A Mixed Heuristic for Generating Cocyclic Hadamard Matrices

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Abstract A way of generating cocyclic Hadamard matrices is described, which combines a new heuristic, coming from a novel notion of fitness, and a peculiar local search, defined as a constraint satisfaction problem. Calculations support the idea that finding a cocyclic Hadamard matrix of order $4 \cdot 47$ might be within reach, for the first time, progressing further upon the ideas explained in this work.

Keywords Hadamard matrices · Cocyclic matrices · Heuristic · Constraint satisfaction problem

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1 Introduction

Square matrices with entries in $\{1, -1\}$ whose rows are pairwise orthogonal are termed *binary* Hadamard matrices, and were introduced in the nineteenth century, related to the maximum determinant problem. They are conjectured to exist for every order 1, 2 or multiple of 4, but nowadays the problem of their existence remains still open. Many constructions of Hadamard matrices are known. Among them, we are particularly interested in *cocyclic* Hadamard matrices, those Hadamard matrices *Hadamard equivalent* (by means of row/columns permutations/negations) to matrices $M_\psi = (\psi(i, j))$ related to some 2-cocycle $\psi \in Z^2(G; \mathbb{Z}_2)$, $\psi : G \times G \to \{\pm 1\}$, i.e. satisfying that

$$\psi(i, j)\ \psi(ij, k)\ \psi(i, jk)\ \psi(j, k) = 1, \quad i, j, k \in G,$$

for G a finite group with given order $\{1, \ldots, 4t\}$, 1 being the identity element.

Precisely, these relations yield a fast cocyclic Hadamard test, consisting in checking whether the summation of each row of M_ψ (but the first) is zero,

$$\sum_{j \in G} \psi(i, j) = 0, \quad i \in G \backslash \{1\}, \tag{1.1}$$

in which case ψ is said to be orthogonal.

Unfortunately, it is far from clear whether it is easier to construct cocyclic Hadamard matrices better than usual binary Hadamard matrices. Some light on this purpose was thrown in [5], where cocyclic Hadamard equivalence classes were identified up to order 40. The interested reader is referred to [6,10–12] for further explanations on (cocyclic) Hadamard matrices and related structures.

The set of cocycles forms an abelian group $Z^2(G, \mathbb{Z}_2)$ under pointwise multiplication. The simplest cocycles are the coboundaries ∂f, defined for any function $f : G \to \{\pm 1\}$ by $\partial f(g, h) = f(g)^{-1} f(h)^{-1} f(gh)$. The subgroup of coboundaries, $B^2(G, \mathbb{Z}_2)$, is naturally generated by the set of elementary coboundaries $\partial_i := \partial \delta_i$, where δ_i is the Kronecker delta function of the ith-element in G in the given ordering. Cocycles may be grouped into cohomological classes, to form $H^2(G; \mathbb{Z}_2) = Z^2(G, \mathbb{Z}_2)/B^2(G, \mathbb{Z}_2)$, so that $[\psi] = [\psi'] \in H^2(G, \mathbb{Z}_2) \Leftrightarrow \psi' = \psi \prod_{i=1}^{|G|} \partial_i^{r_i}$, for $r_i \in \{0, 1\}$. This way, fixed a representative cocycle $[\rho] \in H^2(G; \mathbb{Z}_2)$, in order to look for cocyclic Hadamard matrices M_ψ over G, for $[\psi] = [\rho] \in H^2(G; \mathbb{Z}_2)$, it suffices to look for a subset of elementary coboundaries ∂_{i_j} such that $\psi = \rho \prod_{j=1}^{k} \partial_{i_j}$ is orthogonal.

From (1.1), a cocyclic matrix M_ψ over G is Hadamard if and only if the summation of each row $2 \leq i \leq |G|$ is zero. This fact leads straightforwardly to a naive fitness function $f_1 : Z^2(G; \mathbb{Z}_2) \to \mathbb{Z}^+$, which measures the amount of rows of M_ψ which fail to sum up zero. Searching for a global minimum for f_1 would end providing a cocyclic Hadamard matrix M_ψ over G, if one exists, as soon as $f_1(\psi) = 0$.

Several years ago, some of the authors developed some heuristic procedures searching for cocyclic Hadamard matrices (see [3] and the references therein for details). Most of them used the notion of fitness defined above and seemed to work fine. However, although the procedures converged to a truly cocyclic Hadamard matrix, as desired, unexpectedly we have realized that there is not any apparent correlation between a matrix M_ψ being close to be Hadamard (say, differing just in one coboundary from being a cocyclic Hadamard matrix), and the fitness value $f_1(\psi)$ as it is actually defined (i.e. the number of its rows which fail to sum up zero), which by no means has necessarily to be close to zero as well.

Table 1 (distributed in several pages due to size limitations) shows some examples of cocyclic Hadamard matrices M_ψ over D_{4t}, for odd values of $3 \leq t \leq 17$, so that modified by the action of just one coboundary ∂_i (as indicated in each case), give rise to cocyclic matrices attaining fitness $f_1(\psi \partial_i)$ which almost fulfill the prospective range $[1, t-1]$. Here we assume $D_{4t} = \langle a, b :\ a^{2t} = b^2 = 1, ab = ba^{-1} \rangle$ and the ordering $\{1, a, \ldots, a^{2t-1}, b, ab, \ldots, a^{2t-1}b\}$ in D_{4t}. The representative cocycle ρ is fixed to be

$$M_\rho = \begin{pmatrix} BN_{2t} & BN_{2t} \\ BN_{2t}^s & -BN_{2t}^s \end{pmatrix}$$

Table 1 f_1 does not depend on the Hamming distance from a full Hadamard matrix!

t	$i_j\,:\ \psi=\rho\prod_{j=1}^{k}\partial_{i_j}$ is orthogonal	$i\,:\ \psi'=\psi\cdot\partial_i$	$f_1(\psi\partial_i)$
3	5, 10	2	1
		8	2
5	7, 8, 12, 16, 18	8	1
		2	2
		3	3
		14	4
7	2, 6, 7, 17, 19, 22, 23, 25	5	1
		4	2
		2	3
		7	4
		12	5
7	6, 9, 17, 19, 21, 24, 25	19	6
9	2, 7, 8, 12, 21, 23, 25, 28, 29, 31, 32	32	1
		21	2
		3	3
		2	4
		7	5
		10	6
		23	7
11	2, 3, 6, 8, 9, 10, 11, 12, 13, 15, 18, 19, 21, 22,	25	2
	24, 26, 30, 35, 36, 37, 42	13	3
		4	4
		3	5
		15	6
		2	7
		36	8
11	2, 7, 9, 11, 15, 20,	22	1
	23, 24, 25, 29, 30, 31, 34, 35, 37, 38, 40, 41	11	9
11	3, 5, 7, 10, 11, 13, 14, 15, 16, 20,	5	10
	23, 24, 26, 29, 30, 31, 35, 36,38		
13	3, 4, 6, 8, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 24, 26,	4	2
	28, 31,32, 35, 39, 40, 42, 43, 44, 45, 50	7	3
		2	4
		5	5
		12	6
		6	7
		15	8
		20	9
		17	10
13	2, 5, 6, 7, 9, 10, 11, 16, 17, 19, 21, 22, 23, 25, 26,	28	1
	27, 29, 32, 33, 34, 35, 36, 38, 39, 41, 43, 44, 46, 47, 48, 49, 50	41	11
13	3, 5, 7, 8, 9, 11, 12, 15, 19, 22, 24, 25,	29	12
	27, 29, 31, 32, 33, 34, 35, 36, 40, 41, 42, 45, 46, 47		

Table 1 continued

t	$i_j\,:\ \psi=\rho\prod_{j=1}^{k}\partial_{i_j}$ is orthogonal	$i\,:\ \psi'=\psi\cdot\partial_i$	$f_1(\psi\partial_i)$
15	4, 6, 7, 8, 9, 12, 13, 15, 16, 19, 21, 23, 24, 25, 26, 28, 29, 30,	13	2
	34, 36, 40, 41, 44, 49, 50, 52, 54, 55	50	3
		12	4
		4	5
		3	6
		2	7
		10	8
		8	9
		27	10
		35	11
		28	12
15	2, 4, 5, 7, 8, 10, 11, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28,	32	1
	33, 36, 37, 38, 40, 41, 42, 43, 44, 46, 48, 50, 51, 52, 53, 56, 57, 58	47	13
17	6, 7, 9, 11, 12, 13, 15, 19, 24, 29,	7	2
	38, 39, 40, 41, 43, 47, 50, 51,54, 55, 57, 59, 60, 62, 65	24	14
17	2, 6, 8, 9, 13, 16, 20, 22, 23, 24, 27, 29, 32,33,		
	39, 42, 45, 46,47, 50, 51, 52, 60, 62	44	3
		17	4
		43	5
		25	6
		3	7
		2	8
		4	9
		8	10
		51	11
		34	12
		61	13

for BN_{2t} being the back negacyclic matrix of order $2t$, and BN_{2t}^s being the matrix obtained from BN_{2t} by displaying its rows from bottom to top.

Notice that the only cases which are missed in these tables are fitness 8 for $t=9$, fitness 14 for $t=15$ and fitness 1, 15 and 16 for $t=17$. Anyway, we do not know whether some of them could actually exist, since we have not performed any exhaustive search, that is out of interest for our purposes. What is worth stressing is that, attending to the notion of fitness defined so far, cocyclic matrices $M_{\psi'}$ which fail to be Hadamard in just one coboudary might have any possible value of fitness $f_1(\psi')$, whereas it would have been desirable that they had fitness close to zero.

Actually, this fact has motivated that we look for another way to define a new fitness function f_2, which faithfully reflects how close is a matrix M_ψ to be Hadamard, in terms of a lower bound on the number of coboundaries which have to be modified in the expression defining ψ so that a cocyclic Hadamard matrix might be obtained. This is the main concern of the paper.

2 A New Fitness Function

As introduced in (1.1), we already know that a cocyclic matrix M_ψ over G is Hadamard if and only if the summation of each row $2\leq i\leq |G|$ is zero.

In [2] a way to check these conditions was described, in terms of *paths* and *intersections* of coboundaries, which we reproduce now.

Every 2-cocycle over G may be described as a pointwise product $\psi = \rho \prod_{j=1}^{k} \partial_{i_j}$ of a representative (non trivial) cocycle $\rho \in H^2(G;\mathbb{Z}_2)$ and a subset of elementary coboundaries ∂_{i_j}, for $\partial_{i_j}(h,k) = \delta(i_j,h)\delta(i_j,k)\delta(i_j,hk)$, where δ is the Kronecker delta function. Notice that such a subset $\{\partial_{i_j}\}$ is not unique, in general, until a basis for coboundaries is fixed.

There is empirical evidence (at least, for the groups more intensively studied, see [2,4,8,10] for instance), that there always exists a choice ρ of representative cocycle that tends to be the most successful for providing Hadamard matrices. For this reason, in the remainder of the paper, we will consider that such a ρ is fixed, and hence the only variables to consider are the set of $4t$ elementary coboundaries arising from G. Therefore, when convenient throughout the paper, any cocycle $\psi = \rho \prod_{j=1}^{k} \partial_{i_j}$ (or indistinctly the cocyclic matrix M_ψ beneath, as well) may be naturally displayed as binary tuples $\mathbf{v}_\psi$ of length $4t$, the ith entry being 1 if and only if $i \in \{i_1,\ldots,i_k\}$, and 0 otherwise.

Since negating a row of any coboundary matrix M_{∂_h} has no effect in the validity of the cocyclic Hadamard test (1.1) for ψ, in what follows we consider *generalized* coboundary matrices instead, obtained from the former by just negating the hth row. Nevertheless, for simplicity and convenience of the reader, we will preserve the same notation for both structures.

At any row $2 \le r \le 4t$, every generalized coboundary matrix M_{∂_h} has exactly two negative entries, located at positions (r,h) and $(r,r^{-1}h)$, termed *head* and *tail*, respectively. When the tail of a coboundary is the head of another coboundary at a row r, these coboundaries are said to form a *r-path*. Analogously, when a coboundary share a negative entry (no matter it is a head or a tail) at a row r with the cocycle ρ then they form a *r-intersection*.

Consequently, the cocyclic Hadamard test (1.1) may be reformulated in terms of paths and intersections, so that

$$c_r - I_r = t - \frac{\rho_r}{2}, \quad 2 \le r \le 4t, \tag{2.1}$$

where c_r denotes the number of maximal r-paths in which the subset of coboundaries defining ψ splits into, I_r denotes the number of r-intersections that they give rise to, and ρ_r denotes the amount of negative entries of M_ρ at row r.

Notice that the right-hand sides of the Eq. (2.1) are constant, and do not depend on the subset of coboundaries defining ψ. However, the change of just one coboundary in the expression defining ψ, no matter either eliminated or introduced, reflects in the left-hand sides of Eq. (2.1) as changes of at most 1 unit (either increasing or decreasing) with respect to their former values, as we prove now.

Proposition 2.1 *Given any coboundary ∂_h, the left-hand sides of Eq.* (2.1) *corresponding to the cocycles ψ and $\psi \cdot \partial_h$ differ at most in 1, in absolute value.*

Proof Consider a cocycle $\psi = \rho \prod_{j=1}^{k} \partial_{i_j}$ and a coboundary ∂_h. Fixed a row r, $2 \le r \le 4t$, Table 2 shows the range in which the number c_r of r-paths and I_r of r-intersections vary from ψ to $\psi\partial_h$ depending on whether none, one or even both between the head and the tail of ∂_h are ($\times$) or not ($\circ$) ends of a maximal path in ψ as well. It is worth noting that the states of such head and tail necessarily change from ψ to $\psi\partial_h$, as a straightforward consequence that these are precisely the 2 negative entries that ∂_h contributes with.

Therefore, the left-hand sides of Eq. (2.1) corresponding to ψ and $\psi \cdot \partial_h$ can never differ more than 1, in absolute value, as claimed. □

Table 2 Analyzing how $\Delta(c_r - I_r)$ varies from ψ to $\psi\partial_h$

$\psi \to \psi\partial_h$	Δc_r	ΔI_r	$\Delta(c_r - I_r)$
$\{\circ,\circ\} \to \{\times,\times\}$	$+1$	$0,+1,+2$	$-1,0,+1$
$\{\circ,\times\} \to \{\times,\circ\}$	0	$-1,0,+1$	$-1,0,+1$
$\{\times,\times\} \to \{\circ,\circ\}$	-1	$-2,-1,0$	$-1,0,+1$

The following result, which is a straightforward consequence of Proposition 2.1, includes a natural formulation for a fitness function f_2 meeting the conditions we are expecting for.

Corollary 2.2 *A lower bound on the number of coboundaries needed to transform* $\psi = \rho \prod_{j=1}^{k} \partial_{i_j}$ *into an orthogonal cocycle is*

$$f_2(\psi) = \max_{2 \leq r \leq 4t} \left|(c_r - I_r) - \left(t - \frac{\rho_r}{2}\right)\right|. \tag{2.2}$$

Actually, as commented before, the space of 2^{4t} cocyclic matrices M_ψ may be naturally displayed in terms of the vectors $\mathbf{v}_\psi$ of their coordinates with regards to the set of elementary coboundaries $\{\partial_1, \ldots, \partial_{4t}\}$. Attending to Corollary 2.2, vectors at Hamming distance 1 should have comparable fitness, as desired.

Even more, it is worth stressing that f_2 helps to prune the search space in an impressive way, since it precisely gives a lower bound on the number of coboundaries to modify in the expression defining ψ so that it can be moved to a cocyclic Hadamard matrix. This way, given ψ, we know that no cocyclic Hadamard matrix exists in a Hamming ratio of $f_2(\psi) - 1$ from ψ. Conversely, a further application of this fact is to decide good zones for performing some local searches.

We are now in conditions to design a new heuristic looking for cocyclic Hadamard matrices.

3 The Heuristic

By construction, a cocyclic Hadamard matrix over G corresponds to a global minimum for f_2. This section is devoted to design a heuristic trying to locate such a global minimum. As usual, the main problem to tackle is dealing with local minima.

You can imagine the space of 2^{4t} cocyclic matrices M_ψ organized by their fitness $f_2(\psi)$ as the surface of the sea. No matter the sea is in calm, the number of valleys and hills that the waves originate on its surface tends to be extremely huge. In these circumstances, it is quite easy to locate some local minima, but nevertheless it is extremely difficult to locate points corresponding to minimum height. This is the image we can think of in our minds for the search we are dealing with. Furthermore, the proportion of cocyclic Hadamard matrices with regards to the total space of cocyclic matrices seems to be negligible, as calculations in [2,4,8,10] suggest.

A simple idea would be to perform a step by step search, moving from a given cocycle $\mathbf{v}_\psi$ to some of its $4t$ neighbors $\mathbf{v}_{\psi'}$ at Hamming distance 1, so that $f_2(\psi') < f_2(\psi)$.

Unfortunately, it occurs surprisingly often that such a cocycle ψ' does not exist. In fact, from (2.2), a trivial upper bound for f_2 is $2t + \max(0, \frac{\rho_r}{2} - t) \ll 2^{4t}$, the latter being the total amount of cocyclic matrices. Consequently, there are many different cocyclic matrices meeting each of the possible values in the image of f_2. And it is quite easy that, for any cocyclic matrix M_ψ randomly chosen, none of its neighbors ψ' improves the value $f_2(\psi)$ (in which case, ψ is actually a local minimum for f_2).

For instance, taking ρ and the dihedral group D_{4t} as explained above, Table 3 shows the number $\#_i$ of local minima attaining fitness $f_2 = i$.

A second approach would require to relax the condition considered so far for moving from one vertex to another, in such a way that $f_2(\psi') = f_2(\psi)$ might suffice to move from ψ to ψ'. The question now is determining some subsidiary criteria to break the tie when no neighbor ψ' exists satisfying $f_2(\psi') < f_2(\psi)$, but some do exist such

Table 3 Number of local minima

$t \backslash i$	0	1	2	3
3	72	66		
5	1400	9240	6416	1600

Table 4 D_{4t}-cocyclic Hadamard matrices

t	Iter	Time (s)	$i_j\,:\ \psi=\rho\prod_{j=1}^{k}\partial_{i_j}$ is orthogonal
5	1	0.03	1, 3, 5, 8, 11, 18, 19
7	2	0.17	2, 4, 7, 10, 11, 12, 13, 14, 16, 17, 18, 20, 21, 22, 26, 27
9	4	0.65	2, 3, 4, 7, 9, 11, 12, 13, 14, 15, 16, 17, 20, 24, 25, 30, 31, 33, 36
11	116	31	4, 6, 8, 9, 12, 16, 18, 19, 21, 22, 23, 27, 32, 34, 35, 41, 42, 43
13	242	61	1, 9, 12, 13, 14, 16, 19, 20, 22, 23, 24, 25, 26, 29, 30, 33, 35, 40, 41, 42, 44
15	1374	852	3, 5, 7, 8, 9, 11, 13, 14, 17, 21, 22, 29, 31, 32, 34, 35, 37, 41, 42, 43, 44, 46, 47, 48, 50, 51, 57, 58

that equality holds. This would permit to avoid staying around a local minimum, and to facilitate moving to explore some other promising zones of the search space.

As introduced in (2.2), $f_2(M_\psi)$ gives the maximum from the differences $\left|(c_r-I_r)-(t-\frac{\rho_r}{2})\right|$ along the $4t-1$ rows in the range $2\le r\le 4t$. A way to fine tuning the search in order to break ties may be reached by associating to every cocycle ψ a vector $\mathbf{d}_\psi$ of length $1+f_2(M_\psi)$ which counts the number of times that every integer instance in the range $[0,\, f_2(M_\psi)]$ appears along these $4t-1$ differences. In these circumstances, given two cocycles sharing the same fitness $f_2(M_\psi)=f_2(M_{\psi'})$, one may assume that ψ' is better than ψ if $\mathbf{d}_{\psi'}$ is lesser than $\mathbf{d}_\psi$ in colexicographical order. That is $\mathbf{d}_{\psi'}=(a_1,\ldots a_s)<(b_1,\ldots b_t)=\mathbf{d}_\psi$ if and only if $s<t$ or $s=t$ and $a_i<b_i$ for the last i where a_i and b_i do differ.

When a stationary point (i.e. a local minimum) is reached, in the sense that no neighbor exists at Hamming distance 1 which improves its fitness, then a random move is performed, to reach a new base point at certain prefixed Hamming distance s. This distance s should simultaneously take us close enough to this "potentially" good zone around this local minimum, but at the same time far enough so that we seldom could reach the same stationary point more than once. Empirical results suggest that $s=6$ seems to meet these requirements. This value have been used in the calculations included in Table 4, which improve the results obtained by those heuristics [1,3] using the older and naive fitness notion, f_1, looking for cocyclic Hadamard matrices over D_{4t}. Notice that D_{4t} seems to be the most prolific group for providing cocyclic matrices, see [10] and the references therein.

However, the heuristic fails to give Hadamard matrices for $t\ge 17$ in reasonable time. Actually, the calculations we attempted for $t=17$ were aborted after 70.000 iterations and about 18 hours of computation each. Nevertheless, our determination held on. We turned to think for a local search which helped to improve the heuristic, so that larger matrices might be found.

Undoubtedly, a cocyclic Hadamard matrix is more likely to be found around multiple instances of coyclic matrices $M_{\psi'}$ satisfying $f_2(M_{\psi'})=1$. As a matter of fact, notice that the $4t$ neighbors ψ' of a cocycle ψ defining a cocyclic Hadamard matrix M_ψ satisfy $f_2(\psi')\le 1$.

Experimental results show that, as t increases, it is not as hard as one could expect to find a cocycle ψ satisfying $f_2(\psi)=1$. Actually, after 10 runs in each case, Table 5 shows the average fitness fit of the randomly generated initial cocycle, the average number of iterations $Iter$ as well as the average computing time $Time$ required for reaching a cocycle ψ over D_{4t} satisfying $f_2(\psi)=1$, for odd values of $t\le 47$.

For this reason, every time that a cocycle ψ' meeting $f_2(\psi')=1$ is found, it makes sense to perform a local search, looking exhaustively for cocyclic matrices of fitness at most 1, in the ball $B(\psi',s)$ determined by some fixed Hamming distance s from ψ'. Hopefully, this process might lead to a path of sufficiently close cocycles ψ' all of which meet fitness 1, until a cocyclic Hadamard matrix is finally found.

Nevertheless, the number of matrices in such a ball $B(\psi',s)$ grows exponentially on s, as $\sum_{w=0}^{s}\binom{4t}{w}$. Depending on t, the computation capabilities which are available naturally limit the largest parameter s one might consider.

Table 5 Analyzing the convergence to fitness 1

t	15	17	19	21	23	25	27	29	31
Fit	4.2	4.3	4.4	5.2	5.5	5.4	6.6	5.7	6.9
Iter	1.2	1.4	1.3	2	2.1	3.8	6.1	4.8	9.5
Time (s)	0.46	0.75	0.96	1.98	2.59	7.35	14.78	14.07	33.79
t	33	35	37	39	41	43	45	47	
Fit	6.9	7.7	7.2	6.5	8.5	8	8.4	8.4	
Iter	19.4	20.1	66	245	243.5	650.1	2072	3013.8	
Time (s)	82	101	393	2094	2936	3905	20,703	34,621	

Table 6 The size of $B(\psi, s)$, for $1 \leq s \leq 5$

s	1	2	3	4	5
$\|B(\psi, s)\|$	189	17,767	1,107,603	51,512,518	1,906,413,390

For instance, for $t = 47$, which is actually the first occurrence for which no cocyclic Hadamard matrix is known yet, the sizes of these balls grows as 2^{5s+10}, as Table 6 shows.
So that it could hardly be supported to perform a search for $s \geq 5$. Even the case $s = 4$, which requires managing up to $5 \cdot 10^7$ instances, might be unacceptable for practical purposes. That was the case, until unexpectedly the translation of the problem in terms of a Constraint Satisfaction Problem [7] (CSP in brief, hereafter), made it possible.

Roughly speaking, the idea beyond the constraint satisfaction paradigm is characterizing a problem as a set of constraints to be simultaneously satisfied, and then find the set of solutions by means of a constraint solver. No matter the same solver is used, a given problem might possibly admit many different formulations as a CSP, each of them requiring possibly different amount of resources (not only variables and/or constraints, but also CPU memory) and consequently different running computation times in turn.

An explicit formulation of a CSP looking for all cocycles $\psi' \in B(\psi, 4)$ such that $f_2(\psi') \leq 1$ may be described as follows. We look for binary vectors $\mathbf{v}_{\psi'} = (x_0, \ldots, x_{4t-1})$, which satisfy the conditions:

$$\left|(t-j) - \frac{1}{2}\sum_{i=0}^{4t-1} \sigma_i \left| x_i - x_{2t\lfloor \frac{i}{2t} \rfloor + (i+j \mod 2t)} \right| \right| \leq 1, \tag{3.1}$$

for $\sigma_i = 1$ if $0 \leq i \leq 2t - j$ or $2t \leq i \leq 4t - j$ and -1 elsewhere, and $1 \leq j \leq t - 1$.

Once the model is fixed, the following step consists in carrying it into a solver. We have used MINION [9], one of the fastest and most scalable constraint solvers using the "model and run" methodology. It is a black box from the user point of view, deliberately providing few options, but guaranteeing raw speed in return.

The vector $\mathbf{v}_{\psi'} = (x_0, \ldots, x_{4t-1})$ is naturally codified as a boolean vector `ingred` of $4t$ unknowns. The $t-1$ relations (3.1) may be translated by means of a boolean matrix `camenosin` of size $(t-1) \times 4t$, such that `camenosin`$[j-1, i] = |x_i - x_{2t\lfloor \frac{i}{2t} \rfloor + (i+j \mod 2t)}|$ for $1 \leq j \leq t-1$. In these circumstances, it suffices to impose the constraints

$$\texttt{weightedsumleq}([\overbrace{1,\ldots,1}^{2t-(j-1)}, \overbrace{-1,\ldots,-1}^{j-1}, \overbrace{1,\ldots,1}^{2t-(j-1)}, \overbrace{-1,\ldots,-1}^{j-1}], \texttt{camenosin}[j, \_], 2(t-j))$$

$$\texttt{weightedsumleq}([\overbrace{1,\ldots,1}^{2t-(j-1)}, \overbrace{-1,\ldots,-1}^{j-1}, \overbrace{1,\ldots,1}^{2t-(j-1)}, \overbrace{-1,\ldots,-1}^{j-1}], \texttt{camenosin}[j, \_], 2(t-2-j))$$

for $0 \leq j \leq t-2$.

Table 7 Calculations for $19 \leq t \leq 23$

t	ψ_0	k	ψ_k	MINION (s)
17	$e1c966f450819e555$	1, 12, 48, 68 41, 46, 47, 49 12, 53, 66	$61c966f4500616550$	32
19	$114bd7765e598c2e7c8$	16, 31, 34, 68 29, 47, 55, 57 36, 41, 59, 69	$114ad77c0edb8e8e648$	67
21	$1627dc8eadec977dc78d8$	29, 35, 47, 71 2, 7, 8, 18 8, 25, 28, 33 25, 39, 50, 64 9, 24, 65	$54a79d960feed77c458d8$	122
23	$95ee3e2bbb65ef2df84cfeb$	19, 25, 28, 81 2, 12, 13, 48 12, 17, 79, 92 17, 39,55, 75 7, 43, 49, 77 36, 40, 43, 83 7, 34, 52, 71 31, 38, 40, 83 7, 20, 86, 91 1, 31, 50	$57e60ebbed643d2dfa667a8$	219

From these data a MINION file .min modeling our CSP may be then straightforwardly generated. Running over dihedral groups D_{4t}, for $t = 47$, the procedure takes barely 5 hours of computing time, on a standard Intel(R) Core(TM) i7-5500U CPU, 2.40 GHz, 8 GB RAM.

In order to locate a cocyclic Hadamard matrix, both the heuristic and the local search just described may be combined as follows. Firstly, the heuristic is used until a cocycle ψ_0 is located such that $f_2(\psi_0) = 1$. Starting from ψ_0, a sequence of different cocycles $\{\psi_k\}$ is constructed, such that $\psi_{k+1} \in B(\psi_k, 4)$ and $f_2(\psi_{k+1}) \leq 1$. The process is repeated until either a no valid successor cocycle exists, or hopefully a truly orthogonal cocycle is found.

As an illustration of how the method performs, we now include in Table 7 some calculations which ultimately succeed locating a cocyclic Hadamard matrix over D_{4t}, for some t. We indicate the initial cocycle ψ_0 provided by the heuristic, the chain of subsets of (at most) 4 coboundaries which are changed to move from ψ_i to ψ_{i+1}, the orthogonal cocycle ψ_k obtained as the tail of the chain, as well as the computational time required in every isolated call to MINION. Due to space restrictions, the $4t$-binary tuples representing cocycles have been translated to their equivalent hexadecimal forms, in such a way that every possible nibble (group of 4 bits) from 0000 to 1111 is encoded as its corresponding hexadecimal digit from 0 to f, as usual.

Further examples require an important amount of computational work and time, which is out of the scope of the paper. This effort should be straightforward directed to the case $t = 47$.

Notice that there is not any specific criteria to choose a particular cocycle $\psi_{i+1} \in B(\psi_i, 4)$ among those meeting fitness $f_2(\psi') = 1$. Nevertheless, there is not any guarantee that the process will end providing an orthogonal cocycle, either. Even more, there is not any certainty about the finiteness of such a sequence $\{\psi_i\}$. These questions should require further investigation.

4 Conclusions and Further Work

We have introduced a new fitness function f_2, which faithfully reflects how close is a matrix M_ψ to be Hadamard, in terms of a lower bound (2.2) on the number of coboundaries which have to be modified in the expression defining ψ so that a cocyclic Hadamard matrix might be obtained.

Progressing on this notion, a heuristic has been described looking for cocyclic Hadamard matrices. In order to improve the convergence, a local search has been incorporated to the procedure, in terms of a constraint satisfaction problem, looking for dense zones of cocycles ψ meeting fitness $f_2(\psi) = 1$.

As it has been shown, it is far beyond becoming real that this new heuristic finds out a global minimum in practise, for large t. Some alternative ideas should be considered in order to characterize zones closed to cocyclic Hadamard matrices more reliably, so that a finer tuning might be performed in turn.

Actually, from Proposition 2.1, it follows that $|f_2(\psi') - f_2(\psi)| \leq 1$ for every 1-Hamming distance neighbor ψ' of ψ (that is, defining a vector $\mathbf{v}_{\psi'}$ differing from $\mathbf{v}_\psi$ in just one coordinate). Consequently, given any cocycle ψ' at Hamming distance k from a cocyclic Hadamard matrix ψ, there is a straightforward upper bound for the summation of the fitness values of those cocycles at Hamming distance s from ψ', since they may be organized attending to the number w of coordinates shared simultaneously with both ψ and ψ'. Indeed, given $w \leq s$, it follows that there are $\binom{k}{s-w} \cdot \binom{4t-k}{w}$ cocycles simultaneously at Hamming distance $k - s + 2w$ from ψ and s from ψ'. Let $B(\psi, s)$ denote the set of cocycles at Hamming distance s from ψ.

Corollary 4.1 *A necessary condition for ψ having Hamming distance k from the closest cocyclic Hadamard matrix is that $f_2(\psi) = k$ and*

$$f_{2,s}(\psi) = \sum_{\psi' \in B(\psi,s)} f_2(\psi') \leq \sum_{w=0}^{s} \binom{k}{s-w} \cdot \binom{4t-k}{w} (k - s + 2w) \tag{4.1}$$

for $0 \leq s \leq k$.

In practise, one could take advantage of this result and use $f_{2,s}$ instead of colexicographical order as secondary condition for moving from one cocycle ψ to a neighbor ψ', in case that $f_2(\psi') = f_2(\psi)$. Unfortunately, though we have attempted to perform some runs for $s = 1$ and $t = 47$, it takes too much computational time to proceed from one iteration to another, so that we have limited ourselves to complete some few iterations, with no significant improvements with regards to the calculations developed in the previous section.

In a near future, the effort should be concentrated in order to isolate a cocyclic Hadamard matrix of order $4t = 4 \cdot 47$.

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