just observe that the complement preserves symmetry in the diagram.
□

Remark 4. Note that the operation *complement* modifies the number of coboundaries, substituting the k_0 coboundaries congruent to 2 *mod* 4 by $t - k_0$ coboundaries.

This operation can be observed as an action of the group $\mathbb{Z}_2$ over the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices. For every Hadamard cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, we can consider its orbit. This orbit has exactly 2 elements.

5.2 Rotations. The group $\mathbb{Z}_t$

Given that the Hadamard condition for a cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ depends on the conditions of position and adjacency of the coboundaries defining such matrix, one natural idea is rotating the positions of the coboundaries.

Definition 5.4. *Let $\{\{c_{2j_2}\}, \{c_{3j_3}\}, \{c_{4j_4}\}, \{c_{1j_1}\}\}$ with $j_k \equiv k$ mod 4, be a set of coboundaries. The i-rotated set, $0 \leq i \leq t-1$ of this set of coboundaries, denoted by $\mathrm{T}_i(\{\{c_{2j_2}\}, \{c_{3j_3}\}, \{c_{4j_4}\}, \{c_{1j_1}\}\})$ is the set:*

$$\{\{c_{2j_2-4i}\}, \{c_{3j_3-4i}\}, \{c_{4j_4-4i}\}, \{c_{1j_1-4i}\}\}$$

where additions are mod $4t$.

For instance, for $i = 2$

$$\mathrm{T}_2(\{\{14, 18\}, \{3, 11\}, \{8\}, \{5\}\}) = \{\{6, 10\}, \{3, 15\}, \{20\}, \{17\}\}$$

The i-rotation operation moves the marked positions i places to the right.

As in the complement operation, i-rotation over each of the eight expressions for a Hadamard matrix gives each of the eight expressions of the same cocyclic Hadamard matrix, so we are rigourous when speaking of the i-rotated of a matrix.

This operation can be observed as an action of the group $\mathbb{Z}_t$ over the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices. The element i, $0 \leq i \leq t-1$ acts on every cocyclic matrix by substracting $4i$ *mod* $4t$ to each of the coboundaries defining the matrix.

Theorem 5.5. *The i-rotated set of a set defining a cocyclic Hadamard matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ defines a Hadamard matrix too.*

Proof.

Rotations preserve the relative positions of the coboundaries in the diagram and, thus, the number, length and intersections defined by the paths.

□

Remark 5. There is no need to consider symmetry condition for this proof. For every Hadamard cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, we can consider its orbit. This orbit has exactly t elements.

Remark 6. Notice that the complement and rotation operations commute.

5.3 Swappings. The group S_4

The word *swapping* explains clearly the effect of this operation on a set of coboundaries. The marked positions are permuted among the rows in the diagram.

Given the set of coboundaries $\{c_{ij_i}\}$ for $1 \leq i \leq 4$, we will denote $\{c_{ij_i}\} + k$ the set of coboundaries obtained by adding k to each of their indexes.

Definition 5.6. *For the set of coboundaries $\{\{c_{2j_2}\}, \{c_{3j_3}\}, \{c_{4j_4}\}, \{c_{1j_1}\}\}$ we define the swapping operations:*

- $s_{12}(\{\{c_{2j_2}\}, \{c_{3j_3}\}, \{c_{4j_4}\}, \{c_{1j_1}\}\}) = \{\{c_{1j_1}\} + 1, \{c_{3j_3}\}, \{c_{4j_4}\}, \{c_{2j_2}\} - 1\}$.

- $s_{13}(\{\{c_{2j_2}\},\{c_{3j_3}\},\{c_{4j_4}\},\{c_{1j_1}\}\}) = \{\{c_{2j_2}\},\{c_{1j_1}\}+2,\{c_{4j_4}\},\{c_{3j_3}\}-2\}$.
- $s_{14}(\{\{c_{2j_2}\},\{c_{3j_3}\},\{c_{4j_4}\},\{c_{1j_1}\}\}) = \{\{c_{2j_2}\},\{c_{3j_3}\},\{c_{1j_1}\}+3,\{c_{4j_4}\}-3\}$.

Although we have defined three operations, any composition of them can be considered, obtaining one of the possible 24 permutations on the rows of the diagram.

For instance, for $t = 5$

$$s_{13}(\{\{14,18\},\{3,11\},\{8\},\{5\}\}) = \{\{14,18\},\{7\},\{8\},\{1,9\}\}$$

Theorem 5.7. *If $\{\{c_{2j_2}\},\{c_{3j_3}\},\{c_{4j_4}\},\{c_{1j_1}\}\}$ is a set of coboundaries defining a Hadamard cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, then $s_{ij}(\{\{c_{2j_2}\},\{c_{3j_3}\},\{c_{4j_4}\},\{c_{1j_1}\}\})$, $1 \le i < j \le 4$ is also a Hadamard cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$.*

Proof.

Once again, it suffices to check whether the hypotheses of Theorem 4.3 are satisfied. On one hand, any swapping preserves the number of paths in rows congruent to 1. On the other hand, any permutation of the rows in the diagram preserves symmetry.

□

Remark 7. This result can be proved without using the symmetry condition, one only need to assume that the paths of even length giving 2 and 0 intersections, are equally distributed among the two subsets of components defined in each row, attending to the congruency *mod* 4 of the coboundaries.

This gives us an action of S_4 on diagrams, which commute with complement and rotations. Depending on the diagram (i.e. on the size of the permutation group of its rows), the orbit can have less than 24 elements (actually, 1,4,6,12 or 24).

5.4 Dilatations. The group $\mathbb{Z}_t^*$

As we have seen, the orthogonality of a cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ is determined by the position and adjacency of the coboundaries which define it. This is the reason because of rotating the rows of a diagram preserves the orthogonality of the corresponding $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix. Now we are interested in other kind of geometric transformations: homothecies.

Definition 5.8. *Given a set of coboundaries S determining a diagram with columns $(\mathcal{C}_t, \cdots, \mathcal{C}_1)$, the r-th dilatation applied to S, with $r \in \mathbb{Z}_t^*$, denoted by $V_r(S)$, is the set corresponding to the image of the diagram under the homothecy with ratio r and center the column placed at the right hand-side of the diagram.*

As the column located at the right-hand side corresponds to the coboundaries $2,3,4,1$, a formula for V_r is given by $V_r(\partial_k) = \partial_h$, for

$$h = 4\left(\left(\frac{k-(k \ mod\ 4)}{4}\cdot r\right)\ mod\ t\right) + (k\ mod\ 4). \qquad (8)$$

For instance, for $r = 2$

$$\mathrm{V}_2(\{\{14,18\},\{3,11\},\{8\},\{5\}\}) = \{\{6,14\},\{3,19\},\{12\},\{9\}\}$$

Remark 8. Notice that a dilatation V_r defines a bijection over the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-coboundaries if and only if $r \in \mathbb{Z}_t^*$.

Theorem 5.9. *Dilatations V_r with $r \in \mathbb{Z}_t^*$ preserve the orthogonality of a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix.*

Proof.
It suffices to check whether the hypotheses of Theorem 4.3 are satisfied. In rows congruent to 1, dilatations define some permutation on the number of paths defined (a permutation of the entries in the ingredient), so the total number of paths remains equal to t.
Since the image diagram is symmetric as well, the result holds.

□

Remark 9. The size of the orbit of a symmetric $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix under the action of dilatations is, at most, $\phi(t)$, ϕ being the Euler's totient function.

Remark 10. Depending on the diagrams, the image under dilatations can coincide with the image under some of the previously defined operations; for instance the action on the previous example of the dilatation V_2 coincides with the action of the composition $s_{23}T_4$.

5.5 Orbits and representatives

Definition 5.10. *The total orbit of a cocyclic Hadamard matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ is the union of all orbits under the action of complement, rotations, swappings and dilatations.*

Definition 5.11. *A representative of a total orbit is a set of coboundaries associated to a diagram, with minimal number of coboundaries, symmetric with respect to the central column, and with an increasing number of coboundaries on each row of the diagram.*

Example: One Hadamard matrix for $t = 5$ is given by the following coboundaries $\{\{14,18\},\{3,11\},\{8\},\{5\}\}$, and the representative for its total orbit will be $\{\{10\},\{11\},\{8,16\},\{1,17\}\}$.

□

Table C in Appendix [3], gives us the description of all Hadamard matrices obtained in Table I, split in orbits, which can be generated by its representatives under the action of the four operations previously defined. The second column shows the size of the orbit under complement/rotation/swapping/dilatations, when providing new matrices.

6. THE SYMMETRIC CASE

We have computed the full set of $\mathbf{Z}_t \times \mathbf{Z}_2^2$-cocyclic Hadamard matrices for $t \leq 13$.

There are two special repartitions of coboundaries attending to their congruency, for every Hadamard matrix. One is the repartition given by the expression of the matrix with respect to the basis. And the other is the minimal repartition (this last one can correspond to the matrix, or to its complement). When performing calculations we can restrict to the minimal repartition, in order to simplify the search, avoiding 15 of the 16 possible repartitions for each distribution.

The rotation, swapping and dilatations preserve the number of coboundaries involved. On the other hand, the image of the Hadamard matrix does not depend on the chosen representation (from the possible eight). Thus we can obtain the whole orbit of the matrix under all these operations by only knowing the expression of the matrix with the minimal repartition, so we can restrict the searching to this easier case.

Once this is done, we can compute the complement of each of the obtained matrices, to complete the orbit under all operations.

6.1 The symmetric-diagram case ($t \in [15, 23]$)

The diagrams of all the matrices obtained for $t \leq 13$ were symmetric (symmetry is a sufficient condition to pass the Hadamard test in rows congruent to $2, 3, 0$ module 4, so, assuming symmetry, the Hadamard test reduces to check if the number of paths in rows congruent to 1 is exactly t).

By following this idea, we have computed the set of cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, having a symmetric diagram for $15 \leq t \leq 23$, which are listed in Tables D and E in Appendix [3], in terms of their representatives and the size of their total orbit under complement/rotation/swapping/dilatations, when providing new matrices.

6.2 Williamson type matrices

The first row of Table II shows the number of Williamson type matrices obtained in [4], together with the total number of cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ which we have computed.

t	3	5	7	9	11	13	15	17	19	21	23
$\sharp Will.$	8	24	120	264	240	648	576	–	–	–	–
$\sharp H.$	24	120	840	2376	2640	8424	8640	13056	34200	31248	12144

TABLE II. Williamson type / $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices (only for symmetric diagrams if $t \geq 15$)

It can be observed that the set of Williamson type Hadamard matrices seems to be in proportion $\frac{1}{t}$ with respect to the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices consisting of symmetric diagrams. We now prove the validity of this fact.

By Lemma 3.4 in [4], any Williamson type matrix is Hadamard equivalent to a cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, which is $t \times t$ back-circulant by blocks 4×4,

$$\begin{pmatrix} W_1 & \dots & \dots & W_t \\ W_2 & \iddots & W_t & W_1 \\ \iddots & \iddots & \iddots & \iddots \\ W_t & W_1 & \dots & W_{t-1} \end{pmatrix}, \qquad W_i = \begin{pmatrix} n_i & x_i & y_i & z_i \\ x_i & -n_i & z_i & -y_i \\ y_i & -z_i & -n_i & x_i \\ z_i & y_i & -x_i & -n_i \end{pmatrix}, \tag{9}$$

with $W_{i+1} = W_{t-i+1}$ for $1 \leq i \leq t-1$.

This matrix (9) is a pointwise product of the representative matrix R and a certain set of generalized coboundaries $\{M_{\partial_{d_1}}, \dots, M_{\partial_{d_k}}\}$. The additional condition $W_{i+1} = W_{t-i+1}$ for $1 \leq i \leq t-1$, leaving W_1 alone means that the symmetry column in the diagram representing the matrix (9) is, precisely, the last one to the right, which is associated to the coboundaries $\partial_2, \partial_3, \partial_4, \partial_1$. This fact leads us to conclude that:

Proposition 6.1. *Let $\mathcal{H}$ the set of cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, having a symmetric diagram. Then, the set of Williamson type matrices of type (9) is a subset of $\mathcal{H}$ of size $\frac{|\mathcal{H}|}{t}$. Moreover, for each element $H \in \mathcal{H}$, one and only one of the rotated matrices $T_i H$, $1 \leq i \leq t$, is a Williamson type matrix.*

Proof.

The condition needed tells us that the diagram has to be symmetric with respect to the last column, which is only possible for one of the rotated.

□

So, the predicted number of Williamson type matrices for $t \in [17, 23]$ will be $13056/17 = 768$, $34200/19 = 1800$, $31248/21 = 1488$ and $12144/23 = 528$.

6.3 More Hadamard matrices ($t \in [25, 63]$)

Moreover, the result about Williamson type matrices allows us to go beyond $t = 23$. Actually, we have identified every Williamson type matrix exhibited in the Koukouvinos website [9] (which corresponds to a certain decomposition of $4t$ as a sum of squares) as a pointwise product of coboundaries and computed the whole orbit of matrices for $t \in [25, 39]$. We show the results obtained in Table F in Appendix [3], and give the size of its orbit under the action of complement/rotation/swapping/dilatations, when providing new matrices.

The Koukouvinos website [9] only gives an example of each equivalence class of Williamson type Hadamard matrices for small orders, and not the total number of these matrices. However we have checked that the numbers of Williamson type Hadamard matrices we predict by using our computation of symmetric-diagram cocyclic Hadamard matrices and dividing by t, for $t \in [17, 23]$, is the same as if we take the examples in [9] in each order and compute the sum of their orbit sizes under our operations.

We conjecture that one can obtain the total number of Williamson type Hadamard matrices for $t \in [25, 63]$ from [9] by just computing the sum of the orbit sizes for each order. That is to say, we conjecture that any Williamson type matrix H'

Hadamard equivalent to a given Williamson type matrix H is in the orbit of H under our operations.

Finally we show the total number of Hadamard matrices that we have computed so far

t	3	5	7	9	11	13	15	17	19	21	23
$\sharp H.$	24	120	840	2376	2640	8424	8640	13056	34200	31248	12144
t	25	27	29	31	33	35	37	39	41	43	45
$\sharp H.$	75000	64152	19488	33480	79200	–	53280	7488	4920	14448	12960
t	47	49	51	53	55	57	59	61	63		
$\sharp H.$	–	24696	58752	–	26400	24624	–	21960	9072		

TABLE III. $\mathbb{Z}_t \times \mathbb{Z}_2^2$**-cocyclic Hadamard matrices computed**

6.4 Further questions

After an observation process, some questions come into our minds:

- All $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices obtained so far have a symmetric diagram up to $t = 13$, and we have taken advantage of it when computing them for $t \in [15, 23]$ by restricting ourselves to the symmetric case. Can this symmetry assumption be proved? Does a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrix exist coming from a non symmetric diagram?
- In case that the answer to the second question above is affirmative, then it would be interesting to look for $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices in those orders for which no Williamson type Hadamard matrix is known to exist, such as $t = 35$ [9].
- Now that we have proved that our operations preserve orthogonality, arises the question: do these operations preserve Hadamard equivalence classes? Rotations, dilatations and some of the swappings do, because they can be expressed in terms of the bundle operation defined by Horadam in [6] (this fact will be detailed elsewhere). In addition, for values of t below 23, the number of orbits coincide with the number of non equivalent Williamson type matrices computed in [9], so the result is probably true in general.

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APPENDIX: TABLES AND ALGORITHMS

Algorithm 1: Constructing the set of distributions for t.

INPUT: t

$k = \{ \}$

for i_1 from $\lceil \frac{-1+\sqrt{t}}{2} \rceil$ to $\lfloor \frac{-1+\sqrt{4t-3}}{2} \rfloor$ do

  for i_2 from $\lceil \frac{-1+\sqrt{1+4\frac{t-1-i_1(i_1+1)}{3}}}{2} \rceil$ to

  $\min(i_1, \lfloor \frac{-1+\sqrt{1+4(t-1-i_1(i_1+1))}}{2} \rfloor)$ do

    for i_3 from $\lceil \frac{-1+\sqrt{1+2(t-1-i_1(i_1+1)-i_2(i_2+1))}}{2} \rceil$

     to $\min(i_2, \lfloor \frac{-1+\sqrt{1+4(t-1-i_1(i_1+1)-i_2(i_2+1))}}{2} \rfloor)$

     do

      $x = \frac{-1+\sqrt{1+4(t-1-i_1(i_1+1)-i_2(i_2+1)-i_3(i_3+1))}}{2}$

      if x is an integer, then $k = k \cup \{(x, i_3, i_2, i_1)\}$ fi

    od

  od

od

$l = \{ \}$

for i from 1 to $length(k)$ do

  $l = l \cup \{(\frac{t^2-1}{8} - k_{i,1}, \frac{t^2-1}{8} - k_{i,2}, \frac{t^2-1}{8} - k_{i,3}, \frac{t^2-1}{8} - k_{i,4})\}$

od

OUTPUT: l

Algorithm 2: Exhaustive search for $\mathbb{Z}_t \times \mathbb{Z}_2^2$-Hadamard matrices.

INPUT: t.

Calculate the valid distributions for t.

Calculate all the ingredients associated to every distribution.

Construct the set of recipes corresponding to each distribution.

Determine those subsets of coboundaries defining a recipe.

Check whether these subsets satisfy the balanced distribution of even n-paths for $n \equiv 2, 3, 0 \quad mod \quad 4$, $6 \leq n \leq 2t + 2$.

OUTPUT: the full set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-Hadamard matrices.

Table A shows the total amount $\frac{k(t-k)}{2}$ of paths produced by k coboundaries in the same coset modulo 4, for odd values of t.

$t \backslash k$	1		...			$\frac{t-1}{2}$	$\frac{t+1}{2}$		...			1
3						1	1					
5					2	3	3	2				
7				3	5	6	6	5	3			
9			4	7	9	10	10	9	7	4		
11		5	9	12	14	15	15	14	12	9	5	
13	6	11	15	18	20	21	21	20	18	15	11	6
⋮					⋮	⋮					⋮	

TABLE A. Paths produced from k coboundaries in rows $n \equiv 1$ *mod* 4.

Table B shows the complete set of distributions obtained from Algorithm 1, for $3 \leq t \leq 25$.

t	$\frac{t(t-1)}{2}$	*distributions*	$t_0+t_1+t_2+t_3=\frac{t-1}{2}$
3	3	$(1,1,1,0)$	$0+0+0+1=1$
5	10	$(3,3,2,2)$	$0+0+1+1=2$
7	21	$(6,6,6,3)$ $(6,5,5,5)$	$0+0+0+3=3$ $0+1+1+1=3$
9	36	$(10,10,9,7)$ $(9,9,9,9)$	$0+0+1+3=4$ $1+1+1+1=4$
11	55	$(15,14,14,12)$	$0+1+1+3=5$
13	78	$(21,21,21,15)$ $(21,21,18,18)$ $(20,20,20,18)$	$0+0+0+6=6$ $0+0+3+3=6$ $1+1+1+3=6$
15	105	$(28,28,27,22)$ $(28,27,25,25)$	$0+0+1+6=7$ $0+1+3+3=7$
17	136	$(36,35,35,30)$ $(35,35,33,33)$	$0+1+1+6=8$ $1+1+3+3=8$
19	171	$(45,45,42,39)$ $(45,42,42,42)$ $(44,44,44,39)$	$0+0+3+6=9$ $0+3+3+3=9$ $1+1+1+6=9$
21	210	$(55,55,55,45)$ $(55,54,52,49)$ $(54,52,52,52)$	$0+0+0+10=10$ $0+1+3+6=10$ $1+3+3+3=10$
23	253	$(66,66,65,56)$ $(65,65,63,60)$	$0+0+1+10=11$ $1+1+3+6=11$
25	300	$(78,78,72,72)$ $(78,77,77,68)$ $(78,75,75,73)$ $(75,75,75,75)$	$0+0+6+6=12$ $0+1+1+10=12$ $0+3+3+6=12$ $3+3+3+3=12$

TABLE B. Distributions in terms of decompositions of $\frac{t-1}{2}=t_0+t_1+t_2+t_3$.

t	*orbit*	*representative*
3	$2 \times 3 \times 4 \times 1 = 24$	$\{\{\}, \{7\}, \{8\}, \{5\}\}$
5	$2 \times 5 \times 12 \times 1 = 120$	$\{\{10\}, \{11\}, \{8, 16\}, \{1, 17\}\}$
7	$2 \times 7 \times 24 \times 1 = 336$	$\{\{14\}, \{11, 15, 19\}, \{8, 16, 24\}, \{1, 13, 25\}\}$
	$2 \times 7 \times 12 \times 3 = 504$	$\{\{10, 18\}, \{11, 19\}, \{4, 28\}, \{1, 13, 25\}\}$
9	$2 \times 9 \times 12 \times 3 = 648$	$\{\{14, 22\}, \{15, 19, 23\}, \{4, 16, 24, 36\}, \{1, 13, 21, 33\}\}$
	$2 \times 9 \times 24 \times 3 = 1296$	$\{\{14, 22\}, \{3, 19, 35\}, \{12, 16, 24, 28\}, \{1, 9, 25, 33\}\}$
	$2 \times 9 \times 24 \times 1 = 432$	$\{\{14, 18, 22\}, \{11, 19, 27\}, \{8, 20, 32\}, \{1, 17, 33\}\}$
11	$2 \times 11 \times$ $24 \times 5 = 2640$	$\{\{18, 22, 26\}, \{7, 15, 31, 39\},$ $\{4, 16, 32, 44\}, \{9, 13, 21, 29, 33\}\}$
13	$2 \times 13 \times$ $24 \times 3 = 3744$	$\{\{22, 26, 30\}, \{3, 15, 23, 31, 39, 51\},$ $\{8, 16, 20, 36, 40, 48\}, \{1, 5, 17, 33, 45, 49\}\}$
	$2 \times 13 \times$ $24 \times 3 = 1872$	$\{\{18, 22, 30, 34\}, \{7, 15, 39, 47\},$ $\{8, 12, 24, 32, 44, 48\}, \{1, 5, 21, 29, 45, 49\}\}$
	$2 \times 13 \times$ $24 \times 3 = 1872$	$\{\{14, 18, 34, 38\}, \{15, 23, 27, 31, 39\},$ $\{8, 24, 28, 32, 48\}, \{5, 17, 25, 33, 45\}\}$
	$2 \times 13 \times$ $12 \times 3 = 936$	$\{\{14, 18, 34, 38\}, \{15, 19, 27, 35, 39\},$ $\{4, 12, 28, 44, 52\}, \{1, 9, 25, 41, 49\}\}$

TABLE C. Orbits of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices

$t/orbit/\#H.$			*representative*
15	$2\times15\times24\times4$	2880	$\{\{18,22,38,42\},\{7,19,27,35,43,55\},\{16,20,28,32,36,44,48\},\{1,9,25,29,33,49,57\}\}$
15	$2\times15\times24\times4$	2880	$\{\{10,14,46,50\},\{7,19,27,35,43,55\},\{8,12,24,32,40,52,56\},\{1,5,9,29,49,53,57\}\}$
15	$2\times15\times24\times2$	1440	$\{\{14,26,34,46\},\{11,15,27,35,47,51\},\{16,20,28,32,36,44,48\},\{1,5,21,29,37,53,57\}\}$
15	$2\times15\times12\times4$	1440	$\{\{14,26,30,34,46\},\{15,27,31,35,47\},\{12,16,24,40,48,52\},\{9,13,21,29,37,45,49\}\}$
17	$2\times17\times24\times4$	3264	$\{\{18,30,34,38,50\},\{3,23,31,35,39,47,67\},\{16,20,28,36,44,52,56\},\{1,13,21,25,41,45,53,65\}\}$
17	$2\times17\times24\times4$	3264	$\{\{18,30,34,38,50\},\{3,23,27,35,43,47,67\},\{4,16,28,36,44,56,68\},\{13,17,21,29,37,45,49,53\}\}$
17	$2\times17\times24\times4$	3264	$\{\{18,30,34,38,50\},\{7,15,31,35,39,55,63\},\{12,20,24,36,48,52,60\},\{5,9,13,21,45,53,57,61\}\}$
17	$2\times17\times24\times4$	3264	$\{\{18,26,30,38,42,50\},\{3,19,31,39,51,67\},\{8,24,32,36,40,48,64\},\{9,13,17,33,49,53,57\}\}$
19	$2\times19\times24\times9$	8208	$\{\{18,30,34,42,46,58\},\{15,19,23,39,55,59,63\},\{8,16,28,36,40,44,52,64,72\},\{5,21,25,29,37,45,49,53,69\}\}$
19	$2\times19\times12\times9$	4104	$\{\{10,30,34,42,46,66\},\{11,31,35,39,43,47,67\},\{4,12,16,32,40,48,64,68,76\},\{1,9,13,29,37,45,61,65,73\}\}$
19	$2\times19\times24\times9$	8208	$\{\{10,30,34,42,46,66\},\{15,27,35,39,43,51,63\},\{8,16,32,36,40,44,48,64,72\},\{5,9,25,33,37,41,49,65,69\}\}$
19	$2\times19\times24\times3$	2376	$\{\{6,10,34,42,66,70\},\{11,23,27,35,43,51,55,67\},\{8,16,20,36,44,60,64,72\},\{1,5,9,29,45,65,69,73\}\}$
19	$2\times19\times24\times3$	2736	$\{\{6,10,34,42,66,70\},\{7,15,31,35,43,47,63,71\},\{12,20,24,36,44,56,60,68\},\{1,5,9,25,49,65,69,73\}\}$
19	$2\times19\times24\times9$	8208	$\{\{22,30,34,42,46,54\},\{7,15,19,35,43,59,63,71\},\{4,16,20,36,44,60,64,76\},\{9,13,17,25,49,57,61,65\}\}$

TABLE D. **$\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices [15-19] (only for symmetric diagrams)**

$t/orbit/\#H./representative$		
21	$2\times 21\times 24\times 2$	2016
$\{\{22,26,38,46,58,62\},\{3,15,19,27,39,47,59,67,71,83\},$ $\{8,12,16,24,40,48,64,72,76,80\},\{13,21,25,29,33,49,53,27,61,69\}\}$		
21	$2\times 21\times 24\times 6$	6048
$\{\{10,34,38,42,46,50,74\},\{3,19,23,31,55,63,67,83\},$ $\{12,24,28,36,44,52,60,64,76\},\{5,9,13,21,33,49,61,69,73,77\}\}$		
21	$2\times 21\times 24\times 6$	6048
$\{\{18,26,38,42,46,58,66\},\{19,23,31,35,51,55,63,67\},$ $\{12,16,32,40,44,48,56,72,76\},\{1,5,17,29,37,45,53,65,77,81\}\}$		
21	$2\times 21\times 24\times 6$	6048
$\{\{18,26,38,42,46,58,66\},\{11,23,27,35,51,59,63,75\},$ $\{12,16,32,40,44,48,56,72,76\},\{1,5,9,25,37,45,57,73,77,81\}\}$		
21	$2\times 21\times 12\times 6$	3024
$\{\{6,30,34,38,46,50,54,78\},\{7,31,35,39,47,51,55,79\},$ $\{16,20,36,52,68,72,80\},\{13,17,33,41,49,65,69,77\}\}$		
21	$2\times 21\times 24\times 6$	6048
$\{\{18,30,34,38,46,50,54,66\},\{3,7,31,39,47,55,79,83\},$ $\{8,24,32,36,52,56,64,80\},\{1,17,21,29,41,53,61,65,81\}\}$		
21	$2\times 21\times 24\times 2$	2016
$\{\{14,18,30,38,46,54,66,70\},\{7,15,27,31,55,59,71,79\},$ $\{8,12,20,24,64,68,72,80\},\{13,21,25,37,41,45,57,61,69\}\}$		
23	$2\times 23\times 24\times 11$	12144
$\{\{22,30,38,42,50,54,62,70\},\{15,19,23,35,47,59,71,75,79\},$ $\{12,20,28,40,44,52,56,68,76,84\},\{9,13,25,29,33,57,61,65,77,81\}\}$		

TABLE E. $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices [21-23] (only for symmetric diagrams)

t	$4t = A^2 + B^2 + C^2 + D^2$	$\sharp orbit(compl./rot./swapp./dilat.)$
25	$100 = 9^2 + 3^2 + 3^2 + 1^2$	$2 \times 25 \times 24 \times 10 = 12000$
	$100 = 7^2 + 7^2 + 1^2 + 1^2$	$2 \times 25 \times 12 \times 5 = 3000$
		$2 \times 25 \times 24 \times 5 = 6000$
		$2 \times 25 \times 24 \times 5 = 6000$
	$100 = 7^2 + 5^2 + 5^2 + 1^2$	$2 \times 25 \times 24 \times 10 = 12000$
		$2 \times 25 \times 24 \times 10 = 12000$
		$2 \times 25 \times 24 \times 5 = 6000$
	$100 = 5^2 + 5^2 + 5^2 + 5^2$	$2 \times 25 \times 24 \times 5 = 6000$
		$2 \times 25 \times 24 \times 5 = 6000$
		$2 \times 25 \times 24 \times 5 = 6000$
27	$108 = 9^2 + 5^2 + 1^2 + 1^2$	$2 \times 27 \times 24 \times 9 = 11664$
		$2 \times 27 \times 24 \times 9 = 11664$
	$108 = 7^2 + 7^2 + 3^2 + 1^2$	$2 \times 27 \times 12 \times 9 = 5832$
		$2 \times 27 \times 24 \times 9 = 11664$
		$2 \times 27 \times 24 \times 9 = 11664$
	$108 = 7^2 + 5^2 + 5^2 + 3^2$	$2 \times 27 \times 24 \times 9 = 11664$
29	$116 = 9^2 + 5^2 + 3^2 + 1^2$	$2 \times 29 \times 24 \times 14 = 19488$
31	$124 = 7^2 + 7^2 + 5^2 + 1^2$	$2 \times 31 \times 24 \times 15 = 22320$
	$124 = 7^2 + 5^2 + 5^2 + 5^2$	$2 \times 31 \times 12 \times 15 = 11160$
33	$132 = 11^2 + 3^2 + 1^2 + 1^2$	$2 \times 33 \times 24 \times 10 = 15840$
	$132 = 9^2 + 7^2 + 1^2 + 1^2$	$2 \times 33 \times 24 \times 10 = 15840$
	$132 = 9^2 + 5^2 + 5^2 + 1^2$	$2 \times 33 \times 24 \times 10 = 15840$
		$2 \times 33 \times 24 \times 10 = 15840$
	$132 = 7^2 + 7^2 + 5^2 + 3^2$	$2 \times 33 \times 24 \times 10 = 15840$
37	$148 = 11^2 + 3^2 + 3^2 + 3^2$	$2 \times 37 \times 24 \times 3 = 5328$
	$148 = 9^2 + 7^2 + 3^2 + 3^2$	$2 \times 37 \times 12 \times 18 = 15984$
	$148 = 7^2 + 7^2 + 5^2 + 5^2$	$2 \times 37 \times 24 \times 9 = 15984$
		$2 \times 37 \times 24 \times 9 = 15984$
39	$156 = 9^2 + 5^2 + 5^2 + 5^2$	$2 \times 39 \times 24 \times 4 = 7488$
41	$164 = 9^2 + 9^2 + 1^2 + 1^2$	$2 \times 41 \times 12 \times 5 = 4920$
43	$172 = 7^2 + 7^2 + 7^2 + 5^2$	$2 \times 43 \times 24 \times 7 = 14448$
45	$180 = 9^2 + 7^2 + 5^2 + 5^2$	$2 \times 45 \times 12 \times 12 = 12960$
49	$196 = 9^2 + 9^2 + 5^2 + 3^2$	$2 \times 49 \times 12 \times 21 = 24696$
51	$204 = 11^2 + 9^2 + 1^2 + 1^2$	$2 \times 51 \times 12 \times 16 = 19584$
	$204 = 11^2 + 7^2 + 5^2 + 3^2$	$2 \times 51 \times 24 \times 16 = 39168$
55	$220 = 11^2 + 9^2 + 3^2 + 3^2$	$2 \times 55 \times 12 \times 20 = 26400$
57	$228 = 9^2 + 7^2 + 7^2 + 7^2$	$2 \times 57 \times 12 \times 18 = 24624$
61	$244 = 11^2 + 11^2 + 1^2 + 1^2$	$2 \times 61 \times 12 \times 15 = 21960$
63	$252 = 11^2 + 11^2 + 3^2 + 1^2$	$2 \times 63 \times 12 \times 6 = 9072$

TABLE F. Full orbits from Williamson type matrices