A RIGID LOCAL SYSTEM WITH MONODROMY GROUP $2.J_2$

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Abstract. We exhibit a rigid local system of rank six on the affine line in characteristic $p = 5$ whose arithmetic and geometric monodromy groups are the finite group $2.J_2$ (J_2 the Hall-Janko sporadic group) in one of its two (Galois-conjugate) irreducible representation of degree six.

Contents

1. Introduction: the general setting

We fix a prime number p, a prime number $\ell \neq p$, and a nontrivial $\overline{\mathbb{Q}_\ell}^\times$-valued additive character ψ of $\mathbb{F}_p$. For $k/\mathbb{F}_p$ a finite extension, we denote by ψ_k the nontrivial additive character of k given by $\psi_k := \psi \circ \mathrm{Trace}_{k/\mathbb{F}_p}$. In perhaps more down to earth terms, we fix a nontrivial $\mathbb{Q}(\mu_p)^\times$-valued additive character ψ of $\mathbb{F}_p$, and a field embedding of $\mathbb{Q}(\mu_p)$ into $\overline{\mathbb{Q}_\ell}$ for some $\ell \neq p$.

Given an integer $D \geq 3$ which is prime to p, we form the local system $\mathcal{F}_{p,D}$ on $\mathbb{A}^1/\mathbb{F}_p$ whose trace function, at k-valued points $t \in \mathbb{A}^1(k) = k$, is given by

$$t \mapsto -\sum_{x \in k} \psi_k(x^D + tx).$$

This is a geometrically irreducible rigid local system, being the Fourier Transform of the rank one local system $\mathcal{L}_{\psi(x^D)}$. It has rank $D-1$, and

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each of its $D-1$ $I(\infty)$-slopes is $D/(D-1)$. It is pure of weight one. [It is the local system $\mathcal{F}(\mathbb{F}_p, \psi, \mathbb{1}, D)$ of [Ka-RLSA].]

Let us further fix a choice of $\sqrt{p} \in \overline{\mathbb{Q}_\ell}^\times$. For each finite extension $k/\mathbb{F}_p$, we then use this choice of$\sqrt{p}$ to define $\sqrt{\#k} := \sqrt{p}^{\deg(k/\mathbb{F}_p)}$. We then define the "half"-Tate twisted local system

$$\mathcal{G}_{p,D} := \mathcal{F}_{p,D}(1/2)$$

whose trace function, at at k-valued points $t \in \mathbb{A}^1(k) = k$, is given by

$$t \mapsto -\sum_{x \in k} \psi_k(x^D + tx)/\sqrt{\#k}$$

The local system $\mathcal{G}_{p,D}$ is pure of weight zero.

Lemma 1.1. *The determinant* $\det(\mathcal{G}_{p,D})$ *is arithmetically of finite order. More precisely,* $\det(\mathcal{G}_{p,D})^{\otimes 4p}$ *is arithmetically trivial.*

Proof. It suffices to show that after extension of scalars from $\mathbb{F}_p$ to its quadratic extension $\mathbb{F}_{p^2}$, the $2p$'th power is trivial, i.e., that if $k/\mathbb{F}_p$ is a finite extension of even degree $2d$, then the determinant takes values in μ_{2p}.To see this, note that the twisting factor $\sqrt{p}^{2d} = p^d \in \mathbb{Q}$, so this determinant has values in $\mathbb{Q}(\mu_p)$ which are units at all finite places of residue characteristic not p (use the ℓ-adic incarnations) and which have absolute value 1 at all archimedean places of $\mathbb{Q}(\mu_p)$. Because there is a unique p-adic place of $\mathbb{Q}(\mu_p)$, the product formula shows that the determinant has values which are also units at p, and hence are roots of unity in $\mathbb{Q}(\mu_p)$, i.e., they are $2p$'th roots of unity. $\square$

When we view the local system $\mathcal{G}_{p,D}$ as a representation

$$\rho_{\mathcal{G}_{p,D}} : \pi_1(\mathbb{A}^1/\mathbb{F}_p) \to GL(D-1, \overline{\mathbb{Q}_\ell}),$$

the Zariski closure of the image of $\pi_1(\mathbb{A}^1/\mathbb{F}_p)$ is defined to be the arithmetic monodromy group G_{arith}. The Zariski closure of the image of the normal subgroup $\pi_1^{geom} := \pi_1(\mathbb{A}^1/\overline{\mathbb{F}_p})$ is defined to be the geometric monodromy group G_{geom}. Thus we have inclusions of algebraic groups over $\overline{\mathbb{Q}_\ell}$:

$$G_{geom} \lhd G_{arith} \subset GL(D-1).$$

Applying [Ka-ESDE, 8.14.5, (1) $\iff$ (2) $\iff$ (6)] in the particular case of $\mathcal{G}_{p,D}$, we have

Proposition 1.2. *The following conditions are equivalent.*

(1) $\mathcal{G}_{p,D}$ *has finite* G_{geom}.
(1bis) $\mathcal{F}_{p,D}$ *has finite* G_{geom}.
(2) $\mathcal{G}_{p,D}$ *has finite* G_{arith}.
(3) *All traces of* $\mathcal{G}_{p,D}$ *are algebraic integers.*

When $D \geq 3$ is odd (and prime to p), the local system $\mathcal{F}_{p,D}$ is symplectically self-dual. As shown in [R-L, Proposition 4 and Corollary 6], its G_{geom} is either finite or it is the full symplectic group $\mathrm{Sp}(D-1)$. When $D \geq 3$ is even (and prime to p), the same reference shows that G_{geom} is either finite or is $SL(D-1)$. The proof of [R-L, Proposition 4 and Lemma 5] also shows that when D is not of the form $1+q$ for q a power of p, then $\mathcal{F}_{p,D}$ is not induced (i.e., the given representation of its G_{geom} is not induced). Indeed, by the result [Such, 11.1] of Šuch, if the representation were induced, it would be Artin-Schreier induced, and that is what is ruled out when D is not of the form $1+q$. [When $D = 1+q$, then G_{geom} is, by Pink [Ka-RLSA, 20.3], a finite p-group, and (hence) the representation is induced.]

When $D \geq 3$ is prime to p, the trace function of $\mathcal{G}_{p,D}$ takes values in $\mathbb{Z}[\mu_p]$. If moreover p is 1 mod 4, then we can choose either quadratic Gauss sum, a quantity which itself lies in $\mathbb{Z}[\mu_p]$, as our $\sqrt{p}$, and hence all traces of $\mathcal{G}_{p,D}$ lie in $\mathbb{Z}[\mu_p][1/p]$. If p is not 1 mod 4, this remains true for traces of the pullback of $\mathcal{G}_{p,D}$ to $\mathbb{A}^1/\mathbb{F}_{p^2}$. In either case, the traces of $\mathcal{G}_{p,D}$ in question are algebraic integers if and only if they all have $\mathrm{ord}_p \geq 0$.

Remark 1.3. When $D \geq 3$ is prime to p and **odd**, then the traces of $\mathcal{F}_{p,D}$ lie in the real subfield $\mathbb{Q}(\mu_p)^+$. If in addition p is 1 mod 4, then either quadratic gauss sum is $\pm\sqrt{p}$ also lies in this field, and hence $\mathcal{G}_{p,D}$ has traces in $\mathbb{Q}(\mu_p)^+$.

Results of Kubert, explained in [Ka-RLSA, 4.1,4.2,4.3] and discovered independently in [R-L, Cor. 4, Cor. 5], show that G_{geom} and G_{arith} for $\mathcal{G}_{p,D}$ are finite when q is a power of p and D is any of

$$q+1, \quad \frac{q+1}{2} \text{ with odd } q, \frac{q^n+1}{q+1} \text{ with odd } n.$$

Let us call these the Kubert cases. In [Ka-RLSA, 17.1, 17.2] and [Ka-Ti-RLSMFUG, 3.4] their G_{geom} groups are determined for all odd q.

Both authors have given numerical criteria for $\mathcal{G}_{p,D}$ to have finite G_{geom} and G_{arith}, cf. [Ka-RLSA, first paragraph after 5.1] and [R-L, Thm. 1]. The second author did extensive computer experiments to find other (p, D) than the Kubert cases for which $\mathcal{G}_{p,D}$ seemed to have finite G_{geom} (i.e., where many many traces were all algebraic integers). For primes $p \leq 11$ and $D \leq 10^6$, there was only one other candidate, the case $p = 5,\ D = 7$.

In the first part of this paper, we prove that $\mathcal{F}_{5,7}$ has finite G_{geom} (and hence, by Proposition 1.2, that $\mathcal{G}_{5,7}$ has finite G_{geom} and finite

G_{arith}). In the second part, we show that $G_{geom} = G_{arith} = 2.J_2$ in one of its two six-dimensional irreducible representations. These two representations are symplectic. Their character values lie in $\mathbb{Z}[\sqrt{5}]$ and are Galois-conjugates of each other. As Guralnick and Tiep point out [G-T, Table 1], the group $2.J_2$, sitting inside $Sp(6, \mathbb{C})$, has the exotic property that it has the same moments M_n (dimension of the space of invariants in the n'th tensor power of the given six-dimensional representation) as the ambient group $\mathrm{Sp}(6, \mathbb{C})$ for M_1 through M_{11}; one needs M_{12} to distinguish them.

It is not clear whether there "should" be infinitely many (p, D) other than the Kubert cases for which $\mathcal{G}_{p,D}$ has finite G_{arith}, or finitely many, or just this one $(5, 7)$ case. Much remains to be done.

2. Finiteness of the monodromy

In this section we will prove that the sheaf $\mathcal{F}_{5,7}$ has finite geometric monodromy. We will do so by applying the numerical criterion proven in [R-L, Theorem 1], which we recall here. For a prime p and an integer $x \geq 0$, we define

$$[x]_{p,\infty} := \text{ the sum of the digits of the } p\text{-adic expansion of } x,$$

using the usual digits $\{0, 1, 2, ..., p - 1\}$.

For every $r \geq 1$ we define $[x]_{p,r} = [x]_{p,\infty}$ if $1 \leq x \leq p^r - 1$, and we extend the definition to every integer x by imposing that $[x]_{p,r} = [y]_{p,r}$ if $x \equiv y \pmod{p^r - 1}$.[Thus we are using $\{1, 2, ..., p^r - 1\}$ as representatives of $\mathbb{Z}/(p^r - 1)\mathbb{Z}$.]

From [R-L, Thm. 1], we have

Theorem 2.1. *The sheaf $\mathcal{F}_{p,d}$ has finite geometric monodromy if and only if the inequality*

$$[dx]_{p,r} \leq [x]_{p,r} + \frac{r(p-1)}{2}$$

holds for every $r \geq 1$ and every integer $0 < x < p^r$.

Let us enumerate some basic properties of the functions $[-]_{p,\infty}$ and $[-]_{p,r}$.

Proposition 2.2. *For strictly positive integers x and y, and for $r \in \mathbb{N} \cup \{\infty\}$, we have:*

(1) $[x + y]_{p,r} \leq [x]_{p,r} + [y]_{p,r}$
(2) $[x]_{p,r} \leq [x]_{p,\infty}$
(3) $[px]_{p,r} = [x]_{p,r}$

Proof. We first prove (1) for $r = \infty$. Note that $[x]_{p,\infty}$ is the minimal number of terms in any decomposition of x as a sum of powers of p. By taking the sum of the p-adic expansions of x and y we see that $x + y$ can be written as a sum of $[x]_{p,\infty} + [y]_{p,\infty}$ powers of p, and the inequality follows.

For (2) we proceed by induction on x: for $0 < x < p^r$ it is obvious by definition. If $x \geq p^r$, let s be the largest integer such that $p^s \leq x$. Then $s \geq r$ and $[x - p^s]_{p,\infty} = [x]_{p,\infty} - 1$. Since $x \equiv x - p^s + p^{s-r} \pmod{p^r - 1}$, while $x > x - p^s + p^{s-r} > 0$, we have, by induction on x and $(1)_\infty$,

$$[x]_{p,r} = [x - p^s + p^{s-r}]_{p,r} \leq [x - p^s + p^{s-r}]_{p,\infty} \leq [x - p^s]_{p,\infty} + [p^{s-r}]_{p,\infty} =$$

$$= [x]_{p,\infty} - 1 + 1 = [x]_{p,\infty}.$$

In order to prove (1) for finite r we can assume that $x, y < p^r$. Then by (2) and $(1)_\infty$, we have

$$[x + y]_{p,r} \leq [x + y]_{p,\infty} \leq [x]_{p,\infty} + [y]_{p,\infty} = [x]_{p,r} + [y]_{p,r}.$$

Finally, (3) is obvious for $r = \infty$. For finite r, note that if $x = a_{r-1}p^{r-1} + \cdots + a_1 p + a_0$ is the p-adic expansion of $x < p^r$, then $px \equiv a_{r-2}p^{r-1} + \cdots + a_1 p^2 + a_0 p + a_{r-1} \pmod{p^r - 1}$, so $[px]_{p,r} = [x]_{p,r} = a_{r-1} + \cdots + a_1 + a_0$. $\square$

We now fix $p = 5$ and $d = 7$.

Lemma 2.3. *Let r be a positive integer and $0 \leq x < 5^r$ an integer such that $x \not\equiv 2 \mod 5$. Then $[7x]_{5,\infty} \leq [x]_{5,\infty} + 2r$.*

Proof. We proceed by induction on r. For $r = 1$ and $r = 2$ one checks it by hand.

Now let $r \geq 3$ and $0 \leq x < 5^r$ with $x \not\equiv 2 \mod 5$. If $0 \leq x < 5^{r-1}$ the stronger inequality $[7x]_\infty \leq [x]_\infty + 2r - 2$ holds by induction, so we may assume that $5^{r-1} \leq x < 5^r$. Consider the 5-adic expansion of x, which has r digits, the last one being $\neq 2$ by hypothesis. We distinguish two cases:

Case 1: The constant term is not 2, and there is some other digit $\neq 2$, say the one multiplying 5^s, for some s with $r > s > 0$. Write $x = 5^s y + z$, with $0 \leq z < 5^s$, $0 \leq y < 5^{r-s}$ (that is, split the first $r - s$ and the last s 5-adic digits of x). Then by induction on r we get

$$[7x]_{5,\infty} = [7 \cdot 5^s y + 7z]_{5,\infty} \leq [7 \cdot 5^s y]_{5,\infty} + [7z]_{5,\infty} =$$

$$= [7y]_{5,\infty} + [7z]_{5,\infty} \leq [y]_{5,\infty} + 2(r - s) + [z]_{5,\infty} + 2s = [x]_{5,\infty} + 2r.$$

Case 2: All other digits are $= 2$, that is, $x = (222...22a)_5$ with $a \in \{0, 1, 3, 4\}$. Note that $7 \cdot (222...220)_5 = (322...2140)_5$ (where there are two fewer 2's on the right hand side). Then

$$[7x]_{5,\infty} = [7 \cdot (22...220)_5 + 7a]_{5,\infty} \leq [7 \cdot (22...220)_5]_{5,\infty} + [7a]_{5,\infty} =$$
$$= [(322...2140)_5]_{5,\infty} + [7a]_{5,\infty} = 2(r+1) + [7a]_{5,\infty} \leq 2(r+1) + a + 2 =$$
$$= 2(r-1)+a+6 \leq 2(r-1)+a+2r = |(222...22a)_5]_{5,\infty}+2r = [x]_{5,\infty}+2r.$$

□

Remark 2.4. Although it will not be used, it follows from the lemma that for every $r \geq 1$ and for every integer x with $0 \leq x < 5^r$, we have

$$[7x]_{5,\infty} \leq [x]_{5,\infty} + 2r + 2.$$

Indeed, for $0 \leq x < 5^r$, the quantity $5x$ is $< 5^{r+1}$ and is not 2 mod 5. So by the lemma applied to $5x$ with $r + 1$, we have

$$[7 \cdot 5x]_{5,\infty} \leq [5x]_{5,\infty} + 2r + 2.$$

But $[7 \cdot 5x]_{5,\infty} = [7x]_{5,\infty}$ and $[5x]_{5,\infty} = [x]_{5,\infty}$.

Theorem 2.5. *The geometric monodromy of $\mathcal{F}_{5,7}$ is finite.*

Proof. By Theorem 2.1, we need to show that $[7x]_{5,r} \leq [x]_{5,r} + 2r$ for $r \geq 1$ and $0 < x < 5^r$.

If $x = \frac{5^r-1}{2}$, then $x = (22...22)_5$, so $[x]_{5,r}+2r = 4r$ and the inequality is clear, since $4r$ is an absolute upper bound for the function $[-]_{5,r}$.

Otherwise, some 5-adic digit of x is $\neq 2$. Since multiplying x by 5 cyclically permutes the digits of x modulo $5^r - 1$ and does not change the values of $[x]_{5,r}$ or of $[7x]_{5,r}$, we may assume that the last digit of x is $\neq 2$. Then

$$[7x]_{5,r} \leq [7x]_{5,\infty} \leq [x]_{5,\infty} + 2r = [x]_{5,r} + 2r$$

by Lemma 2.3. □

3. Determination of the monodromy groups

We first give a general descent construction, valid for general $\mathcal{F}_{p,D}$ with $D \geq 3$ prime to p. On $\mathbb{G}_m/\mathbb{F}_p$, consider the rank $D-1$ local system $\mathcal{H}_{p,D}$ whose trace function, for $k/\mathbb{F}_p$ a finite extension, and $t \in \mathbb{G}_m(k) = k^\times$, is

$$t \mapsto -\sum_{x \in k} \psi_k(x^D/t + x).$$

The pullback of $\mathcal{H}_{p,D}$ by the D'th power map $[D]$ is (the restriction to $\mathbb{G}_m$ of) the local system $\mathcal{F}_{p,D}$: simply repace t by t^D and make the change of variable $x \mapsto tx$ inside the ψ.

View $\mathcal{F}_{p,D}$ as the Fourier Transform $FT([D]^\star(\mathcal{L}_{\psi(x)})$. Then we see from [Ka-ESDE, 9.3.2], cf. also [Ka-RLSA, 2.1 (1)], that this $\mathcal{H}_{p,D}$ is geometrically isomorphic to the Kloosterman sheaf formed with all the nontrivial multiplicative characters of order dividing D.

Remark 3.1. Exactly as in Remark 1.3, when $D \geq 3$ is odd and prime to p, and p is 1 mod 4, the field of traces of $\mathcal{H}_{p,D}$ lies in $\mathbb{Q}(\mu_p)^+$.

This descent has all of its $I(\infty)$-slopes equal to $1/(D-1)$ (either from its identification with a Kloosterman sheaf of rank $D-1$, or because its $[D]$-pullback, $\mathcal{F}_{p,D}$, has all its $I(\infty)$-slopes equal to $D/(D-1)$).

Either from the fact that its pullback is geometrically irreducible, or from the Kloosterman description, or just from the fact of having all $I(\infty)$-slopes $1/(D-1)$, we see that $\mathcal{H}_{p,D}$ is geometrically irreducible.

Lemma 3.2. *Let $d \geq 2$, $\ell \neq p$, and M a d-dimensional continuous $\overline{\mathbb{Q}_\ell}$-representation ρ_M of $I(\infty)$ all of whose slopes are $1/d$. Suppose that d is not divisible by p^2. Then there does not exist a factorization of d as $d = ab$ with a, b both $< d$, together with algebraic groups $G_1 \subset SL(a, \overline{\mathbb{Q}_\ell})$ and $G_2 \subset SL(b, \overline{\mathbb{Q}_\ell})$ such that*

$$Image(\rho_M) \subset \text{the image } G_1 \otimes G_2 \text{ of } G_1 \times G_2 \text{ in } SL(ab, \overline{\mathbb{Q}_\ell}).$$

Proof. We argue by contradiction. The map $G_1 \times G_2 \to G_1 \otimes G_2$ has finite kernel, $\mathcal{K}$, which is a subgroup of the group $\mu_{\gcd(a,b)}$ (this being the kernel of $SL(a) \times SL(b) \to SL(ab)$). Because $I(\infty)$ has cohomological dimension one, the group $H^2(I(\infty), \mathcal{K}) = 0$, and therefore there exists a lift of ρ_M to a homomorphism

$$\rho_{a,b} : I(\infty) \to G_1 \times G_2,$$

compare [Ka-ESDE, 7.2.5]. Because the kernel $\mathcal{K}$ has order prime to p, the upper numbering subgroup $I(\infty)^{\frac{1}{d}+}$, which acts trivially on M, lies in the kernel of $\rho_{a,b}$ (simply because $I(\infty)^{\frac{1}{d}+}$ is a pro-p group which maps to the finite group $\mathcal{K}$ which has order prime to p, cf. [Ka-ESDE, 7.1.4]). Then the homomorphisms

$$\rho_a := pr_1 \circ \rho_{a,b} : I(\infty) \to G_1$$

and

$$\rho_b := pr_2 \circ \rho_{a,b} : I(\infty) \to G_2$$

are each trivial on $I(\infty)^{\frac{1}{d}+}$, i.e., each has all slopes $\leq 1/d$. Therefore their Swan conductors have $Swan(\rho_a) \leq a/d < 1$ and $Swan(\rho_b) \leq b/d < 1$. But Swan conductors are nonnegative integers. Therefore both ρ_a and ρ_b have $Swan = 0$, i.e., both are tame. But then M is tame, contradiction. $\square$

When $D \geq 3$ is odd and prime to p, the half-Tate twist $\mathcal{H}_{p,D}(1/2)$ is symplectically selfdual.

We now turn our attention to the particular case of $\mathcal{H}_{5,7}$ and its half-Tate twist $\mathcal{H}_{5,7}(1/2)$. We know from Theorem 2.5 that its G_{geom} (and hence also its G_{arith}, by Proposition 1.2) is a finite irreducible subgroup of $Sp(6, \overline{\mathbb{Q}_\ell})$. By Remark 3.1, the field of traces of $\mathcal{H}_{5,7}(1/2)$ lies in $\mathbb{Q}(\mu_5)^+ = \mathbb{Q}(\sqrt{5})$. Computing the trace at $t = 1 \in \mathbb{F}_5^\times$, we see that its field of traces is in fact $\mathbb{Q}(\sqrt{5})$ (and not just $\mathbb{Q}$).

Lemma 3.3. *The group $G_{geom} \subset Sp(6, \overline{\mathbb{Q}_\ell})$ is primitive, i.e., the given six-dimensional representation is not induced.*

Proof. By Pink's theorem [Ka-MG, Lemma 12], if a Kloosterman sheaf is (geometrically) induced, its list of characters is Kummer induced. So for our Kloosterman sheaf, formed with the nontrivial characters of order 7, being induced would imply that, for some divisor $n \geq 2$ of 6, its characters are all the n'th roots of some collection of $6/n$ characters. In particular, some ratio of distinct characters of order 7 would be a character of order dividing n, for some divisor n of 6, which is not the case: all such ratios have order 7.

Another proof is to observe that if $\mathcal{H}_{5,7}$ were induced, then its pullback $\mathcal{F}_{5,7}$ would be induced (a system of imprimitivity for a group remains one for any subgroup). But by [Such, 11.1], if $\mathcal{F}_{5,7}$ were induced, it would be Artin-Schreier induced, so its rank, 6, would be a multiple of $p = 5$. □

Theorem 3.4. *The local system $\mathcal{H}_{5,7}(1/2)$ has $G_{geom} = G_{arith} = 2.J_2$.*

Proof. Our situation now is that we have a primitive (by Lemma 3.2) irreducible subgroup G (the G_{geom} for $\mathcal{H}_{5,7}(1/2)$) in $Sp(6, \overline{\mathbb{Q}_\ell})$ such that the given six-dimensional representation is not contained in the tensor product of two lower dimensional representations of G. Therefore the larger finite group G_{arith} is a fortiori itself primitive and irreducible inside $Sp(6)$.

We now appeal to the work [Lind, &3, Theorem] of Lindsey, as stated in [C-S, Theorem 3.1]. This gives the list of irreducible primitive subgroups of $SL(6, \mathbb{C})$. Those whose given six-dimensional representation is not contained in a nontrivial tensor product are either

(1) $2.S_5$ or S_7.
(2) a quasisimple group.
(3) a group containing a quasisimple group of index two on which the representation remains irreducible.

The first case is subsumed by the third, as $2.S_5 = 2.A_5.2$ contains $2.A_5$, and S_7 contains A_7. The quasisimple groups in question are

$$2.A_5 = SL(2,5), 3.A_6, 6.A_6, A_7, 3.A_7, 6.A_7, PSL(2,7), SL(2,7), SL(2,11),$$

and

$$SL(2,13), PSp(4,3) \cong PSU(4,2), SU(3,3), 6.PSU(4,3), 2.J_2, 6.PSL(3,4).$$

Of these, those which lie as an index two subgroup of a larger group inside $SL(6,\mathbb{C})$ are

$$SL(2,5), 3.A_6, A_7, PSL(2,7), PSp(4,3),$$

$$SU(3,3), 6.PSU(4,3), 6.PSL(3,4).$$

Of the listed quasisimple groups, the only ones with irreducible symplectic representations of degree six are

$$SL(2,5),\ SL(2,7),\ SL(2,13),\ SU(3,3),\ 2.J_2.$$

.

Of these, the only ones whose field of character values (for any of its six-dimensional irreducible symplectic representations) lies in $\mathbb{Q}(\sqrt{5})$ are $SL(2,5)$, $SU(3,3)$ and $2.J_2$. For $SL(2,5)$ and $SU(3,3)$, the field of traces is $\mathbb{Q}$; for $2.J_2$ it is $\mathbb{Q}(\sqrt{5})$. So the only possibilities for G_{arith} other than $2.J_2$ are the groups $G.2$ for G either $SL(2,5)$ or $SU(3,3)$. But for neither of these two groups does the given representation extend to a symplectic representation (or a selfdual one), as one checks by looking in the Atlas [ATLAS].

Therefore G_{arith} for $\mathcal{H}_{5,7}(1/2)$ must be $2.J_2$. As G_{geom} is a normal subgroup of G_{arith} with cyclic quotient (namely some finite quotient of $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$), we must also have $G_{geom} = G_{arith} = 2.J_2$. □

Corollary 3.5. *For the local system $\mathcal{G}_{5,7}$, we have $G_{geom} = G_{arith} = 2.J_2$.*

Proof. Neither G_{geom} nor G_{arith} changes when we pass from $\mathbb{A}^1$ to the dense open set $\mathbb{G}_m$. Restricted to $\mathbb{G}_m$, $\mathcal{G}_{5,7}$ is the $[7]$ pullback of $\mathcal{H}_{5,7}(1/2)$. This pullback replaces the G_{geom} and G_{arith} of $\mathcal{H}_{5,7}(1/2)$ by normal subgroups of themselves of index dividing 7. But $2.J_2$ has no such proper subgroups. □

4. Appendix: Relation of $[x]_{p,r}$ to Kubert's V function

We denote by $(\mathbb{Q}/\mathbb{Z})_{\text{prime to } p}$ the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of those elements whose order is prime to p. We denote by $\mathbb{Q}_p^{n.r.}$ the fraction field of the Witt vectors of $\overline{\mathbb{F}_p}$. For $\mathbb{F}_q$ a finite extension of $\mathbb{F}_p$, we have the Teichmuller character

$$Teich_{\mathbb{F}_q} : \mathbb{F}_q^\times \cong \mu_{q-1}(\mathbb{Q}_p^{n.r.}),$$

whose reduction mod p is the identity map on $\mathbb{F}_q^\times$. For an integer d, consider the Gauss sum over $\mathbb{F}_q$,

$$G(\psi_{\mathbb{F}_q}, Teich^{-d}) := \sum_{x \in \mathbb{F}_q^\times} \psi_{\mathbb{F}_q}(x) Teich^{-d}(x).$$

If we write $q = p^r$, then by Stickelberger's theorem,

$$\mathrm{ord}_q(G(\psi_{\mathbb{F}_q}, Teich^{-d})) = (1/r) \sum_{j=0}^{r-1} < p^j \frac{d}{p^r - 1} > .$$

As explained in [Ka-G2hyper, p. 206], standard properties of Gauss sums show that there is a unique function

$$V : (\mathbb{Q}/\mathbb{Z})_{\text{prime to } p} \to \text{the real interval } [0, 1)$$

such that for $q = p^r$ and d an integer, we have

$$V\left(\frac{d}{p^r - 1}\right) = (1/r) \sum_{j=0}^{r-1} < p^j \frac{d}{p^r - 1} > .$$

As noted in [R-L, line before Theorem 1], we thus have the identity

$$V\left(\frac{d}{p^r - 1}\right) = \frac{1}{r(p-1)} [d]_{p,r}$$

provided that $1 \le d \le p^r - 2$ (i.e., provided that $\frac{d}{p^r-1}$ is nonzero in $(\mathbb{Q}/\mathbb{Z})_{\text{prime to } p}$). However, for $d = 0$, $V(\frac{d}{p^r-1}) = 0$, while

$$\frac{1}{r(p-1)} [0]_{p,r} = 1.$$

This "reversal" of the values at 0, together with the identity for Kubert's V function

$$V(x) + V(-x) = 1, \quad \text{for } x \neq 0,$$

means precisely that for any integer d and any power p^r of p, we have the identity

$$\frac{1}{r(p-1)} [d]_{p,r} = 1 - V\left(\frac{-d}{p^r - 1}\right).$$

With this identity, one sees easily that the criterion [R-L, Theorem 1] for $\mathcal{F}_{p,D}$ to have finite geometric monodromy, namely that

$$[Dx]_{p,r} \le [x]_{p,r} + r(p-1)/2$$

for all $r \geq 1$ and all integers x, is equivalent to the criterion [Ka-RLSA, first paragraph after 5.1] that for all $y \in (\mathbb{Q}/\mathbb{Z})_{\text{prime to } p}$, we have

$$V(Dy) + 1/2 \geq V(y).$$

References

[ATLAS] Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A. and Wilson, R. A., Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray. Oxford University Press, Eynsham, 1985.

[C-S] Cheltsov, I,, and Shramov, C.,Six-dimensional exceptional quotient singularities. Math. Res. Lett. 18 (20111), no. 06, 1121-1139.

[De-Weil II] Deligne, P., La conjecture de Weil II. Publ. Math. IHES 52 (1981), 313-428.

[G-T] Guranick, R. M. and Tiep, P. H., Symmetric powers and a conjecture of Kollar and Larsen, Invent. Math. 174 (2008), 505–554.

[Ka-ESDE] Katz, N., Exponential sums and differential equations. Annals of Mathematics Studies, 124. Princeton Univ. Press, Princeton, NJ, 1990. xii+430 pp.

[Ka-G2hyper] Katz, N., G_2 and hypergeometric sheaves, Finite Flelds Appl. 13 (2007), no. 2 175-223.

[Ka-MG] Katz, N., On the monodromy groups attached to certain families of exponential sums. Duke Math. J. 54 (1987), no. 1, 41-56.

[Ka-MGcorr] Katz, N., Correction to: "On the monodromy groups attached to certain families of exponential sums". Duke Math. J. 89 (1997), no. 1, 201.

[Ka-RLSA] Katz, N., Rigid local systems on $\mathbb{A}^1$ with finite monodromy, Mathematika 64 (2018) 785-846.

[Ka-Ti-RLSMFUG] Katz, N., and Tiep, P.H., Rigid local systems, moments, and finite unitary groups, available on `https://web.math.princeton.edu/~nmk/kt8_22.pdf`.

[Lind] Lindsey, J.H., Finite linear groups of degree six. Canad. J. Math. 23 (1971) 771-790.

[R-L] Rojas-León, A., Finite monodromy of some families of exponential sums, J. Number. Th. (2018), `https://doi.org/10.1016/j.jnt.2018.06.012`

[Such] Šuch, Ondrej, Monodromy of Airy and Kloosterman Sheaves, Duke Math. J. 103 (2000), no. 3, 397-444.

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