STRONG EXTENSIONS FOR q-SUMMING OPERATORS ACTING IN p-CONVEX BANACH FUNCTION SPACES FOR $1 \leq p \leq q$

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Abstract. Let $1 \leq p \leq q < \infty$ and let X be a p-convex Banach function space over a σ-finite measure μ. We combine the structure of the spaces $L^p(\mu)$ and $L^q(\xi)$ for constructing the new space $S^q_{X_p}(\xi)$, where ξ is a probability Radon measure on a certain compact set associated to X. We show some of its properties, and the relevant fact that every q-summing operator T defined on X can be continuously (strongly) extended to $S^q_{X_p}(\xi)$. This result turns out to be a mixture of the Pietsch and Maurey-Rosenthal factorization theorems, which provide (strong) factorizations for q-summing operators through L^q-spaces when $1 \leq q \leq p$. Thus, our result completes the picture, showing what happens in the complementary case $1 \leq p \leq q$, opening the door to the study of the multilinear versions of q-summing operators also in these cases.

1. Introduction

Fix $1 \leq p \leq q < \infty$ and let $T\colon X \to E$ be a Banach space valued linear operator defined on a saturated order semi-continuous Banach function space X related to a σ-finite measure μ. In this paper we prove an extension theorem for T in the case when T is q-summing and X is p-convex. In order to do this, we first define and analyze a new class

Date: June 18, 2018.

2010 *Mathematics Subject Classification.* 46E30, 47B38.

Key words and phrases. Banach function spaces, extension of operators, order continuity, p-convexity, q-summing operators.

The first author gratefully acknowledge the support of the Ministerio de Economía y Competitividad (project #MTM2012-36732-C03-03) and the Junta de Andalucía (projects FQM-262 and FQM-7276), Spain.

The second author acknowledges with thanks the support of the Ministerio de Economía y Competitividad (project #MTM2012-36740-C02-02), Spain.

of Banach function spaces denoted by $S^q_{X_p}(\xi)$ which have some good properties, mainly order continuity and p-convexity. The space $S^q_{X_p}(\xi)$ is constructed by using the spaces $L^p(\mu)$ and $L^q(\xi)$, where ξ is a finite positive Radon measure on a certain compact set associated to X.

Corollary 5.2 states the desired extension for T. Namely, if T is q-summing and X is p-convex then T can be strongly extended continuously to a space of the type $S^q_{X_p}(\xi)$. Here we use the term "strongly" for this extension to remark that the map carrying X into $S^q_{X_p}(\xi)$ is actually injective; as the reader will notice (Proposition 3.1), this is one of the goals of our result. In order to develop our arguments, we introduce a new geometric tool which we call the family of p-strongly q-concave operators. The inclusion of X into $S^q_{X_p}(\xi)$ turns out to belong to this family, in particular, it is q-concave.

If T is q-summing then it is p-strongly q-concave (Proposition 5.1). Actually, in Theorem 4.4 we show that in the case when X is p-convex, T can be continuously extended to a space $S^q_{X_p}(\xi)$ if and only if T is p-strongly q-concave. This result can be understood as an extension of some well-known relevant factorizations of the operator theory:

(I) Maurey-Rosenthal factorization theorem: If T is q-concave and X is q-convex and order continuous, then T can be extended to a weighted L^q-space related to μ, see for instance [3, Corollary 5]. Several generalizations and applications of the ideas behind this fundamental factorization theorem have been recently obtained, see [1, 2, 4, 5, 9].

(II) Pietsch factorization theorem: If T is q-summing then it factors through a closed subspace of $L^q(\xi)$, where ξ is a probability Radon measure on a certain compact set associated to X, see for instance [6, Theorem 2.13].

In Theorem 4.4, the extreme case $p = q$ gives a Maurey-Rosenthal type factorization, while the other extreme case $p = 1$ gives a Pietsch type factorization. We must say also that our generalization will allow to face the problem of the factorization of several p-summing type of multilinear operators from products of Banach function spaces —a topic of current interest—, since it allows to understand factorization of q-summing operators from p-convex function lattices from a unified point of view not depending on the order relation between p and q.

As a consequence of Theorem 4.4, we also prove a kind of Kakutani representation theorem (see for instance [7, Theorem 1.b.2]) through the spaces $S^q_{X_p}(\xi)$ for p-convex Banach function spaces which are p-strongly q-concave (Corollary 4.5).

2. Preliminaries

Let (Ω, Σ, μ) be a σ-finite measure space and denote by $L^0(\mu)$ the space of all measurable real functions on Ω, where functions which are equal μ-a.e. are identified. By a *Banach function space* (briefly B.f.s.) we mean a Banach space $X \subset L^0(\mu)$ with norm $\|\cdot\|_X$, such that if $f \in L^0(\mu)$, $g \in X$ and $|f| \leq |g|$ μ-a.e. then $f \in X$ and $\|f\|_X \leq \|g\|_X$. In particular, X is a Banach lattice with the μ-a.e. pointwise order, in which the convergence in norm of a sequence implies the convergence μ-a.e. for some subsequence. A B.f.s. X is said to be *saturated* if there exists no $A \in \Sigma$ with $\mu(A) > 0$ such that $f\chi_A = 0$ μ-a.e. for all $f \in X$, or equivalently, if X has a *weak unit* (i.e. $g \in X$ such that $g > 0$ μ-a.e.).

Lemma 2.1. *Let X be a saturated B.f.s. For every $f \in L^0(\mu)$, there exists $(f_n)_{n\geq 1} \subset X$ such that $0 \leq f_n \uparrow |f|$ μ-a.e.*

Proof. Consider a weak unit $g \in X$ and take $g_n = ng/(1+ng)$. Note that $0 < g_n < ng$ μ-a.e., so g_n is a weak unit in X. Moreover, $(g_n)_{n\geq 1}$ increases μ-a.e. to the constant function equal to 1. Now, take $f_n = g_n|f|\chi_{\{\omega\in\Omega:\,|f|\leq n\}}$. Since $0 \leq f_n \leq ng_n$ μ-a.e., we have that $f_n \in X$, and $f_n \uparrow |f|$ μ-a.e. □

The *Köthe dual* of a B.f.s. X is the space X' given by the functions $h \in L^0(\mu)$ such that $\int |hf|\,d\mu < \infty$ for all $f \in X$. If X is saturated then X' is a saturated B.f.s. with norm $\|h\|_{X'} = \sup_{f\in B_X} \int |hf|\,d\mu$ for $h \in X'$. Here, as usual, B_X denotes the closed unit ball of X. Each function $h \in X'$ defines a functional $\zeta(h)$ on X by $\langle \zeta(h), f\rangle = \int hf\,d\mu$ for all $f \in X$. In fact, X' is isometrically order isomorphic (via ζ) to a closed subspace of the topological dual X^* of X.

From now and on, a B.f.s. X will be assumed to be saturated. If for every $f, f_n \in X$ such that $0 \leq f_n \uparrow f$ μ-a.e. it follows that $\|f_n\|_X \uparrow$

$\|f\|_X$, then X is said to be *order semi-continuous*. This is equivalent to $\zeta(X')$ being a *norming subspace* of X^*, i.e. $\|f\|_X = \sup_{h\in B_{X'}} \int |fh|\,d\mu$ for all $f \in X$. A B.f.s. X is *order continuous* if for every $f, f_n \in X$ such that $0 \le f_n \uparrow f$ μ-a.e., it follows that $f_n \to f$ in norm. In this case, X' can be identified with X^*.

For general issues related to B.f.s.' see [7], [8] and [10, Ch. 15] considering the function norm ρ defined as $\rho(f) = \|f\|_X$ if $f \in X$ and $\rho(f) = \infty$ in other case.

Let $1 \le p < \infty$. A B.f.s. X is said to be *p-convex* if there exists a constant $C > 0$ such that

$$\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{1/p}\Big\|_X \le C\Big(\sum_{i=1}^n \|f_i\|_X^p\Big)^{1/p}$$

for every finite subset $(f_i)_{i=1}^n \subset X$. In this case, $M^p(X)$ will denote the smallest constant C satisfying the above inequality. Note that $M^p(X) \ge 1$. A relevant fact is that every p-convex B.f.s. X has an equivalent norm for which X is p-convex with constant $M^p(X) = 1$, see [7, Proposition 1.d.8].

The *p-th power* of a B.f.s. X is the space defined as

$$X_p = \{f \in L^0(\mu) : |f|^{1/p} \in X\},$$

endowed with the quasi-norm $\|f\|_{X_p} = \|\,|f|^{1/p}\,\|_X^p$, for $f \in X_p$. Note that X_p is always complete, see the proof of [8, Proposition 2.22]. If X is p-convex with constant $M^p(X) = 1$, from [3, Lemma 3], $\|\cdot\|_{X_p}$ is a norm and so X_p is a B.f.s. Note that X_p is saturated if and only if X is so. The same holds for the properties of being order continuous and order semi-continuous.

3. The space $S^q_{X_p}(\xi)$

Let $1 \le p \le q < \infty$ and let X be a saturated p-convex B.f.s. We can assume without loss of generality that the p-convexity constant $M^p(X)$ is equal to 1. Then, X_p and $(X_p)'$ are saturated B.f.s.'. Consider the topology $\sigma\big((X_p)', X_p\big)$ on $(X_p)'$ defined by the elements of X_p. Note that the subset $B^+_{(X_p)'}$ of all positive elements of the closed unit ball of $(X_p)'$ is compact for this topology.

Let ξ be a finite positive Radon measure on $B^+_{(X_p)'}$. For $f \in L^0(\mu)$, consider the map $\phi_f \colon B^+_{(X_p)'} \to [0,\infty]$ defined by

$$\phi_f(h) = \Big(\int_\Omega |f(\omega)|^p h(\omega)\, d\mu(\omega) \Big)^{q/p}$$

for all $h \in B^+_{(X_p)'}$. In the case when $f \in X$, since $|f|^p \in X_p$, it follows that ϕ_f is continuous and so measurable. For a general $f \in L^0(\mu)$, by Lemma 2.1 we can take a sequence $(f_n)_{n\geq 1} \subset X$ such that $0 \leq f_n \uparrow |f|$ μ-a.e. Applying monotone convergence theorem, we have that $\phi_{f_n} \uparrow \phi_f$ pointwise and so ϕ_f is measurable. Then, we can consider the integral $\int_{B^+_{(X_p)'}} \phi_f(h) d\xi(h) \in [0,\infty]$ and define the following space:

$$S^q_{X_p}(\xi) = \left\{ f \in L^0(\mu) : \int_{B^+_{(X_p)'}} \Big(\int_\Omega |f(\omega)|^p h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) < \infty \right\}.$$

Let us endow $S^q_{X_p}(\xi)$ with the seminorm

$$\begin{aligned} \|f\|_{S^q_{X_p}(\xi)} &= \left(\int_{B^+_{(X_p)'}} \Big(\int_\Omega |f(\omega)|^p h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) \right)^{1/q} \\ &= \Big\| h \to \big\| f |h|^{1/p} \big\|_{L^p(\mu)} \Big\|_{L^q(\xi)}. \end{aligned}$$

In general, $\|\cdot\|_{S^q_{X_p}(\xi)}$ is not a norm. For instance, if ξ is the Dirac measure at some $h_0 \in B^+_{(X_p)'}$ such that $A = \{\omega \in \Omega : h_0(\omega) = 0\}$ satisfies $\mu(A) > 0$, taking $f = g\chi_A \in X$ with g being a weak unit of X, we have that

$$\|f\|_{S^q_{X_p}(\xi)} = \Big(\int_A |g(\omega)|^p h_0(\omega)\, d\mu(\omega) \Big)^{1/p} = 0$$

and

$$\mu(\{\omega \in \Omega : \ f(\omega) \neq 0\}) = \mu(A \cap \{\omega \in \Omega : \ g(\omega) \neq 0\}) = \mu(A) > 0.$$

Proposition 3.1. *If the Radon measure ξ satisfies*

$$\int_{B^+_{(X_p)'}} \Big(\int_A h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) = 0 \quad \Rightarrow \quad \mu(A) = 0 \tag{3.1}$$

then, $S^q_{X_p}(\xi)$ is a saturated B.f.s. Moreover, $S^q_{X_p}(\xi)$ is order continuous, p-convex (with constant 1) and $X \subset S^q_{X_p}(\xi)$ continuously.

Proof. It is clear that if $f \in L^0(\mu)$, $g \in S^q_{X_p}(\xi)$ and $|f| \leq |g|$ μ-a.e. then $f \in S^q_{X_p}(\xi)$ and $\|f\|_{S^q_{X_p}(\xi)} \leq \|g\|_{S^q_{X_p}(\xi)}$. Let us see that $\|\cdot\|_{S^q_{X_p}(\xi)}$ is a norm. Suppose that $\|f\|_{S^q_{X_p}(\xi)} = 0$ and set $A_n = \{\omega \in \Omega : \, |f(\omega)| > \frac{1}{n}\}$ for every $n \geq 1$. Since $\chi_{A_n} \leq n|f|$ and

$$\int_{B^+_{(X_p)'}} \Big(\int_{A_n} h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) = \big\|\chi_{A_n}\big\|^q_{S^q_{X_p}(\xi)} \leq n^q \|f\|^q_{S^q_{X_p}(\xi)} = 0,$$

from (3.1) we have that $\mu(A_n) = 0$ and so

$$\mu(\{\omega \in \Omega : \, f(\omega) \neq 0\}) = \lim_{n\to\infty} \mu(A_n) = 0.$$

Now we will see that $S^q_{X_p}(\xi)$ is complete by showing that $\sum_{n\geq 1} f_n \in S^q_{X_p}(\xi)$ whenever $(f_n)_{n\geq 1} \subset S^q_{X_p}(\xi)$ with $C = \sum \|f_n\|_{S^q_{X_p}(\xi)} < \infty$. First let us prove that $\sum_{n\geq 1} |f_n| < \infty$ μ-a.e. For every $N, n \geq 1$, taking $A^N_n = \{\omega \in \Omega : \, \sum_{j=1}^n |f_j(\omega)| > N\}$, since $\chi_{A^N_n} \leq \frac{1}{N} \sum_{j=1}^n |f_j|$, we have that

$$\begin{aligned} \int_{B^+_{(X_p)'}} \Big(\int_{A^N_n} h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) &= \|\chi_{A^N_n}\|^q_{S^q_{X_p}(\xi)} \\ &\leq \frac{1}{N^q} \Big\| \sum_{j=1}^n |f_j| \Big\|^q_{S^q_{X_p}(\xi)} \leq \frac{C^q}{N^q}. \end{aligned}$$

Note that, for N fixed, $(A^N_n)_{n\geq 1}$ increases. Taking limit as $n \to \infty$ and applying twice the monotone convergence theorem, it follows that

$$\int_{B^+_{(X_p)'}} \Big(\int_{\cup_{n\geq 1} A^N_n} h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) \leq \frac{C^q}{N^q}.$$

Then,

$$\int_{B^+_{(X_p)'}} \Big(\int_{\cap_{N\geq 1} \cup_{n\geq 1} A^N_n} h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) \leq \lim_{N\to\infty} \frac{C^q}{N^q} = 0,$$

and so, from (3.1),

$$\mu\Big(\Big\{ \omega \in \Omega : \, \sum_{n\geq 1} |f_n(\omega)| = \infty \Big\} \Big) = \mu\Big(\bigcap_{N\geq 1} \bigcup_{n\geq 1} A^N_n \Big) = 0.$$

Hence, $\sum_{n\geq 1} f_n \in L^0(\mu)$. Again applying the monotone convergence theorem, it follows that

$$\begin{aligned} \int_{B^+_{(X_p)'}} \Big(\int_\Omega \Big| \sum_{n\geq 1} f_n(\omega) \Big|^p h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) &\leq \\ \int_{B^+_{(X_p)'}} \Big(\int_\Omega \big(\sum_{n\geq 1} |f_n(\omega)| \big)^p h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) &= \\ \lim_{n\to\infty} \int_{B^+_{(X_p)'}} \Big(\int_\Omega \big(\sum_{j=1}^n |f_j(\omega)| \big)^p h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) &= \\ \lim_{n\to\infty} \Big\| \sum_{j=1}^n |f_j| \Big\|^q_{S^q_{X_p}(\xi)} &\leq C^q \end{aligned}$$

and thus $\sum_{n\geq 1} f_n \in S^q_{X_p}(\xi)$.

Note that if $f \in X$, for every $h \in B^+_{(X_p)'}$ we have that

$$\int_\Omega |f(\omega)|^p h(\omega)\, d\mu(\omega) \leq \| \, |f|^p \, \|_{X_p} \|h\|_{(X_p)'} \leq \|f\|^p_X$$

and so

$$\int_{B^+_{(X_p)'}} \Big(\int_\Omega |f(\omega)|^p h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) \leq \|f\|^q_X \, \xi\big(B^+_{(X_p)'}\big).$$

Then, $X \subset S^q_{X_p}(\xi)$ and $\|f\|_{S^q_{X_p}(\xi)} \leq \xi\big(B^+_{(X_p)'}\big)^{1/q} \|f\|_X$ for all $f \in X$. In particular, $S^q_{X_p}(\xi)$ is saturated, as a weak unit in X is a weak unit in $S^q_{X_p}(\xi)$.

Let us show that $S^q_{X_p}(\xi)$ is order continuous. Consider $f, f_n \in S^q_{X_p}(\xi)$ such that $0 \leq f_n \uparrow f$ μ-a.e. Note that, since

$$\int_{B^+_{(X_p)'}} \Big(\int_\Omega |f(\omega)|^p h(\omega)\, d\mu(\omega) \Big)^{q/p} d\xi(h) < \infty,$$

there exists a ξ-measurable set B with $\xi(B^+_{(X_p)'} \backslash B) = 0$ such that $\int_\Omega |f(\omega)|^p h(\omega)\, d\mu(\omega) < \infty$ for all $h \in B$. Fixed $h \in B$, we have that $|f - f_n|^p h \downarrow 0$ μ-a.e. and $|f - f_n|^p h \leq |f|^p h$ μ-a.e. Then, applying the dominated convergence theorem, $\int_\Omega |f(\omega) - f_n(\omega)|^p h(\omega)\, d\mu(\omega) \downarrow 0$.

Consider the measurable functions $\phi, \phi_n\colon B^+_{(X_p)'} \to [0,\infty]$ given by

$$\begin{aligned} \phi(h) &= \Big(\int_\Omega |f(\omega)|^p h(\omega)\,d\mu(\omega)\Big)^{q/p} \\ \phi_n(h) &= \Big(\int_\Omega |f(\omega)-f_n(\omega)|^p h(\omega)\,d\mu(\omega)\Big)^{q/p} \end{aligned}$$

for all $h \in B^+_{(X_p)'}$. It follows that $\phi_n \downarrow 0$ ξ-a.e. and $\phi_n \le \phi$ ξ-a.e. Again by the dominated convergence theorem, we obtain

$$\|f-f_n\|^q_{S^q_{X_p}(\xi)} = \int_{B^+_{(X_p)'}} \phi_n(h)d\xi(h) \downarrow 0.$$

Finally, let us see that $S^q_{X_p}(\xi)$ is p-convex. Fix $(f_i)_{i=1}^n \subset S^q_{X_p}(\xi)$ and consider the measurable functions $\phi_i\colon B^+_{(X_p)'} \to [0,\infty]$ (for $1 \le i \le n$) defined by

$$\phi_i(h) = \int_\Omega |f_i(\omega)|^p h(\omega)\,d\mu(\omega).$$

for all $h \in B^+_{(X_p)'}$. Then,

$$\begin{aligned} \Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{1/p}\Big\|^q_{S^q_{X_p}(\xi)} &= \int_{B^+_{(X_p)'}} \Big(\int_\Omega \sum_{i=1}^n |f_i(\omega)|^p h(\omega)\,d\mu(\omega)\Big)^{q/p} d\xi(h) \\ &= \int_{B^+_{(X_p)'}} \Big(\sum_{i=1}^n \phi_i(h)\Big)^{q/p} d\xi(h) \\ &\le \Big(\sum_{i=1}^n \|\phi_i\|_{L^{q/p}(\xi)}\Big)^{q/p}. \end{aligned}$$

Since $\|\phi_i\|_{L^{q/p}(\xi)} = \|f_i\|^p_{S^q_{X_p}(\xi)}$ for all $1 \le i \le n$, we have that

$$\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{1/p}\Big\|_{S^q_{X_p}(\xi)} \le \Big(\sum_{i=1}^n \|f_i\|^p_{S^q_{X_p}(\xi)}\Big)^{1/p}.$$

□

Example 3.2. Take a weak unit $g \in (X_p)'$ and consider the Radon measure ξ as the Dirac measure at g. If $A \in \Sigma$ is such that

$$0 = \int_{B^+_{(X_p)'}} \Big(\int_A h(\omega)\,d\mu(\omega)\Big)^{q/p} d\xi(h) = \Big(\int_A g(\omega)\,d\mu(\omega)\Big)^{q/p}$$

then, $g\chi_A = 0$ μ-a.e. and so, since $g > 0$ μ-a.e., $\mu(A) = 0$. That is, ξ satisfies (3.1). In this case, $S^q_{X_p}(\xi) = L^p(gd\mu)$ with equal norms, as

$$\int_{B^+_{(X_p)'}} \Big(\int_\Omega |f(\omega)|^p h(\omega)\, d\mu(\omega)\Big)^{q/p} d\xi(h) = \Big(\int_\Omega |f(\omega)|^p g(\omega)\, d\mu(\omega)\Big)^{q/p}$$

for all $f \in L^0(\mu)$.

Example 3.3. Write $\Omega = \cup_{n\geq 1}\Omega_n$ with $(\Omega_n)_{n\geq 1}$ being a disjoint sequence of measurable sets and take a sequence of strictly positive elements $(\alpha_n)_{n\geq 1} \in \ell^1$. Let us consider the Radon measure $\xi = \sum_{n\geq 1}\alpha_n \delta_{g\chi_{\Omega_n}}$ on $B^+_{(X_p)'}$, where $\delta_{g\chi_{\Omega_n}}$ is the Dirac measure at $g\chi_{\Omega_n}$ with $g \in (X_p)'$ being a weak unit. Note that for every positive function $\phi \in L^0(\xi)$, it follows that $\int_{B^+_{(X_p)'}} \phi\, d\xi = \sum_{n\geq 1}\alpha_n \phi(g\chi_{\Omega_n})$. If $A \in \Sigma$ is such that

$$0 = \int_{B^+_{(X_p)'}} \Big(\int_A h(\omega)\, d\mu(\omega)\Big)^{q/p} d\xi(h) = \sum_{n\geq 1} \alpha_n \Big(\int_{A\cap\Omega_n} g(\omega)\, d\mu(\omega)\Big)^{q/p}$$

then, $\int_{A\cap\Omega_n} g(\omega)\, d\mu(\omega) = 0$ for all $n \geq 1$. Hence,

$$\int_A g(\omega)\, d\mu(\omega) = \sum_{n\geq 1} \int_{A\cap\Omega_n} g(\omega)\, d\mu(\omega) = 0$$

and so $g\chi_A = 0$ μ-a.e., from which $\mu(A) = 0$. That is, ξ satisfies (3.1). For every $f \in L^0(\mu)$ we have that

$$\int_{B^+_{(X_p)'}} \Big(\int_\Omega |f(\omega)|^p h(\omega)\, d\mu(\omega)\Big)^{q/p} d\xi(h) = \sum_{n\geq 1} \alpha_n \Big(\int_{\Omega_n} |f(\omega)|^p g(\omega)\, d\mu(\omega)\Big)^{q/p}.$$

Then, the B.f.s. $S^q_{X_p}(\xi)$ can be described as the space of functions $f \in \cap_{n\geq 1} L^p(g\chi_{\Omega_n} d\mu)$ such that $\big(\alpha_n^{1/q}\|f\|_{L^p(g\chi_{\Omega_n} d\mu)}\big)_{n\geq 1} \in \ell^q$. Moreover, $\|f\|_{S^q_{X_p}(\xi)} = \Big(\sum_{n\geq 1} \alpha_n \,\|f\|^q_{L^p(g\chi_{\Omega_n} d\mu)}\Big)^{1/q}$ for all $f \in S^q_{X_p}(\xi)$.

4. p-strongly q-concave operators

Let $1 \leq p \leq q < \infty$ and let $T\colon X \to E$ be a linear operator from a saturated B.f.s. X into a Banach space E. Recall that T is said to be *q-concave* if there exists a constant $C>0$ such that

$$\Big(\sum_{i=1}^n \|T(f_i)\|_E^q\Big)^{1/q} \le C\Big\|\Big(\sum_{i=1}^n |f_i|^q\Big)^{1/q}\Big\|_X$$

for every finite subset $(f_i)_{i=1}^n \subset X$. The smallest possible value of C will be denoted by $M_q(T)$. For issues related to q-concavity see for instance [7, Ch. 1.d]. We introduce a little stronger notion than q-concavity: T will be called *p-strongly q-concave* if there exists $C>0$ such that

$$\Big(\sum_{i=1}^n \|T(f_i)\|_E^q\Big)^{1/q} \le C \sup_{(\beta_i)_{i\ge 1}\in B_{\ell^r}} \Big\|\Big(\sum_{i=1}^n |\beta_i f_i|^p\Big)^{1/p}\Big\|_X$$

for every finite subset $(f_i)_{i=1}^n \subset X$, where $1<r\le\infty$ is such that $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$. In this case, $M_{p,q}(T)$ will denote the smallest constant C satisfying the above inequality. Noting that $\frac{r}{p}$ and $\frac{q}{p}$ are conjugate exponents, it is clear that every p-strongly q-concave operator is q-concave and so continuous, and moreover $\|T\|\le M_q(T)\le M_{p,q}(T)$. As usual, we will say that X is *p-strongly q-concave* if the identity map $I\colon X\to X$ is so, and in this case, we denote $M_{p,q}(X)=M_{p,q}(I)$.

Our goal is to get a continuous extension of T to a space of the type $S_{X_p}^q(\xi)$ in the case when T is p-strongly q-concave and X is p-convex. To this end we will need to describe the supremum on the right-hand side of the p-strongly q-concave inequality in terms of the Köthe dual of X_p.

Lemma 4.1. *If X is p-convex and order semi-continuous then*

$$\sup_{(\beta_i)_{i\ge 1}\in B_{\ell^r}} \Big\|\Big(\sum_{i=1}^n |\beta_i f_i|^p\Big)^{1/p}\Big\|_X = \sup_{h\in B^+_{(X_p)'}} \Big(\sum_{i=1}^n \Big(\int |f_i|^p h\, d\mu\Big)^{q/p}\Big)^{1/q}$$

for every finite subset $(f_i)_{i=1}^n \subset X$, where $1<r\le\infty$ is such that $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$ and $B^+_{(X_p)'}$ is the subset of all positive elements of the closed unit ball $B_{(X_p)'}$ of $(X_p)'$.

Proof. Given $(f_i)_{i=1}^n \subset X$, since X_p is order semi-continuous, as X is so, and $(\ell^{q/p})^* = \ell^{r/p}$, as $\frac{r}{p}$ is the conjugate exponent of $\frac{q}{p}$, we have that

$$\begin{aligned}
\sup_{(\beta_i)\in B_{\ell^r}} \Big\| \Big(\sum_{i=1}^n |\beta_i f_i|^p \Big)^{1/p} \Big\|_X^p &= \sup_{(\beta_i)\in B_{\ell^r}} \Big\| \sum_{i=1}^n |\beta_i f_i|^p \Big\|_{X_p} \\
&= \sup_{(\beta_i)\in B_{\ell^r}} \sup_{h\in B_{(X_p)'}} \int \sum_{i=1}^n |\beta_i f_i|^p |h| \, d\mu \\
&= \sup_{(\beta_i)\in B_{\ell^r}} \sup_{h\in B^+_{(X_p)'}} \int \sum_{i=1}^n |\beta_i f_i|^p h \, d\mu \\
&= \sup_{h\in B^+_{(X_p)'}} \sup_{(\beta_i)\in B_{\ell^r}} \sum_{i=1}^n |\beta_i|^p \int |f_i|^p h \, d\mu \\
&= \sup_{h\in B^+_{(X_p)'}} \sup_{(\alpha_i)\in B^+_{\ell^{r/p}}} \sum_{i=1}^n \alpha_i \int |f_i|^p h \, d\mu \\
&= \sup_{h\in B^+_{(X_p)'}} \Big(\sum_{i=1}^n \Big(\int |f_i|^p h \, d\mu \Big)^{q/p} \Big)^{p/q}.
\end{aligned}$$

□

In the following remark, from Lemma 4.1, we obtain easily an example of p-strongly q-concave operator.

Remark 4.2. Suppose that X is p-convex and order semi-continuous. For every finite positive Radon measure ξ on $B^+_{(X_p)'}$ satisfying (3.1), it follows that the inclusion map $i\colon X \to S^q_{X_p}(\xi)$ is p-strongly q-concave. Indeed, for each $(f_i)_{i=1}^n \subset X$, we have that

$$\begin{aligned}
\sum_{i=1}^n \|f_i\|^q_{S^q_{X_p}(\xi)} &= \sum_{i=1}^n \int_{B^+_{(X_p)'}} \Big(\int_\Omega |f_i(\omega)|^p h(\omega) \, d\mu(\omega) \Big)^{q/p} d\xi(h) \\
&\le \xi\big(B^+_{(X_p)'}\big) \sup_{h\in B^+_{(X_p)'}} \sum_{i=1}^n \Big(\int_\Omega |f_i(\omega)|^p h(\omega) \, d\mu(\omega) \Big)^{q/p}
\end{aligned}$$

and so, Lemma 4.1 gives the conclusion for $M_{p,q}(i) \le \xi\big(B^+_{(X_p)'}\big)^{1/q}$.

Now let us prove our main result.

Theorem 4.3. *If T is p-strongly q-concave and X is p-convex and order semi-continuous, then there exists a probability Radon measure ξ on $B^+_{(X_p)'}$ satisfying* (3.1) *such that*

$$\|T(f)\|_E \leq M_{p,q}(T)\Big(\int_{B^+_{(X_p)'}}\Big(\int_\Omega |f(\omega)|^p h(\omega)\,d\mu(\omega)\Big)^{q/p} d\xi(h)\Big)^{1/q} \tag{4.1}$$

for all $f \in X$.

Proof. Recall that the stated topology on $(X_p)'$ is $\sigma((X_p)', X_p)$, the one which is defined by the elements of X_p. For each finite subset (with possibly repeated elements) $M = (f_i)_{i=1}^m \subset X$, consider the map $\psi_M \colon B^+_{(X_p)'} \to [0,\infty)$ defined by $\psi_M(h) = \sum_{i=1}^m \big(\int_\Omega |f_i|^p\,h\,d\mu\big)^{q/p}$ for $h \in B^+_{(X_p)'}$. Note that ψ_M attains its supremum as it is continuous on a compact set, so there exists $h_M \in B^+_{(X_p)'}$ such that $\sup_{h\in B^+_{(X_p)'}} \psi_M(h) = \psi_M(h_M)$. Then, the p-strongly q-concavity of T, together with Lemma 4.1, gives

$$\begin{aligned}\sum_{i=1}^m \|T(f_i)\|_E^q &\leq M_{p,q}(T)^q \sup_{h\in B^+_{(X_p)'}} \sum_{i=1}^m \Big(\int_\Omega |f_i|^p h\,d\mu\Big)^{q/p} \\ &\leq M_{p,q}(T)^q \sup_{h\in B^+_{(X_p)'}} \psi_M(h) \\ &= M_{p,q}(T)^q\,\psi_M(h_M).\end{aligned} \tag{4.2}$$

Consider now the continuous map $\phi_M \colon B^+_{(X_p)'} \to \mathbb{R}$ defined by

$$\phi_M(h) = M_{p,q}(T)^q\,\psi_M(h) - \sum_{i=1}^m \|T(f_i)\|_E^q$$

for $h \in B^+_{(X_p)'}$. Take $B = \{\phi_M : M \text{ is a finite subset of } X\}$. Since for every $M = (f_i)_{i=1}^m$, $M' = (f_i')_{i=1}^k \subset X$ and $0 < t < 1$, it follows that $t\phi_M + (1-t)\phi_{M'} = \phi_{M''}$ where $M'' = \big(t^{1/q} f_i\big)_{i=1}^m \cup \big((1-t)^{1/q} f_i'\big)_{i=1}^k$, we have that B is convex. Denote by $\mathcal{C}(B^+_{(X_p)'})$ the space of continuous real functions on $B^+_{(X_p)'}$, endowed with the supremum norm, and by A the open convex subset $\{\phi \in \mathcal{C}(B^+_{(X_p)'}) : \phi(h) < 0 \text{ for all } h \in B^+_{(X_p)'}\}$. By (4.2) we have that $A \cap B = \emptyset$. From the Hahn-Banach separation theorem, there exist $\xi \in \mathcal{C}(B^+_{(X_p)'})^*$ and $\alpha \in \mathbb{R}$ such that $\langle \xi, \phi\rangle < \alpha \leq \langle \xi, \phi_M\rangle$ for all $\phi \in A$ and $\phi_M \in B$. Since every negative constant function is in A, it follows that $0 \le \alpha$. Even more, $\alpha = 0$ as the constant function equal to 0 is just $\phi_{\{0\}} \in B$. It is routine to see that $\langle \xi, \phi \rangle \ge 0$ whenever $\phi \in \mathcal{C}(B^+_{(X_p)'})$ is such that $\phi(h) \ge 0$ for all $h \in B^+_{(X_p)'}$. Then, ξ is a positive linear functional on $\mathcal{C}(B^+_{(X_p)'})$ and so it can be interpreted as a finite positive Radon measure on $B^+_{(X_p)'}$. Hence, we have that

$$0 \le \int_{B^+_{(X_p)'}} \phi_M \, d\xi$$

for all finite subset $M \subset X$. Dividing by $\xi(B^+_{(X_p)'})$, we can suppose that ξ is a probability measure. Then, for $M = \{f\}$ with $f \in X$, we obtain that

$$\|T(f)\|_E^q \le M_{p,q}(T)^q \int_{B^+_{(X_p)'}} \Big(\int_\Omega |f(\omega)|^p h(\omega) \, d\mu(\omega) \Big)^{q/p} d\xi(h)$$

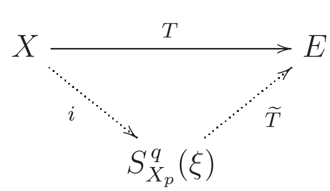
and so (4.1) holds. □

Actually, Theorem 4.3 says that we can find a probability Radon measure ξ on $B^+_{(X_p)'}$ such that $T \colon X \to E$ is continuous when X is considered with the norm of the space $S^q_{X_p}(\xi)$. In the next result we will see how to extend T continuously to $S^q_{X_p}(\xi)$. Even more, we will show that this extension is possible if and only if T is p-strongly q-concave.

Theorem 4.4. *Suppose that X is p-convex and order semi-continuous. The following statements are equivalent:*

(a) *T is p-strongly q-concave.*

(b) *There exists a probability Radon measure ξ on $B^+_{(X_p)'}$ satisfying (3.1) such that T can be extended continuously to $S^q_{X_p}(\xi)$, i.e. there is a factorization for T as*



where $\widetilde{T}$ is a continuous linear operator and i is the inclusion map.

If (a)-(b) holds, then $M_{p,q}(T) = \|\widetilde{T}\|$.

Proof. (a) $\Rightarrow$ (b) From Theorem 4.3, there is a probability Radon measure ξ on $B^+_{(X_p)'}$ satisfying (3.1) such that $\|T(f)\|_E \le M_{p,q}(T)\|f\|_{S^q_{X_p}(\xi)}$ for all $f \in X$. Given $0 \le f \in S^q_{X_p}(\xi)$, from Lemma 2.1, we can take $(f_n)_{n\ge1} \subset X$ such that $0 \le f_n \uparrow f$ μ-a.e. Then, since $S^q_{X_p}(\xi)$ is order continuous, we have that $f_n \to f$ in $S^q_{X_p}(\xi)$ and so $\big(T(f_n)\big)_{n\ge1}$ converges to some element e of E. Define $\widetilde{T}(f) = e$. Note that $\widetilde{T}$ is well defined, since if $(g_n)_{n\ge1} \subset X$ is such that $0 \le g_n \uparrow f$ μ-a.e., then

$$\|T(f_n) - T(g_n)\|_E \le M_{p,q}(T)\|f_n - g_n\|_{S^q_{X_p}(\xi)} \to 0.$$

Moreover,

$$\begin{aligned} \|\widetilde{T}(f)\|_E &= \lim_{n\to\infty} \|T(f_n)\|_E \\ &\le M_{p,q}(T) \lim_{n\to\infty} \|f_n\|_{S^q_{X_p}(\xi)} \\ &= M_{p,q}(T)\|f\|_{S^q_{X_p}(\xi)}. \end{aligned}$$

For a general $f \in S^q_{X_p}(\xi)$, writing $f = f^+ - f^-$ where f^+ and f^- are the positive and negative parts of f respectively, we define $\widetilde{T}(f) = \widetilde{T}(f^+) - \widetilde{T}(f^-)$. Then, $\widetilde{T}\colon S^q_{X_p}(\xi) \to E$ is a continuous linear operator extending T. Moreover $\|\widetilde{T}\| \le M_{p,q}(T)$. Indeed, let $f \in S^q_{X_p}(\xi)$ and take $(f_n^+)_{n\ge1}$, $(f_n^-)_{n\ge1} \subset X$ such that $0 \le f_n^+ \uparrow f^+$ and $0 \le f_n^- \uparrow f^-$ μ-a.e. Then, $f_n^+ - f_n^- \to f$ in $S^q_{X_p}(\xi)$ and

$$T(f_n^+ - f_n^-) = T(f_n^+) - T(f_n^-) \to \widetilde{T}(f^+) - \widetilde{T}(f^-) = \widetilde{T}(f)$$

in E. Hence,

$$\begin{aligned} \|\widetilde{T}(f)\|_E &= \lim_{n\to\infty} \|T(f_n^+ - f_n^-)\|_E \\ &\le M_{p,q}(T) \lim_{n\to\infty} \|f_n^+ - f_n^-\|_{S^q_{X_p}(\xi)} \\ &= M_{p,q}(T)\|f\|_{S^q_{X_p}(\xi)}. \end{aligned}$$

(b) $\Rightarrow$ (a) Given $(f_i)_{i=1}^n \subset X$, we have that

$$\begin{aligned}\sum_{i=1}^n \|T(f_i)\|_E^q &= \sum_{i=1}^n \|\widetilde{T}(f_i)\|_E^q \leq \|\widetilde{T}\|^q \sum_{i=1}^n \|f_i\|^q_{S^q_{X_p}(\xi)} \\ &= \|\widetilde{T}\|^q \sum_{i=1}^n \int_{B^+_{(X_p)'}} \Big(\int_\Omega |f_i(\omega)|^p h(\omega)\, d\mu(\omega)\Big)^{q/p} d\xi(h) \\ &\leq \|\widetilde{T}\|^q \sup_{h\in B^+_{(X_p)'}} \sum_{i=1}^n \Big(\int_\Omega |f_i(\omega)|^p h(\omega)\, d\mu(\omega)\Big)^{q/p}.\end{aligned}$$

That is, from Lemma 4.1, T is p-strongly q-concave with $M_{p,q}(T) \leq \|\widetilde{T}\|$. □

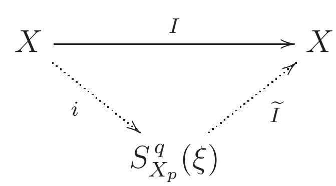
A first application of Theorem 4.4 is the following Kakutani type representation theorem (see for instance [7, Theorem 1.b.2]) for B.f.s.' being order semi-continuous, p-convex and p-strongly q-concave.

Corollary 4.5. *Suppose that X is p-convex and order semi-continuous. The following statements are equivalent:*

(a) *X is p-strongly q-concave.*

(b) *There exists a probability Radon measure ξ on $B^+_{(X_p)'}$ satisfying* (3.1)*, such that $X = S^q_{X_p}(\xi)$ with equivalent norms.*

Proof. (a) $\Rightarrow$ (b) The identity map $I\colon X \to X$ is p-strongly q-concave as X is so. Then, from Theorem 4.4, there exists a probability Radon measure ξ on $B^+_{(X_p)'}$ satisfying (3.1), such that I factors as



where $\widetilde{I}$ is a continuous linear operator with $\|\widetilde{I}\| = M_{p,q}(X)$ and i is the inclusion map. Since ξ is a probability measure, we have that $\|f\|_{S^q_{X_p}(\xi)} \leq \|f\|_X$ for all $f \in X$, see the proof of Proposition 3.1. Let $0 \leq f \in S^q_{X_p}(\xi)$. By Lemma 2.1, we can take $(f_n)_{n\geq 1} \subset X$ such that $0 \leq f_n \uparrow f$ μ-a.e. Since $S^q_{X_p}(\xi)$ is order continuous, it follows that $f_n \to f$ in $S^q_{X_p}(\xi)$ and so $f_n = \widetilde{I}(f_n) \to \widetilde{I}(f)$ in X. Then, there is a subsequence of $(f_n)_{n\geq 1}$ converging μ-a.e. to $\widetilde{I}(f)$ and hence $f=\widetilde{I}(f)\in X$. For a general $f\in S^q_{X_p}(\xi)$, writing $f=f^+-f^-$ where f^+ and f^- are the positive and negative parts of f respectively, we have that $f=\widetilde{I}(f^+)-\widetilde{I}(f^-)=\widetilde{I}(f)\in X$. Therefore, $X=S^q_{X_p}(\xi)$ and $\widetilde{I}$ is de identity map. Moreover, $\|f\|_X=\|\widetilde{I}(f)\|_X\leq\|\widetilde{I}\|\,\|f\|_{S^q_{X_p}(\xi)}=M_{p,q}(X)\|f\|_{S^q_{X_p}(\xi)}$ for all $f\in X$.

(b) ⇒ (a) From Remark 4.2 it follows that the identity map $I\colon X\to X$ is p-strongly q-concave. □

Note that under conditions of Corollary 4.5, if X is p-strongly q-concave with constant $M_{p,q}(X)=1$, then $X=S^q_{X_p}(\xi)$ with equal norms.

5. q-summing operators on a p-convex B.f.s.

Recall that a linear operator $T\colon X\to E$ between Banach spaces is said to be *q-summing* ($1\leq q<\infty$) if there exists a constant $C>0$ such that

$$\Big(\sum_{i=1}^n\|Tx_i\|_E^q\Big)^{1/q}\leq C\sup_{x^*\in B_{X^*}}\Big(\sum_{i=1}^n|\langle x^*,x_i\rangle|^q\Big)^{1/q}$$

for every finite subset $(x_i)_{i=1}^n\subset X$. Denote by $\pi_q(T)$ the smallest possible value of C. Information about q-summing operators can be found in [6].

One of the main relations between summability and concavity for operators defined on a B.f.s. X, is that every q-summing operator is q-concave. This is a consequence of a direct calculation which shows that for every $(f_i)_{i=1}^n\subset X$ and $x^*\in X^*$ it follows that

$$\Big(\sum_{i=1}^n|\langle x^*,f_i\rangle|^q\Big)^{1/q}\leq\|x^*\|_{X^*}\Big\|\Big(\sum_{i=1}^n|f_i|^q\Big)^{1/q}\Big\|_X, \tag{5.1}$$

see for instance [7, Proposition 1.d.9] and the comments below. However, this calculation can be slightly improved to obtain the following result.

Proposition 5.1. *Let $1\leq p\leq q<\infty$. Every q-summing linear operator $T\colon X\to E$ from a B.f.s. X into a Banach space E, is p-strongly q-concave with $M_{p,q}(T)\leq\pi_q(T)$.*

Proof. Let $1 < r \leq \infty$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and consider a finite subset $(f_i)_{i=1}^n \subset X$. We only have to prove

$$\sup_{x^* \in B_{X^*}} \Big(\sum_{i=1}^n |\langle x^*, f_i \rangle|^q \Big)^{1/q} \leq \sup_{(\beta_i)_{i\geq 1} \in B_{\ell^r}} \Big\| \Big(\sum_{i=1}^n |\beta_i f_i|^p \Big)^{1/p} \Big\|_X .$$

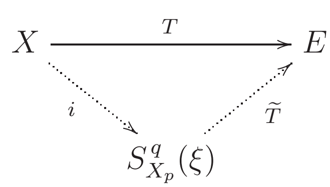
Fix $x^* \in B_{X^*}$. Noting that $\frac{q}{p}$ and $\frac{r}{p}$ are conjugate exponents and using the inequality (5.1), we have

$$\begin{aligned} \Big(\sum_{i=1}^n |\langle x^*, f_i \rangle|^q \Big)^{1/q} &= \sup_{(\alpha_i)_{i\geq 1} \in B_{\ell^{r/p}}} \Big(\sum_{i=1}^n |\alpha_i| |\langle x^*, f_i \rangle|^p \Big)^{1/p} \\ &= \sup_{(\beta_i)_{i\geq 1} \in B_{\ell^r}} \Big(\sum_{i=1}^n |\langle x^*, \beta_i f_i \rangle|^p \Big)^{1/p} \\ &\leq \sup_{(\beta_i)_{i\geq 1} \in B_{\ell^r}} \Big\| \Big(\sum_{i=1}^n |\beta_i f_i|^p \Big)^{1/p} \Big\|_X . \end{aligned}$$

Taking supremum in $x^* \in B_{X^*}$ we get the conclusion. □

From Proposition 5.1, Theorem 4.4 and Remark 4.2, we obtain the final result.

Corollary 5.2. *Set $1 \leq p \leq q < \infty$. Let X be a saturated order semi-continuous p-convex B.f.s. and consider a q-summing linear operator $T \colon X \to E$ with values in a Banach space E. Then, there exists a probability Radon measure ξ on $B^+_{(X_p)'}$ satisfying* (3.1) *such that T can be factored as*



where $\widetilde{T}$ is a continuous linear operator with $\|\widetilde{T}\| \leq \pi_q(T)$ and i is the inclusion map which turns out to be p-strongly q-concave, and so q-concave.

Observe that what we obtain in Corollary 5.2 is a proper extension for T, and not just a factorization as the obtained in the Pietsch theorem for q-summing operators through a subspace of an L^q-space.

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