V Curso Internacional de Análisis Matemático en Andalucía

Optimal extensions of operators on Banach function spaces

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Problem

Given a measure space (Ω,Σ,μ), denote by $L^0(\mu)$ the space of all (classes of) measurable real functions on Ω.

Let E be a Banach space and $X(\mu)$ a Banach function space (B.f.s.), i.e. a Banach space $X(\mu)\subset L^0(\mu)$ such that

$$g\in L^0(\mu),\ f\in X(\mu) \text{ and } |g|\le|f| \ \Rightarrow\ g\in X(\mu) \text{ and } \|g\|_{X(\mu)}\le\|f\|_{X(\mu)}.$$

In particular $X(\mu)$ is a Banach lattice with the μ-a.e. pointwise order.

Consider a linear operator $T\colon X(\mu)\to E$ satisfying a property $(*)$.

Can T be extended to a larger domain in a way that the extension (still with values in E) preserves the property $(*)$? If that is the case, which is the largest of such domains?

Vector measure associated to T

Consider $\mathcal{R}_X=\{A\in\Sigma:\ \chi_A\in X(\mu)\}$ which is a δ-ring (ring closed under countable intersections) and the finitely additive set function

$$\nu_T\colon \mathcal{R}_X \longrightarrow E, \qquad A \longrightarrow \nu_T(A)=T(\chi_A)$$

Assume T is order-w continuous, i.e.

$$0\le f_n\uparrow f \text{ in the order of } X(\mu)\ \Rightarrow\ Tf_n\to Tf \text{ weakly in } E.$$

Then, ν_T is a vector measure, i.e.

$$(A_n)\subset\mathcal{R}_X \text{ disjoint sequence with } \cup A_n\in\mathcal{R}_X\ \Rightarrow\ \nu_T(\cup A_n)=\sum\nu_T(A_n).$$

Integration with respect to ν_T

Take the σ-algebra $\mathcal{R}_X^{loc}=\{A\subset\Omega:\ A\cap B\in\mathcal{R}_X \text{ for all } B\in\mathcal{R}_X\}$.

An $\mathcal{R}_X^{loc}$-measurable function $f\colon\Omega\to\mathbb{R}$ is integrable with respect to ν_T if

(i) $\int|f|\,d|e^*\nu_T|<\infty$ for all $e^*\in E^*$,

(ii) for each $A\in\mathcal{R}_X^{loc}$, there exists $\int_A f\,d\nu_T\in E$ such that

$$e^*\Big(\int_A f\,d\nu_T\Big)=\int_A f\,de^*\nu_T \quad \text{for all } e^*\in E^*.$$

$L^1(\nu_T)=\{\nu_T\text{-a.e. classes of functions } f \text{ integrable with respect to } \nu_T\}$

$L^1_w(\nu_T)=\{\nu_T\text{-a.e. classes of functions } f \text{ satisfying only condition (i)}\}$

Properties

- There exists a measure $\lambda\colon\mathcal{R}_X^{loc}\to[0,\infty]$ such that $\lambda\approx\nu_T$ (i.e. λ has the same null sets as ν_T). Denote $L^0(\nu_T)=L^0(\lambda)$.
- $L^1(\nu_T)$ and $L^1_w(\nu_T)$ are B.f.s.' related to λ with norm $\|f\|_{\nu_T}=\sup_{e^*\in B_{E^*}}\int|f|\,d|e^*\nu_T|$.
- $L^1(\nu_T)$ is order continuous, i.e.
$$f_n,f\in L^1(\nu_T),\ 0\le f_n\uparrow f\ \nu_T\text{-a.e.}\ \Rightarrow\ f_n\to f \text{ in norm of } L^1(\nu_T).$$
- $L^1_w(\nu_T)$ has the Fatou property, i.e.
$$\left.\begin{array}{l}(f_n)\subset L^1_w(\nu_T),\ 0\le f_n\uparrow\ \nu_T\text{-a.e.}\\ \text{and } \sup_n\|f_n\|_{\nu_T}<\infty\end{array}\right\}\Rightarrow\ \begin{array}{l} f=\sup_n f_n\in L^1_w(\nu_T)\\ \text{and } \|f_n\|_{\nu_T}\uparrow\|f\|_{\nu_T}\end{array}$$
- The integration operator $I_{\nu_T}\colon L^1(\nu_T)\to E$ is continuous ($I_{\nu_T}(f)=\int f\,d\nu_T$).

Extension for T

Assume $\mathcal{R}_X^{loc}=\Sigma$. Then, since $\nu_T\ll\mu$ (i.e. every μ-null set is ν_T-null), the map $[i]\colon L^0(\mu)\to L^0(\nu_T)$ which takes a μ-a.e. class in $L^0(\mu)$ represented by f into the ν_T-a.e. class represented by the same f, is well defined.

Theorem 1. ([CDS]). The integration operator I_{ν_T} extends T to $L^1(\nu_T)$, i.e.

$$X(\mu)\xrightarrow{T}E,\qquad X(\mu)\xrightarrow{[i]}L^1(\nu_T)\xrightarrow{I_{\nu_T}}E$$

The extension is optimal in the sense:

$$\left.\begin{array}{l}\text{If } Z(\xi) \text{ is a B.f.s. such that } \xi\ll\mu \text{ and}\\ (1)\quad X(\mu)\xrightarrow{T}E,\quad X(\mu)\xrightarrow{[i]}Z(\xi)\xrightarrow{S}E\\ \text{with } S \text{ order-w continuous such that}\\ (2)\quad \sup_{B\in\mathcal{R}_X\cap\mathcal{P}(A)}|e^*S(\chi_B)|=0\ \Rightarrow\ e^*S(\chi_A)=0\\ \text{for every } e^*\in E^* \text{ and } A\in\Sigma \text{ with } \chi_A\in Z(\xi)\end{array}\right\}\Rightarrow\ \begin{array}{l}[i]\colon Z(\xi)\to L^1(\nu_T)\\ \text{and } S=I_{\nu_T} \text{ on } Z(\xi)\end{array}$$

Note: I_{ν_T} is order-w continuous and satisfies (2).

[CDS] J. M. Calabuig, O. D. & E. A. Sánchez Pérez, Factorizing operators on Banach function spaces through spaces of multiplication operators, J. Math. Anal. Appl. **364** (2010), 88-103.

Particular cases

Case I: $X(\mu)$ has a weak unit (i.e. $g>0$ μ-a.e.).
Case II: $\Sigma=\mathcal{P}(\Omega)$ and $\chi_{\{\omega\}}\in X$ with $T(\chi_{\{\omega\}})\neq0$ for all $\omega\in\Omega$.

In any of these cases, we have that $\mathcal{R}_X^{loc}=\Sigma$ and condition (2) always holds for every order-w continuous operator S satisfying (1). So, Theorem 1 gives:

$L^1(\nu_T)$ is the largest B.f.s. to which T can be extended as an order-w continuous operator (still with values in E).

Also note that, in any of these cases, $L^1_w(\nu_T)$ has the net-Fatou property, i.e. similar to the Fatou property but for nets instead of sequences ([CDJS]).

[CDJS] J. M. Calabuig, O. D., M. A. Juan & E. A. Sánchez Pérez, Banach lattice properties of L^1_w of a vector measure on a δ-ring, preprint.

Further extension for T

Denote by $[L^1(\nu_T)]_F$ the Fatou completion of $L^1(\nu_T)$, i.e. the minimal B.f.s. (related to ν_T) with the Fatou property containing $L^1(\nu_T)$. Note that

$$L^1(\nu_T)\subset[L^1(\nu_T)]_F\subset L^1_w(\nu_T)$$

$$X(\mu)\xrightarrow{T}E,\quad X(\mu)\xrightarrow{[i]}L^1(\nu_T)\xrightarrow{I_{\nu_T}}E,\quad L^1(\nu_T)\xrightarrow{i}[L^1(\nu_T)]_F\xrightarrow{?}E,\quad [L^1(\nu_T)]_F\xrightarrow{i}L^1_w(\nu_T)\xrightarrow{?}E$$

Is it possible to extend T to $[L^1(\nu_T)]_F$ or $L^1_w(\nu_T)$?

Needed notions: Given a Banach lattice Y, we denote by Y_a the order continuous part of Y, i.e. the largest order continuous ideal of Y. An ideal F of Y is order dense in Y if for every $0\le y\in Y$ there exists $(y_n)\subset F$ such that $0\le y_n\uparrow y$ in the order of Y. A linear operator $T\colon Y\to Z$ between Banach lattices is order-order continuous if

$$f_n\uparrow f \text{ in the order of } Y\ \Rightarrow\ Tf_n\uparrow Tf \text{ in the order of } Z.$$

Extension of T to $[L^1(\nu_T)]_F$

Assume E is a Banach lattice and T is positive (i.e. $Tf\ge0$ for all $0\le f\in X(\mu)$).

Theorem 2. If $X(\mu)$ is order continuous and E has the Fatou property, then there exists a linear operator $\overline{I}_{\nu_T}$ such that

$$X(\mu)\xrightarrow{T}E,\qquad X(\mu)\xrightarrow{[i]}[L^1(\nu_T)]_F\xrightarrow{\overline{I}_{\nu_T}}E$$

The extension is optimal in the sense:

$$\left.\begin{array}{l}\text{If } Z(\xi) \text{ is a B.f.s. with } Z(\xi)_a \text{ order dense in } Z(\xi)\\ \text{such that } \xi\ll\mu \text{ and (1) holds with } S \text{ positive,}\\ \text{order-order continuous, satisfying (2) on } Z(\xi)_a\end{array}\right\}\Rightarrow\ \begin{array}{l}[i]\colon Z(\xi)\to[L^1(\nu_T)]_F\\ \text{and } S=\overline{I}_{\nu_T} \text{ on } Z(\xi)\end{array}$$

Note: $\big([L^1(\nu_T)]_F\big)_a=L^1(\nu_T)$ is order dense in $[L^1(\nu_T)]_F$ ([CDJS]). Moreover, $\overline{I}_{\nu_T}$ is positive, order-order continuous and satisfies (2) on $L^1(\nu_T)$.

For Case I (Case II): $[L^1(\nu_T)]_F$ is the largest B.f.s. (having its order continuous part as an order dense subset) to which T can be extended as a positive order-order continuous operator (still with values in E).

Extension of T to $L^1_w(\nu_T)$

For Case I: $L^1_w(\nu_T)=[L^1(\nu_T)]_F$ ([CDJS]), so the problem is solved in Theorem 2.

Notions of net-order density and net-order-order continuity are similar to order density and order-order continuity but for nets instead of sequences.

Theorem 3. If $X(\mu)$ is order continuous and E has the net-Fatou property, then there is a linear operator $\widehat{I}_{\nu_T}$ such that

$$X(\mu)\xrightarrow{T}E,\qquad X(\mu)\xrightarrow{[i]}L^1_w(\nu_T)\xrightarrow{\widehat{I}_{\nu_T}}E$$

If $\nu_T\approx\mu$ and $L^1_w(\nu_T)$ has the net-Fatou property, then we get optimality:

$$\left.\begin{array}{l}\text{If } Z(\xi) \text{ is a B.f.s. with } Z(\xi)_a \text{ net-order dense in } L^0(\xi)\\ \text{such that } \xi\ll\mu \text{ and (1) holds with } S \text{ positive,}\\ \text{net-order-order continuous, satisfying (2) on } Z(\xi)_a\end{array}\right\}\Rightarrow\ \begin{array}{l}[i]\colon Z(\xi)\to L^1_w(\nu_T)\\ \text{and } S=\widehat{I}_{\nu_T} \text{ on } Z(\xi)\end{array}$$

Note: $\big(L^1_w(\nu_T)\big)_a=L^1(\nu_T)$ is net-order dense in $L^0(\nu_T)$ ([CDJS]). Moreover, $\widehat{I}_{\nu_T}$ is positive, net-order-order continuous and satisfies (2) on $L^1(\nu_T)$.

For Case II: $L^1_w(\nu_T)$ is the largest B.f.s. to which T can be extended as a positive net-order-order continuous operator (still with values in E).

Hardy type operators

The following operators satisfy all the hypothesis of Theorems 1, 2 and 3.

- $T\colon L^1[0,\infty)\to L^\infty[0,\infty)$ given by $Tf(x)=\frac{1}{\psi(x)}\int_0^x f(y)\,dy$ for all $f\in L^1[0,\infty)$ and $x\ge0$, where $\psi\colon[0,\infty)\to[0,\infty)$ is such that $\psi>0$ a.e. and $\frac{1}{\psi}\in L^\infty[0,\infty)$.
- $T\colon\ell^1(I)\to\ell^\infty(I)$ given by $T(x)=\big(\sum_{j\in I}x_jK(i,j)\big)_{i\in I}$ for all $x=(x_j)\in\ell^1(I)$, where I is an uncountable set and $K\colon I\times I\to[0,\infty)$ is such that $K(\cdot,j)\neq0$ for all $j\in I$ and $\|K\|_\infty<\infty$.