Generalized binary arrays from quasi-orthogonal cocycles

J. A. Armario and D. L. Flannery

[1] Departamento de Matemática Aplicada I, Universidad de Sevilla, Avda. Reina Mercedes s/n, 41012 Sevilla, Spain
armario@us.es

[2] School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, Galway H91TK33, Ireland
dane.flannery@nuigalway.ie

Abstract. Generalized perfect binary arrays (GPBAs) were used by Jedwab to construct perfect binary arrays. A non-trivial GPBA can exist only if its energy is 2 or a multiple of 4. This paper introduces *generalized optimal binary arrays* (GOBAs) with even energy not divisible by 4, as analogs of GPBAs. We give a procedure to construct GOBAs based on a characterization of the arrays in terms of 2-cocycles. As a further application, we determine negaperiodic Golay pairs arising from generalized optimal binary sequences of small length.

1 Introduction

Let $\phi = (\phi(0), \ldots, \phi(n-1)) \in \{\pm 1\}^n$ be a binary sequence of length n. Reading arguments modulo n,

$$R_\phi(w) := \sum_{k=0}^{n-1} \phi(k)\phi(k+w)$$

is the *periodic autocorrelation of ϕ at shift w*. The *expansion* of ϕ, denoted ϕ', is the concatenation of ϕ and $-\phi$ (in that order). A pair ϕ_1, ϕ_2 of binary sequences, each of length $2t$, such that $R_{\phi_1'}(w) + R_{\phi_2'}(w) = 0$ for $1 \leq w \leq 2t-1$ (equivalently, for $1 \leq w \leq 4t-1$ and $w \neq 2t$), is a *negaperiodic Golay pair* (NGP). Note that the original definition of NGP in [4] coincides with the definition above by [8, Lemma 2].

We seek good sources of NGPs. This objective is connected to several existence problems in algebraic design theory. For example, Egan showed that NGPs of length $2t$ are equivalent to certain relative $(4t, 2, 4t, 2t)$-difference sets in the dicyclic group Q_{8t} of order $8t$ [8, Theorem 3]. Actually, there is a relative $(4t, 2, 4t, 2t)$-difference set in a central extension E of $\mathbb{Z}_2$ by a group G of order $4t$, relative to $\mathbb{Z}_2$, if and only if there is a Hadamard matrix of order $4t$ whose expanded (group-divisible) design admits a special regular action by E: a *cocyclic* Hadamard matrix over G [6, Theorem 2.4]. By way of [9, Theorem 3.3], Ito [13, p. 370] conjectured that Q_{8t} contains such relative difference sets for all t. Schmidt [16] has verified Ito's conjecture up to $t = 46$. Our recent paper [3] initiated the study of *quasi-orthogonal* cocycles over groups G of even order not divisible by 4, in direct analogy with cocyclic Hadamard matrices. The present paper builds on [3].

It is easy to see that

$$\max_{0<w<n} |R_\phi(w)| \geq \begin{cases} 0 & n \equiv 0 \mod 4 \\ 1 & n \equiv 1 \text{ or } 3 \mod 4 \\ 2 & n \equiv 2 \mod 4. \end{cases} \tag{1}$$

The sequence ϕ is *optimal* if equality holds in (1). In particular, ϕ is *perfect* if $R_\phi(w) = 0$ for $0 < w < n$. No perfect binary sequence of length $n > 4$ is known. Attention consequently turns to the larger class of *perfect binary arrays* (PBAs). Jedwab [14] introduced *generalized perfect binary arrays* (GPBAs) to aid in the construction of PBAs. Hughes [11] subsequently demonstrated the cocyclic nature of GPBAs.

A *generalized perfect binary sequence* (GPBS) is a 1-dimensional GPBA; such ϕ have $R_{\phi'}(w) = 0$ for all w. Each pair of GPBSs is obviously an NGP. However, a GPBS exists only if $n = 2$ [14, Result 4.8]. So let $n > 2$ be even; since $R_{\phi'}(w)$ is divisible by 4, and not every $R_{\phi'}(w)$ is 0, some $|R_{\phi'}(w)|$ must be at least 4. Thus, we will say that ϕ of length $2t$ is a *generalized optimal binary sequence* (GOBS) if $\max_{0<w<2t} |R_{\phi'}(w)| = 4$. Equivalently, ϕ is a GOBS if, for $0 < w < 2t$,

$$|R_{\phi'}(w)| = \begin{cases} 0 & w \text{ odd} \\ 4 & w \text{ even} \end{cases}$$

when t is odd, and

$$|R_{\phi'}(w)| = \begin{cases} 4 & w \text{ odd} \\ 0 & w \text{ even} \end{cases}$$

when t is even. We propose searching for NGPs in the set of GOBs of length $2t$, t odd.

Just as the notion of GPBA extends that of GPBS to dimensions greater than 1, a GOBA (*generalized optimal binary array*) is a higher-dimensional version of a GOBS. Section 3 treats GPBAs and GOBAs from the perspective of [3]. We prove a one-to-one correspondence between GOBAs, quasi-orthogonal cocycles over abelian groups, and abelian relative quasi-difference sets. In Section 4, we outline and apply a method to find NGPs among GOBSs that correspond to quasi-orthogonal cocycles over cyclic groups. The concluding Section 5 looks at an important question for cocyclic designs prompted by the analysis in Section 4.

2 Quasi-orthogonal cocycles and related combinatorial structures

Let G and U be finite groups, with U abelian. A map $\psi : G \times G \to U$ such that $\psi(1,1) = 1$ and

$$\psi(g,h)\psi(gh,k) = \psi(g,hk)\psi(h,k) \quad \forall\, g,h,k \in G \tag{2}$$

is a (normalized) *cocycle* over G. If $\phi : G \to U$ is any map that is normalized (i.e., $\phi(1) = 1$) then $\partial\phi(g,h) = \phi(g)\phi(h)\phi(gh)$ defines a cocycle $\partial\phi$, called a *coboundary*. The set of all cocycles over G forms an abelian group $Z^2(G,U)$, whose quotient by the subgroup $B^2(G,U)$ of coboundaries is the *second cohomology group* $H^2(G,U)$. We display $\psi \in Z^2(G,U)$ as a *cocyclic matrix* $M_\psi = [\psi(g,h)]_{g,h\in G}$. If $U = \mathbb{Z}_2 = \langle -1 \rangle$ and M_ψ is Hadamard then ψ is said to be *orthogonal*.

The *row excess* of a $\{\pm 1\}$-matrix $M = [m_{ij}]$ is

$$RE(M) = \textstyle\sum_{i\geq 2} \big| \sum_{j \geq 1} m_{ij} \big|.$$

The cocycle equation (2) guarantees that ψ is orthogonal if and only if $RE(M_\psi)$ is optimal, i.e., zero.

For the rest of this section, $|G| = 4t + 2 > 2$.

Proposition 1. (i) *If $\psi \in Z^2(G,\mathbb{Z}_2)$ then $RE(M_\psi) \geq 4t$.*
(ii) *If $\psi \in B^2(G,\mathbb{Z}_2)$ then $RE(M_\psi) \geq 8t + 2$.*

Proof. See [3, Proposition 1]. □

In analogy with the definition of orthogonal cocycles, we say that ψ is *quasi-orthogonal* if its matrix has least possible row excess: by Proposition 1, either $\psi \notin B^2(G,\mathbb{Z}_2)$ and $RE(M_\psi) = 4t$, or $\psi \in B^2(G,\mathbb{Z}_2)$ and $RE(M_\psi) = 8t + 2$ (coboundaries were excluded from the notion of quasi-orthogonality in [3]).

Lemma 1. *Let $X_m = \{g \in G \mid \sum_{h\in G}\psi(g,h) = m\}$. Then ψ is quasi-orthogonal if and only if $|X_2 \cup X_{-2}| = 4t + 1$ for $\psi \in B^2(G,\mathbb{Z}_2)$, or $|X_0| = 2t + 1$ and $|X_2 \cup X_{-2}| = 2t$ for $\psi \notin B^2(G,\mathbb{Z}_2)$.*

Proof. See [3, Lemma 2.4]. □

It is not known whether quasi-orthogonal cocycles always exist. Indeed, we do not know of a group G such that $Z^2(G,\mathbb{Z}_2)$ does not contain a quasi-orthogonal element (in contrast, there are several non-existence results for orthogonal cocycles, e.g., due to Ito [12]). We have found quasi-orthogonal coboundaries over many abelian G, but none over non-abelian G such as dihedral groups, apart from the dihedral group of order 6. Thirdly, for all t such that $4t + 1$ is a sum of two squares that we tested, we always found a quasi-orthogonal cocycle ψ over some group of order $4t + 2$ with $|\det(M_\psi)|$

attaining the maximum $2(4t+1)(4t)^{2t}$ established by Ehlich-Wojtas. These existence questions all merit deeper investigation.

Let E be a group with a normal subgroup N of order m and index v. A *relative* (v, m, k, λ)*-difference set in* E *relative to* N (the *forbidden subgroup*) is a k-subset R of a transversal for N in E such that

$$|R \cap xR| = \lambda \quad \forall x \in E \setminus N.$$

Relative $(2s, 2, 2s, s)$-difference sets are especially interesting. If s is even then they are equivalent to cocyclic Hadamard matrices [6, Corollary 2.5], whereas none exist if s is odd [10]. In the latter case there is a natural analog of relative difference set. Suppose that $|E| = 8t+4$ and let $Z \cong \mathbb{Z}_2$ be a normal (hence central) subgroup of E. A *relative* $(4t+2, 2, 4t+2, 2t+1)$*-quasi-difference set in* E *with forbidden subgroup* Z is a transversal R for Z in E containing a subset $S \subset R \setminus \{1\}$ of size 0 or $2t+1$ such that, for all $x \in E \setminus Z$,

$$|R \cap xR| = \begin{cases} 2t+1 & x \in SZ \\ 2t \text{ or } 2t+2 & \text{otherwise.} \end{cases}$$

We call R *extremal* if $S = \emptyset$. (This modifies the original definition in [3] of relative quasi-difference set, to allow quasi-orthogonal coboundaries).

The next result is mostly Proposition 4.3 in [3]. For each $\psi \in Z^2(G, \mathbb{Z}_2)$ we have a canonical central extension E_ψ with element set $\{(\pm 1, g) \mid g \in G\}$ and multiplication defined by $(u, g)(v, h) = (uv\psi(g, h), gh)$.

Proposition 2. *The cocycle* ψ *is quasi-orthogonal if and only if* $D = \{(1, g) \mid g \in G\}$ *is a relative* $(4t+2, 2, 4t+2, 2t+1)$*-quasi-difference set in* E_ψ *with forbidden subgroup* $\langle(-1, 1)\rangle$*, where* D *is extremal for* $\psi \in B^2(G, \mathbb{Z}_2)$.

Remark 1. The requisite subset S of D corresponds to the rows of M_ψ with zero sum.

3 Generalized binary arrays with optimal autocorrelation

Jedwab [14] showed that a GPBA is equivalent to an abelian relative difference set, and Hughes [11] identified its underlying orthogonal cocycle. In this section we carry over these ideas into the setting of quasi-orthogonal cocycles.

We start with an adaptation of some material from [11] and [14]. The cyclic group of order m will be written additively, i.e., as $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ under addition modulo m. Let $\mathrm{s} = (s_1, \ldots, s_r)$ be an r-tuple of positive integers greater than 1, and let $G = \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$. A *binary* s*-array* is just a set map $\phi : G \to \{\pm 1\}$; it has *energy* $n := \prod_{i=1}^{r} s_i = |G|$. We view a binary sequence as an s-array with $r = 1$.

Given $\mathbf{s}$ and a *type vector* $\mathbf{z} = (z_1, \ldots, z_r) \in \{0,1\}^r$, let $E = \mathbb{Z}_{(z_1+1)s_1} \times \cdots \times \mathbb{Z}_{(z_r+1)s_r}$. Then

$$H = \{h \in E \mid h_i = 0 \text{ if } z_i = 0, \text{ and } h_i = 0 \text{ or } s_i \text{ if } z_i = 1\},$$
$$K = \{k \in H \mid k \text{ has even weight}\}$$

are elementary abelian 2-subgroups of E. Note that E is a (central) extension of H by G. For $\mathbf{z} \neq \mathbf{0}$ we obtain the short exact sequence

$$1 \longrightarrow \langle -1 \rangle \xrightarrow{\iota} E/K \xrightarrow{\beta} G \longrightarrow 0, \qquad (3)$$

where ι maps -1 to the generator of H/K and $\beta(g + K) = g \mod \mathbf{s}$. This sequence determines a cocycle $f_{\mathbf{z}} \in Z^2(G, \langle -1 \rangle)$ after choice of a transversal map $\tau : G \to E/K$. Specifically, set $\tau(x) = x + K$; then

$$f_{\mathbf{z}}(x, y) = \iota^{-1}(\tau(x) + \tau(y) - \tau(x + y)).$$

We can express $f_{\mathbf{z}}$ as a product of cocycles on cyclic groups. Define $\gamma_m \in Z^2(\mathbb{Z}_m, \langle -1 \rangle)$ by $\gamma_m(j, k) = (-1)^{\lfloor (j+k)/m \rfloor}$, evaluating the exponent as an ordinary integer.

Proposition 3 ([11, Lemma 3.1]).

(i) $f_{\mathbf{z}}(x, y) = \prod_{z_i=1} \gamma_{s_i}(x_i, y_i)$.
(ii) $f_{\mathbf{z}} \in B^2(G, \langle -1 \rangle)$ *if and only if* s_i *is odd for all* i *such that* $z_i = 1$.

Each cocycle $\psi \in Z^2(G, \langle -1 \rangle)$ has an associated short exact sequence

$$1 \longrightarrow \langle -1 \rangle \xrightarrow{\iota'} E_\psi \xrightarrow{\beta'} G \longrightarrow 0, \qquad (4)$$

where $\iota'(u) = (u, 0)$ and $\beta'(u, x) = x$. The following is standard.

Proposition 4. *If* ψ *and* $f_{\mathbf{z}}$ *are cohomologous, say* $\psi = f_{\mathbf{z}} \partial \phi$*, then* (3) *and* (4) *are equivalent short exact sequences: the isomorphism* Γ *defined by* $(u, x) \mapsto \iota(u\phi(x)) + \tau(x)$ *makes the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle -1 \rangle & \xrightarrow{\iota'} & E_\psi & \xrightarrow{\beta'} & G & \longrightarrow & 0 \\ & & \| & & \Gamma \downarrow & & \| & & \\ 1 & \longrightarrow & \langle -1 \rangle & \xrightarrow{\iota} & E/K & \xrightarrow{\beta} & G & \longrightarrow & 0 \end{array}$$

commute.

We broaden concepts defined earlier only for sequences. The *expansion* of a binary $\mathbf{s}$-array ϕ with respect to a type vector $\mathbf{z}$ is the map ϕ' on E given by

$$\phi'(g) = \begin{cases} \phi(a) & g \in a + K \\ -\phi(a) & g \notin a + K \end{cases}$$

where a denotes g modulo $\mathbf{s}$. For any array $\varphi : A \to \{\pm 1\}$ and $x \in A$, let $R_\varphi(x) = \sum_{a \in A} \varphi(a)\varphi(a + x)$.

Lemma 2. *If $h \in H \setminus K$ then $\phi'(h+g) = -\phi'(g)$, and if $h \in K$ then $\phi'(h+g) = \phi'(g)$.*

Corollary 1. *$R_{\phi'}(g) = |H| \sum_{x \in T} \phi'(x)\phi'(x+g)$ where T is any transversal for H in E.*

Lemma 3. *The isomorphism Γ in Proposition 4 maps $\{(1,x) \mid x \in G\} \subseteq E_\psi$ onto $\{g+K \in E/K \mid \phi'(g) = 1\}$.*

Proof. (Cf. [11, p. 330].) Let $\phi'(g) = 1$ and write a for g modulo $\mathbf{s}$; then $g + K = \iota(\phi(a)) + a + K = \Gamma((1,a))$. Conversely, $\Gamma((1,x)) = h + x + K$ where $h + K$ is the generator of H/K if $\phi(x) = -1$ and $h = 0$ otherwise. By Lemma 2, $\phi'(h+x) = 1$. □

The $\mathbf{s}$-array ϕ is a *GPBA*$(\mathbf{s})$ *of type* $\mathbf{z}$ if

$$R_{\phi'}(g) = 0 \quad \forall g \in E \setminus H.$$

When $\mathbf{z} = 0$, this condition becomes (by Corollary 1)

$$R_{\phi}(g) = 0 \quad \forall g \in G \setminus \{0\}.$$

In the latter event ϕ is a PBA; which is equivalent to $\partial\phi$ being orthogonal (we return to this case later in the section). More generally, a GPBA$(\mathbf{s})$ is equivalent to a relative difference set in E/K relative to H/K, hence equivalent also to a cocyclic Hadamard matrix over G: see [11, Theorem 5.3] and [14, Theorem 3.2]. So a GPBA can exist only if its energy n is 2 or a multiple of 4. Theorems 1 and 2 below are analogous results for $n \equiv 2 \bmod 4$.

Assume that $|G| = 4t + 2 > 2$ unless stated otherwise. Let $s_1/2, s_2, \ldots, s_r$ be odd. Thus, if $z_1 = 0$ then E splits over H by Proposition 3, and so $R_{\phi'}$ is never zero by Corollary 1 and Lemma 2.

Definition 1. A *GOBA*$(\mathbf{s})$ *of type* $\mathbf{z}$ is a binary $\mathbf{s}$-array ϕ such that

(i) $R_{\phi'}(g) \in \{0, \pm 2|H|\} \ \ \forall g \in E \setminus H$,

and if $z_1 = 1$ then

(ii) $|\{g \in E \mid R_{\phi'}(g) = 0\}| = |E|/2$.

A GOBS as defined in Section 1 is a GOBA$(\mathbf{s})$ with $r = z_1 = 1$. When $\mathbf{z} = \mathbf{0}$, Definition 1 reduces to

$$R_{\phi}(g) = \pm 2 \quad \forall g \in G \setminus \{0\};$$

we call ϕ satisfying this condition an *optimal binary array* (OBA).

Lemma 4 **([14, Lemma 3.1]).** *For any array $\varphi : A \to \{\pm 1\}$,*

$$R_\varphi(x) = |A| + 4(d_\varphi(x) - |N_\varphi|)$$

where $N_\varphi = \{a \in A \mid \varphi(a) = -1\}$ and $d_\varphi(x) = |N_\varphi \cap (x + N_\varphi)|$.

Proof. Routine counting. □

Theorem 1. *Let ϕ be a binary $\mathbf{s}$-array, $\mathbf{z}$ be a non-zero type vector, and $D = \{g + K \in E/K \mid \phi'(g) = -1\}$. Then ϕ is a GOBA$(\mathbf{s})$ of type $\mathbf{z}$ if and only if D is a relative $(4t+2, 2, 4t+2, 2t+1)$-quasi-difference set in E/K with forbidden subgroup H/K; furthermore, D is extremal if $z_1 = 0$.*

Proof. We continue with the notation of Lemma 4. By Lemma 3, D is a full transversal for H/K in E/K. Also, $|N_{\phi'}| = |E|/2$ by Lemma 2; thus $|D| = |N_{\phi'}|/|K|$.

For each $g \notin H$, denote $|D \cap (g + K + D)|$ by $d_D(g + K)$: this is the number of $x + K \in D$ such that $x - g + K \in D$. Since $d_D(g + K) = d_{\phi'}(g)/|K|$, Lemma 4 implies that

$$\begin{aligned} R_{\phi'}(g) = -2|H| &\Leftrightarrow d_D(g+K) = 2t \\ R_{\phi'}(g) = 0 &\Leftrightarrow d_D(g+K) = 2t+1 \\ R_{\phi'}(g) = 2|H| &\Leftrightarrow d_D(g+K) = 2t+2. \end{aligned} \tag{5}$$

Let $S = \{g + K \in D \mid R_{\phi'}(g) = 0\}$. According to (5), Definition 1 (i) holds if and only if

$$d_D(g+K) = \begin{cases} 2t+1 & g + K \in S + H/K \\ 2t \ \text{ or } \ 2t+2 & \text{otherwise.} \end{cases}$$

Lemma 2 yields

$$|S| = \frac{|\{g + K \in E/K \mid R_{\phi'}(g) = 0\}|}{2} = |R_{\phi'}^{-1}(0)|/2|K|.$$

Thus $|S| = 2t + 1$ for $z_1 = 1$ if and only if Definition 1 (ii) holds. □

Remark 2. Theorem 1 remains valid when D is replaced by its complement $\{g + K \in E/K \mid \phi'(g) = 1\}$.

Theorem 2. *A (normalized) binary $\mathbf{s}$-array ϕ is a GOBA$(\mathbf{s})$ of type $\mathbf{z} \neq \mathbf{0}$ if and only if $f_{\mathbf{z}}\partial\phi$ is quasi-orthogonal.*

Proof. This is a consequence of Theorem 1, Remark 2, Proposition 2, and Lemma 3. □

We proceed to formulate 'base' cases of Theorems 1 and 2. Let $\partial\phi \in B^2(G, \mathbb{Z}_2)$. Since $M_{\partial\phi}$ is Hadamard equivalent to a group-developed matrix, and such a matrix has constant row sum, $\partial\phi$ can be orthogonal only if $|G|$ is square. This situation has been extensively studied.

Theorem 3. *Let $|G| = 4u^2$, and let D be a subset of G of size $2u^2 - u$. Define $R = \{(\phi(g), g) \mid g \in G\} \subset \mathbb{Z}_2 \times G$ where $\phi : G \to \{\pm 1\}$ is the characteristic function of D. Then the following are equivalent.*

(i) *$\partial\phi$ is orthogonal.*
(ii) *D is a Menon-Hadamard difference set in G.*
(iii) *R is a relative $(4u^2, 2, 4u^2, 2u^2)$-difference set in $\mathbb{Z}_2 \times G$ with forbidden subgroup $\mathbb{Z}_2 \times \{1_G\}$.*
(iv) *ϕ is a perfect nonlinear function.*

If G is abelian then (i) – (iv) *are further equivalent to*

(v) *ϕ is a PBA.*

Proof. See [15, Theorem 1] for (iii) $\Leftrightarrow$ (iv). The other equivalences are given by Theorem 2.6 and Lemma 2.10 of [6]. □

Remark 3. In Theorem 3 and Theorem 4 below we may assume that ϕ is normalized, by taking the complement of D (and thus also of R) if necessary.

The next theorem is an analog of the previous one for $|G| \equiv 2 \bmod 4$ (recall that we have not found quasi-orthogonal coboundaries over non-abelian G at orders greater than 6).

Theorem 4. *Let G be abelian of order $4t + 2$, and let D be a k-subset of G with characteristic function $\chi : G \to \mathrm{GF}(2)$. Define $R = \{(\phi(g), g) \mid g \in G\} \subset \mathbb{Z}_2 \times G$ where $\phi(x) = (-1)^{\chi(x)}$. Then the following are equivalent.*

(i) *$\partial\phi$ is quasi-orthogonal.*
(ii) *D is a $(4t + 2, k, k - (t + 1), (4t + 1)(k - t) - k(k - 1))$-almost difference set in G.*
(iii) *R is an extremal relative $(4t + 2, 2, 4t + 2, 2t + 1)$-quasi-difference set in $\mathbb{Z}_2 \times G$ with forbidden subgroup $\mathbb{Z}_2 \times \{1_G\}$.*
(iv) *ϕ is an OBA.*

If a difference set with parameters $(n, \frac{n \pm \sqrt{3n-2}}{2}, \frac{n+2 \pm 2\sqrt{3n-2}}{4})$ does not exist, then (i) – (iv) *are further equivalent to*

(v) *χ has optimal nonlinearity $(t + 1)/(2t + 1)$.*

Proof. Put $|G| = n$.

(i) $\Leftrightarrow$ (iv): Lemma 1 and the fact that $\phi(g)R_\phi(g)$ is the sum of row g in $M_{\partial\phi}$.

(i) $\Leftrightarrow$ (ii): by Lemma 4, $R_\phi(g) = 2$ or -2 if and only if $d_\phi(g) = k - t - 1$ or $k - t$, respectively. Identity (19) of [5] then accounts for this part.

(i) $\Leftrightarrow$ (iii): Proposition 2 together with the isomorphism $E_{\partial\phi} \to \mathbb{Z}_2 \times G$ defined by $(u, g) \mapsto (u\phi(g), g)$; cf. Proposition 4.

(ii) $\Leftrightarrow$ (v): see [5, Theorem 25]. □

Remark 4. The condition attached to (v) is only needed for (v) $\Rightarrow$ (ii). No difference sets with the stated parameters are known; see [5, Remark II, p. 224].

We end this section with a discussion of calculating GOBAs. Label the elements of G as $g_1 = 0, g_2, \ldots, g_{4t+2}$, and let $\delta_k\colon G \to \{\pm 1\}$ be the characteristic function of $\{g_k\}$. Up to relabeling, $\{\partial_2, \ldots, \partial_{4t+1}\}$ is a basis of $B^2(G, \langle -1\rangle)$, where $\partial_k := \partial\delta_k$ is an *elementary coboundary*. Choose $\mathbf{z} \neq \mathbf{0}$. We first try to find quasi-orthogonal $\psi \in Z^2(G, \langle -1\rangle)$ such that $f_{\mathbf{z}}\psi \in B^2(G, \langle -1\rangle)$. Straightforward linear algebra gives the decomposition $\psi = f_{\mathbf{z}} \prod_k \partial_k^{i_k}$. Then $\phi = \prod_k \delta_k^{i_k}$ is a GOBA($\mathbf{s}$) of type $\mathbf{z}$ over G.

Example 1. The maps $\phi_1 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $\phi_2 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, $\phi_3 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ on $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$ are GOBA$(2,3)$s of type $\mathbf{z}_1 = (1,0)$, $\mathbf{z}_2 = (0,1)$, $\mathbf{z}_3 = (1,1)$, respectively. We display each quasi-orthogonal cocycle $f_{\mathbf{z}_i}\partial\phi_i$ as a Hadamard (componentwise) product:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

Note that $f_{\mathbf{z}_2}\partial\phi_2$ is a quasi-orthogonal coboundary; as are all the $\partial\phi_i$.

Example 2. The map $\begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$ on $\mathbb{Z}_6 \times \mathbb{Z}_3 = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is a GOBA$(6,3)$ of type $\mathbf{z} = (1,0)$. Its quasi-orthogonal cocycle is $f_{\mathbf{z}}\partial_4\partial_8\partial_{10}\partial_{13}$.

4 Negaperiodic Golay pairs

In this section we explore how GOBSs can be used to construct NGPs.

Proposition 5 ([8, Theorem 3]). *Binary sequences ϕ_1, ϕ_2 of length $2t$ form an NGP if and only if $\{x^i \mid \phi'_1(i) = 1\} \cup \{x^i y \mid \phi'_2(i) = 1\}$ is a relative $(4t, 2, 4t, 2t)$-difference set in the dicyclic group $Q_{8t} = \langle x, y \,|\, x^{2t} = y^2, y^4 = 1,\, y^{-1}xy = x^{-1}\rangle$.*

Remark 5. By Proposition 5 and [2, Theorems 5.6 and 5.7], NGPs of length $(q+1)/2$ exist for all prime powers $q \equiv 3 \mod 4$.

Proposition 5 ties NGPs into the mainstream theory of cocyclic Hadamard matrices: by [9, Proposition 6.5], existence of a $(4t, 2, 4t, 2t)$-difference set in Q_{8t} is equivalent to existence of certain orthogonal cocycles over the dihedral group D_{4t} of order $4t$. (Incidentally, this gives another justification of Remark 5, via Ito's Hadamard groups of quadratic residue type [12, pp. 986–987].) These cocycles lie in a single cohomology class, with representative labeled $(A, B, K) = (1, -1, -1)$ in [9]; A, B are 'inflation' variables and K is the 'transgression' variable in a Universal Coefficients theorem decomposition of $H^2(D_{4t}, \mathbb{Z}_2)$.

The next theorem makes Proposition 5 more explicit. It shows how to translate directly between cocycles and NGPs. When the latter are complementary GOBSs, this implies existence of orthogonal cocycles if there exist quasi-orthogonal cocycles at half the order (unfortunately, the process does not reverse).

Theorem 5. *Let $G = \langle a, b \,|\, a^n = b^2 = 1, a^b = a^{-1}\rangle \cong D_{2n}$ with elements ordered as $1, a, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b$. Also let ϕ_1, ϕ_2 be binary sequences of length n, and define $j_{k,i}$ to be 1 or 0 depending on whether $\phi_i(k) = -1$ or 1, respectively. Then (ϕ_1, ϕ_2) is an NGP if and only if $\lambda \prod_{k=1}^{n} \partial_k^{j_{k,1}} \partial_{n+k}^{j_{k,2}}$ is an orthogonal cocycle over G, where λ is the cohomology class representative labeled $(A, B, K) = (1, -1, -1)$ in* [9, Section 6].

Proof. The center of $\langle x, y \,|\, x^n = y^2, y^4 = 1,\, y^{-1}xy = x^{-1}\rangle \cong Q_{4n}$ is $\langle x^n\rangle$. Since $G \cong Q_{4n}/\langle x^n\rangle$, we may define a transversal map $\sigma : G \to Q_{4n}$ by

$$a^i \mapsto x^{i+n\delta_{\phi_1(i),-1}}, \qquad a^i b \mapsto x^{i+n\delta_{\phi_2(i),-1}} y$$

where δ is the Kronecker delta. Assuming that ϕ_1 and ϕ_2 are normalized, let ψ be the cocycle for σ, i.e., $\psi(g, h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$. By Proposition 5 and [6, Corollary 2.5], ψ is orthogonal if and only if (ϕ_1, ϕ_2) is an NGP.

Set $\varphi(a^i) = \phi_1(i)$ and $\varphi(a^i b) = \phi_2(i)$. Then $\lambda = \psi\partial\varphi$ has matrix

$$\begin{bmatrix} A & A \\ B & -B \end{bmatrix}$$

where $A = [(-1)^{\lfloor (i+j)/n \rfloor}]_{0 \le i,j \le n-1}$ is back negacyclic, and B is A with rows r and $n - r + 1$ swapped for $1 \le r \le n$. Furthermore, $\partial\varphi = \prod_{k=1}^{n} \partial_k^{j_{k,1}} \partial_{n+k}^{j_{k,2}}$ under the stipulated ordering of G. □

We now undertake a case study of quasi-orthogonal cocycles over cyclic groups. Let $G = \mathbb{Z}_{4t+2}$ and index matrices by $1, \ldots, 4t+2$ in this order. The set $\mathcal{B} = \{\gamma, \partial_i \mid 2 \le i \le 4t+2\}$ where $\gamma = \gamma_{4t+2}$ (as defined before Proposition 3) is a basis of $Z^2(G, \mathbb{Z}_2)$. We get an elementary coboundary matrix $M_i := M_{\partial_i}$ by normalizing the back circulant matrix whose first row is 1s except for the ith entry. Also, M_γ is the back negacyclic matrix N of order $4t+2$.

Lemma 5. *Let $\psi \in Z^2(G, \mathbb{Z}_2) \setminus B^2(G, \mathbb{Z}_2)$, say $M_\psi = M_{i_1} \circ \cdots \circ M_{i_w} \circ N$. Then*

(i) *up to sign, M_ψ has ith row sum equal to its $(4t+4-i)$th row sum.*
(ii) *The $(2t+2)$th row sum of M_ψ is 0.*
(iii) *ψ is quasi-orthogonal if and only if the ith row sum of M_ψ is 0 for even i and ± 2 for odd $i > 1$.*

Proof. If $\psi \in \mathcal{B}$ then row $i > 2t+2$ of M or its negation is row $(4t+4-i)$ cycled $4t+4-i-1$ positions to the right. Part (i) then follows. For (ii), observe that row $2t+2$ in N is $[1 \overset{2t+1}{\cdots} 1 \; -1 \overset{2t+1}{\cdots} -1]$, whereas the first half of row $2t+2$ in M_i is identical to the second half. Finally, (iii) holds because the number of -1s in any row of M_i is even; and the rows of N indexed by an even (respectively, odd) integer have an odd (respectively, even) number of -1s. □

We use an approach borrowed from [1] to count the negative entries in a G-cocyclic matrix. Negating row i of M_i gives a *generalized coboundary matrix* $\overline{M}_i$, with exactly two -1s in each non-initial row r: these are in columns i and $[i-r+1]_{4t+2}$, where $[m]_n \in \{1, \ldots, n\}$ denotes the residue of m modulo n. (Although $\overline{M}_i$ is not cocyclic, row negation preserves row excess.) Hence the two generalized coboundary matrices with -1 in position (r, c) are $\overline{M}_c$ and $\overline{M}_{[r+c-1]_{4t+2}}$.

A set $\{\overline{M}_{i_j} : 1 \le j \le w\}$ defines an *r-walk* if there is an ordering $\overline{M}_{l_1}, \ldots, \overline{M}_{l_w}$ of its elements such that $\overline{M}_{l_i}$ and $\overline{M}_{l_{i+1}}$ both have -1 in row r and column l_{i+1}, for $1 \le i \le w$. The walk is an *r-path* if its initial (equivalently, final) element shares a -1 in row r with a generalized coboundary matrix not in the walk itself. Clearly, the number of -1s in row r of $\overline{M}_{i_1} \circ \cdots \circ \overline{M}_{i_w}$ is $2\mathcal{C}_r$ where $\mathcal{C}_r$ is the number of maximal r-paths in $\{\overline{M}_{i_1}, \ldots, \overline{M}_{i_w}\}$. To calculate $\mathcal{C}_r$ we set up a bipartite graph on vertex sets $S = \{i_1, \ldots, i_w\}$ and $T = \{[i_1 - r + 1]_{4t+2}, \ldots, [i_w - r + 1]_{4t+2}\}$. Draw an edge between $i_j \in S$ and $l \in T$ if $i_j = l$ or $l = [i_j - r + 1]_{4t+2} \in S$. The number of maximal paths in this bipartite graph is $\mathcal{C}_r$.

Next, let $\mathcal{I}_r$ be the number of columns where N and $\overline{M}_{i_1} \circ \cdots \circ \overline{M}_{i_w}$ share a -1 in row r. These column indices comprise the intersection of $\{4t+4-r, \ldots, 4t+2\}$ and the set of endpoints of the previously calculated maximal r-paths.

Theorem 6 (cf. [1, Proposition 1]). *A $\mathbb{Z}_{4t+2}$-cocyclic matrix $M_{i_1} \circ \cdots \circ M_{i_w} \circ N$ is quasi-orthogonal if and only if, for $2 \leq r \leq 2t+1$,*

$$\begin{array}{ll} \mathcal{C}_r \in \{\mathcal{I}_r + t + \frac{1-r}{2}, \mathcal{I}_r + t + \frac{3-r}{2}\} & r \text{ odd} \\ \mathcal{C}_r = \mathcal{I}_r + t + 1 - \frac{r}{2} & r \text{ even.} \end{array}$$

Proof. The number of -1s in row r of $\overline{M}_{i_1} \circ \cdots \circ \overline{M}_{i_w} \circ N$ is $2\mathcal{C}_r + r - 1 - 2\mathcal{I}_r$, so Lemma 5 gives the result. □

Corollary 2. *Let $\psi = \gamma \prod_{j=1}^{w} \partial_{i_j}$ with $\partial_{i_j} \in \mathcal{B}$. If ψ is quasi-orthogonal then $t \leq w \leq 3t+1$.*

Proof. We have $\mathcal{I}_2 = 0$, and $\mathcal{C}_2 = t$ by Theorem 6. Thus $t \leq w$. On the other hand, since the basis of coboundaries forms a 2-path, at least $t-1$ coboundaries must be removed to get t 2-paths. Hence $w \leq 4t - (t-1)$. □

Corollary 2 is equivalent to

Lemma 6. *If $\phi\colon \mathbb{Z}_{4t+2} \to \{\pm 1\}$ is a GOBS containing w occurrences of -1 then $t \leq w \leq 3t+1$.*

Proof. Negating all odd index entries or all even index entries of a GOBS produces another GOBS. So it may be assumed that $\phi(0) = \phi(4t+1) = 1$. □

We search for NGPs in the set of quasi-orthogonal cocycles over $\mathbb{Z}_{4t+2}$, motivated by the ubiquity of these cocycles and the optimal autocorrelation of each map in the resulting pair. Computer-aided searches found the NGPs in Table 1.

k	ϕ_1	ϕ_2
3	$1^2, 4$	$2, 1, 3$
5	$2, 1^3, 5$	$3, 1, 2, 1, 3$
7	$2, 1, 5, 1^3, 3$	$2, 1, 4, 2, 1^2, 3$
9	$3, 1, 2, 1^3, 3, 1, 5$	$2, 1, 2, 3, 2, 1^3, 5$
13	$3, 3, 2, 2, 1, 2, 1, 2, 1^4, 6$	$3, 3, 1, 3, 1, 2, 1, 2, 1^4, 6$
15	$3, 2, 4, 1^2, 2, 2, 1, 2, 1^5, 7$	$3, 2, 3, 2, 1, 2, 2, 1, 2, 1^5, 7$

Table 1. NGPs (ϕ_1, ϕ_2) from quasi-orthogonal cocycles over $\mathbb{Z}_{2k}$

Each sequence in Table 1 starts with 1 and is designated by an integer string, where i in the string means a run of i identical entries in the sequence, and 1^j is an alternating subsequence of length j. There are no NGPs among the sequences coming from quasi-orthogonal cocycles over $\mathbb{Z}_{22}$ (however, as we know, NGPs of length 22 exist). This gap could be related to the maximal determinant problem: the Ehlich-Wojtas bound is not attainable because 21 is not a sum of two squares.

Egan [8] classified NGPs of length $2k$ for $k \leq 10$ up to equivalence with respect to five elementary operations as defined in [4]. The set of NGPs that come from GOBSs is invariant under each elementary operation. Table 2 records the number $\hat{n}(k)$ of such NGPs of length $2k$, and the number $\hat{d}(k)$ of their equivalence classes. To compare against [8, Table 2], we have included the total number $n(k)$ of NGPs of length $2k$ and the number $d(k)$ of their equivalence classes.

k	$n(k)$	$\hat{n}(k)$	$d(k)$	$\hat{d}(k)$
3	576	576	1	1
5	11200	4800	3	2
7	90944	18816	5	1
9	1041984	62208	20	2

Table 2. Enumeration of NGPs and their equivalence classes

5 Normal cocyclic matrices

This section is essentially independent of the main thrust of the paper. Nonetheless, it addresses a fundamental question in algebraic design theory, which we answer in special cases that were the focus of Section 4.

A matrix M is *normal* if it commutes with its transpose (possibly up to row or column permutations), i.e., $\mathrm{Gr}(M) = \mathrm{Gr}(M^\top)$, where $\mathrm{Gr}(M)$ denotes the Grammian $MM^\top$. Many kinds of pairwise combinatorial designs are normal matrices (the defining pairwise constraint on rows implies the same constraint on columns; see [7, Chapter 7]). We also note that the matrix of a quasi-orthogonal cocycle is normal [3, Remark 6]. Thus, by the following lemma derived from (2), a cocycle ψ is quasi-orthogonal if and only if M_ψ has optimal column excess.

Lemma 7. *For any group G and $\psi \in Z^2(G, \mathbb{Z}_2)$,*

$$\mathrm{Gr}(M_\psi)_{ij} = \psi(g_i g_j^{-1}, g_j) \sum_{g \in G} \psi(g_i g_j^{-1}, g)$$

and

$$\mathrm{Gr}(M_\psi^\top)_{ij} = \psi(g_i, g_i^{-1}g_j)\sum_{g\in G}\psi(g, g_i^{-1}g_j).$$

We use Lemma 7 to prove that cocyclic matrices for two familiar classes of indexing groups are normal.

Proposition 6. *Let G be abelian or dihedral of order $2m$, m odd, and let $\psi \in Z^2(G,\mathbb{Z}_2)$ where $\psi \notin B^2(G,\mathbb{Z}_2)$ if G is dihedral. Then M_ψ is normal (under the same indexing of rows and columns by the elements of G).*

Proof. We suppose that G is generated by a and b, with $a^m = b^2 = 1$, and index rows and columns by the elements of G under the ordering $1, a, \ldots, a^{m-1}, b, ab, \ldots, a^{m-1}b$. A representative β for the non-identity element of $H^2(G,\mathbb{Z}_2)$ has matrix

$$\begin{bmatrix} J & J \\ J & -J \end{bmatrix}.$$

Thus, if G is abelian then M_ψ is symmetric and so trivially normal.

Henceforth G is dihedral. Let $\psi = \beta\partial\phi$. We collect together some basic properties of M_ψ.

(i) For each i, $\{\partial\phi(a^ib, a^j) \mid 1 \le j \le m\} = \{\partial\phi(a^ib, a^jb) \mid 1 \le j \le m\}$; and for each j, $\{\partial\phi(a^i, a^jb) \mid 1 \le i \le m\} = \{\partial\phi(a^ib, a^jb) \mid 1 \le i \le m\}$. Thus, if $k > m$ then the kth row sum and kth column sum of M_ψ are zero.

(ii) Since $\{\partial\phi(a^i, a^jb) \mid 1 \le j \le m\} = \{\partial\phi(a^jb, a^i) \mid 1 \le j \le m\}$, the kth row sum of M_ψ equals its kth column sum for $k \le m$.

Now we consider the Grammian quadrants in turn.

If $1 \le i \le m$ and $m+1 \le j \le 2m$ then

$$\mathrm{Gr}(M_\psi)_{ij} = \psi(a^{i+j-2}b, a^{j-1}b)\sum_{g\in G}\psi(a^{i+j-2}b, g) = 0$$

by Lemma 7 and (i); $\mathrm{Gr}(M_\psi^\top)_{ij} = 0$ similarly.

Let $1 \le i \le m$ and $1 \le j \le m$. Then

$$\mathrm{Gr}(M_\psi)_{ij} = \partial\phi(a^{i-j}, a^{j-1})\sum_{g\in G}\partial\phi(a^{i-j}, g) = \phi(a^{j-1})\phi(a^{i-1})\sum_{g\in G}\phi(g)\phi(a^{i-j}g)$$

and

$$\mathrm{Gr}(M_\psi^\top)_{ij} = \phi(a^{j-1})\phi(a^{i-1})\sum_{g\in G}\phi(g)\phi(ga^{j-i}).$$

These entries are equal by the identity $\sum_{k=1}^m \phi(a^k)\phi(a^{k+1}) = \sum_{k=1}^m \phi(a^k)\phi(a^{k-1})$.

Finally, let $m+1 \le i, j \le 2m$. Then

$$\mathrm{Gr}(M_\psi)_{ij} = \psi(a^{i-j}, a^{j-1}b) \sum_{g\in G} \psi(a^{i-j}, g)$$

and

$$\mathrm{Gr}(M_\psi^\top)_{ij} = \psi(a^{i-1}b, a^{i-j}) \sum_{g\in G} \psi(g, a^{i-j}).$$

Since $\psi(a^{i-1}b, a^{i-j}) = \partial\phi(a^{i-1}b, a^{i-j}) = \partial\phi(a^{i-j}, a^{j-1}b) = \psi(a^{i-j}, a^{j-1}b)$, we are done by (ii). □

Remark 6. There are plenty of examples of non-normal cocyclic matrices M_ψ for $\psi \notin B^2(G, \mathbb{Z}_2)$ and $|G|$ divisible by 4.

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