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Leaps of the chain of m-integrable derivations in the sense of Hasse-Schmidt

Author
María de la Paz Tirado Hernández

Supervisor
Prof. Dr. *Luis Narváez Macarro*

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Resumen

Sea k un anillo conmutativo. Los módulos de las k-derivaciones m-integrables (en el sentido de Hasse-Schmidt) de una k-álgebra conmutativa forman una cadena decreciente cuyas inclusiones pueden ser estrictas. Decimos que un entero $s > 1$ es un *leap* de una k-álgebra conmutativa si la $(s-1)$-ésima inclusión en la cadena anterior es propia. En esta tesis, estudiamos el conjunto que forman los leaps en diferentes contextos.

En primer lugar, consideramos k un anillo de característica positiva y probamos que los leaps de cualquier k-álgebra conmutativa sólo ocurren en las potencias de la característica.

Luego, nos centramos en estudiar el comportamiento de los módulos de las k-derivaciones m-integrables de una k-álgebra conmutativa finitamente generada bajo cambios de base y probamos que si consideramos extensiones de cuerpos trascendentes puras y k-álgebras conmutativas finitamente presentadas, entonces el conjunto de los leaps no cambia bajo el cambio de base. Lo mismo ocurre si consideramos extensiones separables de anillos sobre un cuerpo de característica positiva y k-álgebras conmutativas finitamente generadas.

Por último calculamos el módulo de las k-derivaciones m-integrables en diferentes curvas planas. Principalmente, damos los generadores de los módulos de las k-derivaciones m-integrables, donde k es un anillo reducido de característica p, del cociente del anillo de polinomios en dos variables con coeficientes en k sobre un ideal generado por la ecuación $x^n - y^q$ donde n o q no es múltiplo de p.

Abstract

Let k be a commutative ring. The modules of m-integrable k-derivations (in the sense of Hasse-Schmidt) of a commutative k-algebra form a decreasing chain whose inclusions could be strict. We say that an integer $s > 1$ is a *leap* of a commutative k-algebra if the $s-1$-th inclusion of the previous chain is proper. In this thesis, we study the set of leaps in different contexts.

First, we consider a commutative ring k of positive characteristic and we prove that leaps of any commutative k-algebra only happen at powers of the characteristic.

Thereafter, we focus on studying the behavior of the modules of m-integrable k-derivations of a commutative finitely generated k-algebra under base change and we prove that if we consider pure transcendental field extensions and commutative finitely presented k-algebras, then the set of leaps does not change under the base change. The same happens if we consider separable ring extensions over a field of positive characteristic and commutative finitely generated k-algebras.

Finally, we compute the modules of m-integrable k-derivations of different plane curves. Mainly, we give the generators of the modules of m-integrable k-derivations, where k is a reduced ring of characteristic $p > 0$, of the quotient of the polynomial ring in two variables with coefficients in k over the ideal generated by the equation $x^n - y^q$ where n or q is not multiple of p.

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Contents

Introduction

Background

Let k be a commutative ring and A a commutative k-algebra. A Hasse-Schmidt derivation of A over k of length $m \geq 0$ (or $m = \infty$) is a sequence $D = (D_0, D_1, \ldots, D_m)$ (or $D = (D_0, D_1, \ldots)$) of k-linear endomorphisms of A such that D_0 is the identity map and the following Leibniz identity holds:

$$D_r(xy) = \sum_{i+j=r} D_i(x) D_j(y)$$

for all $x, y \in A$ and all $r \geq 0$. We write $\mathrm{HS}_k(A; m)$ for the set of Hasse-Schmidt derivations of A over k of length $m \geq 1$ or $m = \infty$.

Any Hasse-Schmidt derivation $D = (\mathrm{Id}, D_1, D_2, \ldots) \in \mathrm{HS}_k(A; m)$ can be associated with a k-algebra homomorphism $\varphi : A \to A[|\mu|]_m = A[|\mu|]/\langle \mu^{m+1} \rangle$ such that $\varphi(x) \equiv x \mod \mu$ for all $x \in A$ given by $\varphi(x) = x + D_1(x)\mu + \cdots + D_m(x)\mu^m$. A group structure (non-commutative in general) can be defined on the set of Hasse-Schmidt derivations $\mathrm{HS}_k(A; m)$. Namely, if $D, D' \in \mathrm{HS}_k(A; m)$, $D'' := D \circ D' \in \mathrm{HS}_k(A; m)$ such that $D''_r = \sum_{i+j=r} D_i \circ D'_j$. Moreover, for all $r \geq 1$, the rth component D_r of a Hasse-Schmidt derivation turns out to be a k-linear differential operator of order $\leq r$ vanishing at 1. In particular, D_1 is a k-derivation of A (in classical sense) and we can identify the additive group of k-derivations of A, which is denoted by $\mathrm{Der}_k(A)$, with the group of Hasse-Schmidt derivations of length 1.

An important notion related with the theory of Hasse-Schmidt derivations is m-integrability for $m \geq 1$ or $m = \infty$. We say that a k-derivation $\delta \in \mathrm{Der}_k(A)$ is m-integrable if there exists $D \in \mathrm{HS}_k(A; m)$, which is called an m-integral of δ, such that $D_1 = \delta$, or in other words, if the k-algebra map $\varphi_\delta : a \in A \mapsto a + \delta(a)\mu \in A[|\mu|]_1$ can be lifted to a k-algebra map $\varphi : A \to A[|\mu|]_m$. The set of all m-integrable k-derivations is an A-submodule of $\mathrm{Der}_k(A)$ for all $m \geq 1$ or $m = \infty$, which is denoted by $\mathrm{IDer}_k(A; m)$ and it is clear that

$$\mathrm{Der}_k(A) = \mathrm{IDer}_k(A; 1) \supseteq \mathrm{IDer}_k(A; 2) \supseteq \cdots \supseteq \mathrm{IDer}_k(A; \infty).$$

If k is a ring of characteristic 0, i.e. if $\mathbb{Q} \subseteq k$, then any k-derivation δ is ∞-integrable, since we can take $D = (\delta^i/i!)$ as an ∞-integral of δ. The same property holds if A is 0-smooth over k (cf. [Ma2, §25] and [Ma2, Th. 27.1]) or if A is a "normal crossing singularity", i.e. $A = k[x_1, \ldots, x_d]/\langle x_1 \cdots x_e \rangle$ or $A = k[|x_1, \ldots, x_d|]/\langle x_1 \cdots x_e \rangle$ for $e \leq d$. However, in [Ma1] we can already find examples of k-derivations that are not ∞-integrable in the case where k is a ring of positive characteristic $p > 0$ ($\mathbb{F}_p \subseteq k$). These examples also implicitly prove the existence of k-derivations which are $(m-1)$-integrable but not m-integrable for some

$m \geq 1$. That implies that, if k has positive characteristic, the chain of modules of m-integrable derivations could have strict inclusions, i.e. there could be some positive integers m for which $\mathrm{IDer}_k(A; m-1) \neq \mathrm{IDer}_k(A; m)$. In that case, we say that A has a *leap* at m and we denote by $\mathrm{Leaps}_k(A)$ the set of leaps of A over k.

The notion of Hasse-Schmidt derivation was introduced by H. Hasse and F.K. Schmidt in [H-S] in the case where k is a field of positive characteristic and A a field of algebraic functions over k. P. Ribemboin studied them in a general setting in [Ri1] and [Ri2] and they have been used in different contexts, for instance W. Traves proved in [Tr] that if A is a smooth algebra of finite type over a field k, then the ring of differential operators of A over k equals the Hasse-Schmidt algebra of A over k, i.e. the subalgebra of $\mathrm{End}_k(A)$ generated by all components of all Hasse-Schmidt derivations, or P. Vojta used Hasse-Schmidt derivations to describe jet spaces in [Vo].

The problem of deciding when a derivation is ∞-integrable or not has been studied by several authors such as W.C. Brown in [Br]. One of the first results we can find about the modules of integrable derivations is due to A. Seidenberg. In [Se], he proved the following result: Let A be a domain and Σ its quotient field. Let us denote A' the ring of all elements of Σ which are quasi-integral i.e. $\alpha \in A'$ if there is $d \in A$, $d \neq 0$ such that $d\alpha^s \in A$ for all $s \geq 0$ (note that if A is notherian then A' coincides with the integral closure of A) and we consider $D \in \mathrm{HS}(\Sigma; \infty)$. Then, if $D_r(A) \subseteq A$ for all $r \geq 0$, then $D_r(A') \subseteq A'$ for all $r \geq 0$. Hence, we can deduce that any ∞-integrable derivation of A can be extended to an ∞-integrable derivation of A'. However, this result is not true when A has positive characteristic and we consider $D \in \mathrm{Der}(\Sigma)$ instead of a Hasse-Schmidt derivation of length ∞.

Another interesting result about integrability is due to S. Molinelli. She showed that if $(A; \mathfrak{m})$ is a local domain of characteristic $p > 0$, $\hat{A}$ its completion and k a coefficient field of $\hat{A}$ then, we have that $\mathrm{rank}\left(\{\delta \in \mathrm{Der}(A) \mid \hat{\delta} \in \mathrm{IDer}_k(\hat{A})\}\right) \leq \dim A$ (see [Mo, Corollary 2.3]) although the rank of $\mathrm{Der}_k(\hat{A})$ could be strictly greater than $\dim(A) = \dim(\widehat{A})$. In [Ma1], H. Matsumura, in addition to giving the aforementioned examples, proved some sufficient conditions for ∞-integrability, for instance if $k \to A$ is a separable field extension, all k-derivations are ∞-integrable.

Later, M. Fernández Lebrón and L. Narváez Macarro used the module of ∞-integrable derivations to generalize a result of M. Nomura ([Ma2, Th. 30.6]) in [F-N] and in [Na1], L. Narváez Macarro proved that there is a canonical map of graded A-algebras $\upsilon : \Gamma_A \mathrm{IDer}_k(A) \to \mathrm{gr}\,\mathrm{Diff}_{A/k}$, where $\Gamma_A(*)$ denotes the divided power algebra functor and $\mathrm{gr}\,\mathrm{Diff}_{A/k}$ is the graded ring of the filtered ring of k-linear differential operators of A, such that υ equals to the canonical map of graded A-algebras $\mathrm{Sym}_A \mathrm{Der}_k(A) \to \mathrm{gr}\,\mathrm{Diff}_{A/k}$ if k has characteristic zero and υ is an isomorphism whenever $\mathrm{IDer}_k(A) = \mathrm{Der}_k(A)$ and $\mathrm{Der}_k(A)$ is a finitely generated projective A-module.

More recently, some results about finite integrability have been given. In [Na2], L. Narváez Macarro showed that if A is a finitely presented k-algebra and m is an integer, the property of being m-integrable for a k-derivation δ of A is a local property, i.e. δ is m-integrable if and only if the induced derivation $\delta_{\mathfrak{p}} : A_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is m-integrable for each prime ideal $\mathfrak{p} \subseteq A$ ([Na2, Th. 3.2.6]).

We would also like to highlight the work of D. Hoffmann and P. Kowalski in [H-K1] and [H-K2] where they generalized, among others, a result of H. Matsumura ([Ma1]). Namely, they proved that if k is a field of characteristic $p > 0$ and $k \subseteq A$ is a separable (not necessarily algebraic) field extension, then any iterative Hasse-Schmidt derivation $D \in \mathrm{HS}_k(A; p^m)$ for some $m \geq 1$ (that is, for all $i, j \in \mathbb{N}$, we have that $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$) can be extended to an iterative Hasse-Schmidt derivation of A (over k) of infinite length.

The chain of modules of m-integrable derivations of a k-algebra A seems to reflect some specific properties of singularities in non-zero characteristic, and so, in our opinion, it deserves to be studied.

Results of this thesis

In this work we focus on the study of leaps of a commutative k-algebra, where k is a commutative ring. Let us state now the main results of this thesis.

I (Theorem 2.5.1). *If k is a commutative ring of characteristic $p > 0$ and A is a commutative k-algebra, leaps of A only occur at powers of $p > 0$, i.e.* $\mathrm{Leaps}_k(A) \subseteq \{p^\tau \mid \tau \geq 1\}$.

Let us consider a ring extension $k \to L$ and a commutative finitely generated k-algebra A, we give a $(L \otimes_k A)$-linear base change map for m-integrable k-derivations, which we denote by $\Phi_m^{L,A} : L \otimes_k \mathrm{IDer}_k(A; m) \to \mathrm{IDer}_L(L \otimes_k A; m)$ (see section 3.2.2) and we prove the following result (see Corollary 3.2.23 and Corollary 3.2.33):

II *Let us consider a ring extension $k \to L$ and a commutative finitely generated k-algebra A. The map $\Phi_m^{L,A}$ is an isomorphism for all $m \geq 1$ if some of the following conditions holds:*

1. *$L = k[t_i \mid i \in \mathcal{I}]$ a polynomial ring in an arbitrary number of variables.*

2. *k is a field of characteristic $p > 0$ and L is a separable k-algebra.*

Therefore, in both cases, $\mathrm{Leaps}_k(A) = \mathrm{Leaps}_L(L \otimes_k A)$.

We also give a counterexample (3.2.14) for the surjectivity of $\Phi_m^{L,A}$ and from 1., we deduce that if $k \to L$ is a pure transcendental field extension and A is a finitely presented k-algebra, then $\Phi_m^{L,A}$ is also bijective for all $m \geq 1$ (Corollary 3.2.25).

Finally, we give some explicit computations of generators of modules of m-integrable derivations of some "plane curves singularities". Our main computations concern the curve $x^n - y^q$. We show the following results.

III (Corollary 4.1.4). *Let k be a commutative reduced ring of characteristic $p > 0$ and $A = k[x, y]/\langle h \rangle$ where $h = x^n - y^q$ with $n, q \neq 0$. Let $\alpha := \mathrm{val}_p(n)$ be the p-adic valuation of n, $s = n/p^\alpha$, m the remainder of the division of q by p and $\beta := \mathrm{val}_p(q - m)$. Then, we have the following properties.*

1. *If $n, q \neq 0 \mod p$ then,* $\mathrm{Leaps}_k(A) = \emptyset$.

2. *If $n = 0 \mod p$ and $q = 1$ then,* $\mathrm{Leaps}_k(A) = \emptyset$.

3. *If* $\alpha, m \geq 1$ *and* $q \geq 2$*, then*

$$\mathrm{Leaps}_k(A) = \begin{cases} \{p^\alpha, p^{\alpha+\beta}\} & \textit{if } s = 1,\ \alpha \leq \beta,\ m = 1 \\ \{p^\alpha\} & \textit{otherwise} \end{cases}$$

4. *If* $\alpha = 0$ *(i.e.* $n \neq 0 \mod p$*) and* $m = 0$ *(i.e.* $q = 0 \mod p$*) then,* $\mathrm{Leaps}_k(A) = \mathrm{Leaps}_k(A')$ *where* $A' = k[x,y]/\langle x^q - y^n \rangle$*.*

Moreover, if k *is a unique factorization domain,* $m = 0$*,* $\alpha, \beta \geq 1$ *and we denote* $\tau = \min\{\alpha, \beta\} \geq 1$*,* $n' = n/p^\tau$ *and* $q' = q/p^\tau$*, we have that*

$$\mathrm{Leaps}_k(A) = \{p^\tau\} \cup \{ip^\tau \mid i \in \mathrm{Leaps}_k(B)\} \textit{ where } B = k[x,y]/\left\langle x^{n'} - y^{q'} \right\rangle$$

From the computation of leaps of $k[|x,y|]/\langle x^3 - y^5 + x^2y^2 \rangle$ and the previous result we can deduce the following result:

Proposition 4.3.3. *Leaps of irreducible algebroid plane curve over an algebraically closed field are not determined by the semigroup of the curve.*

In addiction, from the computation of leaps of $k[x,y]/I$ and $k[x,y]/\overline{I}$ where $I = \langle x^2, y^2 \rangle$ and $\overline{I}$ is its integral closure we have the following result.

Lemma 4.3.4. *Leaps are not the same up integral closure of ideals.*

The last two results answer two questions proposed by Professor H. Mourtada.

Contents of the chapters

This text is organized as follows: In chapter 1, we recall main definitions of the theory of the Hasse-Schmidt derivations. In the first section, we give the definition of Hasse-Schmidt derivations and some properties. We also define what is meant by logarithmic Hasse-Schmidt derivations and we recall the main object of our work: modules of m-integrable derivations. In section 1.2, we consider a polynomial ring and we see some properties of Hasse-Schmidt derivations in this particular case. Moreover, we prove the relationship between integrable derivations of the quotient of a polynomial ring over $\langle h \rangle$ and over $\langle h^p \rangle$ where h is a polynomial, when k is a unique factorization domain. In section 1.3, we recall a generalization of Hasse-Schmidt derivations that we use to describe a special Hasse-Schmidt derivation in section 2. In the last section of this chapter we talk about substitution maps and how they act on Hasse-Schmidt derivations.

Chapter 2 is devoted to prove **I**. We start this chapter with some numerical and technical results that will be useful in the rest of the chapter. In section 2.2 we associate with any Hasse-Schmidt derivation a special Hasse-Schmidt derivation that we use to prove the main theorem of this chapter. In section 2.3 we prove that any k-algebra does not have leaps at certain integers. Namely, if k is any commutative ring and A any commutative k-algebra, we show that A does not have leaps at any integers invertible in k; If the characteristic of k is

$p = 2$, then we prove that A does not have a leap at 6, and if the characteristic of k is $p \neq 2$, then we prove that A does not have a leap at $2p$. In section 2.4, we give an integral of the first component of a Hasse-Schmidt derivation that might not be zero and in the last section, we prove our main result of this chapter, namely that if k is a commutative ring of characteristic $p > 0$ and A is any commutative k-algebra, A only has leaps at powers of p.

The aim of chapter 3 is to prove **II**. In the first section of this chapter, we see that an I-logarithmic Hasse-Schmidt derivation of a polynomial ring $R = k[x_1, \ldots, x_d]$ over a ring k of positive characteristic (where $I \subseteq R$ is an ideal) can be decomposed in two Hasse-Schmidt derivations if its first component is zero. In the next section, we recall some classical results of base change maps for k-derivations and we generalize these maps for integrable k-derivations, which is denoted by $\Phi_m^{L,A} : L \otimes \mathrm{IDer}_k(A; m) \to \mathrm{IDer}_L(L \otimes_k A; m)$ where $k \to L$ is a commutative ring extension and A a commutative k-algebra. We see that $\Phi_m^{L,A}$ is not surjective in general giving a counterexample when $k \to L$ is an algebraic non-separable field extension and we prove that if $k \to L$ is a pure transcendental field extension and A is a finitely presented k-algebra, then $\Phi_m^{L,A}$ is surjective and the same happens if L is a separable k-algebra where k is a field of positive characteristic and A is a finitely generated k-algebra.

In chapter 4, we prove **III**. Namely, in section 4.1 we compute the modules of m-integrable k-derivations, where k is a reduced ring of characteristic $p > 0$, of the quotients of the polynomial ring in two variables over the ideal generated by the equation $x^n - y^q$ when n or q is not a multiple of p. Thanks to this, we can describe the integrable derivations of $k[x, y]/\langle x^n - y^q \rangle$ when n and q are both multiples of p and k is a unique factorization domain. In section 4.2 we compute the modules of integrable derivations in three examples taken from [Gr] assuming that k is a domain of positive characteristic and showing that there exist singular curves with no leaps. In the last section of this chapter we prove Proposition 4.3.3 and Lemma 4.3.4.

Further developments

To conclude this introduction, we would like to comment some of the problems related with m-integrability that remain open and that we would like to study in the near future.

As we have already said, the study of the chain of modules of m-integrable derivations could help us with singularities in positive characteristic but there are still many questions to solve, for instance: are leaps related with some known invariant of singularities? or, how leaps behave under geometric constructions, such as blowing-ups? It could also be interesting to know more about the relationship between Hasse-Schmidt derivations and jet spaces (see [Vo]), and therefore with arc spaces. In addition, we would also like to understand the meaning of the absence of leaps for a singularity in positive characteristic: does it mean that the behavior of such a singularity is "closer" (in some sense) to the behavior of singularities in characteristic 0?

One of the main problems in the theory of Hasse-Schmidt derivations is to compute where leaps occur. We know that in characteristic zero and if A is 0-smooth over k, $\mathrm{Leaps}_k(A) = \emptyset$ and we have proven that if k has positive characteristic then, $\mathrm{Leaps}_k(A) \subseteq \{p^\tau \mid \tau \geq 1\}$ (Theorem

2.5.1). However, we do not know what happens when the base ring is not any one of these types, for instance if $k = \mathbb{Z}$. We also do not know if any set of leaps is possible, that is, if given $m \geq 1$ positive integer or infinity, there is a ring with m leaps. In view of the results of chapter 4, we can say that there are rings with 1, 2 or 3 leaps but we can expect that longer sets of leaps are possible, and to check this, new algorithms are needed.

Another interesting question is the one proposed by L. Narváez Macarro in [Na2, Q. 3.6.5]: *Assume that the base ring k is a field of positive characteristic or $\mathbb{Z}$, or perhaps a more general noetherian ring, and A a finitely generated k-algebra. Is there an integer $n \geq 1$ such that* $\mathrm{IDer}_k(A; n) = \mathrm{IDer}_k(A; \infty)$*? Or at least, is the descending chain of A-modules* $\mathrm{IDer}_k(A; 1) \supseteq \mathrm{IDer}_k(A; 2) \supseteq \cdots$ *stationary?* And, what about more general base rings k? This problem can be seen as a problem of "extensions of k-linear maps" in the following way:

Let A be a finitely generated k-algebra. Then, A can be seen as a quotient of a polynomial ring $k[x_1, \ldots, x_d]$ over an ideal $I \subseteq R$. Let us assume that $I = \langle f_1, \ldots, f_r \rangle$ and we have $\delta \in \mathrm{IDer}_k(A; m)$. By definition of m-integrability, we have that there is a k-linear map $\varphi : A \to A[|\mu|]_m$ associated with an m-integral of δ and $f_i(\varphi(x_1), \ldots, \varphi(x_d)) = 0 \mod \mu^{m+1}$. Then, to prove that $\delta \in \mathrm{IDer}_k(A; \infty)$ it would be enough to extend φ to $\varphi' : A \to A[|\mu|]$, i.e. to find $\varphi'(x_j) \in A[|\mu|]$ for all $j = 1, \ldots, d$ such that $f_i(\varphi'(x_1), \ldots, \varphi'(x_d)) = 0$. This extension seems to be related with Artin's Approximation Theorem, and deserves futher study.

It may also be interesting to know necessary and sufficient conditions for a Hasse-Schmidt derivation to be extended to a Hasse-Schmidt derivation of higher length. For instance, we can study the relationship between the integrability of the Hasse-Schmidt derivation D and the integrability of the derivations associated with D, $\varepsilon_i(D)$ for $i \geq 1$ (see [Na4] and section 1.1.2). In characteristic zero, any Hasse-Schmidt derivation is determined by these derivations so, does the integrability of a Hasse-Schmidt derivation D depend, or is related with the integrability of the $\varepsilon_i(D)$?

On the other hand, we want to continue studying the base change map $\Phi_m^{L,A} : L \otimes_k \mathrm{IDer}_k(A; m) \to \mathrm{IDer}_L(A_L; m)$ for any k-algebra A, any ring extension $k \to L$ and any $m \geq 1$ or $m = \infty$ that we have defined in chapter 3, specially the surjectivity of $\Phi_m^{L,A}$. Although we know that $\Phi_m^{L,A}$ is not surjective in general, we have seen that if L is a polynomial ring in an arbitrary number of variables or $k \to L$ is a separable field extension where k is a field of characteristic $p > 0$ and A is finitely generated k-algebra, then $\Phi_m^{L,A}$ is surjective. Both cases are examples of 0-smooth base change, so a natural question would be whether the base change map is an isomorphism under this general hypothesis.

Finally, if A is a finitely presented k-algebra we know that m-integrability with $m \geq 1$ an integer is a local property (see [Na2]) but we do not know how the modules of m-integrable derivation of a local ring behave under completion. We know that if $\delta \in \mathrm{Der}_k(A)$ is m-integrable, then its induced $\hat{\delta} : \hat{A} \to \hat{A}$ is m-integrable but, is it true its converse?

Chapter 1

Hasse-Schmidt derivations

Hasse-Schmidt derivations were introduced by H. Hasse and F.K. Schmidt in [H-S]. In this text, we are interested in a particular notion originated in the theory of Hasse-Schmidt derivations: The module of m-integrable derivations, where $m \in \mathbb{N}$ or $m = \infty$. In this chapter we will recall its definition and we will give necessary properties for the rest of the chapters.

1.1 Introduction to Hasse-Schmidt derivations

In this section we recall the main definitions and properties of Hasse-Schmidt derivations. Most of the results presented in this section can be found in [Ma2, §27], [Na2] and [Na3]. In this chapter, k will be a commutative ring and A a commutative k-algebra.

We will start by setting the following notation: We denote $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and, for each integer $m \geq 1$, we will write $A[|\mu|]_m := A[|\mu|]/\langle \mu^{m+1} \rangle$ and $A[|\mu|]_\infty := A[|\mu|]$.

Definition 1.1.1 *A Hasse-Schmidt derivation (HS-derivation for short) of A (over k) of length $m \geq 1$ (resp. of length ∞) is a sequence $D := (D_0, D_1, \ldots, D_m)$ (resp. $D = (D_0, D_1, \ldots)$) of k-linear maps $D_r : A \to A$, satisfying the conditions:*

$$D_0 = \mathrm{Id}_A, \quad D_r(xy) = \sum_{\beta+\gamma=r} D_\beta(x) D_\gamma(y)$$

for all $x, y \in A$ and for all r. We write $\mathrm{HS}_k(A; m)$ (resp. $\mathrm{HS}_k(A; \infty) = \mathrm{HS}_k(A)$) for the set of HS-derivations of A (over k) of length m (resp. ∞).

The D_r component is a k-linear differential operator of order $\leq r$ vanishing at 1 if $r \geq 1$. In particular, D_1 is a k-derivation.

Any HS-derivation $D \in \mathrm{HS}_k(A; m)$ is determined by the k-algebra homomorphism

$$\begin{array}{rccl} \varphi_D : & A & \longrightarrow & A[|\mu|]_m \\ & x & \longmapsto & \displaystyle\sum_{r \geq 0}^{m} D_r(x) \mu^r \end{array}$$

satisfying $\varphi_D(x) \equiv x \mod \mu$. If we denote

$$\mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[|\mu|]_m) := \{f \in \mathrm{Hom}_{k-\mathrm{alg}}(A, A[|\mu|]_m) \mid f(x) \equiv x \mod \mu \ \forall x \in A\},$$

we have a bijection

$$D \in \mathrm{HS}_k(A;m) \longmapsto \varphi_D \in \mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[|\mu|]_m).$$

Moreover, any $\varphi \in \mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[|\mu|]_m)$ can be uniquely extended to a k-algebra automorphism $\varphi^\mu : A[|\mu|]_m \to A[|\mu|]_m$ with $\varphi^\mu(\mu) = \mu$. Hence, we can define a group structure on $\mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[|\mu|]_m)$ given by the composition. That is, for each $\varphi, \varphi' \in \mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[|\mu|]_m)$,

$$\varphi \circ \varphi' := \varphi^\mu \circ \varphi' \in \mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[|\mu|]_m).$$

Therefore, $\mathrm{HS}_k(A;m)$ inherits a canonical group structure (non-commutative in general) where the identity is $\mathbb{I} = (\mathrm{Id}, 0, \ldots, 0)$ and we denote by $D^* \in \mathrm{HS}_k(A;m)$ the inverse of $D \in \mathrm{HS}_k(A;m)$. Namely, for each $D, D' \in \mathrm{HS}_k(A;m)$, $D'' := D \circ D'$ is the HS-derivation of length m associated with the k-algebra homomorphism $\varphi_{D''} = \varphi^\mu_D \circ \varphi_{D'}$ which is explicitly given by

$$D''_r = \sum_{\beta+\gamma=r} D_\beta \circ D'_\gamma$$

for all r. Observe that $(\mathrm{Id}, D_1) \in \mathrm{HS}_k(A;1) \mapsto D_1 \in \mathrm{Der}_k(A)$ is a group isomorphism. We have the following result:

Lemma 1.1.2 [Na3, §4] *Let k be a ring, A a k-algebra and $m \in \overline{\mathbb{N}}$. Then, the map*

$$D \in \mathrm{HS}_k(A;m) \longmapsto \left[\varphi_D : x \in A \mapsto \sum_{r=0}^{m} D_r(x)\mu^r\right] \in \mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[|\mu|]_m)$$

is a group isomorphism.

Moreover, we can obtain an expression for composition of several HS-derivations.

Lemma 1.1.3 *Let $D^a \in \mathrm{HS}_k(A;m)$ be an ordered family of HS-derivations for $a = 1, \ldots, t$. We denote $D := \circ_{a=1}^t D^a = D^1 \circ D^2 \circ \cdots \circ D^t \in \mathrm{HS}_k(A;m)$. Then, $D_r = \sum_{|\beta|=r} D^1_{\beta_1} \circ \cdots \circ D^t_{\beta_t}$ for all $0 \le r \le m$ where $|\beta| := \beta_1 + \cdots + \beta_t$.*

Proof. We prove the result by induction on $t \ge 2$. If $t = 2$, we have the lemma thanks to the definition of the composition. Let us suppose that the result is true for $t-1$ and we will prove it for t. In this case, for all $r \ge 0$,

$$D_r = \left(\left(\circ_{a=1}^{t-1} D^a\right) \circ D^t\right)_r = \sum_{\beta+\gamma=r} \left(\circ_{a=1}^{t-1} D^a\right)_\beta \circ D^t_\gamma = \sum_{\beta+\gamma=r} \left(\sum_{\beta_1+\cdots+\beta_{t-1}=\beta} D^1_{\beta_1} \circ \cdots \circ D^{t-1}_{\beta_{t-1}}\right) \circ D^t_\gamma$$

$$= \sum_{\beta_1+\cdots+\beta_{t-1}+\beta_t=r} D^1_{\beta_1} \circ \cdots \circ D^{t-1}_{\beta_{t-1}} \circ D^t_{\beta_t}$$

and the lemma has been proved.

$\square$

Any HS-derivation D of A over k of length m can be understood as a power series

$$\sum_{r=0}^{m} D_r \mu^r \subseteq \mathrm{End}_k(A)[|\mu|]_m$$

and so we can consider $\mathrm{HS}_k(A;m)$ as a subgroup of the group of units of $\mathrm{End}_k(A)[|\mu|]_m$ and we can give a explicitly expression of the inverse (with respect to the group structure) of any HS-derivation.

Lemma 1.1.4 [Na3, **Prop. 9**] *For each $D \in \mathrm{HS}_k(A;m)$, its inverse D^* is given by $D^*_0 = \mathrm{Id}$ and, for all $r \geq 1$*

$$D^*_r = \sum_{d=1}^{r} (-1)^d \sum_{\beta \in \mathcal{P}(r,d)} D_{\beta_1} \circ \cdots \circ D_{\beta_d}$$

where $\mathcal{P}(r,d) := \{\beta = (\beta_1, \ldots, \beta_d) \mid \beta_i \in \mathbb{N},\ \beta_i \neq 0,\ |\beta| = r\}$.

Observe that if B is a commutative k-algebra such that A is isomorphic to B (as k-algebra), then $\mathrm{HS}_k(A;m)$ is isomorphic to $\mathrm{HS}_k(B,m)$ (as group) for all $m \geq 1$. Namely,

Lemma 1.1.5 *Let $f : A \to B$ be a k-algebra isomorphism. Then, the map*

$$\begin{array}{ccc} \mathrm{HS}_k(A;m) & \longrightarrow & \mathrm{HS}_k(B;m) \\ (D_r)_r & \longmapsto & D^f := (f \circ D_r \circ f^{-1})_r \end{array}$$

is a group isomorphism.

In this text we mainly use three operations on HS-derivations: Let $D \in \mathrm{HS}_k(A;m)$ be a HS-derivation of length $m \in \overline{\mathbb{N}}$.

1. For each $x \in A$, the sequence $x \bullet D = (x^r D_r)_r \in \mathrm{HS}_k(A;m)$.
2. Let $1 \leq n \leq m$ be an integer, the truncation $\tau_{mn}(D)$ is given by $\tau_{mn}(D) = (\mathrm{Id}, D_1, \ldots, D_n) \in \mathrm{HS}_k(A;n)$.
3. For each integer $n \geq 1$, we define $D[n] \in \mathrm{HS}_k(A;mn)$ as

$$D[n]_r = \begin{cases} D_{r/n} & \text{if } r = 0 \mod n \\ 0 & \text{otherwise} \end{cases}$$

It is easy to prove the following relationships between these operations:

Lemma 1.1.6 [Na2, **§1.2**] *Let $D \in \mathrm{HS}_k(A;m)$ be a HS-derivation of length $m \in \overline{\mathbb{N}}$, $n \geq 1$ and $q \leq m$. The following properties hold:*

1. *$(x^n \bullet D)[n] = x \bullet (D[n])$ for all $x \in A$.*

2. $\tau_{mn,m'n}(D[n]) = (\tau_{mm'}(D))[n]$ *for all* $1 \leq m' \leq m$.

3. $\tau_{mq}(x \bullet D) = x \bullet (\tau_{mq}(D))$ *for all* $x \in A$.

The following lemma is clear.

Lemma 1.1.7 *Let* $D \in \mathrm{HS}_k(A;m)$ *be a HS-derivation of length* $m \in \mathbb{N}$ *and* $\delta \in \mathrm{Der}_k(A)$. *Then,* $D \circ (\mathrm{Id}, \delta)[m] = (\mathrm{Id}, \delta)[m] \circ D$.

Definition 1.1.8 *For each HS-derivation* $D \in \mathrm{HS}_k(A;m)$ *such that* $D \neq \mathbb{I}$, *we denote*

$$\ell(D) := \min\{h \geq 1 \mid D_h \neq 0\} = \mathrm{ord}(D - \mathbb{I})$$

and for $D = \mathbb{I}$, $\ell(D) = \infty$.

It is easy to see the following lemma (see [Na3, §5]).

Lemma 1.1.9 *If* $D, E \in \mathrm{HS}_k(A;m)$, *then* $\ell(D \circ E) \geq \min\{\ell(D), \ell(E)\}$. *In particular, if* $\ell(D), \ell(E) \geq n$, *then* $\ell(D \circ E) \geq n$ *and* $(D \circ E)_n = D_n + E_n$.

Let us recall the following result.

Proposition 1.1.10 [Na3, **Prop. 7**] *For each* $D \in \mathrm{HS}_k(A;m)$ *we have that* D_r *is a* k*-linear differential operator of order* $\leq \lfloor r/\ell(D) \rfloor$ *for all* $0 \leq r \leq m$.

Definition 1.1.11 *For each* $D \in \mathrm{HS}_k(A;m)$ *and* $e \in \mathbb{N}$ *such that* $1 < e \leq m$, *if* $D_j = 0$ *for all* $j \neq 0 \mod e$, *we denote* $\ell(D;e) = \lceil m/e \rceil$ *if* $m < \infty$ *and* $\ell(D;e) = \infty$ *if* $m = \infty$. *Otherwise,*

$$\ell(D;e) := \min\{h \geq 0 \mid D_{he+\alpha} \neq 0 \text{ for some } \alpha \in \{1, \ldots, e-1\}\}.$$

Lemma 1.1.12 *Let* $D, E \in \mathrm{HS}_k(A;m)$ *and* $e \in \mathbb{N}$ *such that* $1 < e \leq m$. *The following properties hold.*

1. $\ell(D) \geq e$ *if and only if* $\ell(D;e) \geq 1$.

2. $\ell(D[e];e) = m$ *if* $m < \infty$ *and* $\ell(D[e];e) = \infty$ *when* $m = \infty$.

3. *If* $\ell(D;e) = i \geq 1$ *and* $\ell(E;je) \geq i/j$ *where* $1 \leq j \leq i$, *then* $\ell(D \circ E;e) \geq i$.

Proof. The first two statements are obvious, we will prove the third one. We denote $D' = D \circ E$. To show that $\ell(D';e) \geq i$, we have to see that $D'_r = 0$ for all $r < ie$ such that $r \neq 0 \mod e$. Let us consider r with these properties. Since $1 \leq i/j \leq \ell(E;je)$, we have that $ie \leq \ell(E;je)je$, so we have that $E_\gamma = 0$ for all $\gamma \neq 0 \mod je$ such that $\gamma \leq ie$. Thanks to this,

$$D'_r = \sum_{\beta+\gamma=r} D_\beta \circ E_\gamma = \sum_{\gamma=0}^{r} D_{r-\gamma} \circ E_\gamma = \sum_{\gamma=0}^{\lfloor r/je \rfloor} D_{r-je\gamma} \circ E_{je\gamma}.$$

Note that $r - je\gamma \neq 0 \mod e$ and $r - je\gamma < ie - je\gamma \leq ie$. Then, $D_{r-je\gamma} = 0$ because $\ell(D;e) = i$. Hence, $D'_r = 0$ and $\ell(D';e) \geq i$.

□

Lemma 1.1.13 *Let $D \in \mathrm{HS}_k(A;m)$ be a HS-derivation of length $m \in \overline{\mathbb{N}}$ and $1 < e \leq m$ an integer. Let us assume that $\ell(D;e) = i \geq 1$. Then, for all $\alpha = 0, \ldots, e-1$ such that $ie+\alpha \leq m$, $D_{ie+\alpha} \in \mathrm{Der}_k(A)$.*

Proof. From the definition of HS-derivation,

$$D_{ie+\alpha}(xy) = \sum_{\beta+\gamma=ie+\alpha} D_\beta(x)D_\gamma(y) = \sum_{\beta=0}^{ie} D_\beta(x)D_{ie+\alpha-\beta}(y) + \sum_{\beta=1}^{\alpha} D_{ie+\beta}(x)D_{\alpha-\beta}(y).$$

In the second term, $D_{\alpha-\beta} = 0$ for all $\beta \neq \alpha$ because $0 < \alpha - \beta < e$ and $\ell(D;e) \geq 1$. In the first one, since $\ell(D;e) = i$, if $\beta \neq 0 \mod e$, then $D_\beta = 0$, so we can write the previous equation as:

$$D_{ie+\alpha}(xy) = \sum_{\beta=0}^{i} D_{\beta e}(x)D_{ie+\alpha-\beta e}(y) + D_{ie+\alpha}(x)y.$$

Note that if $\beta \neq 0$, then $ie+\alpha-\beta e < ie$. Moreover, $ie+\alpha-\beta e \neq 0 \mod e$, so $D_{ie+\alpha-\beta e} = 0$. Then,

$$D_{ie+\alpha}(xy) = xD_{ie+\alpha}(y) + D_{ie+\alpha}(x)y$$

i.e. $D_{ie+\alpha}$ is a k-derivation of A for all $\alpha = 0, \ldots, e-1$.

$\square$

Lemma 1.1.14 *Let $m > 1$ be an integer and $n \in \overline{\mathbb{N}}$. If $D \in \mathrm{HS}_k(A;mn)$ is a HS-derivation such that $\ell(D;m) = n$ then, there exists $D' \in \mathrm{HS}_k(A;n)$ such that $D'_r = D_{mr}$ for all $r \leq n$.*

Proof. We have to prove that $D' = (D_{mr})_r$ is a HS-derivation. It is obvious that D'_r are k-linear maps. Moreover, $D'_0 = D_0 = \mathrm{Id}$ and

$$D'_r(xy) = D_{mr}(xy) = \sum_{\beta+\gamma=mr} D_\beta(x)D_\gamma(y) = \sum_{m\beta+m\gamma=mr} D_{m\beta}(x)D_{m\gamma}(y) = \sum_{\beta+\gamma=r} D'_\beta(x)D'_\gamma(y)$$

where the third equality holds thanks to $\ell(D;m) = n$. Hence, $D' \in \mathrm{HS}_k(A;n)$.

$\square$

1.1.1 Logarithmic derivations

Let us consider k a commutative ring, A a commutative k-algebra and $I \subseteq A$ an ideal. Remember that a k-derivation $\delta : A \to A$ is called I-logarithmic if $\delta(I) \subseteq I$. The set of I-logarithmic k-derivations is an A-submodule of $\mathrm{Der}_k(A)$ and will be denoted by $\mathrm{Der}_k(\log I)$. This concept can be generalized for the HS-derivations as can be seen in [Na2]. In this section, we recall this generalization and give some technical results.

Definition 1.1.15 *Let $D \in \mathrm{HS}_k(A;m)$ where $m \in \overline{\mathbb{N}}$ and $I \subseteq A$ an ideal.*

- *We say that D is I-logarithmic if $D_r(I) \subseteq I$ for all r. The set of I-logarithmic HS-derivations is denoted by $\mathrm{HS}_k(\log I;m)$ and $\mathrm{HS}_k(\log I) := \mathrm{HS}_k(\log I;\infty)$. In particular we have that $\mathrm{Der}_k(\log I) \equiv \mathrm{HS}_k(\log I;1)$.*

- *More generally, for $n \leq m$, D is $n - I$-logarithmic if $\tau_{mn}(D) \in \mathrm{HS}_k(\log I; n)$.*

The following lemma is clear.

Lemma 1.1.16 *$\mathrm{HS}_k(\log I; m)$ is a subgroup of $\mathrm{HS}_k(A; m)$ for all $m \in \overline{\mathbb{N}}$.*

Definition 1.1.17 *Let $I \subseteq A$ be an ideal. An I-differential operator is a (k-linear) differential operator $H : A \to A$ such that $H(I) \subseteq I$.*

Lemma 1.1.18 *Let $D \in \mathrm{HS}_k(A; m)$ be a HS-derivation with length $m \in \overline{\mathbb{N}}$ and $n, s \geq 1$ positive integers such that $n \leq m$. If D is $(n-1) - I$-logarithmic then, $D[s] \in \mathrm{HS}_k(A; ms)$ is $(ns-1) - I$-logarithmic.*

Proof. Let us consider $r < ns$. By definition $D[s]_r = 0$ if $r \neq 0 \mod s$ and $D[s]_r = D_{r/s}$ if $r = 0 \mod s$. Since $r/s < n$, $D_{r/s}$ is an I-differential operator. So, $D[s]$ is $(ns-1) - I$-logarithmic.

□

Lemma 1.1.19 *Let $I \subseteq A$ be an ideal and let us consider an ordered family $D^1, \ldots, D^t \in \mathrm{HS}_k(A; m)$ of $(m-1)-I$-logarithmic HS-derivations. We denote $D := D^1 \circ \cdots \circ D^t \in \mathrm{HS}_k(A; m)$. Then, D is $(m-1) - I$-logarithmic and*

$$D_m = \sum_{a=1}^{t} D_m^a + H_m$$

where H_m is an I-differential operator of order $\leq m$.

Proof. Thanks to Lemma 1.1.3, we have that

$$\left(D^1 \circ \cdots \circ D^t\right)_r = \sum_{|\beta|=r} D^1_{\beta_1} \circ \cdots \circ D^t_{\beta_t} = \sum_{a=1}^{t} D^a_r + H_r \text{ where } H_r = \sum_{\substack{|\beta|=r \\ \beta_i < r}} D^1_{\beta_1} \circ \cdots \circ D^t_{\beta_t}.$$

For each $\beta \in \mathbb{N}^t$ such that $|\beta| = r$, we have that $D^1_{\beta_1} \circ \cdots \circ D^t_{\beta_t}$ is a differential operator of order $\leq r$ and, since $r \leq m$, this term is an I-differential operator. So, H_r is also an I-differential operator or order $\leq r$ for all $r \leq m$. If $r < m$, then the first summand is an I-differential operator, so D is $(m-1) - I$-logarithmic and we have the result.

□

Corollary 1.1.20 *Let $I \subseteq A$ be an ideal. If $D \in \mathrm{HS}_k(A; m)$ is $(m-1) - I$-logarithmic and $E \in \mathrm{HS}_k(\log I; m)$, then $D \circ E \in \mathrm{HS}_k(A; m)$ is $(m-1)-I$-logarithmic and $(D \circ E)_m = D_m + H_m$ where H_m is an I-differential operator of order $\leq m$.*

Proof. By Lemma 1.1.19, we have that $D \circ E$ is $(m-1) - I$-logarithmic and $(D \circ E)_m = D_m + E_m + H'_m$ where H'_m is an I-differential operator of order $\leq m$. Since E_m is an I-differential operator of order $\leq m$, we have the corollary.

□

1.1.2 Euler derivation

In this section we recall Euler derivation defined in [Na4] associated with a HS-derivation that will allow us to prove certain property about integrability in the sense of Hasse-Schmidt. From now on, k will be a commutative ring, A a commutative k-algebra and we denote $T = \mathrm{End}_k(A)$.

Let us denote the Euler derivation as $\chi = \mu \dfrac{\partial}{\partial \mu} : k[|\mu|] \to k[|\mu|]$ and

$$\begin{array}{rccc} \chi_T : & T[|\mu|]_m & \longrightarrow & T[|\mu|]_m \\ & \displaystyle\sum_{r\geq 0}^{m} D_r \mu^r & \longmapsto & \displaystyle\sum_{r\geq 0}^{m} D_r \chi(\mu^r) = \sum_{r\geq 0}^{m} r D_r \mu^r. \end{array}$$

Definition 1.1.21 [Na4, **Def. 1.2.11**] *Let $D \in \mathrm{HS}_k(A; m)$, i.e. $D = \sum_r D_r \mu^r \in T[|\mu|]_m$. Then,*

$$\varepsilon(D) := D^* \chi_T(D) = \sum_{r>0} \left(\sum_{\beta+\gamma=r} \gamma D^*_\beta \circ D_\gamma \right) \mu^r.$$

If we consider the expression of D^* given in Lemma 1.1.4, we can see that:

$$\varepsilon(D) = \sum_{r>0} \left(\sum_{d=1}^{r} (-1)^{d-1} \left(\sum_{\beta \in \mathcal{P}(r,d)} \beta_d D_{\beta_1} \circ \cdots \circ D_{\beta_d} \right) \right) \mu^r. \tag{1.1}$$

Proposition 1.1.22 [Na4, **Prop. 3.1.2**] *If $D \in \mathrm{HS}_k(A; m)$, then $\varepsilon(D) \in \mathrm{Der}_k(A)[|\mu|]_m \cap T[|\mu|]_{m,+}$ where $T[|\mu|]_{m,+} = \ker(\tau_{m,0} : \sum a_i \mu^i \in T[|\mu|]_m \mapsto a_0 \in T)$.*

For each $0 < r \leq m$, we denote $\varepsilon_r(D) = \sum_{d=1}^{r} (-1)^{d-1} \left(\sum_{\beta \in \mathcal{P}(r,d)} \beta_d D_{\beta_1} \circ \cdots \circ D_{\beta_d} \right)$. The previous proposition tells us that $\varepsilon_r(D) \in \mathrm{Der}_k(A)$.

Lemma 1.1.23 *Let us consider $D \in \mathrm{HS}_k(A; m)$. For all $r > 0$, there exists a differential operator H_r of order $\leq r$ such that*

$$\varepsilon_r(D) = r D_r + H_r.$$

Moreover, if D is $(m-1)-I$-logarithmic, then H_r is an I-differential operator for all $0 < r \leq m$.

Proof. Remember that

$$\mathcal{P}(r, d) = \{\beta \in \mathbb{N}^d \mid \beta_i \neq 0,\ |\beta| = r\}.$$

Hence, $\mathcal{P}(r, 1) = \{r\}$ and, if we take $\beta \in \mathcal{P}(r, d)$ with $d \geq 2$, we have that $\beta_i < r$ for all $i = 1, \ldots, d$. Taking into account the equation (1.1), we have that

$$\varepsilon_r(D) = r D_r + H_r \text{ where } H_r = \sum_{d=2}^{r} \left(\sum_{\beta \in \mathcal{P}(r,d)} \beta_d D_{\beta_1} \circ \cdots \circ D_{\beta_d} \right).$$

Then, H_r is a differential operator of order $\leq r$ which depends on D_i for all $i < r$. Thanks to this, if D is $(m-1) - I$-logarithmic, H_r is an I-differential operator for all $0 < r \leq m$ and we have the lemma.

$\square$

1.1.3 Integrable derivations

In this section, we recall the notion of n-integrable derivation (see [Ma1], [Na2]). Modules of n-integrable derivations will be the main object of study in the following chapters and sections. We will see, among other things, where leaps occur or how a change of base ring affects them. From now on, k will be a commutative ring, A a commutative k-algebra and $I \subseteq A$ an ideal.

Definition 1.1.24 *Let $D \in \mathrm{HS}_k(A;m)$ where $m \in \overline{\mathbb{N}}$ and $n \geq m$.*

- *D is n-integrable if there exists $E \in \mathrm{HS}_k(A,n)$ such that $\tau_{nm}(E) = D$. Any such E will be called an n-integral of D. If D is ∞-integrable we simply say that D is integrable. If $m = 1$, we write $\mathrm{IDer}_k(A;n)$ for the set of n-integrable derivations and $\mathrm{IDer}_k(A) := \mathrm{IDer}_k(A;\infty)$.*
- *If $D \in \mathrm{HS}_k(\log I;m)$, we say that D is I-logarithmically n-integrable if there exists $E \in \mathrm{HS}_k(\log I;n)$ such that E is an n-integral of D. We denote $\mathrm{IDer}_k(\log I;n)$ the set of I-logarithmically n-integrable derivations (i.e. for $m = 1$) and $\mathrm{IDer}_k(\log I) := \mathrm{IDer}_k(\log I,\infty)$.*

The following lemma is clear thanks to the group structure of HS-derivations and operation 1.

Lemma 1.1.25 *The set $\mathrm{IDer}_k(A;n)$ (resp. $\mathrm{IDer}_k(\log I;n)$) is an A-submodule of $\mathrm{Der}_k(A)$ (resp. $\mathrm{Der}_k(\log I)$) for all $n \in \overline{\mathbb{N}}$.*

Moreover, if we have a k-algebra isomorphism, $A \cong B$, then there exists a bijection between $\mathrm{IDer}_k(A;n)$ and $\mathrm{IDer}_k(B;n)$ for all $n \in \overline{\mathbb{N}}$. Namely,

Lemma 1.1.26 *If $f : A \to B$ is an isomorphism of k-algebras. Then, the map*

$$\begin{array}{rcl} \gamma_{f,n} : \mathrm{IDer}_k(A;n) & \to & \mathrm{IDer}_k(B;n) \\ \delta & \mapsto & \delta^f := (f \circ \delta \circ f^{-1}) \end{array}$$

is a bijection for all $n \in \overline{\mathbb{N}}$.

Proof. If $\delta \in \mathrm{IDer}_k(A;n) \subseteq \mathrm{Der}_k(A)$, it is obvious that $\delta^f \in \mathrm{Der}_k(B;n)$. Moreover, by definition, there exists $D \in \mathrm{HS}_k(A;n)$ such that $\delta = D_1$. By Lemma 1.1.5, $D^f := (f \circ D_r \circ f^{-1}) \in \mathrm{HS}_k(B;n)$ and $D_1^f = f \circ D_1 \circ f^{-1} = \delta^f$. So, $\delta^f \in \mathrm{IDer}_k(B;n)$. Hence, $\gamma_{f,n}$ is well-defined and its inverse is $\gamma_{f^{-1},n}$.

$\square$

Let us suppose that $\delta \in \mathrm{IDer}_k(A;n)$. Then, there exists $D \in \mathrm{HS}_k(A;n)$ such that $D_1 = \delta$. So, $\tau_{n,n-1}(D) \in \mathrm{HS}_k(A;n-1)$ is an $(n-1)$-integral of δ. Hence, $\mathrm{IDer}_k(A;n) \subseteq \mathrm{IDer}_k(A;n-1)$

(the same occur when we consider I-logarithmically n-integrable derivations). Then, we obtain the chain of A-modules

$$\mathrm{Der}_k(A) = \mathrm{IDer}_k(A;1) \supseteq \mathrm{IDer}_k(A;2) \supseteq \mathrm{IDer}_k(A;3) \supseteq \cdots .$$

However, equality is not true in general, that is, there may be an $(n-1)$-integrable derivation that is not n-integrable. For example, let us consider $k = \mathbb{F}_p$, $A = k[x]/\langle x^p \rangle$ and $\overline{\partial}_x$ the derivation induced by the derivative with respect to x in A. It is easy to see that $\overline{\partial}_x \in \mathrm{IDer}_k(A; p-1)$, it is enough to consider the HS-derivation associated with the k-algebra homomorphism $x \in A \mapsto x + \mu \in A[|\mu|]_{p-1}$. But, $\overline{\partial}_x \notin \mathrm{IDer}_k(A;p)$, otherwise there would exist a well-defined k-algebra homomorphism of the form

$$\varphi : x \in A \mapsto x + \mu + a_2\mu^2 + \cdots + a_p\mu^p \in A[|\mu|]_p$$

but $\varphi(x^p) = x^p + \mu^p \not\equiv 0 \mod \langle x^p \rangle$!!!. Actually, $\mathrm{IDer}_k(A;p) = \langle x\overline{\partial}_x \rangle$ (it is enough to consider the HS-derivation $x \in A \mapsto x + x\mu \in A[|\mu|]_p$). Then,

$$\mathrm{IDer}_k(A; p-1) \supsetneq \mathrm{IDer}_k(A;p)$$

and we say that A has a *leap* at p.

Definition 1.1.27 *Let $s > 1$ be an integer. We say that the k-algebra A has a leap at $s > 1$ if the inclusion $\mathrm{IDer}_k(A; s-1) \supsetneq \mathrm{IDer}_k(A;s)$ is proper. The set of leaps of A over k is denoted by $\mathrm{Leaps}_k(A)$.*

Let k be a ring of characteristic 0 (i.e. $k \supseteq \mathbb{Q}$) and A a k-algebra. Then, $\mathrm{IDer}_k(A;n) = \mathrm{Der}_k(A)$ for all $n \in \overline{\mathbb{N}}$ (if $\delta \in \mathrm{Der}_k(A)$, it is enough to take $D := (\delta^r/r!)_r \in \mathrm{HS}_k(A)$ as an integral). So, $\mathrm{Leaps}_k(A) = \emptyset$. If k is a ring of characteristic $p > 0$ (i.e. $\mathbb{F}_p \subseteq k$) we will prove, in chapter 2, that leaps only occur at powers of p. For the moment, we have the following results related with the integrability of a HS-derivation over a ring k of any characteristic.

We recall that a k-algebra A is 0-*smooth* over k if it has the following property: for any k-algebra C, any ideal N of C satisfying $N^2 = 0$, and any k-algebra homomorphism $u : A \to C/N$, there exists a lifting $v : A \to C$ of u to C, as a k-algebra homomorphism. In terms of diagrams, we have that

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[dr, "v"] & C/N \\ k \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

We have the following results.

Theorem 1.1.28 [Ma2, **Th. 27.1**] *If A is 0-smooth over k, then any HS-derivation of length $m < \infty$ over k is ∞-integrable.*

Proposition 1.1.29 [Na2, **Ex. 2.1.11**]*(normal crossings). Let us take $h = \prod_{i=1}^{e} x_i \in R = k[x_1, \ldots, x_d]$. Then $\mathrm{IDer}_k(R/\langle h \rangle) = \mathrm{Der}_k(R/\langle h \rangle)$.*

Lemma 1.1.30 *Let $m > 1$ and $n > 0$ be two integers and $D \in \mathrm{HS}_k(A; mn)$ (resp. $D \in \mathrm{HS}_k(\log I; mn)$) a HS-derivation such that $\ell(D; m) = n$. Then, D is $(n+1)m-1$-integrable (resp. I-logarithmically $(n+1)m-1$-integrable) and there is an integral $D' \in \mathrm{HS}_k(A; (n+1)m-1)$ (resp. $D' \in \mathrm{HS}_k(\log I; (n+1)m-1)$) of D such that $\ell(D'; m) = n+1$.*

Proof. Let $\delta_1, \ldots, \delta_{m-1} \in \mathrm{Der}_k(A)$ be k-derivations and let us consider the sequence

$$D' = (\mathrm{Id}, D'_1, \ldots, D'_{mn}, D'_{mn+1}, \ldots, D'_{mn+m-1}) := (\mathrm{Id}, D_1, \ldots, D_{mn}, \delta_1, \ldots, \delta_{m-1}).$$

We claim that $D' \in \mathrm{HS}_k(A; (n+1)m-1)$. If this is true, D' is an $(n+1)m-1$-integral of D.

To prove this claim we have to show that the following equality holds for all $\alpha = 1, \ldots, m-1$:

$$D'_{mn+\alpha}(xy) := \delta_\alpha(xy) = \sum_{\beta=0}^{mn+\alpha} D'_\beta(x) D'_{mn+\alpha-\beta}(y).$$

By hypothesis, $D_\beta = 0$ for all $\beta \neq 0 \mod m$ and $\beta \leq mn$. Since $D'_\beta = D_\beta$ for all $\beta \leq mn$,

$$\begin{aligned}\sum_{\beta=0}^{mn+\alpha} D'_\beta(x) D'_{mn+\alpha-\beta}(y) &= \sum_{\beta=0}^{mn} D_\beta(x) D'_{mn+\alpha-\beta}(y) + \sum_{\gamma=mn+1}^{mn+\alpha} D'_\gamma(x) D'_{mn+\alpha-\gamma}(y) \\ &= \sum_{\beta=0}^{n} D_{\beta m}(x) D'_{(n-\beta)m+\alpha}(y) + \sum_{\gamma=1}^{\alpha} D'_{mn+\gamma}(x) D'_{\alpha-\gamma}(y).\end{aligned}$$

In the first term, if $\beta > 0$, then $0 < (n-\beta)m + \alpha < mn$ and $(n-\beta)m + \alpha \neq 0 \mod m$, so $D'_{(n-\beta)m+\alpha} = D_{(n-\beta)m+\alpha} = 0$. In the second one, if $\gamma \neq \alpha$, then $D'_{\alpha-\gamma} = D_{\alpha-\gamma} = 0$ because $0 < \alpha - \gamma < m$. So,

$$\sum_{\beta=0}^{mn+\alpha} D'_\beta(x) D'_{mn+\alpha-\beta}(y) = x D'_{mn+\alpha}(y) + D'_{mn+\alpha}(x) y = x\delta_\alpha(y) + \delta_\alpha(x) y = \delta_\alpha(xy).$$

Observe that, for each $\alpha = 1, \ldots, m-1$, we can choose any k-derivation to be δ_α. In particular, we can put $\delta_\alpha = 0$ for all α. In that case, $\ell(D'; m) = n+1$. Thanks to this, we can deduce the lemma for $D \in \mathrm{HS}_k(\log I; mn)$.

□

1.2 Hasse-Schmidt derivations on polynomial rings

Let us consider $R = k[x_i \mid i \in \mathcal{I}]$ the polynomial ring over a commutative ring k in an arbitrary number of variables and $I \subseteq R$ an ideal. In this section, we recall some general results about integrability of k-derivations in polynomial rings. Moreover, if k is a unique factorization domain of characteristic $p > 0$ (i.e. $\mathbb{F}_p \subseteq k$), we give the relationship between $\langle h \rangle$-logarithmically n-integrable derivations and $\langle h^p \rangle$-logarithmically n-integrable derivations where h is a polynomial of R. In this text, we denote by $\partial_i : R \to R$ the partial derivative with respect to x_i.

The following result is a straightforward consequence of Theorem 1.1.28.

Proposition 1.2.1 *Any HS-derivation of R (over k) of length $m \geq 1$ is integrable.*

Let us consider a k-algebra A, $I \subseteq A$ an ideal and $m \in \overline{\mathbb{N}}$. We denote by $\Pi^I_{\mathrm{HS},m}$: $\mathrm{HS}_k(\log I; m) \to \mathrm{HS}_k(A/I; m)$ the map defined as:

$$D \in \mathrm{HS}_k(\log I; m) \longmapsto \Pi^I_{\mathrm{HS},m}(D) = \overline{D} = \left(\overline{D}_r\right) \in \mathrm{HS}_k(A/I; m),\ \overline{D}_r(x+I) = D_r(x)+I \ \ \forall x \in A$$

and we denote by $\Pi^I_m : \mathrm{IDer}_k(\log I; m) \to \mathrm{IDer}_k(A/I; m)$ the map given by:

$$\delta \in \mathrm{IDer}_k(\log I; m) \longmapsto \Pi^I_m(\delta) = \overline{\delta} \in \mathrm{IDer}_k(A/I; m),\ \overline{\delta}(x+I) = \delta(x) + I \ \ \forall x \in A.$$

The proof of the following proposition is analogous to that of Proposition 1.3.4 of [Na2].

Proposition 1.2.2 *If $R = k[x_i \mid i \in \mathcal{I}]$ and $I \subseteq R$ is an ideal, then the map $\Pi^I_{\mathrm{HS},m}$: $\mathrm{HS}_k(\log I; m) \to \mathrm{HS}_k(R/I; m)$ is a surjective group homomorphism for all $m \in \overline{\mathbb{N}}$.*

The following result generalizes Corollary 2.1.9 of [Na2] for integrable derivations.

Corollary 1.2.3 *If $R = k[x_i \mid i \in \mathcal{I}]$ and $I \subseteq R$ is an ideal, then the map $\Pi^I_m : \mathrm{IDer}_k(\log I; m) \to \mathrm{IDer}_k(R/I; m)$ is a surjective homomorphism of R-modules for all $m \in \overline{\mathbb{N}}$.*

Proof. Let $\delta \in \mathrm{IDer}_k(R/I; m)$ be an m-integral derivation. From the definition, there exists $E \in \mathrm{HS}_k(R/I; m)$ an m-integral of δ. By Proposition 1.2.2, there exists $D \in \mathrm{HS}_k(\log I; m)$ such that $\Pi^I_{\mathrm{HS},m}(D) = E$. Then, $D_1 \in \mathrm{IDer}_k(\log I; m)$ and $\Pi^I_m(D_1) = \overline{D}_1 = E_1 = \delta$. □

Corollary 1.2.4 *Let I be an ideal of $R = k[x_i \mid i \in \mathcal{I}]$. Then, R/I has a leap at $s > 1$ if and only if the inclusion $\mathrm{IDer}_k(\log I; s-1) \supsetneq \mathrm{IDer}_k(\log I; s)$ is proper.*

Remark 1.2.5 *If $A = k[|x_1, \dots, x_d|]$ is the formal power series ring over k and $I \subseteq A$ is an ideal then, $\Pi^I_{\mathrm{HS},m}$ and Π^I_n are surjective in a similar way to Proposition 1.2.2 and Corollary 1.2.3 and we have Corollary 1.2.4.*

Let us consider $R = k[x_1, \dots, x_d]$ a polynomial ring in a finite number of variables. Then, it is clear that the following short sequence of R-modules is exact:

$$0 \to I(\mathrm{Der}_k(R)) \to \mathrm{Der}_k(\log I) \xrightarrow{\Pi^I_1} \mathrm{Der}_k(R/I) \to 0.$$

The same occurs when we consider integrable derivations:

Proposition 1.2.6 *Let $m \in \overline{\mathbb{N}}$, $R = k[x_1, \dots, x_d]$ and $I \subseteq R$ an ideal. Then, the following short sequence of R-modules is exact:*

$$0 \to I(\mathrm{Der}_k(R)) \to \mathrm{IDer}_k(\log I; m) \xrightarrow{\Pi^I_m} \mathrm{IDer}_k(R/I; m) \to 0.$$

Proof. First, note that if $\delta \in \mathrm{Der}_k(R)$ and $h \in I$, then $h\delta \in \mathrm{IDer}_k(\log I; m)$ (it is enough to consider $h \bullet D$ where $D \in \mathrm{HS}_k(R; m)$ is an integral of δ) and it is clear that $I(\mathrm{Der}_k(R)) \to \mathrm{IDer}_k(\log I; m)$ is injective. From Corollary 1.2.3, Π^I_m is surjective. So, we have to prove that $\ker \Pi^I_m = I(\mathrm{Der}_k(R))$.

Let $h \in I$ and $\delta \in \mathrm{Der}_k(R)$. Then, $\Pi^I_m(h\delta) = \overline{h\delta}$ where $\overline{h\delta}(r+I) = (h\delta)(r) + I = 0$. So, $I(\mathrm{Der}_k(R)) \subseteq \ker \Pi^I_m$. Now, let us consider $\delta \in \ker \Pi^I_m$. Then $\overline{\delta} = 0$ i.e. $\delta(r) \in I$ for all $r \in R$. Since $\mathrm{Der}_k(R)$ is finitely generated by $\partial_i : R \to R$ for all $i = 1, \ldots, d$, we have that $\delta = \sum_{i=1}^d b_i \partial_i$. Hence, $\delta(x_i) = b_i \in I$ for all $i = 1, \ldots, d$ and $\delta \in I(\mathrm{Der}_k(R))$. Therefore, $\ker \Pi^I_m = I(\mathrm{Der}_k(R))$ and the proposition is proved.

□

Let us recall the following two results.

Proposition 1.2.7 [Na2, **Prop. 2.2.4**] *Let R be $k[x_1, \ldots, x_d]$ or $k[|x_1, \ldots, x_d|]$. Let us consider $f \in R$, $I = \langle f \rangle$, and $J^0 = \langle \partial_1(f), \ldots, \partial_d(f) \rangle$ the gradient ideal. If $\delta : R \to R$ is an I-logarithmic k-derivation with $\delta \in J^0\,\mathrm{Der}_k(R)$, then δ admits an I-logarithmic integral $D \in \mathrm{HS}_k(\log I)$ with $D_i(f) = 0$ for all $i > 1$. In particular, if $\delta(f) = 0$, the integral D can be taken with $\varphi_D(f) = f$.*

Theorem 1.2.8 [Tr, **Th. 1.2**] *Let R be $k[x_1, \ldots, x_d]$ or $k[|x_1, \ldots, x_d|]$. Let us consider $I \subseteq R$ an ideal generated by quasi-homogeneous polynomials with respect to the weights $w(x_r) \geq 0$. Then, the Euler vector field $\chi = \sum_{r=0}^d w(x_r) x_r \partial_r$ is I-logarithmically (∞-)integrable. In fact, an I-logarithmic integral of χ is the HS-derivation associated with the map $R \to R[|\mu|]$ given by*

$$x_r \longmapsto x_r \left(\frac{1}{1-\mu} \right)^{w(x_r)}, \quad r = 1, \ldots, d.$$

1.2.1 I^p-logarithmic derivations

In this section let us consider $R = k[x_1, \ldots, x_d]$ the polynomial ring in d variables over a unique factorization domain (UFD) k of characteristic $p > 0$ (i.e. $\mathbb{F}_p \subseteq k$) and $h \in R$ a polynomial. We want to describe the module of n-integrable derivations of $A = R/\langle h^p \rangle$ for all $n \in \overline{\mathbb{N}}$ from the modules of n-integrable derivations of $R/\langle h \rangle$. Thanks to Corollary 1.2.3, it is enough to study the relationship between $\langle h \rangle$-logarithmically n-integrable derivations and $\langle h^p \rangle$-logarithmically n-integrable derivations. From now on, k will be a commutative ring and $R = k[x_1, \ldots, x_d]$. We start with two general results.

Lemma 1.2.9 *Let k be a ring of characteristic $p > 0$, A a commutative k-algebra and $h \in A$. Consider $D \in \mathrm{HS}_k(A; m)$ with $m \in \overline{\mathbb{N}}$ and $\tau \geq 0$. Then, for all $i \leq m$, the following identity holds:*

$$D_i\left(h^{p^\tau}\right) = \begin{cases} 0 & \text{if} \quad p^\tau \nmid i \\ D_{i/p^\tau}(h)^{p^\tau} & \text{if} \quad p^\tau | i. \end{cases}$$

Proof. Let $\varphi : A \to A[|\mu|]_m$ be the k-algebra homomorphism determined by D. Then,

$$\sum_{i \geq 0}^m D_i\left(h^{p^\tau}\right) \mu^i = \varphi\left(h^{p^\tau}\right) = \varphi(h)^{p^\tau} = \sum_{j \geq 0}^m D_j(h)^{p^\tau} \mu^{jp^\tau} \mod \langle \mu^{m+1} \rangle$$

and we obtain the result by equating the coefficients in the above equation. □

Lemma 1.2.10 *Let k be a commutative ring, A a commutative k-algebra, $n \in \overline{\mathbb{N}}$ and $m \leq n$. Consider $g \in A$ and $D \in \mathrm{HS}_k(A;n)$. Suppose that D is $m-\langle g\rangle$-logarithmic. Then, for all $r \geq 1$, D is $m-\langle g^r\rangle$-logarithmic and, if $m \in \mathbb{N}$, we have that*

$$D_{m+1}(g^r) \in rg^{r-1}D_{m+1}(g) + \langle g^r\rangle. \tag{1.2}$$

Proof. First, we will prove that D is $m-\langle g^r\rangle$-logarithmic for all $r \geq 1$. We proceed by induction on $r \geq 1$. When $r = 1$, the result is obvious from the hypothesis. Let us suppose that D is $m-\langle g^{r-1}\rangle$-logarithmic, i.e. $D_\beta(g^{r-1}) \in \langle g^{r-1}\rangle$ for all $\beta \leq m$. From the definition of HS-derivation, for all $j \leq m$,

$$D_j\left(g^r\right) = \sum_{\beta+\gamma=j} D_\beta\left(g^{r-1}\right) D_\gamma(g) \in \langle g^r\rangle.$$

So, D is $m-\langle g^r\rangle$-logarithmic for all $r \geq 1$. Now, we will prove (1.2) by induction on $r \geq 1$. It is obvious for $r = 1$, let us suppose that $D_{m+1}(g^{r-1}) \in (r-1)g^{r-2}D_{m+1}(g) + \langle g^{r-1}\rangle$. From the definition of HS-derivation,

$$D_{m+1}\left(g^r\right) = D_{m+1}\left(g^{r-1}\right)g + D_{m+1}(g)g^{r-1} + \sum_{\substack{\beta+\gamma=m+1\\ \beta,\gamma\neq 0}} D_\beta\left(g^{r-1}\right) D_\gamma(g) \in rg^{r-1}D_{m+1}(g) + \langle g^r\rangle$$

and the lemma is proved. □

From now on, k will be a unique factorization domain and $R = k[x_1,\ldots,x_d]$.

Proposition 1.2.11 *If $f, g \in R$ are coprime then, for all $n \in \overline{\mathbb{N}}$, we have that*

$$\mathrm{HS}_k(\log fg;n) = \mathrm{HS}_k(\log f;n) \cap \mathrm{HS}_k(\log g;n).$$

Proof.

- $\supseteq$. Let $D \in \mathrm{HS}_k(\log f;n) \cap \mathrm{HS}_k(\log g;n)$. By definition, $D_i(f) \in \langle f\rangle$ and $D_i(g) \in \langle g\rangle$ for all $i \leq n$. Then $D_i(fg) = \sum_{\beta+\gamma=i} D_\beta(f)D_\gamma(g) \in \langle fg\rangle$, so $D \in \mathrm{HS}_k(\log fg;n)$.
- $\subseteq$. Let $D \in \mathrm{HS}_k(\log fg;n)$. This implies that $D_i(fg) \in \langle fg\rangle$ for all $i \leq n$. We will prove the result by induction on i. When $i = 1$, then $D_1(fg) = D_1(f)g + D_1(g)f \in \langle fg\rangle \subseteq \langle f\rangle, \langle g\rangle$. So, $D_1(f)g \in \langle f\rangle$. Since g and f are coprime, $D_1(f) \in \langle f\rangle$. For g is analogous.

  Now let us assume that $D_i(f) \in \langle f\rangle$ and $D_i(g) \in \langle g\rangle$ for all $i < n$. By definition,

  $$D_n(fg) = D_n(f)g + D_n(g)f + \sum_{\substack{\beta+\gamma=n\\ \beta,\gamma\neq 0}} D_\beta(f)D_\gamma(g) \in \langle fg\rangle \Rightarrow D_n(f)g + D_n(g)f \in \langle fg\rangle$$

  and we can proceed as in case $i = 1$.

□

Corollary 1.2.12 *If $f, g \in R$ are coprime then, for all $n \in \overline{\mathbb{N}}$,*

$$\mathrm{IDer}_k(\log fg; n) \subseteq \mathrm{IDer}_k(\log f; n) \cap \mathrm{IDer}_k(\log g; n).$$

Proof. If $\delta \in \mathrm{IDer}_k(\log fg; n)$ then, there exists $D \in \mathrm{HS}_k(\log fg; n)$ an n-integral of δ. By Proposition 1.2.11, $D \in \mathrm{HS}_k(\log f; n) \cap \mathrm{HS}_k(\log g; n)$ so, $\delta \in \mathrm{IDer}_k(\log f; n) \cap \mathrm{IDer}_k(\log g; n)$. □

In general, equality in Corollary 1.2.12 does not hold. For example, consider $k = \mathbb{F}_2$ and $f = y$ and $g = x^2 - y$ two polynomials of $k[x, y]$. Then $\partial_x \in \mathrm{IDer}_k(\log f; 2) \cap \mathrm{IDer}_k(\log g; 2)$, it is enough to consider the k-algebra homomorphisms:

$$\begin{array}{ccc} R & \to & R[|\mu|]_2 \\ x & \mapsto & x+\mu \\ y & \mapsto & y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R & \to & R[|\mu|]_2 \\ x & \mapsto & x+\mu \\ y & \mapsto & y+\mu^2 \end{array}$$

The first one is an f-logarithmic 2-integral of ∂_x and, the second one is a g-logarithmic 2-integral of this derivation. However, $\partial_x \notin \mathrm{IDer}_k(\log fg; 2)$. To see this, let us consider a generic 2-integral of ∂_x:

$$\begin{array}{cccl} \varphi: & R & \to & R[|\mu|]_2 \\ & x & \mapsto & x+\mu+u_2\mu^2 \\ & y & \mapsto & y \quad + \quad v_2\mu^2 \end{array}$$

Then,

$$\varphi(fg) = y(x^2 - y) + (x^2 v_2 + y)\mu^2.$$

In order for φ to be fg-logarithmic, $x^2 v_2 + y \in \langle fg \rangle$. So, it should exist $F \in k[x, y]$ such that $x^2 v_2 + y = F(x^2 - y)y$ but, if we consider the coefficient of y in this equality, we have that $1 = 0$!!!.

Corollary 1.2.13 *Let $f_1, \ldots, f_m \in R$. If f_i, f_j are coprime whenever $i \neq j$ then, for all $n \in \overline{\mathbb{N}}$ we have:*

$$\mathrm{HS}_k(\log f_1 \cdots f_m; n) = \textstyle\bigcap_i \mathrm{HS}_k(\log f_i; n) \quad \textit{and} \quad \mathrm{IDer}_k(\log f_1 \cdots f_m; n) \subseteq \textstyle\bigcap_i \mathrm{IDer}_k(\log f_i; n).$$

Proof. The result is obtained thanks to Proposition 1.2.11 and Corollary 1.2.12 by induction on m. □

From now on, k will be a UFD of characteristic $p > 0$ and $R = k[x_1, \ldots, x_d]$.

Lemma 1.2.14 *Let f be an irreducible polynomial, $a \geq 1$ and $n \in \overline{\mathbb{N}}$. Let us consider $D \in \mathrm{HS}_k(R; n)$ such that $D_i(f^a)^p \in \langle f^{ap} \rangle$ for all $i \leq n$. Then, $D \in \mathrm{HS}_k(\log f^a; n)$.*

Proof. We write $a = sp^\alpha$ where $\alpha = \mathrm{val}_p(a) \geq 0$ is the p-adic valuation of a and $s \geq 1$. By Lemma 1.2.9,

$$D_i\left(f^{sp^\alpha}\right) = \begin{cases} 0 & \text{if} \quad p^\alpha \nmid i \\ D_{i/p^\alpha}(f^s)^{p^\alpha} & \text{if} \quad p^\alpha | i. \end{cases}$$

Hence, if $n < p^\alpha$, we have the lemma. So, let us consider $n \geq p^\alpha$. Moreover, we can focus on the case $i = jp^\alpha \leq n$. It is enough to show that D is $m - \langle f \rangle$-logarithmic where $m = \lfloor n/p^\alpha \rfloor$

if $n \in \mathbb{N}$ and $m = \infty$ if $n = \infty$ because, if this is true, we have that D is $m - \langle f^s \rangle$-logarithmic by Lemma 1.2.10, and $D_i(f^a) = D_i(f^{sp^\alpha}) = D_j(f^s)^{p^\alpha} \in \langle f^a \rangle$ for all $i = jp^\alpha \leq n$ and we deduce that $D \in \mathrm{HS}_k(\log f^a; n)$.

Let us consider $j \leq m$ an integer. Since $jp^\alpha \leq n$ we have that

$$D_j \left(f^s\right)^{p^{\alpha+1}} = D_{jp^\alpha} \left(f^{sp^\alpha}\right)^p \in \left\langle f^{sp^{\alpha+1}} \right\rangle. \tag{1.3}$$

We proceed by induction on $j \geq 1$. If $j = 1$, $D_1\left(f^s\right) = sf^{s-1}D_1(f)$ by definition of derivation. Taking into account the previous expression, we have that

$$D_1 \left(f^s\right)^{p^{\alpha+1}} = sf^{(s-1)p^{\alpha+1}} D_1(f)^{p^{\alpha+1}} \in \left\langle f^{sp^{\alpha+1}} \right\rangle. \tag{1.4}$$

Since R is UFD and $f, s \neq 0$, $D_1(f)^{p^{\alpha+1}} \in \left\langle f^{p^{\alpha+1}} \right\rangle \subseteq \langle f \rangle$ and hence $D_1(f) \in \langle f \rangle$.

Let us assume that $D_l(f) \in \langle f \rangle$ for all $l < j \leq m$, i.e. D is $(j-1) - \langle f \rangle$-logarithmic. Thanks to the hypothesis, we can use Lemma 1.2.10, and we have

$$D_j \left(f^s\right) = sf^{s-1}D_j(f) + Ff^s$$

for some $F \in R$. Taking into account (1.3),

$$sf^{(s-1)p^{\alpha+1}} D_j(f)^{p^{\alpha+1}} + F^{p^{\alpha+1}} f^{sp^{\alpha+1}} \in \left\langle f^{sp^{\alpha+1}} \right\rangle \Rightarrow sf^{(s-1)p^{\alpha+1}} D_j(f)^{p^{\alpha+1}} \in \left\langle f^{sp^{\alpha+1}} \right\rangle.$$

Observe that it is the same condition that (1.4), so we can deduce that $D_j(f) \in \langle f \rangle$.

□

Proposition 1.2.15 *Let k be a UFD of characteristic $p > 0$ and $R = k[x_1, \ldots, x_d]$ the polynomial ring over k. Let h be a polynomial of R. For all $n \in \overline{\mathbb{N}}$, we have that*

$$\mathrm{IDer}_k(\log h; n) = \mathrm{IDer}_k\left(\log h^p; np\right).$$

Proof.

- $\subseteq$. Let $D_1 \in \mathrm{IDer}_k(\log h; n)$ and $D \in \mathrm{HS}_k(\log h; n)$ an integral of D_1. If $n < \infty$, from Proposition 1.2.1, D is np-integrable, so let D' be an np-integral of D. If $n = \infty$, we put $D' = D$. Observe that $D'_1 = D_1$ so, if $D' \in \mathrm{HS}_k\left(\log h^p; np\right)$ then $D_1 \in \mathrm{IDer}_k(\log h^p; np)$. We have to see that $D'_i(h^p) \in \langle h^p \rangle$ for all $i \leq np$.

  By Lemma 1.2.9,
  $$D'_i\left(h^p\right) = \begin{cases} 0 & \text{if} \quad p \nmid i \\ D'_{i/p}(h)^p & \text{if} \quad p | i. \end{cases}$$

  Then, we can focus on $i = jp$ where $1 \leq j \leq n$. Note that $D'_j = D_j$ for all $1 \leq j \leq n$, so $D'_i\left(h^p\right) = D'_j(h)^p = D_j(h)^p \in \langle h^p \rangle$. Therefore, $D'_i(h^p) \in \langle h^p \rangle$ for all $i \leq np$ and we have the inclusion.

$\supseteq$. Let $D_1 \in \mathrm{IDer}_k(\log h^p; np)$ and $D \in \mathrm{HS}_k(\log h^p; np)$ an np-integral of D_1. Let $h = h_1^{a_1} \cdots h_m^{a_m}$ be the factorization of h in irreducible factors i.e., h_i is irreducible and $a_i \geq 1$ for all $i = 1, \ldots, m$ and $h_i \neq h_j$ if $i \neq j$. Then, $h_i^{a_i}$ and $h_j^{a_j}$ are coprime whenever $i \neq j$, and therefore, $h_1^{a_1 p}, \ldots, h_m^{a_m p}$ are coprime too. By Corollary 1.2.13,

$$D \in \mathrm{HS}_k(\log h^p; np) = \bigcap_i \mathrm{HS}_k(\log h_i^{a_i p}; np).$$

Hence,

$$D_j(h_i^{a_i})^p = D_{jp}(h_i^{a_i p}) \in \langle h_i^{a_i p} \rangle$$

for all $0 \leq j \leq n$. By Lemma 1.2.14, $\tau_{np,n}(D) \in \mathrm{HS}_k(\log h_i^{a_i}; n)$ for all $i = 1, \ldots, m$. So,

$$\tau_{np,n}(D) \in \bigcap \mathrm{HS}_k(\log h_i^{a_i}; n) = \mathrm{HS}_k(\log h; n)$$

Therefore $D_1 \in \mathrm{IDer}_k(\log h; n)$.

□

Corollary 1.2.16 *For all $\tau \geq 0$ and $n \in \overline{\mathbb{N}}$, we have that*

$$\mathrm{IDer}_k(\log h; n) = \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau\right).$$

Proof. By induction on τ using Proposition 1.2.15. □

Proposition 1.2.17 *Let k be a UFD of characteristic $p > 0$, $R = k[x_1, \ldots, x_d]$ the polynomial ring over k, $h \in R$ and $\tau \geq 1$. We denote $A := R/\left\langle h^{p^\tau}\right\rangle$ and $A' := R/\langle h \rangle$. Then,*

$$\mathrm{Leaps}_k(A) = \begin{cases} \{np^\tau \mid n \in \mathrm{Leaps}_k(A')\} & \text{if } \mathrm{Der}_k(\log h) = \mathrm{Der}_k(R) \\ \{np^\tau \mid n \in \mathrm{Leaps}_k(A')\} \cup \{p^\tau\} & \text{if } \mathrm{Der}_k(\log h) \neq \mathrm{Der}_k(R). \end{cases}$$

Proof. By Corollary 1.2.4, $s \in \mathrm{Leaps}_k(A)$ if and only if the inclusion $\mathrm{IDer}_k\left(\log h^{p^\tau}; s-1\right) \supsetneq \mathrm{IDer}_k\left(\log h^{p^\tau}; s\right)$ is proper. First of all, we will prove the next two equalities:

1. For $s < p^\tau$, $\mathrm{IDer}_k\left(\log h^{p^\tau}; s\right) = \mathrm{Der}_k(R)$.

The inclusion $\subseteq$ is always true. Let $D_1 \in \mathrm{Der}_k(R) = \mathrm{IDer}_k(R)$ and $D \in \mathrm{HS}_k(R)$ an integral. Since $s < p^\tau$, for all $j \leq s$, $p^\tau \nmid j$. By Lemma 1.2.9, $D_j\left(h^{p^\tau}\right) = 0 \in \left\langle h^{p^\tau}\right\rangle$ for all $j \leq s$. Then, any derivation D_1 has a h^{p^τ}-logarithmic s-integral and the other inclusion holds. So, A does not have a leap at s.

2. Let s be an integer such that $np^\tau < s < (n+1)p^\tau$ for some $n \geq 1$. Then, $\mathrm{IDer}_k\left(\log h^{p^\tau}; s\right) = \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau\right)$.

Since $s > np^\tau$, the inclusion $\subseteq$ is true. Let $D_1 \in \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau\right)$. By definition there exists an integral $D \in \mathrm{HS}_k\left(\log h^{p^\tau}; np^\tau\right)$ of D_1. By Proposition 1.2.1, we can consider $D' \in \mathrm{HS}_k(R; s)$ an integral of D. Hence, for all j such that $np^\tau < j \leq s < (n+1)p^\tau$, $p^\tau \nmid j$ and, by Lemma

1.2.9, $D'_j\left(h^{p^\tau}\right) = 0 \in \left\langle h^{p^\tau}\right\rangle$. Since $D'_l = D_l$ for all $l \leq np^\tau$, $D' \in \mathrm{HS}_k\left(\log h^{p^\tau}; s\right)$. Therefore, $D_1 \in \mathrm{IDer}_k\left(\log h^{p^\tau}; s\right)$ and A does not have a leap at s.

Thanks to these two equalities we know that the leaps are given on $s = np^\tau$ for some $n \geq 1$. Let us suppose that $s = p^\tau$. By Corollary 1.2.16 and the point 1.,

$$\mathrm{Der}_k(R) = \mathrm{IDer}_k\left(\log h^{p^\tau}; s-1\right) \supseteq \mathrm{IDer}_k\left(\log h^{p^\tau}; p^\tau = s\right) = \mathrm{Der}_k(\log h).$$

Hence, A has a leap at p^τ if and only if $\mathrm{Der}_k(\log h) \neq \mathrm{Der}_k(R)$. Now, let us consider $s = np^\tau$ for $n \geq 2$. By Corollary 1.2.16 and the point 2.

$$\begin{aligned}\mathrm{IDer}_k(\log h; n-1) = {} & \mathrm{IDer}_k\left(\log h^{p^\tau}; (n-1)p^\tau\right) = \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau - 1\right) \\ \supseteq {} & \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau\right) = \mathrm{IDer}_k(\log h; n).\end{aligned}$$

Then, A has a leap at $s = np^\tau$ if and only if n is a leap of $R/\langle h\rangle$ and we have proved the result. □

1.3 Multivariate Hasse-Schmidt derivations

In this section we recall a generalization of the HS-derivations and its group structure. This generalization will be used in chapter 2. We also remember a particular multivariate HS-derivation called external product of HS-derivations. Most of the result of this section can be found in [Na3].

Throughout this section, k will be a commutative ring and A a commutative k-algebra. Let $q \geq 1$ be an integer and let us call $\mathbf{s} = \{s_1, \ldots, s_q\}$ a set of q variables.

The monoid $\mathbb{N}^q$ is endowed with a natural partial ordering. Namely, for $\alpha, \beta \in \mathbb{N}^q$, we define

$$\alpha \leq \beta \Leftrightarrow \exists \gamma \in \mathbb{N}^q \text{ such that } \beta = \alpha + \gamma \Leftrightarrow \alpha_i \leq \beta_i \ \forall i = 1, \ldots, q.$$

The support of a series $a = \sum_\alpha a_\alpha \mathbf{s}^\alpha \in A[|\mathbf{s}|]$ is $\mathrm{Supp}(a) := \{\alpha \in \mathbb{N}^q \mid a_\alpha \neq 0\}$. The order of a non-zero series $a = \sum_\alpha a_\alpha \mathbf{s}^\alpha \in A[|\mathbf{s}|]$ is

$$\mathrm{ord}(a) := \min\{|\alpha| \mid \alpha \in \mathrm{Supp}(a)\}$$

and if $a = 0$ we define $\mathrm{ord}(a) := \infty$.

Definition 1.3.1 *We say that a subset $\Delta \subseteq \mathbb{N}^q$ is a co-ideal of $\mathbb{N}^q$ if whenever $\alpha \in \Delta$ and $\alpha' \leq \alpha$, then $\alpha' \in \Delta$.*

For example, for $\beta \in \mathbb{N}^q$, $\mathfrak{n}_\beta := \{\alpha \in \mathbb{N}^q \mid \alpha \leq \beta\}$ is a co-ideal of $\mathbb{N}^q$.

Definition 1.3.2 *For each co-ideal $\Delta \subset \mathbb{N}^q$, we denote by Δ_A the ideal of $A[|\mathbf{s}|]$ whose elements are the series $\sum_{\alpha\in\mathbb{N}^q} a_\alpha \mathbf{s}^\alpha$ such that $a_\alpha = 0$ if $\alpha \in \Delta$ i.e. $\Delta_A = \{a \in A[|\mathbf{s}|] \mid \mathrm{Supp}(a) \subseteq \Delta^c\}$.*

Let us denote $A[|\mathbf{s}|]_\Delta := A[|s|]/\Delta_A$. Note that if $q = 1$ and $\Delta = \{i \in \mathbb{N} \mid i \leq m\}$, then $A[|\mathbf{s}|]_\Delta = A[|s|]_m$. From now on, Δ will be a non-empty co-ideal.

Definition 1.3.3 *A (q,Δ)-variate Hasse-Schmidt derivation ((q,Δ)-variate HS-derivation for short) of A (over k) is a family $D=(D_\alpha)_{\alpha\in\Delta}$ of k-linear maps $D_\alpha : A \to A$, satisfying the conditions:*

$$D_0 = \mathrm{Id}_A,\ \ D_\alpha(xy) = \sum_{\beta+\gamma=\alpha} D_\beta(x) D_\gamma(y)$$

for all $x,y \in A$ and for all $\alpha \in \Delta$. We denote by $\mathrm{HS}^q_k(A;\Delta)$ the set of all (q,Δ)-variate HS-derivations of A (over k) and $\mathrm{HS}^q_k(A)$ for $\Delta = \mathbb{N}^q$.

For $q=1$ and $\Delta = \{i \in \mathbb{N} \mid i \leq m\}$, a $(1,\Delta)$-variate HS-derivation is a HS-derivation of length m in the usual way. Moreover, as in this case, for each $q \geq 1$ and $\Delta \subseteq \mathbb{N}^q$ a co-ideal, any (q,Δ)-variate HS-derivation D of A over k can be understood as a power series

$$\sum_{\alpha\in\Delta} D_\alpha \mathbf{s}^\alpha \subseteq \mathrm{End}_k(A)[|\mathbf{s}|]_\Delta$$

and so we can consider $\mathrm{HS}^q_k(A;\Delta) \subseteq \mathrm{End}_k(A)[|\mathbf{s}|]_\Delta$.

Lemma 1.3.4 [Na3, **Corollary 1**] *Let k be a commutative ring, A a commutative k-algebra, $q \geq 1$ an integer and $\Delta \subseteq \mathbb{N}^q$ a non-empty co-ideal. Then, $\mathrm{HS}^q_k(A;\Delta)$ is a group.*

Namely, the group operation on $\mathrm{HS}^q_k(A;\Delta)$ is explicitly given by

$$(D,E) \in \mathrm{HS}^q_k(A;\Delta) \times \mathrm{HS}^q_k(A;\Delta) \mapsto D \circ E \in \mathrm{HS}^q_k(A;\Delta)$$

with

$$(D\circ E)_\alpha = \sum_{\beta+\gamma=\alpha} D_\beta \circ E_\gamma .$$

Let us denote

$$\mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[|\mathbf{s}|]_\Delta) := \{ f \in \mathrm{Hom}_{k-\mathrm{alg}}(A, A[|\mathbf{s}|]_\Delta) \mid f(x) \equiv x \mod (\mathfrak{n}_0)_A \ \forall x \in A\}.$$

Lemma 1.3.5 [Na3, **§5**] *Let k be a commutative ring, A a commutative k-algebra, $q \geq 1$ an integer, $\mathbf{s} = \{s_1,\dots,s_q\}$ a set of q variables and Δ a non-empty co-ideal. Then, the map*

$$D \in \mathrm{HS}^q_k(A;\Delta) \longmapsto \left[x \in A \mapsto \sum_{\alpha\in\Delta} D_\alpha(x)\mathbf{s}^\alpha \right] \in \mathrm{Hom}^\circ_{k-\mathrm{alg}}\left(A, A[|\mathbf{s}|]_\Delta\right)$$

is a group isomorphism.

Definition 1.3.6 *Let R be a ring, $q,m \geq 1$, $\mathbf{s} = \{s_1,\dots,s_q\}$, $\mathbf{t} = \{t_1,\dots,t_m\}$ disjoint sets of variables and $\Delta \subseteq \mathbb{N}^q$ and $\nabla \subseteq \mathbb{N}^m$ non-empty co-ideals. For each $r \in R[|\mathbf{s}|]_\Delta$, $r' \in R[|\mathbf{t}|]_\nabla$, the external product $r \boxtimes r' \in R[|\mathbf{s} \sqcup \mathbf{t}|]_{\Delta\times\nabla}$ is defined as*

$$r \boxtimes r' := \sum_{(\alpha,\beta)\in\Delta\times\nabla} r_\alpha r'_\beta \mathbf{s}^\alpha \mathbf{t}^\beta .$$

Proposition 1.3.7 [Na3, **Prop. 6**] *Let $D \in \mathrm{HS}_k^q(A;\Delta)$, $E \in \mathrm{HS}_k^m(A;\nabla)$ be HS-derivations. Then, its external product $D \boxtimes E$ is a $(\mathbf{s} \sqcup \mathbf{t}, \Delta \times \nabla)$-variate HS-derivation.*

Let us consider $D, E \in \mathrm{HS}_k(A)$. Then, its external product $D \boxtimes E \in \mathrm{HS}_k^2(A)$ is given by $(D \boxtimes E)_{(i,j)} = D_i \circ E_j$ for all $(i,j) \in \mathbb{N}^2$ and it is easy to prove the following result about its inverse.

Lemma 1.3.8 *Let $D, E \in \mathrm{HS}_k(A)$. Then, $(D \boxtimes E)^*_{(i,j)} = E^*_j \circ D^*_i$ for all $(i,j) \in \mathbb{N}^2$.*

1.4 The action of substitution maps

In this section we recall the definition of substitution maps and its action on the group of HS-derivations. Most of the result of this section can be found in [Na3, §6].

Let k be a commutative ring, A a commutative k-algebra, $\mathbf{s} = \{s_1, \ldots, s_q\}$, $\mathbf{t} = \{t_1, \ldots, t_d\}$, $\mathbf{u} = \{u_1, \ldots, u_l\}$ three sets of variables where $q, d, l \geq 1$ and $\Delta \subseteq \mathbb{N}^q$, $\nabla \subseteq \mathbb{N}^d$ and $\Omega \subseteq \mathbb{N}^l$ non-empty co-ideals.

Definition 1.4.1 *An A-algebra map $\phi : A[|\mathbf{s}|]_\Delta \to A[|\mathbf{t}|]_\nabla$ will be called a substitution map if $\mathrm{ord}(\phi(s_i)) \geq 1$ for all $i = 1, \ldots, q$.*

Definition 1.4.2 *We say that a substitution map $\phi : A[|\mathbf{s}|]_\Delta \to A[|\mathbf{t}|]_\nabla$ has constant coefficients if $c^s_\beta \in k$ for all $s \in \mathbf{s}$ and all $\beta \in \nabla$ where*

$$\phi(s) = \sum_{\beta \in \nabla, 0 < |\beta|} c^s_\beta \mathbf{t}^\beta \in \mathfrak{n}_0^A(\mathbf{t})/\nabla_A \subseteq A[|\mathbf{t}|]_\nabla$$

with $\mathfrak{n}_0^A(\mathbf{t}) = \ker\left(\sum_\alpha a_\alpha \mathbf{t}^\alpha \in A[|\mathbf{t}|] \mapsto a_0 \in A\right)$. In particular, $\phi : A[|\mu|]_m \to A[|\mu|]_n$ has constant coefficient if $\phi(\mu) = \sum_{i \geq 1}^n a_i \mu^i$ with $a_i \in k$ for all i.

It is clear that composition of substitution maps (of constant coefficients) are also substitution maps (of constant coefficients).

Proposition 1.4.3 [Na3, **Prop. 10**] *For any substitution map $\phi : A[|\mathbf{s}|]_\Delta \to A[|\mathbf{t}|]_\nabla$, we have that if $f \in \mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[|\mathbf{s}|]_\Delta)$, then $\phi \circ f \in \mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[|\mathbf{t}|]_\nabla)$.*

Notation 1.4.4 *Let $\phi : A[|\mathbf{s}|]_\Delta \to A[|\mathbf{t}|]_\nabla$ be a substitution map and $D \in \mathrm{HS}_k^q(A;\Delta)$ a (q,Δ)-variate HS-derivation. We denote by $\phi \bullet D \in \mathrm{HS}_k^d(A;\nabla)$ the (d,∇)-variate HS-derivation determined by $\varphi_{\phi \bullet D} = \phi \circ \varphi_D$. In terms of power series, we have:*

$$\phi \bullet D = \phi \bullet \left(\sum_{\alpha \in \Delta} D_\alpha \mathbf{s}^\alpha\right) = \sum_{\alpha \in \Delta} \phi(\mathbf{s})^\alpha D_\alpha.$$

Remark 1.4.5 *Thanks to the previous expression, it is easy to see that, if $\phi : A[|\mathbf{s}|]_\Delta \to A[|\mathbf{t}|]_\nabla$ is a substitution map and $D \in \mathrm{HS}_k^q(\log I;\Delta)$ for any $I \subseteq A$ an ideal, i.e. $D_\alpha(I) \subset I$ for all $\alpha \in \Delta$, then $\phi \bullet D \in \mathrm{HS}_k^d(\log I;\nabla)$.*

Examples 1.4.6 *The operations defined in 1.1 are examples of substitution maps. Namely, let $D \in \mathrm{HS}_k(A;m)$ be a HS-derivation of length $m \in \overline{\mathbb{N}}$.*

1. *For each $x \in A$, $x \bullet D = \phi \bullet D$ where $\phi : A[|\mu|]_m \to A[|\mu|]_m$ is given by $\phi(\mu) = x\mu$.*
2. *Let $1 \leq n \leq m$ be an integer and let us consider the projection $\pi_{mn} : A[|\mu|]_m \to A[|\mu|]_n$ ($\pi_{mn}(\mu) = \mu$). Then, $\tau_{mn}(D) = \pi_{mn} \bullet D$.*
3. *For each integer $n \geq 1$, $D[n] = \phi \bullet D$ where $\phi : A[|\mu|]_m \to A[|\mu|]_{mn}$ is the substitution map given by $\phi(\mu) = \mu^n$.*

Substitution maps of type 2. and 3. of Example 1.4.6 have constant coefficients. Moreover, if $a \in k$, the substitution map $a \bullet (*)$ of type 1. has constant coefficients too.

The following lemma comes from 8. and Prop. 11 of [Na3, §6].

Lemma 1.4.7 *Let $\phi : A[|\mathbf{s}|]_\Delta \to A[|\mathbf{t}|]_\nabla$ and $\psi : A[|\mathbf{t}|]_\nabla \to A[|\mathbf{u}|]_\Omega$ be substitution maps and $D, D' \in \mathrm{HS}^q_k(A;\Delta)$ HS-derivations. We have the following properties:*

1. *If ϕ has constant coefficient, then $\phi \bullet (D \circ D') = (\phi \bullet D) \circ (\phi \bullet D')$.*
2. *$\psi \bullet (\phi \bullet D) = (\psi \circ \phi) \bullet D$.*

As a straightforward consequence we obtain the following corollary.

Corollary 1.4.8 *Let $D, D^1, \ldots, D^t \in \mathrm{HS}_k(A,m)$ be HS-derivations of length $m \in \overline{\mathbb{N}}$. The following properties hold:*

1. *If $\eta \in k$, then $\eta \bullet (D^1 \circ \cdots \circ D^t) = (\eta \bullet D^1) \circ \cdots \circ (\eta \bullet D^t)$.*
2. *$\tau_{mn}(D^1 \circ \cdots \circ D^t) = \tau_{mn}(D^1) \circ \cdots \circ \tau_{mn}(D^t)$ for any $1 \leq n \leq m$ integer.*
3. *$(D^1 \circ \cdots \circ D^t)[n] = D^1[n] \circ \cdots \circ D^t[n]$ for any $n \geq 1$.*
4. *$D[nn'] = (D[n])[n']$ for any $n, n' \geq 1$.*

Proposition 1.4.9 [Na3, **Prop. 11**] *Let $\phi : A[|\mathbf{s}|]_\Delta \to A[|\mathbf{t}|]_\nabla$ be a substitution map of constant coefficients. Then, $(\phi \bullet D)^* = \phi \bullet D^*$ for each $D \in \mathrm{HS}^q_k(A;\Delta)$.*

Thanks to this result we can easily show the following results:

Lemma 1.4.10 *Let $D, E \in \mathrm{HS}_k(A;m)$ be two HS-derivations of length $m \in \mathbb{N}$ such that $\tau_{m,m-1}(D) = \tau_{m,m-1}(E)$. Then, there exists $\delta \in \mathrm{Der}_k(A)$ such that $D = E \circ (\mathrm{Id}, \delta)[m]$.*

Proof. Let $E^* \in \mathrm{HS}_k(A;m)$ be the inverse of E. From Proposition 1.4.9 we have that $\tau_{m,m-1}(E^*) = (\tau_{m,m-1}(E))^* = (\tau_{m,m-1}(D))^* = \tau_{m,m-1}(D^*)$. So, $E^* \circ D = (\mathrm{Id}, 0, \ldots, 0, \delta) \in \mathrm{HS}_k(A;m)$ with $\delta \in \mathrm{Der}_k(A)$ (by definition of HS-derivation) and hence, $D = E \circ (\mathrm{Id}, \delta)[m]$. □

Lemma 1.4.11 *Let $D \in \mathrm{HS}_k(A;m)$ be a HS-derivation of length $m \in \overline{\mathbb{N}}$, $n, s \geq 1$ positive integers such that $n \leq m$ and $I \subseteq A$ an ideal. The following properties hold:*

a. *If D is $n-I$-logarithmic, then D^* is $n-I$-logarithmic too.*

b. *If D is $(n-1)-I$-logarithmic, then $D^*[s] \in \mathrm{HS}_k(A;ms)$ is $(ns-1)-I$-logarithmic.*

c. *If D is $(n-1)-I$-logarithmic, then $D^*_n = -D_n + H_n$ where H_n is an I-differential operator of order $\leq n$.*

Proof.

a. We have that $\tau_{mn}(D) \in \mathrm{HS}_k(\log I; n)$. From Lemma 1.4.9, $\tau_{mn}(D^*) = (\tau_{mn}(D))^* \in \mathrm{HS}_k(\log I; n)$. Hence D^* is $n-I$-logarithmic.

b. From $a.$, D^* is $(n-1)-I$-logarithmic and by Lemma 1.1.18, $D^*[s]$ is $(ns-1)-I$-logarithmic for all $s \geq 1$.

c. From $a.$, D^* is $(n-1)-I$-logarithmic. Then, by Lemma 1.1.19, there exists H_n an I-differential operator of order $\leq n$ such that $(D \circ D^*)_n = D^*_n + D_n + H_n = 0$. So, $D^*_n = -D_n - H_n$ and we have the result.

□

Lemma 1.4.12 *Let $I \subseteq A$ be an ideal and $\phi : A[|\mu|]_m \to A[|\mu|]_n$ a substitution map. Let us denote $B = A/I$ and $\phi^B : B[|\mu|]_m \to B[|\mu|]_n$ the substitution map induced by ϕ. Then, for each $D \in \mathrm{HS}_k(\log I; m)$ we have that*

$$\phi^B \bullet \left(\Pi^I_{\mathrm{HS},m}(D)\right) = \Pi^I_{\mathrm{HS},n}\left(\phi \bullet D\right).$$

Proof. Let us write $D = \sum_{i=0}^m D_i\mu^i \in \mathrm{End}(A)[|\mu|]_m$ and $\phi(\mu) = \sum_{j=1}^n a_j\mu^j$. Then,

$$\phi \bullet D = \sum_{i=0}^m \phi(\mu)^i D_i = \sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=i}} \binom{i}{\beta} a_1^{\beta_1}\cdots a_n^{\beta_n} D_i \mu^{\beta_1+\cdots+n\beta_n} \mod \mu^{n+1}$$

where $\binom{i}{\beta} = i!/(\beta_1!\cdots\beta_n!)$. So, if we denote $\mathcal{J}_j = \{\beta = (\beta_1,\ldots,\beta_n) \in \mathbb{N}^n \mid \sum_{s=1}^n s\beta_s = j\}$, we obtain that

$$\phi \bullet D = \sum_{j=0}^n \left(\sum_{\beta\in\mathcal{J}_j} \binom{|\beta|}{\beta} a_1^{\beta_1}\cdots a_n^{\beta_n} D_{|\beta|}\right)\mu^j \in \mathrm{HS}_k(\log I; n).$$

Let us denote $\overline{a} = a + I$ for all $a \in A$. From the definition of the map $\Pi^I_{\mathrm{HS},n}$, $\Pi^I_{\mathrm{HS},n}(\phi \bullet D) = \overline{\phi \bullet D} \in \mathrm{HS}_k(B; n)$ where

$$\begin{aligned}(\overline{\phi \bullet D})_j(a+I) = {} & (\phi \bullet D)_j(a) + I = \left(\sum_{\beta\in\mathcal{J}_j} \binom{|\beta|}{\beta} a_1^{\beta_1}\cdots a_n^{\beta_n} D_{|\beta|}(a)\right) + I \\ = {} & \sum_{\beta\in\mathcal{J}_j} \binom{|\beta|}{\beta} \overline{a_1}^{\beta_1}\cdots \overline{a_n}^{\beta_n} \overline{D}_{|\beta|}(a)\end{aligned}$$

where $\Pi^I_{\mathrm{HS},m}(D) = \sum_{i=0}^m \overline{D}_i \mu^i \in \mathrm{HS}_k(B;m)$. So,

$$\overline{\phi \bullet D} = \sum_{j=0}^{n} \left(\sum_{\beta \in \mathcal{J}_j} \binom{|\beta|}{\beta} \overline{a_1}^{\beta_1} \cdots \overline{a_n}^{\beta_n} \overline{D}_{|\beta|} \right) \mu^j.$$

On the other hand, $\phi^B(\mu) = \sum_{j=1}^n \overline{a_j} \mu^j$ and, analogously, we have that

$$\phi^B \bullet \left(\Pi^I_{\mathrm{HS},m}(D)\right) = \phi^B \left(\sum_{i=0}^{m} \overline{D}_i \mu^i \right) = \sum_{i=0}^{m} \phi^B(\mu)^i \overline{D}_i = \sum_{j=0}^{n} \left(\sum_{\beta \in \mathcal{J}_j} \binom{|\beta|}{\beta} \overline{a_1}^{\beta_1} \cdots \overline{a_n}^{\beta_n} \overline{D}_{|\beta|} \right) \mu^j.$$

Hence, we have the lemma.

$\square$

Chapter 2

Leaps of modules of integrable derivations

One of the main problems of the theory of HS-derivations is to know when a k-derivation is n-integrable for some $n \in \overline{\mathbb{N}}$. We know that if k is a ring of characteristic 0, i.e. $\mathbb{Q} \subseteq k$, then any k-derivation of a k-algebra is (∞-)integrable. The same happens when we consider derivations of a 0-smooth algebra over any commutative ring k (Theorem 1.1.28). In this chapter, we will assume that k is a commutative ring of characteristic $p > 0$, i.e. $\mathbb{F}_p \subseteq k$ and A is a commutative k-algebra and we will prove that leaps of A over k only occur at powers of p. We start this chapter with some previous and technical results.

2.1 Previous results

2.1.1 Numerical results

The aim of this section is to expose all the numerical results used in this chapter to facilitate the reading of its content.

Lemma 2.1.1 *Let p be a prime. Then, for all m such that $1 < m < p$, there exists a finite number of elements $a_i \in \mathbb{F}_p^*$ (multiplicative group) such that*

$$\begin{cases} \sum_i a_i = 1 \mod p \\ \sum_i a_i^m = 0 \mod p. \end{cases}$$

Proof. Note that $p > 2$. Since $\mathbb{F}_p^*$ is a cyclic group, there exists $g \in \mathbb{F}_p^*$ a generator of $\mathbb{F}_p^* = \{g, g^2, \ldots, g^{p-1} = 1\}$. So, $g \neq g^m$ for all $m = 2, \ldots, p-1$. We call $a_0' = g$ and let us consider $h = g^m \mod p$ with $0 < h < p$. Then, we put $a_i' = 1$ for $i = 1, \ldots, p-h$. In this case,

$$\sum_{i=0}^{p-h} (a_i')^m = g^m + \sum_{i=1}^{p-h} 1 = g^m + p - h = 0 \mod p$$

and

$$\sum_{i=0}^{p-h} a_i' = g + \sum_{i=1}^{p-h} 1 = g + p - h = g - h \neq 0 \mod p$$

because $h = g^m$ and, if $g = h \mod p$ then $g = g^m \mod p$!!!. If we define $a_i = a'_i/(g-h)$, we have the result.

$\square$

Let us consider p a prime and $n = e_s p^s + \cdots + e_t p^t$ a positive integer expressed in base p expansion where $1 \leq t \leq s$ and $0 \leq e_i < p$ with $e_s, e_t \neq 0$ (note that s and t can be the same and n is a multiple of p).

Lemma 2.1.2 *Let p, n, t be as above. Then,*

$$p^t = \min\left\{ m \in \mathbb{N}_+ \mid \binom{n}{m} \neq 0 \mod p \right\}.$$

Proof. We know that

$$\binom{n}{p^t} = \binom{e_s}{0} \cdots \binom{e_t}{1} = e_t \neq 0 \mod p$$

so, p^t is in the set described in the lemma. Now, consider $0 < m < p^t$. If we express m in base p expansion, then $m = m_l p^l + \cdots + m_0$ where $l < t$ and $m_l \neq 0$. In this case,

$$\binom{n}{m} = \binom{e_s}{0} \cdots \binom{e_t}{0} \cdots \binom{0}{m_l} \cdots \binom{0}{m_0} = 0 \mod p$$

because $\binom{0}{m_1} = 0 \mod p$.

$\square$

Lemma 2.1.3 *Let p, n, t be as above and let us suppose that n is not a power of p. Then, $2p^t \leq n$.*

Proof. Let us consider $p = 2$. Since n is not a power of 2, we have $s > t$. Hence $2p^t = p^{t+1} \leq p^s \leq p^s + e_{s-1}p^{s-1} + \cdots + p^t = n$. Let us assume that $p \geq 3$. If $s > t$ then, $2p^t < p^s \leq e_s p^s \leq n$ and we have the inequality. Otherwise, if $s = t$, we have $e_t \geq 2$ because n is not a power of p. Therefore, $2p^t \leq e_t p^t = n$.

$\square$

2.1.1.1 Definition of digital root in base p

In this section we recall the definition of digital root of a positive number n in base p where p is a prime. Although this construction is known we have not found any reference in books or journals. From now on n will be a positive integer.

Definition 2.1.4 *Let $n = e_s p^s + \cdots + e_0$ be a positive integer expressed in base p expansion where $e_s \neq 0$. We define $s_p(n) := \sum_{i=0}^{s} e_i$.*

Is is clear that if $1 \leq n \leq p-1$, then $s_p(n) = n$ and if $n \geq p$, then $s_p(n) < n$.

Definition 2.1.5 *For each $j \geq 0$, we define* $s_p^j(n) := \underbrace{s_p(s_p(\cdots(s_p}_{j \text{ times}}(n))\cdots)$.

Lemma 2.1.6 *If p is a prime and n a positive integer, there exists $j \geq 0$ such that $s_p^j(n) = s_p^{j+1}(n)$. Moreover, if $s_p^j(n) = s_p^{j+1}(n)$ then, $s_p^j(n) = s_p^J(n)$ for all $J \geq j$.*

Proof. If $n \leq p-1$, $n = s_p(n)$. Hence, the lemma holds for $j = 0$. If $n \geq p$, then $s_p(n) < n$. So, if $s_p(n) \leq p-1$, then $s_p^2(n) = s_p(n)$ and the lemma holds for $j = 1$. Otherwise, $s_p^2(n) < s_p(n) < n$. By performing this process recursively, we obtain that $s_p^j(n) \leq p-1$ for some j. So, $s_p^j(n) = s_p^{j+1}(n)$ and the lemma holds for this j. Moreover, if $s_p^j(n) = s_p^{j+1}(n)$, then $s_p^j(n) \leq p-1$, so $s_p^J(n) = s_p^j(n)$ for all $J \geq j$.

□

Definition 2.1.7 *Let p be a prime and n a positive integer. Let us consider $j = \min\{l \geq 0 \mid s_p^l(n) = s_p^{l+1}(n)\}$. The digital root of n in base p is $T_p(n) := s_p^j(n)$.*

For example, $T_2(10) = 1$, $T_3(10) = 2$, $T_5(10) = 2$, $T_7(10) = 4$ and $T_p(10) = 10$ for all $p \geq 11$.

Lemma 2.1.8 *Under the above conditions, $T_p(n) = n \mod p-1$.*

Proof. Let us write $n = e_s p^s + \cdots + e_1 p + e_0$. Taking this expression module $p-1$, we have that $n = e_s + \cdots + e_1 + e_0 = s_p(n) \mod p-1$. So, doing this process recursively, we obtain that, for all $j \geq 0$, $n = s_p^j(n) \mod p-1$. Hence, $T_p(n) = n \mod p-1$.

□

Lemma 2.1.9 *For all $x \in \mathbb{F}_p$ and $n \geq 1$, we have that $x^n = x^{T_p(n)} \mod p$.*

Proof. Since $T_p(n) = n \mod p-1$ (Lemma 2.1.8), there exists $s \in \mathbb{N}$ such that $n = T_p(n) + s(p-1)$. Hence,

$$x^n = x^{T_p(n)+s(p-1)} = x^{T_p(n)} \mod p.$$

□

2.1.1.2 Definition of $C^p_{m,e,s}$

Throughout this section p, s, e, m will be integers such that $p, s \geq 1$. Although in principle we do not impose any restrictions on e and m in the rest of the chapter they will always be positive integers.

Definition 2.1.10 *Let p, s, m, e be integers such that $p, s \geq 1$. Then, we define*

$$C^p_{m,e,s} := \{j \in \mathbb{N} \mid mp^j < ep^s\}.$$

Lemma 2.1.11 *If $e \leq m < ep^s$, then $C^p_{m,e,s}$ is not empty and $0 \leq \max C^p_{m,e,s} < s$.*

Proof. $C^p_{m,e,s} \neq \emptyset$ because $j = 0$ holds the inequality, so $\max C^p_{e,m,s} \geq 0$. On the other hand, let us consider $r \geq s$, then

$$ep^s \leq ep^r \leq mp^r$$

so, $r \notin C^p_{m,e,s}$ and we can conclude that $0 \leq \max C^p_{m,e,s} < s$.

$\square$

Lemma 2.1.12 *Let us assume that $e < m < ep^s$ with $m \neq 0 \mod e$ and we denote $r = \max C^p_{m,e,s}$. Then, $mp^{r+1} - 1 \geq ep^s$.*

Proof. Since $r = \max C^p_{m,e,s}$, we have that $mp^{r+1} \geq ep^s$. We will see that the equality never holds. Suppose that $mp^{r+1} = ep^s$. From Lemma 2.1.11, $r + 1 \leq s$, so $m = ep^{s-(r+1)}$ but m is not a multiple of e by hypothesis. Therefore, $mp^{r+1} > ep^s$ and we have the result.

$\square$

Let us consider p a prime and $n = e_s p^s + \cdots + e_t p^t$ a positive integer expressed in base p expansion where $1 \leq t \leq s$ and $0 \leq e_i < p$ with $e_s, e_t \neq 0$.

Lemma 2.1.13 *Let p, n, t be as above. For all $m \in \mathbb{N}$ such that $2p^t + 1 \leq m < n + 1$, we have $0 \leq \max C^p_{m,n+1,t} \leq s$.*

Proof. Observe that $0 \in C^p_{m,n+1,t}$, so these sets are not empty. Consider $r > s$, then

$$(2p^t + 1)p^r < (n + 1)p^t \Leftrightarrow (2p^t + 1)p^{r-t} = 2p^r + p^{r-t} < n + 1.$$

The last inequality is false because $n < p^{s+1} \leq p^r$, so $n + 1 < p^r + 1 \leq 2p^r + p^{r-t}$. Hence, $\max C^p_{2p^t+1,n+1,t} \leq s$. Now, we consider $m > 2p^t + 1$ and, as before, $r > s$, then,

$$(n + 1)p^t \leq (2p^t + 1)p^r < mp^r$$

where the first inequality holds because $\max C^p_{2p^t+1,n+1,t} \leq s$. So, $r \notin C^p_{m,n+1,t}$ for $r > s$. That implies that $\max C^p_{m,n+1,t} \leq s$.

$\square$

Lemma 2.1.14 *With the above notation, let us assume that n is not a power of p (note that n is a multiple of p). For each integer m such that $2p^t+1 \leq m < n+1$, we denote $r_m = \max C^p_{m,n+1,t}$. Then, $mp^{r_m+1} - 1 \geq (n + 1)p^t$.*

Proof. By definition, $mp^{r_m+1} \geq (n + 1)p^t$. We will see that the equality never holds. Let us suppose that $mp^{r_m+1} = (n + 1)p^t$. Since $m < n + 1$, we have that $r_m + 1 > t$. Then, $mp^{r_m+1-t} = n + 1$ so, $n + 1$ has to be a multiple of p!!! Hence, $mp^{r_m+1} - 1 \geq (n + 1)p^t$.

$\square$

To illustrate the set $C^p_{m,e,s}$ we give some examples for different values of p, e, s and m:

$p = 2, s = 1$

$e \backslash m$	1	2	3
1	$\{0\}$	$\emptyset$	$\emptyset$
2	$\{0, 1\}$	$\{0\}$	$\{0\}$
3	$\{0, 1, 2\}$	$\{0, 1\}$	$\{0\}$

$p = 2, s = 2$

$e \backslash m$	1	2	3
1	$\{0, 1\}$	$\{0\}$	$\{0\}$
2	$\{0, 1, 2\}$	$\{0, 1\}$	$\{0, 1\}$
3	$\{0, 1, 2, 3\}$	$\{0, 1, 2\}$	$\{0, 1\}$

$p = 3, s = 1$

e \ m	1	2	3
1	$\{0\}$	$\{0\}$	$\emptyset$
2	$\{0,1\}$	$\{0\}$	$\{0\}$
3	$\{0,1\}$	$\{0,1\}$	$\{0\}$

$p = 3, s = 2$

e \ m	1	2	3
1	$\{0,1\}$	$\{0,1\}$	$\{0\}$
2	$\{0,1,2\}$	$\{0,1\}$	$\{0,1\}$
3	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1\}$

2.1.2 Some technical lemmas

Throughout this section, k will be a commutative ring, A a commutative k-algebra and $I \subseteq A$ an ideal.

Lemma 2.1.15 *Let $D \in \mathrm{HS}_k(A; n)$ be a HS-derivation of length $n \in \mathbb{N}$. For each $m > 1$, there exists $E \in \mathrm{HS}_k(A; (n+1)m-1)$ such that $E_m = -D_1$ and $\ell(E; m) = n+1$. Moreover, if D is I-logarithmic, then E is I-logarithmic.*

Proof. We know that $D' := ((-1) \bullet D)\,[m]$ is a HS-derivation of length mn such that $D'_m = -D_1$ and $\ell(D'; m) = n$. By Lemma 1.1.30, there exists an integral $E \in \mathrm{HS}_k(A; (n+1)m-1)$ of D' with $\ell(E; m) = n+1$. So, this derivation satisfies the lemma. Moreover, if D is I-logarithmic then D' is also I-logarithmic and, by Lemma 1.1.30, E is I-logarithmic too.

□

Definition 2.1.16 *For each $D \in \mathrm{HS}_k(A; n)$ and $m > 1$, we denote by $E^{D,m} \in \mathrm{HS}_k(A; (n+1)m-1)$ a HS-derivation holding Lemma 2.1.15.*

Lemma 2.1.17 *Let e, i, m be integers such that $e > 1$, $i \geq 1$ and $m \geq ie$. Let $D, E \in \mathrm{HS}_k(A; m)$ be two HS-derivations such that $\ell(D; e) = i \geq 1$ and $\ell(E) > ie$. Then, for $r \leq m$, we have*

$$(D \circ E)_r = \begin{cases} D_r & \text{if } r \leq ie \\ D_r + E_r & \text{if } r = ie+1, \ldots, ie+(e-1). \end{cases}$$

Proof. We denote $D' = D \circ E \in \mathrm{HS}_k(A; m)$. If $0 < \gamma \leq ie$, then $E_\gamma = 0$, so

$$D'_r = \sum_{\beta+\gamma=r} D_\beta \circ E_\gamma = D_r + \sum_{\gamma=ie+1}^{r} D_{r-\gamma} \circ E_\gamma.$$

Hence, if $r \leq ie$, $D'_r = D_r$. Let us consider $r = ie + \alpha \leq m$ where $\alpha \in \{1, \ldots, e-1\}$. Then, the previous equation can be written as

$$D'_{ie+\alpha} = D_{ie+\alpha} + \sum_{\gamma=1}^{\alpha} D_{\alpha-\gamma} \circ E_{ie+\gamma}.$$

Note that if $\gamma \neq \alpha$, then $0 < \alpha - \gamma < e$ and, since $\ell(D; e) \geq 1$, $D_{\alpha-\gamma} = 0$. That implies that $D'_{ie+\alpha} = D_{ie+\alpha} + E_{ie+\alpha}$ for all α.

□

Lemma 2.1.18 *Let e, j, m be integers such that $e > 1$, $j > 0$ and $m \geq je$ and let us consider $D, E \in \mathrm{HS}_k(A; m)$ two HS-derivations such that $\ell(D) = je$ and $\ell(E; je) = \lceil m/je \rceil$. Let us denote $D' := D \circ E \in \mathrm{HS}_k(A; m)$. Then, $\ell(D') \geq je$, $\ell(D'; e) \geq \ell(D; e)$, $D'_{je} = D_{je} + E_{je}$ and for each $i \in \mathbb{N}$ such that $j \leq i \leq \ell(D; e)$, we have that, for $r \leq m$:*

$$D'_r = D_r \text{ if } r = ie + 1, \ldots, ie + (e - 1).$$

Proof. Since $\ell(E; je) \geq 1$, $\ell(E) \geq je$. From Lemma 1.1.9, $l(D') \geq je$. Let us denote $\ell(D; e) = s \geq j$. Then, $(s-1)e < m$, so $(s-1)/j = (s-1)e/je < \lceil m/je \rceil$. Then, $s - 1 < \lceil m/je \rceil j$, i.e., $s/j \leq \ell(E; je)$. Hence, by Lemma 1.1.12, $\ell(D'; e) \geq \ell(D; e)$.

By hypothesis, $E_\gamma = 0$ for all $\gamma \neq 0 \mod je$ so,

$$D'_r = \sum_{\beta+\gamma=r} D_\beta \circ E_\gamma = \sum_{\beta+je\gamma=r} D_\beta \circ E_{je\gamma}. \tag{2.1}$$

If $r = je$, then γ can only take the values 0 and 1, so $D'_r = D_{je} + E_{je}$. Let us consider i such that $j \leq i \leq \ell(D; e)$ and $r = ie + \alpha \leq m$ where $\alpha \in \{1, \ldots, e-1\}$. Then, in the equation (2.1), $\beta = r - je\gamma = (i - j\gamma)e + \alpha$. Hence, when $\gamma > 0$, $\beta < ie$ and it is not a multiple of e, so $D_\beta = 0$ and the only non-zero term is when $\gamma = 0$. That means $D'_{ie+\alpha} = D_{ie+\alpha}$ for all α.

□

Lemma 2.1.19 *Let $e, i \geq 1$ be integers and $m \geq ie + e - 1$. Let us consider $D^1, \ldots, D^{e-1} \in \mathrm{HS}_k(A; m)$ an ordered family of HS-derivations such that $\ell(D^a; ie+a) \geq 2$ for all $a = 1, \ldots, e-2$ and $\ell(D^{e-1}; ie+e-1) \geq 1$. We denote $D := D^1 \circ D^2 \circ \cdots \circ D^{e-1} \in \mathrm{HS}_k(A; m)$. Then $\ell(D) \geq ie+1$ and*

$$D_{ie+a} = D^a_{ie+a} \text{ where } a = 1, \ldots, e-1.$$

Proof. Since $\ell(D^a; ie + a) \geq 1$ for all $a = 1, \ldots, e-1$, we have that $\ell(D^a) \geq ie + a \geq ie + 1$ and, by Lemma 1.1.9, we can deduce that $\ell(D) \geq ie + 1$. Suppose now that $r = ie + a \leq m$ where $a \in \{1, \ldots, e-1\}$. From Lemma 1.1.3, we have that

$$D_r = \sum_{|\beta|=r} D^1_{\beta_1} \circ \cdots \circ D^{e-1}_{\beta_{e-1}}.$$

Let us consider $\beta = (\beta_1, \ldots, \beta_{e-1})$ such that $|\beta| = r$. If there is $b \in \{1, \ldots, e-1\}$ such that $0 < \beta_b < ie+b$, then the term associated with β is zero so, we can consider $\beta_b = 0$ or $\beta_b \geq ie+b$ for all $b = 1, \ldots, e-1$.

Let us suppose that there exist $b, b' \in \{1, \ldots, e-1\}$ such that $\beta_b, \beta_{b'} > 0$, then,

$$ie + a = r \geq \beta_b + \beta_{b'} \geq ie + b + ie + b' > 2ie > r!!!$$

Hence, there is only one $b \in \{1, \ldots, e-1\}$ such that $\beta_b \neq 0$. Since $\ell(D^b; ie + b) \geq 2$ for all $b = 1, \ldots, e-2$, we have that $D^b_\gamma = 0$ for all $\gamma = ie + b + 1, \ldots, 2i(e + b) - 1$ (or until m if $m \leq 2i(e+b)-1$). So, in order for the term associated with β to be not zero, if $b \in \{1, \ldots, e-2\}$, $\beta_b = ie + b$ or $\beta_b = 0$. On the other hand, if $b = e-1$ and $\beta_b > ie + b = (i+1)e - 1$, then

$r = ie + a \leq (i+1)e - 1 < \beta_b$!!! So, $\beta_b = ie + b$ or $\beta_b = 0$. Hence, we can conclude that, if $\beta_b \neq 0$, then $\beta_b = ie + b$ and

$$\beta_b = ie + b = ie + a = r \Leftrightarrow b = a.$$

Therefore, the only summand which is not zero is the one associated with $\beta = (0, \dots, 0, ie + a, 0, \dots, 0)$ where $ie + a$ is in the a-th position. So, $D_{ie+a} = D^a_{ie+a}$ for all $a = 1, \dots, e-1$.

□

2.2 A special Hasse-Schmidt derivation

In this section, we consider k a commutative ring, A a commutative k-algebra and $I \subseteq A$ an ideal. We define a HS-derivation associated with another HS-derivation that will allow us to prove that leaps only occur at powers of p.

Notation 2.2.1 *Let $D \in \mathrm{HS}_k(A)$ be a HS-derivation. We denote $B^D := \phi \bullet D \in \mathrm{HS}^2_k(A)$ where $\phi : A[|\mu|] \to A[|\mu_1, \mu_2|]$ is the substitution map of constant coefficients given by $\phi(\mu) = \mu_1 + \mu_2$.*

Lemma 2.2.2 *Let $D \in \mathrm{HS}_k(A)$ be a HS-derivation. Then, $B^D_{(i,j)} = \binom{i+j}{i} D_{i+j}$ for all $(i,j) \in \mathbb{N}^2$.*

Proof. We can write $D = \sum_{\alpha \geq 0} D_\alpha \mu^\alpha \subseteq \mathrm{End}_k(A)[|\mu|]$. Then,

$$B^D = \phi \bullet \left(\sum_{\alpha \geq 0} D_\alpha \mu^\alpha \right) = \sum_{\alpha \geq 0} D_\alpha (\mu_1 + \mu_2)^\alpha = \sum_{\alpha \geq 0} D_\alpha \sum_{i+j=\alpha} \binom{\alpha}{j} \mu_1^i \mu_2^j = \sum_{i+j \geq 0} \binom{i+j}{j} D_{i+j} \mu_1^i \mu_2^j.$$

So,

$$B^D_{(i,j)} = \binom{i+j}{j} D_{i+j}.$$

□

Lemma 2.2.3 *Let $D \in \mathrm{HS}_k(A)$ be an $(n-1) - I$-logarithmic HS-derivation. If $i + j < n$, then $B^D_{(i,j)}(I) \subseteq I$.*

Proof. If $i + j < n$, then $D_{i+j}(I) \subseteq I$, so $B^D_{(i,j)}(I) = \binom{i+j}{i} D_{i+j}(I) \subseteq I$. □

Notation 2.2.4 *Let $D \in \mathrm{HS}_k(A)$ be a HS-derivation. We denote $F^D = D \boxtimes D \in \mathrm{HS}^2_k(A)$, the external product of D, and $\left(F^D\right)^* \in \mathrm{HS}^2_k(A)$ its inverse. Recall that $\left(F^D\right)^*_{(i,j)} = D^*_j \circ D^*_i$ for all $(i,j) \in \mathbb{N}^2$ (see Lemma 1.3.8).*

Lemma 2.2.5 *Let $D \in \mathrm{HS}_k(A)$ be an $(n-1) - I$-logarithmic HS-derivation. If $i, j < n$, then $\left(F^D\right)^*_{(i,j)}(I) \subseteq I$.*

Proof. Since D is $(n-1) - I$-logarithmic, D^* is $(n-1) - I$-logarithmic too by Lemma 1.4.11, *a*. So, $\left(F^D\right)^*_{(i,j)}(I) = D^*_j \circ D^*_i(I) \subseteq I$.

□

Notation 2.2.6 *For each $D \in \mathrm{HS}_k(A)$, we define $G^D := B^D \circ \left(F^D\right)^* \in \mathrm{HS}^2_k(A)$.*

From now on, we fix $D \in \mathrm{HS}_k(A)$ and we will omit the superscript in the HS-derivations defined before, so we will write $G := G^D$, $B := B^D$ and $F := F^D$.

Lemma 2.2.7 *For each $m > 0$, we have that $G_{(m,0)} = G_{(0,m)} = 0$ and $G_{(1,m)}, G_{(m,1)} \in \mathrm{Der}_k(A)$.*

Proof. First, we compute $G_{(m,0)}$:

$$G_{(m,0)} = \sum_{\alpha+\beta=(m,0)} B_\alpha \circ F^*_\beta = \sum_{\alpha_1+\beta_1=m} B_{(\alpha_1,0)} \circ F^*_{(\beta_1,0)} = \sum_{\alpha_1+\beta_1=m} D_{\alpha_1} \circ D^*_{\beta_1} = 0.$$

The computation of $G_{(0,m)}$ is analogous. Now, by definition of multivariate HS-derivation:

$$\begin{aligned} G_{(1,m)}(xy) &= \sum_{\substack{\alpha_1+\beta_1=1\\ \alpha_2+\beta_2=m}} G_{(\alpha_1,\alpha_2)}(x) G_{(\beta_1,\beta_2)}(y) \\ &= \sum_{\alpha_2+\beta_2=m} G_{(0,\alpha_2)}(x) G_{(1,\beta_2)}(y) + \sum_{\alpha_2+\beta_2=m} G_{(1,\alpha_2)}(x) G_{(0,\beta_2)}(y) \\ &= x G_{(1,m)}(y) + G_{(1,m)}(x) y. \end{aligned}$$

To obtain the third equality, recall that $G_{(0,0)} = \mathrm{Id}$ and, thanks to the previous computation, $G_{(0,m)} = 0$ for all $m \geq 1$. It is analogous for $G_{(m,1)}$.

□

Lemma 2.2.8 *Let us suppose that $D \in \mathrm{HS}_k(A)$ is $(n-1) - I$-logarithmic. We have the following properties:*

1. *If $0 \leq i + j < n$, then $G_{(i,j)}(I) \subseteq I$.*
2. *If i and j are not zero and $i + j = n > 0$, then $G_{(i,j)} = \binom{n}{i} D_n + H$ where H is an I-differential operator of order $\leq n$.*

Proof.

1. If $i + j = 0$, then $G_{(i,j)} = \mathrm{Id}$ and, if $i = 0$ or $j = 0$ then, $G_{(i,j)} = 0$ so the result is obvious and we can suppose that $i, j > 0$. We have that

$$G_{(i,j)} = \sum_{\substack{\alpha_1+\beta_1=i\\ \alpha_2+\beta_2=j}} B_{(\alpha_1,\alpha_2)} \circ F^*_{(\beta_1,\beta_2)}.$$

Since i and j are not zero, $1 \leq i, j < n-1$ so, $\beta_1, \beta_2 < n-1$. Moreover, $\alpha_1+\beta_1+\alpha_2+\beta_2 = i+j < n$, so $\alpha_1+\alpha_2 < n$. By Lemmas 2.2.3 and 2.2.5, the terms of the sum is I-logarithmic and $G_{(i,j)}$ is an I-differential operator.

2. By definition,

$$
\begin{aligned}
G_{(i,j)} &= \sum_{\alpha+\beta=(i,j)} B_\alpha \circ F^*_\beta = B_{(i,j)} + \sum_{\substack{\alpha_1+\beta_1=i\\ \alpha_2+\beta_2=j\\ \alpha\neq(i,j)}} B_{(\alpha_1,\alpha_2)} \circ F^*_{(\beta_1,\beta_2)} \\
&= \binom{n}{i} D_n + \sum_{\substack{\alpha_1+\beta_1=i\\ \alpha_2+\beta_2=j\\ \alpha\neq(i,j)}} \binom{\alpha_1+\alpha_2}{\alpha_1} D_{\alpha_1+\alpha_2} \circ D^*_{\beta_2} \circ D^*_{\beta_1}.
\end{aligned}
$$

If $\alpha \neq (i,j)$, then $\alpha_1 < i$ or $\alpha_2 < j$ so, $\alpha_1+\alpha_2 < i+j = n$ and, by Lemma 2.2.3, $B_\alpha(I) \subseteq I$. Moreover, B_α is a differential operator of order $\leq |\alpha|$. On the other hand, $\beta_1, \beta_2 < n$ because $i, j < n$. Hence, $F^*_\beta(I) \subseteq I$ (Lemma 2.2.5) and, since $D^*_{\beta_i}$ is a differential operator of order β_i for $i = 1, 2$, we have that F^*_β is an I-differential operator of order $\leq |\beta|$. Hence, we can conclude that, the sum is an I-differential operator of order $\leq n$.

$\square$

In the rest of this section, k will be a commutative ring of characteristic $p > 0$ (i.e. $\mathbb{F}_p \subseteq k$), A a commutative k-algebra and I an ideal of A. Let $n = e_s p^s + \cdots + e_t p^t$ be a positive integer expressed in base p expansion where $1 \leq t \leq s$ and $0 \leq e_i < p$ with $e_s, e_t \neq 0$ (note that t and s could be equal). Thanks to Lemma 2.1.2, we can prove the next result.

Lemma 2.2.9 *With the above notation, let us consider $i, j \geq 0$ such that $i+j = n$ and $i < p^t$. If $D \in \mathrm{HS}_k(A)$ is $(n-1)-I$-logarithmic then, $G_{(i,j)}(I) \subseteq I$.*

Proof. By Lemma 2.2.7, if $i = 0$ or $j = 0$, then $G_{(i,j)} = 0$ so, it is an I-differential operator. If $i, j \geq 1$, by Lemma 2.2.8, $G_{(i,j)} = \binom{n}{i} D_n + H$ where H is an I-differential operator. By Lemma 2.1.2, $\binom{n}{i} = 0$ and we have the result.

$\square$

Let us consider the following substitution map of constant coefficients:

$$
\begin{array}{rccc}
\varphi^r: & A[|\mu_1,\mu_2|] & \longrightarrow & A[|\mu|] \\
& \mu_1 & \longmapsto & \mu^{r+1} \\
& \mu_2 & \longmapsto & \mu^r
\end{array}
$$

Notation 2.2.10 *Let p be a prime and $n = e_s p^s + \cdots + e_t p^t$ a positive integer expressed in base p expansion where $1 \leq t \leq s$ and $0 \leq e_i < p$ with $e_s, e_t \neq 0$. Let $D \in \mathrm{HS}_k(A)$ be a HS-derivation and let us consider $G^D \in \mathrm{HS}^2_k(A)$ defined in 2.2.6. We define $G^{D,p^t} = \tau_{\infty,(n+1)p^t}\left(\varphi^{p^t} \bullet G^D\right) \in \mathrm{HS}_k\left(A;(n+1)p^t\right)$.*

Lemma 2.2.11 *Under the condition of Notation 2.2.10, $\ell\left(G^{D,p^t}\right) \geq 2p^t + 1$. Moreover, if $D \in \mathrm{HS}_k(A)$ is $(n-1)-I$-logarithmic then, G^{D,p^t} is $((n+1)p^t - 1) - I$-logarithmic and $G^{D,p^t}_{(n+1)p^t} = \binom{n}{p^t} D_n + H$ where H is an I-differential operator of order $\leq n$.*

Proof. Note that

$$\varphi^{p^t}\bullet G^D=\varphi^{p^t}\left(\sum_{(i,j)\in\mathbb{N}^2}G^D_{(i,j)}\mu_1^i\mu_2^j\right)=\sum_{(i,j)\in\mathbb{N}^2}G^D_{(i,j)}\mu^{(p^t+1)i+p^tj}=\sum_{\alpha\geq 0}\left(\sum_{(i,j):(p^t+1)i+p^tj=\alpha}G^D_{(i,j)}\right)\mu^\alpha.$$

Since $G^D_{(i,j)}=0$ if i or j is zero (Lemma 2.2.7), we have that $G_0^{D,p^t}=\mathrm{Id}$ and for all $\alpha\geq 1$,

$$G^{D,p^t}_\alpha=\sum_{\substack{(i,j):(p^t+1)i+p^tj=\alpha\\ i,j\neq 0}}G^D_{(i,j)}.$$

If $\alpha<2p^t+1$ then there is not $(i,j)\in\mathbb{N}^2$ with $i,j\neq 0$ such that $(p^t+1)i+p^tj=\alpha$, so $G^{D,p^t}_\alpha=0$. Hence, $\ell\left(G^{D,p^t}\right)\geq 2p^t+1$. Now, we will suppose that D is $(n-1)-I$-logarithmic and will prove the rest of the lemma.

Let us consider a pair $(i,j)\in\mathbb{N}^2$ with $i,j\neq 0$. If $i+j<n$, then $G_{(i,j)}$ is an I-differential operator by Lemma 2.2.8. So, we have to focus on the case when $i+j=n+l$ where $l\geq 0$. We have

$$(p^t+1)i+p^tj=p^t(i+j)+i=p^t(n+l)+i.$$

If $l>0$, then $p^t(n+l)+i>p^t(n+l)\geq p^t(n+1)$. So, $G^D_{(i,j)}$ does not appear in any component of G^{D,p^t}.

If $l=0$, then $p^tn+i\leq(n+1)p^t$ if and only if $i\leq p^t$. So, $G^D_{(i,j)}$ appears in some component of G^{D,p^t} if $i\leq p^t$. By Lemma 2.2.9, $G^D_{(i,j)}(I)\subseteq I$ if $i<p^t$. On the other hand, if $i=p^t$, then $j=n-p^t$ and $(p^t+1)p^t+p^t(n-p^t)=(n+1)p^t$. Hence, $G^D_{(p^t,n-p^t)}$ is a term of $G^{D,p^t}_{(n+1)p^t}$ and it is the only component that could be not I-logarithmic. So, G^{D,p^t} is $((n+1)p^t-1)-I$-logarithmic and

$$G^{D,p^t}_{(n+1)p^t}=G^D_{(p^t,n-p^t)}+\textit{some } I\textit{-diff. op. of order }\leq n=\binom{n}{p^t}D_n+\textit{some } I\textit{-diff. op. of order }\leq n$$

where the last equality holds because of Lemma 2.2.8.

□

2.3 Some partial integrability results

In this section, k will be a commutative ring of characteristic $p>0$ (i.e. $\mathbb{F}_p\subseteq k$) and A a commutative k-algebra. We will give some results about leaps of A over k. Namely, we prove that A does not have leaps at integers which are not a multiple of p and either at the first multiple of p which is not a power of p.

Lemma 2.3.1 *Let k be a commutative ring and m an integer invertible in k. Then, any HS-derivation of length $m-1$ is m-integrable.*

Proof. Since A is a k-algebra, we can write $A := R/I$ where R is a polynomial ring (in an arbitrary number of variables) and $I \subseteq R$ an ideal. Let $D \in \mathrm{HS}_k(A; m-1)$ be a HS-derivation of A of length $m-1$. Then, there exists $\widetilde{D} \in \mathrm{HS}_k(\log I; m-1)$ such that $\Pi^I_{\mathrm{HS},m-1}(\widetilde{D}) = D$ (see Proposition 1.2.2). Thanks to Proposition 1.2.1, we can integrate $\widetilde{D}$. So, we have $E \in \mathrm{HS}_k(R; m)$ such that $\tau_{m,m-1}(E) = \widetilde{D}$. From Lemma 1.1.23, there exists an I-differential operator H of order $\leq m$ such that $\varepsilon_m(E) = mE_m + H \in \mathrm{Der}_k(R)$. Then,

$$E' := E \circ ((-1/m) \bullet (\mathrm{Id}, \varepsilon_m(E)))[m] = (\mathrm{Id}, E_1, \ldots, E_{m-1}, -(1/m)H) \in \mathrm{HS}_k(\log I; m).$$

So, $\Pi^I_{\mathrm{HS},m}(E') \in \mathrm{HS}_k(A; m)$ is an m-integral of D.

□

Remark 2.3.2 *If k is a ring of characteristic 0 then, this lemma allows us to prove, in a different way, that* $\mathrm{Leaps}_k(A) = \emptyset$ *for any k-algebra A.*

Corollary 2.3.3 *If k has characteristic $p > 0$ and $m \neq 0 \mod p$ then,* $\mathrm{IDer}_k(A; m-1) = \mathrm{IDer}_k(A; m)$*, i.e. $m \notin \mathrm{Leaps}_k(A)$.*

Proof. If $m \neq 0 \mod p$, then m is not a multiple of p, so m is invertible in k. Let us consider $\delta \in \mathrm{IDer}_k(A; m-1)$. By definition, there exists $D \in \mathrm{HS}_k(A; m-1)$ such that $D_1 = \delta$. From Lemma 2.3.1, D is m-integrable, so there exists $D' \in \mathrm{HS}_k(A; m)$ such that $\tau_{m,m-1}(D') = D$, in particular $D'_1 = \delta$. Hence, $\delta \in \mathrm{IDer}_k(A; m)$, so $\mathrm{IDer}_k(A; m-1) \subseteq \mathrm{IDer}_k(A; m)$. Since the other inclusion always holds, we have the equality.

□

Proposition 2.3.4 *Let k be a commutative ring of characteristic $p = 2$ and A a commutative k-algebra. Then,* $\mathrm{IDer}_k(A; 5) = \mathrm{IDer}_k(A; 6)$.

Proof. We can write $A := R/I$ where R is a polynomial ring and $I \subseteq R$ an ideal. By Corollary 1.2.4, $\mathrm{IDer}_k(A; 5) = \mathrm{IDer}_k(A; 6)$ if and only if $\mathrm{IDer}_k(\log I; 5) = \mathrm{IDer}_k(\log I; 6)$. The inclusion $\mathrm{IDer}_k(\log I; 6) \subseteq \mathrm{IDer}_k(\log I; 5)$ is always true, so let $\delta \in \mathrm{IDer}_k(\log I; 5)$ be an I-logarithmically 5-integrable k-derivation and let us consider $D \in \mathrm{HS}_k(\log I; 5)$ a 5-integral of δ. By Proposition 1.2.1, we can integrate D up to ∞. So, we have $D = (\mathrm{Id}, D_1, \ldots, D_5, D_6, \ldots) \in \mathrm{HS}_k(R)$ an integral of δ which is $5-I$-logarithmic. Let us consider $G := G^D \in \mathrm{HS}^2_k(R)$ defined in 2.2.6. By Lemma 2.2.8, $G_{(i,j)}(I) \subseteq I$ for all $i + j \leq 5$. Moreover, $G_{(2,4)} = \binom{6}{2} D_6 + H = D_6 + H$ where H is an I-differential operator of order ≤ 6.

On the other hand, by definition of multivariate HS-derivation and Lemma 2.2.7:

$$\begin{aligned}
G_{(2,4)}(xy) = & \sum_{\alpha+\beta=(2,4)} G_\alpha(x) G_\beta(y) = \sum_{\substack{\alpha_1+\beta_1=2 \\ \alpha_2+\beta_2=4}} G_{(\alpha_1,\alpha_2)}(x) G_{(\beta_1,\beta_2)}(y) \\
= & \sum_{\alpha_2+\beta_2=4} G_{(2,\alpha_2)}(x) G_{(0,\beta_2)}(y) + \sum_{\alpha_2+\beta_2=4} G_{(1,\alpha_2)}(x) G_{(1,\beta_2)}(y) + \\
+ & \sum_{\alpha_2+\beta_2=4} G_{(0,\alpha_2)}(x) G_{(2,\beta_2)}(y) \\
= & G_{(2,4)}(x) y + G_{(1,1)}(x) G_{(1,3)}(y) + G_{(1,3)}(x) G_{(1,1)}(y) + G_{(1,2)}(x) G_{(1,2)}(y) + x G_{(2,4)}(y).
\end{aligned}$$

Since $G_{(1,j)} \in \mathrm{Der}_k(A)$ by Lemma 2.2.7,

$$\begin{aligned} G_{(1,1)}G_{(1,3)}(xy) = & \ G_{(1,1)}\left(G_{(1,3)}(x)y + xG_{(1,3)}(y)\right) \\ = & \ G_{(1,1)}G_{(1,3)}(x)y + G_{(1,3)}(x)G_{(1,1)}(y) + G_{(1,1)}(x)G_{(1,3)}(y) + xG_{(1,1)}G_{(1,3)}(y). \end{aligned}$$

Let us consider $D'_2 = G_{(2,4)} - G_{(1,1)}G_{(1,3)}$. Then, $D'_2(xy) = D'_2(x)y + G_{(1,2)}(x)G_{(1,2)}(y) + xD'_2(y)$. So, since $G_{(1,2)} \in \mathrm{Der}_k(R)$ by Lemma 2.2.7, we have that

$$D' = (\mathrm{Id}, G_{(1,2)}, G_{(2,4)} - G_{(1,1)}G_{(1,3)}) \in \mathrm{HS}_k(R; 2).$$

Moreover, D' is $1 - I$-logarithmic and $G_{(1,1)}G_{(1,3)}$ is an I-differential operator of order ≤ 6, so $D'_2 = D_6 + H'$ where H' is an I-differential operator of order ≤ 6. Then,

$$\begin{aligned} D'' = \tau_{\infty,6}(D) \circ D'[3] = & \ (\mathrm{Id}, D_1, \ldots, D_6 + D_3G_{(1,2)} + D_6 + H') \\ = & \ (\mathrm{Id}, D_1, \ldots, D_3G_{(1,2)} + H') \in \mathrm{HS}_k(\log I; 6). \end{aligned}$$

Hence, $\mathrm{IDer}_k(\log I; 5) = \mathrm{IDer}_k(\log I; 6)$ and we have the proposition.

$\square$

Theorem 2.3.5 *Let k be a commutative ring of characteristic $p > 0$ and A a commutative k-algebra. Let $n \geq 1$ be an integer such that $T_p(n) \neq 1$ (see Definition 2.1.7). Then,* $\mathrm{IDer}_k(A; n-1) = \mathrm{IDer}_k(A, n)$.

Proof. Since A is a k-algebra, we can see $A = R/I$ where R is a polynomial ring (in an arbitrary number of variables) and $I \subseteq R$ an ideal. By Corollary 1.2.4, A does not have leap at n if and only if $\mathrm{IDer}_k(\log I; n-1) = \mathrm{IDer}_k(\log I; n)$. The inclusion $\mathrm{IDer}_k(\log I; n-1) \supseteq \mathrm{IDer}_k(\log I; n)$ is always true. Let us consider $\delta \in \mathrm{IDer}_k(\log I; n-1)$ and $D \in \mathrm{HS}_k(\log I; n-1)$ an integral of δ. By Proposition 1.2.1, we can integrate D up to n. So, we redefine $D = (\mathrm{Id}, D_1, \ldots, D_{n-1}, D_n) \in \mathrm{HS}_k(R; n)$ as an integral of the previous D and we obtain an integral of δ which is $(n-1) - I$-logarithmic.

Let us consider $(a_i)_i$ a solution of the system of Lemma 2.1.1 where $m = T_p(n)$. Then,

$$E := \circ_i (a_i \bullet D) = \left(\mathrm{Id}, \sum_i a_i D_1, \ldots, \sum_i a_i^n D_n + H\right) \text{ where } H := \sum_{\substack{|\beta|=n: \\ \beta_i < n \ \forall i}} \circ_i \left(a_i^{\beta_i} D_{\beta_i}\right).$$

By Lemma 2.1.9, $\sum a_i^n = \sum a_i^{T_p(n)} = 0 \mod p$. Moreover, since $D_\beta(I) \subseteq I$ for all $\beta < n$, we have that H is an I-differential operator of order $\leq n$. So, since $\sum a_i = 1 \mod p$,

$$E = (\mathrm{Id}, D_1, \ldots, H) \in \mathrm{HS}_k(\log I; n).$$

Therefore, $\delta \in \mathrm{IDer}_k(\log I; n)$ and, by Corollary 1.2.4, $\mathrm{IDer}_k(A; n-1) = \mathrm{IDer}_k(A; n)$.

$\square$

Corollary 2.3.6 *Let k be a commutative ring of characteristic $p \geq 3$ and A a commutative k-algebra. Then,* $\mathrm{IDer}_k(A; 2p-1) = \mathrm{IDer}_k(A; 2p)$.

Proof. Since $T_p(2p) = 2$, we have the result by Theorem 2.3.5. $\square$

2.4 Integrating the first non-vanishing component of a Hasse-Schmidt derivation

In this section, k will be a commutative ring, A a commutative k-algebra and $I \subseteq A$ an ideal. We calculate an integral of the first component of a HS-derivation that could be not zero which will be the key to prove the main theorem of section 2.5.

Hypothesis 2.4.1 *Let $a \geq 1$ and $p \geq 2$ be integers and $I \subseteq A$ an ideal. We say that A satisfies the condition $H^I_{p,a}$ if for all $M \in \mathbb{N}_+$ not a power of p with $1 < M < p^a$, then $\mathrm{IDer}_k(\log I; M-1) = \mathrm{IDer}_k(\log I; M)$.*

Remark 2.4.2 *Note that I can be A. In this case, the condition in Hypothesis 2.4.1 is $\mathrm{IDer}_k(A; M-1) = \mathrm{IDer}_k(A; M)$.*

Lemma 2.4.3 *1. If A satisfies $H^I_{p,a}$ for some $a \geq 1$, then A satisfies $H^I_{p,s}$ for all $1 \leq s \leq a$.*

2. If $\mathrm{char}(k) = p > 0$, then A satisfies $H^A_{p,1}$.

3. If $\mathrm{char}(k) = p > 0$ and $A = k[x_i \mid i \in \mathcal{I}]$, the polynomial ring in an arbitrary number of variables, then A satisfies $H^I_{p,1}$ for all ideals $I \subseteq A$.

Proof.

1. It is obvious.

2. If $1 < M < p$, then M can not be a multiple of p, so $M \neq 0 \mod p$. From Corollary 2.3.3, $\mathrm{IDer}_k(A; M-1) = \mathrm{IDer}_k(A; M)$ and we deduce that A satisfies $H^A_{p,1}$.

3. From Corollary 1.2.4, we have that $\mathrm{IDer}_k(\log I; M-1) = \mathrm{IDer}_k(\log I; M)$ if and only if A/I does not have a leap at M, i.e. if $\mathrm{IDer}_k(A/I; M-1) = \mathrm{IDer}_k(A/I; M)$. Since A/I satisfies $H^{A/I}_{p,1}$, for all $M \in \mathbb{N}_+$ with $1 < M < p$, we have the last equality, so A satisfies $H^I_{p,1}$.

$\square$

From now on, k will be a commutative ring, A a commutative k-algebra, $I \subseteq A$ an ideal and $p \geq 2$ an integer.

Lemma 2.4.4 *Let us assume that A satisfies $H^I_{p,1}$. Let $e > 1$ and $0 < i < p$ be integers. For each $(ep-1)-I$-logarithmic HS-derivation $D \in \mathrm{HS}_k(A; ep)$ such that $\ell(D; e) = i$, there exists an $(ep-1)-I$-logarithmic HS-derivation $D' \in \mathrm{HS}_k(A; ep)$ and an I-differential operator H of order $\leq ep$ such that $\ell(D'; e) \geq i+1$, $D'_r = D_r$ for all $r \leq ie$ and $D'_{ep} = D_{ep} + H$.*

Proof. Since $\ell(D; e) = i \geq 1$, from Lemma 1.1.13, we have that $D_{ie+\alpha} \in \mathrm{Der}_k(\log I)$ for all $\alpha = 1, \ldots, e-1$ and, thanks to the condition $H^I_{p,1}$, we know that all derivations are I-logarithmically $(p-1)$-integrable. Let $D^\alpha \in \mathrm{HS}_k(\log I; p-1)$ be an integral of $D_{ie+\alpha}$, i.e. $D^\alpha_1 = D_{ie+\alpha}$, and consider $E^{D^\alpha, ie+\alpha} \in \mathrm{HS}_k(\log I; (ie+\alpha)p-1)$, defined in 2.1.16, for all

$\alpha = 1, \ldots, e-1$. By Lemma 2.1.15, $E^{D^\alpha, ie+\alpha}_{ie+\alpha} = -D_{ie+\alpha}$ and $\ell(E^{D^\alpha, ie+\alpha}; ie+\alpha) = p$. That means that $E^{D^\alpha, ie+\alpha}_j = 0$ for all $j \neq 0 \mod (ie+\alpha)$.

Since $(ie+\alpha)p - 1 > iep \geq ep$, we can truncate all these derivations until length ep. We denote

$$E^\alpha := \tau_{(ie+\alpha)p-1, ep}\left(E^{D^\alpha, ie+\alpha}\right).$$

Note that $E^\alpha_{ie+\alpha} = -D_{ie+\alpha}$ and $E^\alpha_j = 0$ for all $j \neq 0 \mod (ie+\alpha)$, so $\ell(E^\alpha; ie+\alpha) = \lceil ep/ie+\alpha \rceil$. Moreover, $\ell(E^\alpha, ie+\alpha) \geq 2$ for all α because $ie+\alpha < (i+1)e \leq ep$.

By Lemma 2.1.19, if we denote $E := E^1 \circ E^2 \circ \cdots \circ E^{e-1} \in \mathrm{HS}_k(\log I; ep)$, we have that $\ell(E) > ie$ and $E_{ie+\alpha} = E^\alpha_{ie+\alpha} = -D_{ie+\alpha}$ for all $\alpha = 1, \ldots, e-1$. Let us consider $D' = D \circ E \in \mathrm{HS}_k(A; ep)$. From Corollary 1.1.20, D' is $(ep-1) - I$-logarithmic and $D'_{ep} = D_{ep} + H$ where H is an I-differential operator of order $\leq ep$. Moreover, from Lemma 2.1.17, $D'_r = D_r$ for all $r \leq ie$ and $D'_{ie+\alpha} = D_{ie+\alpha} + E_{ie+\alpha} = 0$ for $\alpha = 1, \ldots, e-1$, so $\ell(D'; e) \geq i+1$. Hence, D' satisfies the lemma.

□

Lemma 2.4.5 *Let us assume that A satisfies $H^I_{p,1}$. Let $e \geq 1$ be an integer and $D \in \mathrm{HS}_k(A; ep)$ an $(ep-1) - I$-logarithmic HS-derivation such that $\ell(D) \geq e$. Then, D_e is p-integrable and there exists a $(p-1) - I$-logarithmic integral $D' \in \mathrm{HS}_k(A; p)$ of D_e and an I-differential operator H of order $\leq p$ such that $D'_p = D_{ep} + H$.*

Proof. First note that D_{ep} is a differential operator of order $\leq p$ by Proposition 1.1.10. This result is trivial if $e = 1$, so we will suppose that $e > 1$. We proceed by decreasing induction on $\ell(D; e)$. Note that $1 \leq \ell(D; e) \leq p$ because $\ell(D) \geq e$ and, by definition, $\ell(D; e) \leq \lceil ep/e \rceil = p$.

If $\ell(D; e) = p$, by Lemma 1.1.14, there exists $D' \in \mathrm{HS}_k(A; p)$ such that $D'_r = D_{re}$ for all $r = 1, \ldots, p$. Since D is $(ep-1) - I$-logarithmic, $D'_r(I) = D_{re}(I) \subseteq I$ for all $r < p$, so D' is $(p-1) - I$-logarithmic. Moreover, $D'_p = D_{ep}$ so, D' satisfies the lemma. Now, let us assume that any HS-derivation with $\ell(*) \geq e$ and $\ell(*; e) \geq i+1$ where $1 \leq i < p$ holds the result and we take a HS-derivation D such that $\ell(D) \geq e$ and $\ell(D; e) = i$.

By Lemma 2.4.4, there exists an $(ep-1) - I$-logarithmic HS-derivation $D' \in \mathrm{HS}_k(A; ep)$ and an I-differential operator H of order $\leq ep$ such that $\ell(D'; e) \geq i+1$, $D'_r = D_r$ for all $r \leq ie$ and $D'_{ep} = D_{ep} + H$. Since $\ell(D') \geq e$, because $\ell(D'; e) \geq i+1 \geq 1$, we have that D'_{ep} is a differential operator of order $\leq p$ and, since D_{ep} has also order $\leq p$, H has order $\leq p$. Moreover, we can apply the induction hypothesis, so there exists an I-differential operator H' of order $\leq p$ and a $(p-1) - I$-logarithmic integral $D'' \in \mathrm{HS}_k(A; p)$ of $D'_e = D_e$ such that $D''_p = D'_{ep} + H' = D_{ep} + H + H'$. Hence, we have the lemma.

□

Lemma 2.4.6 *Let us assume that A satisfies $H^I_{p,a}$ for some $a \geq 1$. Let e, s, m be integers such that $1 \leq s \leq a$ and $1 < e \leq m < ep^s$. We denote $r := \max C^p_{m,e,s}$ (see Definition 2.1.10) and we consider $\delta \in \mathrm{IDer}_k(\log I; p^r)$. We have the following properties:*

1 If $m = 0 \mod e$, then there exists $E \in \mathrm{HS}_k(\log I; ep^s - 1)$ such that $E_m = -\delta$ and $\ell(E; m) = \lceil ep^s - 1/m \rceil$.

2 *If* $m \neq 0 \mod e$*, then there exists* $E \in \mathrm{HS}_k(\log I; ep^s)$ *such that* $E_m = -\delta$ *and* $\ell(E;m) = \lceil ep^s/m \rceil$.

Proof. By Lemma 2.1.11, we have that $0 \leq r < s \leq a$, so $p^{r+1} \leq p^a$. Thanks to the condition $H^I_{p,a}$, we have that $\delta \in \mathrm{IDer}_k(\log I; p^r) = \mathrm{IDer}_k(\log I; p^{r+1}-1)$. Let $D \in \mathrm{HS}_k(\log I; p^{r+1}-1)$ be an integral of δ. Let us consider $E^{D,m} \in \mathrm{HS}_k(\log I, mp^{r+1}-1)$ where $E^{D,m}_m = -\delta$ and $\ell\left(E^{D,m}; m\right) = p^{r+1}$, i.e. $E^{D,m}_\alpha = 0$ for all $\alpha \neq 0 \mod m$.

On the other hand, from the definition of r, $mp^{r+1}-1 \geq ep^s - 1$. Hence, if $m = 0 \mod e$, then $E = \tau_{mp^{r+1}-1, ep^s-1}(E^{D,m})$ satisfies the lemma. Otherwise, if $m \neq 0 \mod e$, by Lemma 2.1.12, $mp^{r+1}-1 \geq ep^s$. So, $E = \tau_{mp^{r+1}-1, ep^s}(E^{D,m})$ satisfies the lemma.

□

Lemma 2.4.7 *Let us assume that* A *satisfies* $H^I_{p,a}$ *for some* $a \geq 1$*. Let* e, s, m *be integers such that* $1 \leq s \leq a$ *and* $1 < e \leq m < ep^s$ *and we denote* $r := \max C^p_{m,e,s}$*. Let* $D \in \mathrm{HS}_k(\log I; ep^s - 1)$ *be a HS-derivation such that* $\ell(D) \geq m$ *and* $D_m \in \mathrm{IDer}_k(\log I; p^r)$*. Then, there exists* $D' \in \mathrm{HS}_k(\log I; ep^s-1)$ *such that* $\ell(D') \geq m+1$ *and* $D'_\alpha = D_\alpha$ *for all* $\alpha = m+1, \ldots, 2m-1$.

Proof. If $D_m = 0$, we put $D' = D$ and we have the lemma. Let us assume that $D_m \neq 0$. If $m = 0 \mod e$, by Lemma 2.4.6, we have $E \in \mathrm{HS}_k(\log I, ep^s-1)$ such that $E_m = -D_m$ and $\ell(E, m) = \lfloor ep^s - 1/m \rfloor$. If $m \neq 0 \mod e$, by Lemma 2.4.6, we have $E' \in \mathrm{HS}_k(\log I, ep^s)$ such that $E'_m = -D_m$ and $\ell(E'; m) = \lfloor ep^s/m \rfloor$, that means $E'_j = 0$ for all $j \neq 0 \mod m$. So, let us consider $E = \tau_{ep^s, ep^s-1}(E') \in \mathrm{HS}_k(\log I; ep^s-1)$. Then, $E_m = -D_m$ and $E_j = 0$ for all $j \neq 0 \mod m$, i.e. $\ell(E;m) = \lfloor ep^s - 1/m \rfloor$.

Hence, we can apply Lemma 2.1.18 to $D' = D \circ E \in \mathrm{HS}_k(\log I; ep^s-1)$ in both cases. Then, $\ell(D') \geq m$ and

$$D'_\alpha = \begin{cases} D_m + E_m & \text{if } \alpha = m \\ D_\alpha & \text{if } \alpha = m+1, \ldots, 2m-1. \end{cases}$$

Since $E_m = -D_m$, $D'_m = 0$ and hence, $\ell(D') \geq m+1$ and D' satisfies the lemma.

□

Theorem 2.4.8 *Let us suppose that* A *satisfies* $H^I_{p,a}$ *for some* $a \geq 1$*. Let* $e, s \geq 1$ *be two integers such that* $s \leq a$ *and let us consider an* $(ep^s-1) - I$*-logarithmic HS-derivation* $D \in \mathrm{HS}_k(A; ep^s)$ *with* $\ell(D) \geq e$*. Then, there exists an integral* $D' \in \mathrm{HS}_k(A; p^s)$ *of* D_e *and an* I*-differential operator* H *of order* $\leq p^s$ *such that* D' *is* $(p^s-1)-I$*-logarithmic and* $D'_{p^s} = D_{ep^s} + H$.

Proof. We prove the result by induction on $s \geq 1$. Observe that if $s = 1$, we have the theorem from Lemma 2.4.5. So, let us assume that the theorem is true for all j such that $1 \leq j < s \leq a$. Moreover, we can suppose that $e > 1$ (if $e = 1$ the theorem is trivial). We will divide this proof in several lemmas:

Lemma 2.4.9 *Let* $D \in \mathrm{HS}_k(\log I; ep^s-1)$ *such that* $\ell(D) \geq m$ *with* $1 < e \leq m < ep^s$*. Then,* $D_m \in \mathrm{IDer}_k(\log I; p^r)$ *with* $r = \max C^p_{m,e,s} < s$.

Proof. By Lemma 2.1.11, we have that $0 \leq r < s$. We rewrite $D := \tau_{ep^s-1, mp^r}(D) \in \mathrm{HS}_k(\log I; mp^r)$ (note that $mp^r \leq ep^s - 1$ by definition of $C^p_{m,e,s}$). If $r = 0$, then it is obvious that D_m is I-logarithmically p^r-integrable. Let us suppose that $r \geq 1$. Then, since $1 \leq r <$

$s \leq a$, by the induction hypothesis of the theorem, there exists $D' \in \mathrm{HS}_k(A;p^r)$ an integral of D_m such that D' is $(p^r-1)-I$-logarithmic and $D'_{p^r} = D_{mp^r} + (\textit{some } I\textit{-diff. op.})$. But D_{mp^r} is an I-differential operator, so D' is I-logarithmic too and D_m is I-logarithmically p^r-integrable.

◇

Lemma 2.4.10 *Let $D \in \mathrm{HS}_k(\log I; ep^s-1)$ such that $\ell(D) \geq e > 1$ and $\ell(D;e) = i < p^s$. Then, there exists $D' \in \mathrm{HS}_k(\log I; ep^s-1)$ such that $\ell(D') > ie$ and $D'_\alpha = D_\alpha$ for all $\alpha = ie+1,\ldots,ie+e-1$.*

Proof. Note that the only components that can be not zero before $ie+1$ are those that are in the multiples of e. If $\ell(D) > ie$ then the lemma is obvious, otherwise $\ell(D) = je$ for some $1 \leq j \leq i$. We will prove the result by decreasing induction on $1 \leq j \leq i$.

Let us assume that $\ell(D) = ie$. By Lemma 2.4.9, $D_{ie} \in \mathrm{IDer}_k(\log I; p^r)$ where $r = \max C^p_{ie,e,s} < s$. From Lemma 2.4.7, there exists $D' \in \mathrm{HS}_k(\log I; ep^s-1)$ such that $\ell(D') \geq ie+1$ and $D'_\alpha = D_\alpha$ for all $\alpha = ie+1,\ldots,\min\{ep^s-1, 2ie-1\}$. Note that $ie+e-1 \leq \min\{ep^s-1, 2ie-1\}$, so D' satisfies the lemma. Let us suppose now that the lemma is true for all derivations with $\ell(*) > je$ and we will prove it for $1 \leq j < i$.

By Lemma 2.4.9, $D_{je} \in \mathrm{IDer}_k(\log I; p^r)$ where $r = \max C^p_{je,e,s} < s$. From Lemma 2.4.6, there exists $E \in \mathrm{HS}_k(\log I; ep^s-1)$ such that $E_{je} = -D_{je}$ and $\ell(E; je) = \lceil ep^s - 1/je \rceil \geq 1$. We can apply Lemma 2.1.18 to D and E and we obtain $D' = D \circ E \in \mathrm{HS}_k(\log I, ep^s-1)$ such that $\ell(D') \geq je$, $\ell(D';e) \geq \ell(D;e) = i$ and

$$D'_\alpha = \begin{cases} D_{je} + E_{je} & \text{if } \alpha = je \\ D_\alpha & \text{for all } \alpha = ie+1,\ldots,ie+e-1. \end{cases}$$

Since $\ell(D;e) = i$, there exists $a \in \{1,\ldots,e-1\}$ such that $D_{ie+a} \neq 0$ and, since $D'_{ie+a} = D_{ie+a}$, we have that $\ell(D';e) = i$. Moreover, $E_{je} = -D_{je}$, so $\ell(D') \geq je+1$, but $\ell(D';e) > j$, therefore $\ell(D') \geq (j+1)e$. Now, we can apply the induction hypothesis. Hence, there exists $D'' \in \mathrm{HS}_k(\log I; ep^s-1)$ such that $\ell(D'') > ie$ and $D''_\alpha = D'_\alpha = D_\alpha$ for all $\alpha = ie+1,\ldots,ie+e-1$ and we have the lemma.

◇

Lemma 2.4.11 *Let $D \in \mathrm{HS}_k(\log I; ep^s-1)$ be a HS-derivation such that $\ell(D) > ie$ with $1 \leq i < p^s$. Then, for all $\alpha = 1,\ldots,e-1$ there exists $E^\alpha \in \mathrm{HS}_k(\log I; ep^s)$ such that $E^\alpha_{ie+\alpha} = -D_{ie+\alpha}$ and $\ell(E^\alpha; ie+\alpha) = \lceil ep^s/ie+\alpha \rceil$.*

Proof. If $\ell(D) \geq (i+1)e$, then $D_{ie+\alpha} = 0$ for all $\alpha = 1,\ldots,e-1$ and we have the result, it is enough to put $E^\alpha = \mathbb{I}$. Let us suppose that $\ell(D) = (i+1)e-1$. By Lemma 2.4.9, $D_{(i+1)e-1} \in \mathrm{IDer}_k(\log I; p^{r_{e-1}})$ where $r_{e-1} = \max C^p_{(i+1)e-1,e,s}$. Since $(i+1)e-1 \neq 0$ mod e, Lemma 2.4.6 give us the result. Let us assume that the lemma is true for all HS-derivations such that $\ell(*) = ie+\beta$ with $1 \leq j < \beta \leq e-1$ and we take a HS-derivation $D \in \mathrm{HS}_k(\log I; ep^s-1)$ such that $\ell(D) = ie+j$.

As before, from Lemmas 2.4.9 and 2.4.6, there exists $E^j \in \mathrm{HS}_k(\log I; ep^s)$ such that $E^j_{ie+j} = -D_{ie+j}$ and $\ell(E^j; ie+j) = \lceil ep^s/ie+j \rceil$. We can apply Lemma 2.1.18 to D and $E := \tau_{ep^s,ep^s-1}(E^j)$ obtaining $D' = D \circ E \in \mathrm{HS}_k(\log I; ep^s - 1)$ such that $\ell(D') \geq ie+j$ and

$$D'_\alpha = \begin{cases} D_\alpha + E_\alpha & \text{if } \alpha = ie+j \\ D_\alpha & \text{if } \alpha = ie+j+1, \ldots, \min\{ep^s - 1, 2(ie+j) - 1\}. \end{cases}$$

Note that $ie+e-1 \leq \min\{ep^s-1, 2(ie+j)-1\}$, so $D'_\alpha = D_\alpha$ for all $\alpha = ie+j+1, \ldots, ie+e-1$. Since $E_{ie+j} = -D_{ie+j}$, $\ell(D') > ie+j$ and we can use the induction hypothesis on D' obtaining that, for all $\alpha = j+1, \ldots, e-1$, there exists $E^\alpha \in \mathrm{HS}_k(\log I; ep^s)$ such that $E^\alpha_{ie+\alpha} = -D'_{ie+\alpha} = -D_{ie+\alpha}$ and $\ell(E^\alpha) = \lceil ep^s/ie+\alpha \rceil$. So, we have the lemma.

◇

Lemma 2.4.12 *Let $D \in \mathrm{HS}_k(A; ep^s)$ be an $(ep^s - 1) - I$-logarithmic HS-derivation with $1 \leq \ell(D; e) = i < p^s$. Then, there exists an $(ep^s - 1) - I$-logarithmic HS-derivation $D' \in \mathrm{HS}_k(A; ep^s)$ and an I-differential operator H of order $\leq ep^s$ such that $\ell(D'; e) \geq i+1$, $D'_{je} = D_{je}$ for all $j \leq i$ and $D'_{ep^s} = D_{ep^s} + H$.*

Proof. Since $\ell(D; e) = i$, there exists $D_{ie+\alpha} \neq 0$ for some $\alpha \in \{1, \ldots, e-1\}$ and $D_j = 0$ for all $j \not\equiv 0 \mod e$ with $j \leq ie$. Hence, if we consider $D^\tau = \tau_{ep^s,ep^s-1}(D) \in \mathrm{HS}_k(\log I; ep^s - 1)$, we have that $D^\tau_j = 0$ for all $j \not\equiv 0 \mod e$ with $j \leq ie$ and $D^\tau_{ie+\alpha} = D_{ie+\alpha} \neq 0$. So, $\ell(D^\tau; e) = i \geq 1$. In this case, $\ell(D^\tau) \geq e$.

By Lemma 2.4.10, there exists $D' \in \mathrm{HS}_k(\log I; ep^s - 1)$ such that $\ell(D') > ie$ and $D'_{ie+\alpha} = D_{ie+\alpha}$ for all $\alpha = 1, \ldots, e-1$. By Lemma 2.4.11, for each $\alpha = 1, \ldots, e-1$, there exists $E^\alpha \in \mathrm{HS}_k(\log I; ep^s)$ such that $E^\alpha_{ie+\alpha} = -D'_{ie+\alpha} = -D_{ie+\alpha}$ and $\ell(E^\alpha; ie+\alpha) = \lceil ep^s/ie+\alpha \rceil$. Note that $ie+\alpha < ep^s$ so, $\lceil ep^s/ie+\alpha \rceil \geq 2$. By Lemma 2.1.19, if we denote $E = E^1 \circ \cdots \circ E^{e-1} \in \mathrm{HS}_k(\log I; ep^s)$, then $\ell(E) \geq ie+1$ and $E_{ie+\alpha} = E^\alpha_{ie+\alpha} = -D_{ie+\alpha}$.

Now, we consider $D'' = D \circ E \in \mathrm{HS}_k(A; ep^s)$. By Corollary 1.1.20, D'' is $(ep^s - 1) - I$-logarithmic and there exists an I-differential operator H of order $\leq ep^s$ such that $D''_{ep^s} = D_{ep^s} + H$. On the other hand, by Lemma 2.1.17, we have that

$$D''_\beta = \begin{cases} D_\beta & \text{if } \beta \leq ie \\ D_\beta + E_\beta & \text{if } \beta = ie+1, \ldots, ie+e-1. \end{cases}$$

Hence, $D''_\beta = 0$ for all $\beta = ie+1, \ldots, ie+e-1$ so, $\ell(D''; e) \geq i+1$. Therefore, D'' satisfies the lemma.

◇

Now, with the help of the previous lemmas we will finish the proof of Theorem 2.4.8. We show this result by decreasing induction on $1 \leq \ell(D; e) \leq p^s$.

If $\ell(D; e) = p^s$, by Lemma 1.1.14, there is $D' \in \mathrm{HS}_k(A; p^s)$ such that $D'_\alpha = D_{\alpha e}$ for all $\alpha \leq p^s$. Then, D' is a $(p^s - 1) - I$-logarithmic p^s-integral of D_e with $D'_{p^s} = D_{ep^s}$ and we have the result in this case. Let us assume that the theorem is true for HS-derivation with $\ell(*; e) > i$ for $1 \leq i < p^s$ and we take a HS-derivation $D \in \mathrm{HS}_k(A; ep^s)$ with $\ell(D; e) = i$.

By Lemma 2.4.12, there exists an I-differential operator H of order $\leq ep^s$ and $(ep^s-1)-I$-logarithmic HS-derivation $D' \in \mathrm{HS}_k(A; ep^s)$ such that $\ell(D'; e) \geq i+1$ (so $\ell(D') \geq e$), $D'_{ej} = D_{ej}$ for all $j \leq i$ and $D'_{ep^s} = D_{ep^s} + H$. Observe that D_{ep^s} and D'_{ep^s} are differential operators of order $\leq p^s$ because $\ell(D), \ell(D') \geq e$ (see Proposition 1.1.10). So, H has order $\leq p^s$. By induction hypothesis, there exists an integral $D'' \in \mathrm{HS}_k(A; p^s)$ of $D'_e = D_e$ and an I-differential operator H' of order $\leq p^s$ such that D'' is $(p^s-1)-I$-logarithmic and $D''_{p^s} = D'_{ep^s} + H' = D_{ep^s} + H + H'$. Hence, we have the result.

□

Corollary 2.4.13 *Let us suppose that A satisfies $H^I_{p,a}$ for some $a \geq 1$. Let $e, s \geq 1$ be two integers such that $s \leq a$ and let us consider $D \in \mathrm{HS}_k(\log I; ep^s)$ with $\ell(D) \geq e$. Then, D_e is I-logarithmically p^s-integrable.*

Proof. Since D is $(ep^s-1)-I$-logarithmic, we can apply Theorem 2.4.8. Then, there exists an integral $D' \in \mathrm{HS}_k(A; p^s)$ of D_e and an I-differential operator H such that D' is $(p^s-1)-I$-logarithmic and $D'_{p^s} = D_{ep^s} + H$. Since $D_{ep^s}(I) \subseteq I$, we have that $D' \in \mathrm{HS}_k(\log I; p^s)$ and we have the result.

□

2.5 Leaps in positive characteristic

In this section we prove the main theorem of this chapter, we show that, any k-algebra, where k is a ring of characteristic $p > 0$, only has leaps at powers of p.

Theorem 2.5.1 *Let k be a commutative ring of characteristic $p > 0$ and A a commutative k-algebra. Then,* $\mathrm{Leaps}_k(A) \subseteq \{p^\tau \mid \tau \geq 1\}$.

Proof. It is enough to show that $n \notin \mathrm{Leaps}_k(A)$ for n a multiple of p, not a power of p because, if $n \neq 0 \mod p$, by Corollary 2.3.3, we have that $\mathrm{IDer}_k(A; n-1) = \mathrm{IDer}_k(A; n)$. We will prove this theorem by induction on n multiple of p, not a power of p. We have two different base cases, when $p = 2$ and $p \neq 2$. In the first case, we have to prove that $\mathrm{IDer}_k(A; 5) = \mathrm{IDer}_k(A; 6)$, which is Proposition 2.3.4. In the second one, we have to prove that $\mathrm{IDer}_k(A; 2p-1) = \mathrm{IDer}_k(A; 2p)$, which is Corollary 2.3.6. This concludes the base step. Let us assume that for all $m < n$ not a power of p, $\mathrm{IDer}_k(A; m-1) = \mathrm{IDer}_k(A; m)$ and we will prove the equality for n, a multiple of p, not a power of p.

Since A is a k-algebra, we can express $A = R/I$ where $R = k[x_i \mid i \in \mathcal{I}]$ is a polynomial ring whose variables x_i are indexed by the set $\mathcal{I}$ depending on A and $I \subseteq R$ an ideal. Then, by Corollary 1.2.4, we have that $\mathrm{IDer}_k(\log I; m-1) = \mathrm{IDer}_k(\log I; m)$ for all $m < n$ not a power of p and it is enough to prove that $\mathrm{IDer}_k(\log I; n-1) = \mathrm{IDer}_k(\log I; n)$.

Let us express $n = e_s p^s + \cdots + e_t p^t$ in base p expansion where $1 \leq t \leq s$ and $0 \leq e_i < p$ with $e_s, e_t \neq 0$. By induction hypothesis, we have that R satisfies $H^I_{p,s}$ (2.4.1).

Let $\delta \in \mathrm{IDer}_k(\log I; n-1)$ be a k-derivation and $D \in \mathrm{HS}_k(\log I; n-1)$ an integral of δ. We can integrate D up to infinite length (see Proposition 1.2.1), so we redefine $D \in \mathrm{HS}_k(R)$ the integral of D. Note that $D_1 = \delta$ and D is $(n-1)-I$-logarithmic. Now, we consider

$G := G^{D,p^t} \in \mathrm{HS}_k(R;(n+1)p^t)$ the HS-derivation defined in 2.2.10. From Lemma 2.2.11, G is $((n+1)p^t-1)-I$-logarithmic, $\ell(G) \geq 2p^t+1$ and $G_{(n+1)p^t} = \binom{n}{p^t} D_n + H$ for some I-differential operator H.

By Lemma 2.1.3, we have that $2p^t+1 \leq n+1$. If $n+1 = 2p^t+1$, from Theorem 2.4.8, we obtain a $(p^t-1)-I$-logarithmic HS-derivation $T \in \mathrm{HS}_k(R;p^t)$ and an I-differential operator H' such that $T_{p^t} = G_{(n+1)p^t} + H' = \binom{n}{p^t} D_n + H + H'$ where $H+H'$ is an I-differential operator.

Let us suppose now that $2p^t+1 < n+1$ and we denote $r = \max C^p_{2p^t+1,n+1,t}$. By Lemma 2.1.13, $0 \leq r \leq s$ and by definition of $C^p_{2p^t+1,n+1,t}$, $(2p^t+1)p^r < (n+1)p^t$. Hence, we can consider $\tau_{(n+1)p^t,(2p^t+1)p^r}(G) \in \mathrm{HS}_k(\log I;(2p^t+1)p^r)$. If $r=0$, then $G_{2p^t+1} \in \mathrm{IDer}_k(\log I;p^r)$. Otherwise, $r \geq 1$ and applying Corollary 2.4.13 to this HS-derivation, we have that G_{2p^t+1} is I-logarithmically p^r-integrable. So, in both cases, we have that $G_{2p^t+1} \in \mathrm{IDer}_k(\log I;p^r)$. We have two cases:

- If $r < s$, from induction hypothesis, $G_{2p^t+1} \in \mathrm{IDer}_k(\log I;p^{r+1}-1)$, i.e. there exists an integral $D' \in \mathrm{HS}_k(\log I;p^{r+1}-1)$ of G_{2p^t+1} and we can consider $E^{D',2p^t+1} \in \mathrm{HS}_k\left(\log I;(2p^t+1)p^{r+1}-1\right)$.

  By Lemma 2.1.14, we have $T = \tau_{(2p^t+1)p^{r+1}-1,(n+1)p^t}\left(E^{D',2p^t+1}\right) \in \mathrm{HS}_k(\log I;(n+1)p^t)$ where $T_{2p^t+1} = -G_{2p^t+1}$ and $\ell(T) \geq 2p^t+1$ (recall that $\ell(E^{D',2p^t+1};2p^t;1) = p^{r+1}$).

- If $r = s$, then $G_{2p^t+1} \in \mathrm{IDer}_k(\log I;p^s)$. Since $p^s < n < p^{s+1}$, $G_{2p^t+1} \in \mathrm{IDer}_k(\log I;n-1)$. Let $D' \in \mathrm{HS}_k(\log I;n-1)$ be an integral of G_{2p^t+1} and let us consider $E^{D';2p^t+1} \in \mathrm{HS}_k(\log I;(2p^t+1)n-1)$. Note that

  $$n(2p^t+1)-1 > (n+1)p^t \Leftrightarrow np^t+n-1 > p^t.$$

  Since the last inequality always holds, we have $T = \tau_{(2p^t+1)n-1,(n+1)p^t}\left(E^{D',2p^t+1}\right) \in \mathrm{HS}_k(\log I;(n+1)p^t)$ where $T_{2p^t+1} = -G_{2p^t+1}$ and $\ell(T) \geq 2p^t+1$.

Therefore, in both cases, we can compose G and T obtaining an $((n+1)p^t-1)-I$-logarithmic HS-derivation

$$G^{(2p^t+2)} := T \circ G^{p^t} = \left(\mathrm{Id},0,\ldots,0,G^{(2p^t+2)}_{2p^t+2},\ldots,G^{(2p^t+2)}_{n+1},\ldots,G^{(2p^t+2)}_{(n+1)p^t}\right) \in \mathrm{HS}_k(R;(n+1)p^t)$$

where $G^{(2p^t+1)}_{(n+1)p^t} = \binom{n}{p^t} D_n + H$ for some I-differential operator H.

We will prove that we can obtain an $((n+1)p^t-1)-I$-logarithmic HS-derivation $G^{(n+1)} \in \mathrm{HS}_k(R;(n+1)p^t)$ such that $\ell(G^{(n+1)}) \geq n+1$ and $G^{(n+1)}_{(n+1)p^t} = \binom{n}{p^t} D_n + H$ for some I-differential operator H by induction. Suppose that, by doing the previous process, we obtain an $((n+1)p^t-1)-I$-logarithmic HS-derivation:

$$G^{(j)} = (\mathrm{Id},0,\ldots,0,G^{(j)}_j,\ldots,G^{(j)}_{n+1},\ldots,\binom{n}{p^t} D_n + H) \in \mathrm{HS}_k\left(R;(n+1)p^t\right)$$

with H an I-differential operator and $2p^t+1 < j < n+1$. We denote $r = \max C^p_{j,n+1,t}$. By Lemma 2.1.13, $0 \leq r \leq s$. Since $jp^r < (n+1)p^t$, we have $\tau_{(n+1)p^t,jp^r}\left(G^{(j)}\right) \in \mathrm{HS}_k(\log I;jp^r)$ and we can deduce that $G^{(j)}_j$ is I-logarithmically p^r-integrable in the same way as above. We have two cases:

- If $r < s$, then $G_j^{(j)} \in \mathrm{IDer}_k(\log I; p^r) = \mathrm{IDer}_k(\log I; p^{r+1}-1)$. Let us consider $D' \in \mathrm{HS}_k(\log I; p^{r+1}-1)$ an integral of $G_j^{(j)}$ and $E^{D',j} \in \mathrm{HS}_k(\log I; jp^{r+1}-1)$. By Lemma 2.1.14, $jp^{r+1}-1 \geq (n+1)p^t$. So, we have $T = \tau_{jp^{r+1}-1,(n+1)p^t}\left(E^{D',j}\right) \in \mathrm{HS}_k(\log I; (n+1)p^t)$ where $T_j = -G_j^{(j)}$ and $\ell(T) \geq j$ (recall that $\ell(E^{D',j}; j) = p^{r+1}$).

- If $r = s$, then $G_j^{(j)} \in \mathrm{IDer}_k(\log I; p^s) = \mathrm{IDer}_k(\log I; n-1)$. Then, there exists $D' \in \mathrm{HS}_k(\log I; n-1)$ an integral of $G_j^{(j)}$ and we can consider $E^{D',j} \in \mathrm{HS}_k(\log I; jn-1)$. Since $jn-1 > (2p^t+1)n-1 > (n+1)p^t$, we can define $T = \tau_{jn-1,(n+1)p^t}\left(E^{D',j}\right) \in \mathrm{HS}_k(\log I; (n+1)p^t)$ where $T_j = -G_j^{(j)}$ and $\ell(T) \geq j$.

Therefore, we can obtain an $((n+1)p^t-1) - I$-logarithmic HS-derivation:

$$G^{(j+1)} := T \circ G^{(j)} = (\mathrm{Id}, 0, \ldots, 0, G_{j+1}^{(j+1)}, \ldots, G_{n+1}^{(j+1)}, \ldots, \binom{n}{p^t} D_n + H') \in \mathrm{HS}_k\left(R; (n+1)p^t\right)$$

where H' is an I-differential operator. So, we can do this process for all j such that $2p^t+1 \leq j < n+1$ and we obtain an $((n+1)p^t-1) - I$-logarithmic HS-derivation:

$$G^{(n+1)} = (\mathrm{Id}, 0, \ldots, 0, G_{n+1}^{(n+1)}, \ldots, \binom{n}{p^t} D_n + H') \in \mathrm{HS}_k\left(R; (n+1)p^t\right)$$

where H' is an I-differential operator. Then, we can apply Theorem 2.4.8 to $G^{(n+1)}$. So, in both cases, when $n+1 = 2p^t+1$ or not, we have that there exists a $(p^t-1) - I$-logarithmic HS-derivation $T \in \mathrm{HS}_k(R; p^t)$ and an I-differential operator H' such that $T = (\mathrm{Id}, T_1, \ldots, \binom{n}{p^t} D_n + H')$.

Let $f \in \mathbb{F}_p^*$ be the inverse of $\binom{n}{p^t}$. So that,

$$D \circ (-f \bullet T)\,[n/p^t] =$$
$$\left(\mathrm{Id}, D_1, \ldots, D_n + (-f)^{p^t}\binom{n}{p^t} D_n - f^{p^t} H' + \sum_{\alpha+\beta=n, \alpha,\beta\neq 0} D_\alpha \circ \left((-f \cdot T)\,[n/p^t]\right)_\beta\right) =$$
$$\left(\mathrm{Id}, D_1, \ldots, \sum_{\alpha+\beta=n,\alpha,\beta\neq 0} D_\alpha \circ \left((-f \cdot T)\,[n/p^t]\right)_\beta - fH'\right) \in \mathrm{HS}_k(\log I; n).$$

Hence, $D_1 = \delta \in \mathrm{IDer}_k(\log I; n)$ and A does not have a leap at n.

□

Chapter 3

On the behavior of integrability under base change

The behavior of the module of k-derivations of a finitely generated k-algebra under base change is well-known. In this chapter, we generalize the base change map for modules of k-derivations to the modules of m-integrable k-derivations for $m \geq 1$.

In this chapter we will use the following notations: Let k be a commutative ring and L a ring extension of k. We denote $R := k[x_1, \ldots, x_d]$ the polynomial ring over k in d variables and, if A is a finitely generated k-algebra, we assume that A is the quotient of R by some ideal I of R. For any k-algebra B, we denote $B_L := L \otimes_k B$.

3.1 A decomposition of logarithmic Hasse-Schmidt derivation in characteristic $p > 0$

Let us consider k a commutative ring of characteristic $p > 0$ (i.e. $\mathbb{F}_p \subseteq k$), $R = k[x_1, \ldots, x_d]$ and $I \subseteq R$ an ideal. In this section we will see that any I-logarithmic HS-derivation with some properties can be decomposed in two HS-derivations.

Notation 3.1.1 *Let $l \geq 1$ be an integer and $D \in \mathrm{HS}_k(R; p^l)$. We define:*

$$\mathcal{J}(l, D) := \{j \in \mathbb{N} \mid \ell(D) \leq j \leq p^l,\ p \nmid j\}.$$

Note that if $E \in \mathrm{HS}_k(R, p^l)$ such that $\ell(E) \leq \ell(D)$, then $\mathcal{J}(l, D) \subseteq \mathcal{J}(l, E)$ and $\mathcal{J}(l, E) \setminus \mathcal{J}(l, D) = \{j \in \mathbb{N} \mid \ell(E) \leq j < \ell(D),\ p \nmid j\}$. For each family $F^j \in \mathrm{HS}_k(R; m)$, $j \in \mathcal{J}(l, D)$, we will write:

$$\circ_{j \in \mathcal{J}(l,D)} F^j = F^{p^l - 1} \circ \cdots \circ F^{\ell(D)}$$

(observe that we have chosen the decreasing ordering) where $F^j = \mathbb{I}$ if $j \notin \mathcal{J}(l, D)$.

The proof of the following lemma is clear.

Lemma 3.1.2 *Let i, l be two positive integers such that $i < p^l$ and i is not a power of p. If we denote $s = \max C^p_{i,1,l}$ (see Definition 2.1.10), then $ip^{s+1} > p^l$.*

Remark 3.1.3 *Thanks to Theorem 2.5.1 and Corollary 1.2.4, we have that R and I satisfies $H^I_{p,a}$ for all $a \in \mathbb{N}$, so we can always apply Theorem 2.4.8 and Corollary 2.4.13.*

Proposition 3.1.4 *Let $l \geq 1$ be an integer and let us denote $s_j = \max C^p_{j,1,l}$ for each integer j with $1 \leq j \leq p^l$. Then, for any $\left(p^l - 1\right) - I$-logarithmic HS-derivation $D \in \mathrm{HS}_k(R; p^l)$ with $\ell(D) > 1$, there exists:*

- *a $\left(p^{l-1} - 1\right) - I$-logarithmic HS-derivation $T \in \mathrm{HS}_k(R; p^{l-1})$,*
- *a $(p^{s_j+1} - 1) - I$-logarithmic HS-derivation $F^j \in \mathrm{HS}_k(R; p^{s_j+1})$, for each $j \in \mathcal{J}(l, D)$, and*
- *an I-differential operator H of order $\leq p^l$*

such that $T_{p^{l-1}} = D_{p^l} + H$ and

$$D = T[p] \circ \left(\circ_{j \in \mathcal{J}(l,D)} \left(\psi^j \bullet F^j\right)\right)$$

where $\psi^j : R[|\mu|]_{p^{s_j+1}} \to R[|\mu|]_{p^l}$ is the substitution map given by $\psi^j(\mu) = \mu^j$.

Proof. First, note that ψ^j is well-defined for all $j \in \mathcal{J}(l, D)$ because $jp^{s_j+1} \geq p^l$ by definition of s_j. Moreover, observe that $\psi^j \bullet E = \tau_{jp^{s_j+1}, p^l}(E[j])$ for any $E \in \mathrm{HS}_k(R; p^{s_j+1})$. If $\ell(D) = \infty$, then $D = \mathbb{I}$, $\mathcal{J}(l, D) = \emptyset$ and we may take $T = \mathbb{I}$ to obtain the result. Let us suppose that $\ell(D)$ is finite, i.e. $1 < \ell(D) \leq p^l$. We proceed by decreasing induction on $\ell(D)$.

Assume that $\ell(D) = p^l$. Then, $\mathcal{J}(l, D) = \emptyset$ and, by Corollary 1.4.8,

$$D = (\mathrm{Id}, \delta)\left[p^l\right] = (\mathrm{Id}, \delta)\left[p^{l-1}\right][p]$$

So, if we put $T := (\mathrm{Id}, \delta)\left[p^{l-1}\right]$, we have the result. Let us suppose that the proposition is true for all HS-derivations such that $\ell(*) > i$ and let us take a $(p^l - 1) - I$-logarithmic HS-derivation $D \in \mathrm{HS}_k(R; p^l)$ with $1 < \ell(D) = i < p^l$. We divide the proof in two cases:

1. *If i is a power of p.*

Let us write $i = p^t$ where $t < l$. Since $\ell(D) > 1$, then $t \geq 1$ and we can see $D \in \mathrm{HS}_k(R; p^t p^{l-t})$. By Theorem 2.4.8, there exists an integral $F \in \mathrm{HS}_k(R; p^{l-t})$ of D_{p^t} and an I-differential operator H of order $\leq p^{l-t}$ such that F is $(p^{l-t} - 1) - I$-logarithmic and $F_{p^{l-t}} = D_{p^l} + H$. Then, by Proposition 1.4.9 and Lemma 1.4.11, b., $F^*[p^t] = (F[p^t])^* \in \mathrm{HS}_k(R; p^l)$ is $(p^l - 1) - I$-logarithmic. Moreover, $(F[p^t])^*_{p^t} = F^*_1 = -D_{p^t}$ and, by Lemma 1.4.11, c.,

$$(F[p^t])^*_{p^l} = F^*_{p^{l-t}} = -D_{p^l} - H + E$$

where E is an I-differential operator of order $\leq p^{l-t}$. We define

$$D' := \left(F\left[p^t\right]\right)^* \circ D.$$

By Lemma 1.1.9, $D'_{p^t} = (F[p^t])^*_{p^t} + D_{p^t} = 0$ so, $\ell(D') > i = p^t$ and, by Lemma 1.1.19, D' is $(p^l - 1) - I$-logarithmic and $D'_{p^l} = F^*_{p^{l-t}} + D_{p^l} + \textit{some } I\textit{-diff. op. of order } \leq p^l = D_{p^l} - D_{p^l} + H' =$

H' where H' is an I-differential operator of order $\leq p^l$. So, $D' \in \mathrm{HS}_k(\log I; p^l)$. We apply the induction hypothesis to D' and we obtain that

$$D' = T'[p] \circ \left(\circ_{j\in\mathcal{J}(l,D')}\left(\psi^j \bullet F^j\right)\right)$$

where $F^j \in \mathrm{HS}_k(R; p^{s_j+1})$ is $(p^{s_j+1}-1) - I$-logarithmic for $j \in \mathcal{J}(l, D')$ and $T' \in \mathrm{HS}_k\left(R; p^{l-1}\right)$ is $\left(p^{l-1}-1\right) - I$-logarithmic with

$$T'_{p^{l-1}} = D'_{p^l} + \textit{some } I\textit{-diff. op. of order} \leq p^l.$$

Since $D' \in \mathrm{HS}_k(\log I; p^l)$, we have that $T' \in \mathrm{HS}_k(\log I; p^{l-1})$. We put $F^j = \mathbb{I} \in \mathrm{HS}_k(\log I; p^{s_j+1})$ for all $j \in \mathcal{J}(l, D) \setminus \mathcal{J}(l, D')$. By Lemma 1.4.8,

$$D = F[p^t] \circ T'[p] \circ \left(\circ_{j\in\mathcal{J}(l,D)}\left(\psi^j \bullet F^j\right)\right) = \left(F\left[p^{t-1}\right] \circ T'\right)[p] \circ \left(\circ_{j\in\mathcal{J}(l,D)}\left(\psi^j \bullet F^j\right)\right).$$

By Lemma 1.1.18, $F\left[p^{t-1}\right]$ is $(p^{l-1}-1) - I$-logarithmic. Moreover, $F[p^{t-1}]_{p^{l-1}} = F_{p^{l-t}} = D_{p^l} + H$ (recall that H is an I-differential operator of order $\leq p^{l-t}$). So, by Corollary 1.1.20, $T := F\left[p^{t-1}\right] \circ T' \in \mathrm{HS}_k(R; p^{l-1})$ is $(p^{l-1}-1) - I$-logarithmic and $T_{p^{l-1}} = F[p^{t-1}]_{p^{l-1}} + \textit{some } I\textit{-diff. op. of order } \leq p^l = D_{p^l} + \textit{some } I\textit{-diff. op. of order } \leq p^l$ and we have the proposition in this case.

2. *If i is not a power of p.*

Since i is not a power of p, by Lemma 3.1.2, $ip^{s_i+1} > p^l$ where $s_i = \max C^p_{i,1,l}$. Then, we can consider $\tau_{p^l,ip^{s_i}}(D) \in \mathrm{HS}_k(\log I; ip^{s_i})$. If $s_i \geq 1$, then D_i is I-logarithmically p^{s_i}-integrable by Corollary 2.4.13. If $s_i = 0$, then $D_i \in \mathrm{Der}_k(\log I)$. In both cases, since leaps only occur at powers of p (Theorem 2.5.1 and Corollary 1.2.4), we have that D_i is I-logarithmically $(p^{s_i+1}-1)$-integrable. Thanks to Proposition 1.2.1, we can integrate any I-logarithmic $(p^{s_i+1}-1)$-integral of D_i so, there exists $F \in \mathrm{HS}_k(R; p^{s_i+1})$ a $(p^{s_i+1}-1) - I$-logarithmic integral of D_i. Then, by Lemma 1.4.11, b., $F^*[i] \in \mathrm{HS}_k(R; ip^{s_i+1})$ is $(ip^{s_i+1}-1) - I$-logarithmic. By Proposition 1.4.9, $\psi^i \bullet F^* = (\psi^i \bullet F)^* \in \mathrm{HS}_k(\log I; p^l)$ and $(\psi^i \bullet F)^*_i = F[i]^*_i = -D_i$.

a. If $i \not\equiv 0 \mod p$, by Corollary 1.1.20, and Lemma 1.1.9, $D' := D \circ (\psi^i \bullet F)^*$ is $(p^l-1) - I$-logarithmic with $\ell(D') > i$ and $D'_{p^l} = D_{p^l} + H$ with H an I-differential operator of order $\leq p^l$. We apply the induction hypothesis to D' and we obtain that

$$D' = T[p] \circ \left(\circ_{j\in\mathcal{J}(l,D')}\left(\psi^j \bullet F^j\right)\right) \Rightarrow D = T[p] \circ \left(\circ_{j\in\mathcal{J}(l,D')}\left(\psi^j \bullet F^j\right)\right) \circ (\psi^i \bullet F)$$

where $T \in \mathrm{HS}_k(R; p^{l-1})$ is $(p^{l-1}-1) - I$-logarithmic with $T_{p^{l-1}} = D'_{p^l} + \textit{some } I\textit{-diff. op. of order} \leq p^l = D_{p^l} + H'$ where H' is an I-differential operator of order $\leq p^l$. Then, we put $F^i = F \in \mathrm{HS}_k(R; p^{s_i+1})$ and $F^j = \mathbb{I} \in \mathrm{HS}_k(\log I; p^{s_j+1})$ for $j \in \mathcal{J}(l, D) \setminus (\mathcal{J}(l, D') \cup \{i\})$ and we have the result.

b. If i is a multiple of p, by Lemmas 1.1.19 and 1.1.9, $D' := (\psi^i \bullet F)^* \circ D$ is $(p^l-1) - I$-logarithmic with $\ell(D') > i$ and $D'_{p^l} = D_{p^l} + H$ where H is an I-differential operator of order $\leq p^l$. Then, we apply the induction hypothesis to D' and we have that

$$D = \left(\psi^i \bullet F\right) \circ T'[p] \circ \left(\circ_{j\in\mathcal{J}(l,D')}\left(\psi^j \bullet F^j\right)\right)$$

where $T' \in \mathrm{HS}_k(R; p^{l-1})$ is $(p^{l-1}-1) - I$-logarithmic with $T'_{p^{l-1}} = D'_{p^l} + \textit{some } I\textit{-diff. op. of order} \leq p^l = D_{p^l} + H'$ where H' is an I-differential operator of order $\leq p^l$. We put $F^j = \mathbb{I}$ for all $j \in \mathcal{J}(l, D) \setminus \mathcal{J}(l, D')$. On the other hand, by Corollary 1.4.8 and Lemma 1.1.6,

$$\psi^i \bullet F = \tau_{ip^{s_i+1},p^l}(F[i]) = \tau_{ip^{s_i+1},p^l}(F[i/p][p]) = \tau_{ip^{s_i},p^{l-1}}(F[i/p])[p].$$

Since F is $(p^{s_i+1}-1) - I$-logarithmic, $F[i/p]$ is $(ip^{s_i}-1) - I$-logarithmic by Lemma 1.1.18 and, since $ip^{s_i} > p^{l-1}$, $\tau_{ip_i^s,p^{l-1}}(F[i/p]) \in \mathrm{HS}_k(\log I, p^{l-1})$. By Corollary 1.1.20, $T := \tau_{ip^{s_i},p^{l-1}}(F[i/p]) \circ T'$ is $(p^{l-1}-1) - I$-logarithmic and $T_{p^{l-1}} = T'_{p^{l-1}} + H' + \textit{some } I\textit{-diff. op. of order} \leq p^l = D_{p^l} + H''$ where H'' is an I-differential operator of order $\leq p^l$. Since $D = T[p] \circ \left(\circ_{j\in\mathcal{J}(l,D)} \left(\psi^j \bullet F^j\right)\right)$, we have the proposition.

□

Corollary 3.1.5 *Let $l \geq 1$ be an integer and $D \in \mathrm{HS}_k(R; p^l)$ a $(p^l-1) - I$-logarithmic HS-derivation with $\ell(D) > 1$. Then, there exists $F \in \mathrm{HS}_k(\log I; p^l)$ with $\ell(F) > 1$ and a $(p^{l-1}-1) - I$-logarithmic HS-derivation $T \in \mathrm{HS}_k(R; p^{l-1})$ such that $D = T[p] \circ F$.*

Proof. From Proposition 3.1.4, we have that

$$D = T[p] \circ \left(\circ_{j\in\mathcal{J}(l,D)} \left(\psi^j \bullet F^i\right)\right)$$

for some $(p^{l-1}-1) - I$-logarithmic HS-derivation $T \in \mathrm{HS}_k(R; p^{l-1})$ and some $(p^{s_j+1}-1) - I$-logarithmic HS-derivation $F^j \in \mathrm{HS}_k(R; p^{s_j+1})$, for $j \in \mathcal{J}(l, D)$ and $s_j = \max C^p_{j,1,l}$. Since $\psi^j \bullet F^j = \tau_{jp^{s_j+1},p^l}(F^j[j])$ and $F^j[j]$ is $(jp^{s_j+1}-1) - I$-logarithmic by Lemma 1.1.18, we have that $\psi^j \bullet F^j \in \mathrm{HS}_k(\log I; p^l)$ because $j \not\equiv 0 \mod p$ and, by Lemma 3.1.2, $jp^{s_j+1} > p^l$. Hence, $F := \circ_{j\in\mathcal{J}(l,D)} \left(\psi^j \bullet F^j\right) \in \mathrm{HS}_k(\log I; p^l)$. Moreover, $\ell(F^j[j]) > 1$ for all $j \in \mathcal{J}(l, D)$, so $\ell(\psi^j \bullet F^j) > 1$ and $\ell(F) > 1$ by Lemma 1.1.9.

□

3.2 Base change

Let k be a commutative ring, $k \to L$ a ring extension, $R = k[x_1, \ldots, x_d]$ the polynomial ring and $A = R/I$ a commutative finitely generated k-algebra. Recall that we denote $B_L := L \otimes_k B$ for any k-algebra B. In this section, we study the relationship between $\mathrm{IDer}_k(A; m)$ and $\mathrm{IDer}_L(A_L; m)$ under suitable hypotheses on the ring extension $k \to L$. We start recalling some classical results on derivations.

3.2.1 Base change for derivations

Let k be a commutative ring, $k \to L$ a ring extension and A a commutative k-algebra. For each k-derivation $\delta : A \to A$, let us denote by $\widetilde{\delta} : A_L \to A_L$ the natural L-linear extension given by $\widetilde{\delta}(c \otimes a) = c \otimes \delta(a)$ for all $c \in L$ and all $a \in A$. It is clear that $\widetilde{\delta} \in \mathrm{Der}_L(A_L)$. The map $\delta \in \mathrm{Der}_k(A) \mapsto \widetilde{\delta} \in \mathrm{Der}_L(A_L)$, being A-linear, gives rise to an A_L-linear base change map:

$$\begin{array}{rccc} \Phi_1^{L,A}: & L \otimes_k \mathrm{Der}_k(A) = A_L \otimes_A \mathrm{Der}_k(A) & \longrightarrow & \mathrm{Der}_L(A_L) \\ & c \otimes \delta & \longmapsto & c\widetilde{\delta} \end{array}$$

If $R = k[x_1, \ldots, x_d]$ the polynomial ring, then $R_L = L[x_1, \ldots, x_d]$ is also a polynomial ring. It is easy to see that, if $\partial_i : R \to R$ is the partial derivative of R with respect to x_i, then $\widetilde{\partial}_i : R_L \to R_L$ is the partial derivative of R_L with respect to x_i. Since the modules of derivations of a polynomial ring in a finite number of variable is free with basis the partial derivatives, if $\delta = \sum_{i=1}^d b_i \partial_i$ then $\widetilde{\delta} = \sum_{i=1}^d b_i \widetilde{\partial}_i$. Hence, we can deduce the following result.

Lemma 3.2.1 *Let $R = k[x_1, \ldots, x_d]$ be the polynomial ring and $k \to L$ a ring extension. Then, $\Phi_1^{L,R}$ is an R_L-module isomorphism.*

Moreover, if L is free over k (as k-module), then any k-basis of L is an R-basis of R_L and we have the following lemma.

Lemma 3.2.2 *Let $k \to L$ be a free ring extension and $\mathcal{B} = \{a_i,\ i \in \mathcal{I}\}$ a k-basis of L. Let us consider $\delta \in \mathrm{Der}_L(R_L)$. Then, there exists a finite subset $\mathcal{J} \subseteq \mathcal{I}$ and a k-derivation $\delta_j \in \mathrm{Der}_k(R)$ for each $j \in \mathcal{J}$ such that $\delta = \sum_{j \in \mathcal{J}} a_j \widetilde{\delta}_j$.*

Proof. Let us consider $\delta = \sum_{i=1}^d b_i \widetilde{\partial}_i \in \mathrm{Der}_L(R_L)$ where $b_i \in R_L$. Since $\mathcal{B}$ is an R-basis of R_L, there exists a finite subset $\mathcal{J} \subseteq \mathcal{I}$ and unique elements $b_{ij} \in R$, $1 \le i \le d$ and $j \in \mathcal{J}$, such that $b_i = \sum_{j \in \mathcal{J}} a_j b_{ij}$. Hence, we have

$$\delta = \sum_{i=1}^d b_i \widetilde{\partial}_i = \sum_{i=1}^d \sum_{j \in \mathcal{J}} a_j b_{ij} \widetilde{\partial}_i = \sum_{j \in \mathcal{J}} a_j \widetilde{\delta}_j, \text{ with } \delta_j = \left(\sum_{i=1}^d b_{ij} \partial_i \right) \in \mathrm{Der}_k(R)$$

and the lemma is proved.

□

We denote $I^e = IR_L = IL[x_1, \ldots, x_d]$ the extended ideal of I in R_L. It is clear that $\Phi_1^{L,R}$ induce two R_L-module homomorphisms:

$$\Phi_1^{L,R,I} : L \otimes \mathrm{Der}_k(\log I) \longrightarrow \mathrm{Der}_k(\log I^e)$$

and

$$\Phi_{\mathrm{ind},I}^{L,R} : L \otimes_k I(\mathrm{Der}_k(R)) \to I^e\, \mathrm{Der}_L(R_L).$$

Lemma 3.2.3 *Let $k \to L$ be a ring extension and $I \subseteq R$ an ideal. We have the following properties:*

a. *$\Phi_{\mathrm{ind},I}^{L,R}$ is surjective.*

b. *If L is flat over k, then $\Phi_{\mathrm{ind},I}^{L,R}$ is bijective and $\Phi_1^{L,R,I}$ is injective.*

Proof.

a. Let $\delta \in I^e \operatorname{Der}_L(R_L) \subseteq \operatorname{Der}_L(R_L)$. Then $\delta = \sum_{i=1}^d b_i \widetilde{\partial}_i$ where $b_i \in I^e$ for all $i = 1, \ldots, d$. Since $b_i \in I^e = IL[x_1, \ldots, x_d]$, there is a finite set $\mathcal{J}$ and elements $h_{ij} \in I$ and $l_{ij} \in L$, $1 \leq i \leq d$ and $j \in \mathcal{J}$, such that $b_i = \sum_{j \in \mathcal{J}} l_{ij} h_{ij}$. Then,

$$\delta = \sum_{i=1}^{j} \sum_{j \in \mathcal{J}} l_{ij} h_{ij} \widetilde{\partial}_i = \sum_{j \in \mathcal{J}} l_{ij} \widetilde{\delta}_j \quad \text{with} \quad \delta_j = \sum_{i=1}^{d} h_{ij} \partial_i \in I(\operatorname{Der}_k(R)).$$

Hence, we can deduce that $\Phi^{L,R}_{\mathrm{ind},I}$ is surjective.

b. $\Phi^{L,R}_{\mathrm{ind},I}$ is always surjective thanks to previous point. Since L is flat over k and $\Phi^{L,R}_1$ is bijective, then it is clear that $\Phi^{L,R}_{\mathrm{ind},I}$ and $\Phi^{L,R,I}_1$ are both injective.

□

Let A be a finitely generated k-algebra, i.e. $A = R/I$ where $R = k[x_1, \ldots, x_d]$ and $I \subseteq R$ an ideal. Then, with the previous notation, we have the following commutative diagram:

$$\begin{tikzcd}
 & L \otimes_k (I(\operatorname{Der}_k(R))) \arrow[r] \arrow[d, "\Phi^{L,R}_{\mathrm{ind},I}"] & L \otimes_k \operatorname{Der}_k(\log I) \arrow[r] \arrow[d, "\Phi^{L,R,I}_1"] & L \otimes_k \operatorname{Der}_k(A) \arrow[r] \arrow[d, "\Phi^{L,A}_1"] & 0 \\
0 \arrow[r] & I^e \operatorname{Der}_L(R_L) \arrow[r] & \operatorname{Der}_L(\log I^e) \arrow[r] & \operatorname{Der}_L(A_L) \arrow[r] & 0.
\end{tikzcd} \tag{3.1}$$

From Proposition 1.2.6, this diagram has exact rows and if L is flat over k, then the top row is also left exact.

The proof of the following proposition follows from the diagram (3.1), the previous lemma and (cf. [Gro, **Prop. 16.5.11**]).

Proposition 3.2.4 *Under the above hypotheses, if $k \to L$ is a flat ring extension, then the following properties are equivalent:*

a. The map $\Phi^{L,R,I}_1 : L \otimes_k \operatorname{Der}_k(\log I) \to \operatorname{Der}_L(\log I^e)$ is an isomorphism.

b. The map $\Phi^{L,A}_1 : L \otimes_k \operatorname{Der}_k(A) \to \operatorname{Der}_L(A_L)$ is an isomorphism.

Moreover, both properties hold if I is finitely generated (i.e. if A is finitely presented over k).

We also have the following result for any commutative finitely generated k-algebra $A = R/I$, with $I \subseteq R$ an ideal.

Proposition 3.2.5 *Under the above hypotheses, if $k \to L$ is a free ring extension (L is a free k-module) and $A = R/I$ is a finitely generated k-algebra, then properties a. and b. in Proposition 3.2.4 hold.*

Proof. Since L is a flat extension of k, after Proposition 3.2.4 we only need to prove that the map $\Phi_1^{L,R,I} : L \otimes_k \mathrm{Der}_k(\log I) \to \mathrm{Der}_L(\log I^e)$ is surjective. Let $\mathcal{B} = \{a_i, i \in \mathcal{I}\}$ be a k-basis of L and $\delta : R_L \to R_L$ an I^e-logarithmic derivation. By Lemma 3.2.2, there exists a finite subset $\mathcal{J}$ of $\mathcal{I}$ and $\delta_j \in \mathrm{Der}_k(R)$ for each $j \in \mathcal{J}$ such that

$$\delta = \sum_{j \in \mathcal{J}} a_j \widetilde{\delta}_j.$$

Let us consider $h \in I$. Since $\delta \in \mathrm{Der}_k(\log I^e)$, we have that $\delta(h) \in I^e$. Hence, there is a subset $\mathcal{I}_0$ of $\mathcal{I}$ and $g_l \in I$ for all $l \in \mathcal{I}_0$ such that

$$\delta(h) = \sum_{j \in \mathcal{J}} a_j \delta_j(h) = \sum_{l \in \mathcal{I}_0} a_l g_l.$$

Then, $\delta_j(h) = g_j \in I$ if $j \in \mathcal{I}_0$ and $\delta_j(h) = 0$ otherwise. Therefore, $\delta_j \in \mathrm{Der}_k(\log I)$ and, since $\delta = \Phi_1^{L,R,I}\left(\sum_{j \in \mathcal{J}} (a_j \otimes \delta_j)\right)$, $\Phi_1^{L,R,I}$ is surjective.

□

3.2.2 Base change for integrable derivations

Let $k \to L$ be a ring extension and A a k-algebra. In the previous section, we recalled the base change map $\Phi_1^{L,A} : L \otimes \mathrm{Der}_k(A) \to \mathrm{Der}_L(A_L)$. In this section we want to generalize this map to the modules of m-integrable derivations for all $m \in \overline{\mathbb{N}}$. To do this, we will start extending HS-derivations of A over k to HS-derivations of A_L over L.

Proposition 3.2.6 *Let A be a k-algebra, $I \subseteq A$ an ideal, $k \to L$ a ring extension, $I^e = IA_L$, the extended ideal and $m \in \overline{\mathbb{N}}$. For any HS-derivation $D \in \mathrm{HS}_k(A; m)$, there exists a unique HS-derivation $\widetilde{D} \in \mathrm{HS}_L(A_L; m)$ such that the following diagram is commutative:*

$$\begin{tikzcd} A \arrow[r, "\varphi_D"] \arrow[d, "\text{nat.}"'] & A[|\mu|]_m \arrow[d, "\text{nat.}"] \\ A_L \arrow[r, "\varphi_{\widetilde{D}}"] & A_L[|\mu|]_m. \end{tikzcd} \tag{3.2}$$

Moreover, if D is I-logarithmic, then $\widetilde{D}$ is I^e-logarithmic.

Proof. Let us denote $t : A \to A_L$ and $t_m : A[|\mu|]_m \to A_L[|\mu|]_m$ the natural maps. Then, we define $\varphi_{\widetilde{D}}(c \otimes a) = ct_m(\varphi_D(a))$ for all $c \in L$ and all $a \in A$. Observe that this map is well-defined. Moreover, since t_m and φ_D are k-algebra homomorphisms and it is clear that $\varphi_{\widetilde{D}}$ is L-linear, $\varphi_{\widetilde{D}}$ is an L-algebra homomorphism. In order for $\varphi_{\widetilde{D}}$ to be a HS-derivation, $\varphi_{\widetilde{D}} \equiv \mathrm{Id} \mod \mu$, and this property is obtained thanks to φ_D is a HS-derivation because $\varphi_{\widetilde{D}}(c \otimes a) = ct_m(\varphi_D(a)) = ct(a) \mod \mu = c \otimes a \mod \mu$ for all $c \in L$ and all $a \in A$.

Let us consider $\varphi_E : A_L \to A_L[|\mu|]$ another L-algebra homomorphism such that the diagram (3.2) commutes. Then, since φ_E is L-linear, for all $c \in L$ and all $a \in A$,

$$\varphi_E(c \otimes a) = c\varphi_E(1 \otimes a) = c\varphi_E(t(a)) = ct_m(\varphi_D(a)) = \varphi_{\widetilde{D}}(c \otimes a).$$

Hence, $\widetilde{D}$ is unique. Now, we assume that D is I-logarithmic. Let us consider $g \in I$. Then, $\varphi_{\widetilde{D}}(1 \otimes g) = t_m(\varphi_D(g)) \subseteq I^e$ so, for all $c \in L$ and all $a \in A$, $\varphi_{\widetilde{D}}(g(c \otimes a)) = \varphi_{\widetilde{D}}(1 \otimes g)\varphi_{\widetilde{D}}(c \otimes a) \in I^e$, i.e. $\widetilde{D}$ is I^e-logarithmic.

□

Observe that if $m = 1$, we know that $\mathrm{Der}_k(A) \cong \mathrm{HS}_k(A;1)$ and the extension process $D \mapsto \widetilde{D}$ described in Proposition 3.2.6 coincides with the usual extension $\delta \mapsto \widetilde{\delta}$ of derivations.

Remark 3.2.7 *If $k \to L$ is a free ring extension, then this map is injective. In this case, if $R = k[x_1, \ldots, x_d]$, $R_L = L[x_1, \ldots, x_d]$, $D \in \mathrm{HS}_k(R;m)$ and $\widetilde{D} = (\widetilde{D}_i)_i \in \mathrm{HS}_L(R_L;m)$ is the extension of D, then $\widetilde{D}_{i|R} = D_i$ because $R \to R_L$ can be seen as an inclusion.*

Lemma 3.2.8 *Let A be a k-algebra, $I \subset A$ an ideal, $k \to L$ a ring extension, $m \in \overline{\mathbb{N}}$, $n \leq m$ an integer, $D \in \mathrm{HS}_k(A;m)$ a HS-derivation and $\psi : A[|\mu|]_m \to A[|\mu|]_n$ a substitution map. The following properties hold:*

- *a. The map $D \in \mathrm{HS}_k(A;m) \mapsto \widetilde{D} \in \mathrm{HS}_L(A_L;m)$ is a group homomorphism.*
- *b. $\widetilde{\psi \bullet D} = \widetilde{\psi} \bullet \widetilde{D}$, where $\widetilde{\psi} : A_L[|\mu|]_m \to A_L[|\mu|]_n$ is the substitution map induced by ψ.*
- *c. If D is $n - I$-logarithmic, then $\widetilde{D}$ is $n - I^e$-logarithmic.*

Proof.

a. Let us consider $D, E \in \mathrm{HS}_k(A;m)$. Then, $D \circ E$ is the HS-derivation associated with $\varphi^\mu_D \circ \varphi_E$ (remember that $\varphi^\mu_D : A[|\mu|]_m \to A[|\mu|]_m$ is the unique k-algebra automorphism which extend φ_D and $\varphi^\mu_D(\mu) = \mu$). Let us consider $\widetilde{D}, \widetilde{E} \in \mathrm{HS}_L(A_L;m)$ defined in Proposition 3.2.6, $\varphi^\mu_{\widetilde{D}}$ the L-algebra automorphism of $\widetilde{D}$ and $\varphi_{\widetilde{E}}$ the L-algebra homomorphism of $\widetilde{E}$. Then, we have the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{\varphi_E} & A[|\mu|]_m & \xrightarrow{\varphi^\mu_D} & A[|\mu|]_m \\ \downarrow{\scriptstyle \text{nat.}} & & \downarrow{\scriptstyle \text{nat.}} & & \downarrow{\scriptstyle \text{nat.}} \\ A_L & \xrightarrow{\varphi_{\widetilde{E}}} & A_L[|\mu|]_m & \xrightarrow{\varphi^\mu_{\widetilde{D}}} & A_L[|\mu|]_m \end{array}$$

It is easy to see that the external square is commutative. By Proposition 3.2.6, $\varphi_{\widetilde{D \circ E}}$ is the unique L-algebra homomorphism with that property so, $\varphi_{\widetilde{D \circ E}} = \varphi^\mu_{\widetilde{D}} \circ \varphi_{\widetilde{E}}$. From the definition of composition of two HS-derivation, we can deduce that $\widetilde{D \circ E} = \widetilde{D} \circ \widetilde{E}$.

b. Let us consider φ_D the k-algebra homomorphism of $D \in \mathrm{HS}_k(A;m)$. Then, $\psi \bullet D$ is the HS-derivation associated with $\psi \circ \varphi_D$. It is easy to prove that the following diagram is commutative.

$$\begin{array}{ccccc} A & \xrightarrow{\varphi_D} & A[|\mu|]_m & \xrightarrow{\psi} & A[|\mu|]_n \\ \downarrow{\scriptstyle \text{nat.}} & & \downarrow{\scriptstyle \text{nat.}} & & \downarrow{\scriptstyle \text{nat.}} \\ A_L & \xrightarrow{\varphi_{\widetilde{D}}} & A_L[|\mu|]_m & \xrightarrow{\widetilde{\psi}} & A_L[|\mu|]_n \end{array}$$

As in the previous point, $\varphi_{\widetilde{\psi \bullet D}}$ is the unique L-algebra homomorphism such that the external square is commutative, so $\varphi_{\widetilde{\psi \bullet D}} = \widetilde{\psi} \circ \varphi_{\widetilde{D}}$ i.e. $\widetilde{\psi \bullet D} = \widetilde{\psi} \bullet \widetilde{D}$.

c. If D is $n - I$-logarithmic, $\tau_{mn}(D) \in \mathrm{HS}_k(\log I; n)$, so $\widetilde{\tau_{mn}(D)} \in \mathrm{HS}_k(\log I^e; n)$ by Proposition 3.2.6. From the previous point, $\widetilde{\tau_{mn}(D)} = \tau_{mn}(\widetilde{D})$. So, $\widetilde{D} \in \mathrm{HS}_L(A_L; m)$ is $n - I^e$-logarithmic.

□

Lemma 3.2.9 *Let $I \subseteq A$ be an ideal, $B = A/I$ and $I^e = IA_L$ the extended ideal. Then, for each $D \in \mathrm{HS}_k(\log I; m)$,*

$$\widetilde{\Pi^I_{\mathrm{HS},m}(D)} = \Pi^{I^e}_{\mathrm{HS},m}\left(\widetilde{D}\right)$$

(observe that $\widetilde{\Pi^I_{\mathrm{HS},m}(D)}$ is the extension of $\Pi^I_{\mathrm{HS},m}(D) \in \mathrm{HS}_k(B; m)$ to $B_L = A_L/I^e$ and $\widetilde{D} \in \mathrm{HS}_L(\log I^e; m) \subseteq \mathrm{HS}_L(A_L; m)$).

Proof. From the proof of Proposition 3.2.6, for all $c \in L$ and all $a \in A$,

$$\varphi_{\widetilde{D}}(c \otimes a) = \sum_{i=0}^{m} (c \otimes D_i(a))\mu^i = \sum_{i=0} (\mathrm{Id}_L \otimes D_i)(c \otimes a)\mu^i.$$

So, $\widetilde{D} = (\mathrm{Id}_L \otimes D_i)_{i=0}^m$. Then,

$\Pi^{I^e}_{\mathrm{HS},m}\left(\widetilde{D}\right) = \left(\overline{\widetilde{D}}_i\right)_i$ where $\overline{\widetilde{D}}_i((c \otimes a) + I^e) = (c \otimes D_i(a)) + I^e$ for all $(c \otimes a) + I^e \in B_L = A_L/I^e$.

To prove this result it is enough to show that the following diagram is commutative.

$$\begin{tikzcd} B \arrow[r, "\varphi_{\overline{D}}"] \arrow[d, "t^B"'] & A[|\mu|]_m \arrow[d, "t^B_m"] \\ B_L \arrow[r, "\varphi_{\Pi^{I^e}_{\mathrm{HS},m}(\widetilde{D})}"] & B_L[|\mu|]_m \end{tikzcd}$$

where $\overline{D} = \Pi^I_{\mathrm{HS},m}(D)$, $t^B(a + I) = (1 \otimes a) + I^e$ for all $a \in A$ and $t^B_m : B[|\mu|]_m \to B_L[|\mu|]_m$ is the map induced by t^B. Let us consider $a \in A$, then

$$\varphi_{\Pi^{I^e}_{\mathrm{HS},m}(\widetilde{D})} \circ t^B(a + I) = \sum_{i=0}^{m} \overline{\widetilde{D}}_i((1 \otimes a) + I^e)\mu^i = \sum_{i=0}^{m} ((1 \otimes D_i(a)) + I^e)\,\mu^i.$$

On the other hand,

$$t^B_m \circ \varphi_{\overline{D}}(a + I) = t^B_m\left(\left(\sum_{i=0}^{m} D_i(a) + I\right)\mu^i\right) = \sum_{i=0}^{m} ((1 \otimes D_i(a)) + I^e)\,\mu^i.$$

Therefore, the lemma is proved.

□

Corollary 3.2.10 *Under the hypotheses of Lemma 3.2.8, let $\delta : A \to A$ be a k-derivation (resp. an I-logarithmic k-derivation). If δ is m-integrable (resp. I-logarithmically m-integrable), then $\widetilde{\delta}$ is also m-integrable (resp. I^e-logarithmically m-integrable).*

Proof. Let us suppose that $\delta \in \operatorname{IDer}_k(A;m)$ and let us consider an m-integral $D \in \operatorname{HS}_k(A;m)$ of δ, i.e. $D_1 = \delta$. From Proposition 3.2.6, $\widetilde{D} \in \operatorname{HS}_L(A_L;m)$ is an m-integral of $\widetilde{D}_1 = \widetilde{\delta}$, i.e. $\widetilde{\delta} \in \operatorname{IDer}_k(A;m)$. Moreover, if $\delta \in \operatorname{IDer}_k(\log I;m)$, then we can consider $D \in \operatorname{HS}_k(\log I;m)$ and, by Proposition 3.2.6, $\widetilde{D} \in \operatorname{HS}_L(\log I^e;m)$. Hence, $\widetilde{\delta} \in \operatorname{IDer}_L(\log I^e;m)$.

□

In view of the proof of this result, if $D \in \operatorname{HS}_k(A;m)$ is an m-integral of $\delta \in \operatorname{Der}_k(A)$, then $\widetilde{D} \in \operatorname{HS}_L(A_L;m)$ is an m-integral of $\widetilde{\delta}$.

As a consequence of the above corollary, base change map $\Phi_1^{L,A} : L \otimes_k \operatorname{Der}_k(A) \to \operatorname{Der}_L(A_L)$ induce, for each $m \in \overline{\mathbb{N}}$, new A_L-linear base change maps:

$$\Phi_m^{L,A} : L \otimes_k \operatorname{IDer}_k(A;m) \longrightarrow \operatorname{IDer}_L(A_L;m), \quad \Phi_m^{L,A,I} : L \otimes_k \operatorname{IDer}_k(\log I;m) \to \operatorname{IDer}_L(\log I^e;m).$$

From now on, we assume that L is flat over k and A a finitely generated k-algebra. Then, we can put $A = R/I$ where $R = k[x_1, \ldots, x_d]$ is a polynomial ring and $I \subset R$ an ideal.

From the exact sequence in Proposition 1.2.6, we obtain for each $m \in \overline{\mathbb{N}}$ a commutative diagram with exact rows (compare with (3.1)):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L \otimes_k (I(\operatorname{Der}_k(R))) & \longrightarrow & L \otimes_k \operatorname{IDer}_k(\log I;m) & \xrightarrow{\operatorname{Id}\otimes\Pi_m^I} & L \otimes_k \operatorname{IDer}_k(A;m) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi_{\mathrm{ind},I}^{L,R}} & & \downarrow{\scriptstyle \Phi_m^{L,R,I}} & & \downarrow{\scriptstyle \Phi_m^{L,A}} & & \\
0 & \longrightarrow & I^e \operatorname{Der}_L(R_L) & \longrightarrow & \operatorname{IDer}_L(\log I^e;m) & \xrightarrow{\Pi_m^{I^e}} & \operatorname{IDer}_L(A_L;m) & \longrightarrow & 0.
\end{array} \tag{3.3}$$

Moreover the left vertical arrow is bijective (see Lemma 3.2.3) and, since L is flat over k, the middle vertical arrow is injective.

The proof of the following lemma is clear.

Lemma 3.2.11 *Under the above hypotheses, the following properties hold:*

1. *$\Phi_m^{L,A}$ is injective.*
2. *$\Phi_m^{L,R,I}$ is surjective if and only if $\Phi_m^{L,A}$ is surjective.*

Remark 3.2.12 *If k is a ring of characteristic 0 and L is free over k, then $\Phi_m^{L,A}$ is bijective for any finitely generated k-algebra thanks to Proposition 3.2.5 and equality $\operatorname{IDer}_k(A) = \operatorname{Der}_k(A)$.*

Moreover, we have the following result about leaps.

Lemma 3.2.13 *Assume that L is faithfully flat over k and A a finitely generated k-algebra. If $\Phi_m^{L,A}$ is surjective for all $m \geq 1$ then,*

$$\operatorname{Leaps}_k(A) = \operatorname{Leaps}_L(A_L).$$

Proof. Since L is flat over k, we have that $\Phi_m^{L,A}$ is bijective so, $\operatorname{IDer}_L(A_L;m) = \operatorname{IDer}_L(A_L;m-1)$ if and only if

$$\operatorname{IDer}_L(A_L;m-1)/\operatorname{IDer}_L(A_L;m) = 0 \Leftrightarrow L\otimes(\operatorname{IDer}_k(A;m-1)/\operatorname{IDer}_k(A;m)) = 0.$$

Since L is faithfully flat over k, the last equality holds if and only if $\operatorname{IDer}_k(A;m-1)/\operatorname{IDer}_k(A;m) = 0$ and we have the result.

$\square$

In the rest of this section we will study the surjectivity of $\Phi_m^{L,A}$.

3.2.2.1 Algebraic non-separable extensions

In this section we prove that $\Phi_m^{L,A}$ is not surjective in general giving an example and we could deduce that $\Phi_m^{L,A}$ is not surjective when $k\to L$ is a non-separable algebraic field extension.

Counterexample 3.2.14 *Let $k=\mathbb{F}_2(s,t)$ be the quotient field of $\mathbb{F}_2[s,t]$ and $L=\overline{k}$ the perfect closure of k. Let us consider the irreducible polynomial $h=x^2+y^2+tx^4+sy^4\in k[x,y]$ and we denote $A:=k[x,y]/\langle h\rangle$. Then, $\Phi_4^{L,A}$ is not surjective.*

Proof. We need to calculate the 4-integrable derivations of A (resp. A_L) over k (resp. over L). We will follow the same step of Example 7 of [Ma1]. Let us suppose that there exists $\delta\in\operatorname{IDer}_k(A;m)$ and $D\in\operatorname{HS}_k(A;m)$ an integral of δ. Let us consider

$$\begin{array}{rccl} \varphi_D: & A & \longrightarrow & A[|\mu|] \\ & x & \longmapsto & x+u_1\mu+u_2\mu^2+\cdots \\ & y & \longmapsto & y+v_1\mu+v_2\mu^2+\cdots \end{array}$$

where $u_i,v_i\in A$. To φ_D be well-defined, $\varphi_D(h)=0$, i.e.

$$(x+u_1\mu+u_2\mu^2+\cdots)^2+(y+v_1\mu+v_2\mu^2+\cdots)^2+t(x+u_1\mu+u_2\mu^2+\cdots)^4+s(y+v_1\mu+v_2\mu^2+\cdots)^4=0$$

The coefficient of μ^2 in the previous equation is $u_1^2+v_1^2=(u_1+v_1)^2=0$. Since A is a domain, $u_1=v_1$. Let us consider the coefficient of μ^4, then $u_2^2+v_2^2+tu_1^4+sv_1^4=0$. We can write $w=u_2+v_2$ and $u=u_1=v_1$, and we obtain the equation:

$$w^2+(t+s)u^4=0.$$

Let W and U be elements of $k[x,y]$ such that $W+\langle h\rangle=w$ and $U+\langle h\rangle=u$. Then, thanks to the previous equation:

$$W^2+(t+s)U^4=hG \tag{3.4}$$

for some $G\in k[x,y]$. Let ∂_s and ∂_t be the derivations that extend the partial derivations with respect to s and t, respectively, in $\mathbb{F}_2[s,t,x,y]$ to $k[x,y]$ and we apply those derivations to (3.4), obtaining:

$$\begin{array}{rl} \partial_t: & U^4=x^4G+h\partial_t(G) \\ \partial_s: & U^4=y^4G+h\partial_s(G). \end{array}$$

Then, if $g := G + \langle h\rangle$, we have the following equalities in A:

$$\begin{array}{ll}\partial_t : & u^4 = x^4 g \\ \partial_s : & u^4 = y^4 g\end{array} \Rightarrow (x^4 - y^4)g = 0.$$

Since A is a domain and $x^4 \neq y^4$, $g = 0$, so $u = u_1 = v_1 = 0$. Then, we can not integrate any non-zero derivation until length 4, i.e. $\mathrm{IDer}_k(A;4) = 0$ and $L \otimes_k \mathrm{IDer}_k(A;4) = 0$.

To prove that $\mathrm{IDer}_L(A_L;4)$ is not zero, we calculate $\mathrm{IDer}_L(\log\langle h\rangle^e;4)$. Thanks to Proposition 1.2.15, it is enough to calculate $\mathrm{IDer}_L(\log H;2)$ where $H = x + y + t^{1/2}x^2 + s^{1/2}y^2$. Note that $J^0 = \langle 1\rangle$ so, by Proposition 1.2.7, any I-logarithmic k-derivation is integrable. It is easy to see that $\mathrm{Der}_L(\log H) = \langle \widetilde{\partial}_x + \widetilde{\partial}_y, H\widetilde{\partial}_x\rangle$. Hence, thanks to Corollary 1.2.3, $\mathrm{IDer}_L(A_L;4) = \langle \overline{\delta}_1, \overline{\delta}_2\rangle \neq 0$ where $\overline{\delta}_1$ (resp. $\overline{\delta}_2$) is the derivation induced by $\widetilde{\partial}_x + \widetilde{\partial}_y$ (resp. $H\widetilde{\partial}_x$) in the quotient. Therefore, $\Phi_4^{L,A}$ is not surjective.

□

As straightforward consequence of this example, we have the following result.

Lemma 3.2.15 *Let $k \to L$ be a non-separable algebraic field extension, A a finitely generated k-algebra and $m \geq 1$. Then, $\Phi_m^{L,A}$ is not a surjective A_L-module homomorphism, in general.*

3.2.2.2 Pure transcendental extensions

In this section, we will study the surjectivity of $\Phi_m^{L,A} : L \otimes_k \mathrm{IDer}_k(A;m) \to \mathrm{IDer}_k(A_L;m)$ when $k \to L$ is a pure transcendental field extension and A is a finitely presented k-algebra.

From now on, $k \to L$ will be a ring extension where $L := k[t_i \mid i \in \mathcal{I}]$ is a polynomial ring in an arbitrary number of variables and A a finitely generated k-algebra. We define $\mathbb{N}^{(\mathcal{I})} = \{\alpha := (\alpha_i)_{i\in\mathcal{I}} \mid \alpha_i \in \mathbb{N},\ \alpha_i = 0 \textit{ except for a finite number of } i \in \mathcal{I}\}$ and, for $\alpha \in \mathbb{N}^{(\mathcal{I})}$, we put $t^\alpha = \prod_{i\in\mathcal{I}} t_i^{\alpha_i}$. We start with some numerical results.

Lemma 3.2.16 *Let $n \leq m$ be two positive integers. We have the following properties.*

a. $(\lfloor m/n\rfloor + 1)\, n - 1 \geq m$.

b. If $m \neq 0 \mod n$, then $\lfloor m/n\rfloor = \lfloor (m-1)/n\rfloor$. Otherwise, $\lfloor m/n\rfloor = \lfloor (m-1)/n\rfloor + 1$.

c. If $n < m$ such that $m = 0 \mod n$. Then, there exists a prime factor of m which divides m/n.

Proof.

a. Let us write $m = cn + r$ where $0 \leq r < n$ and $c = \lfloor m/n\rfloor$, then $(c+1)n = cn + r + (n-r) > m$ and we have the result.

b. If $m \neq 0 \mod n$, then $m = cn + r$ where $1 \leq r < n$. So, $m - 1 = cn + (r-1)$ where $0 \leq r - 1 < n$. Hence, $\lfloor m/n\rfloor = c = \lfloor (m-1)/n\rfloor$. If $m = 0 \mod n$, then $m = cn$. So, $(c-1)n \leq m - 1 < cn$. If we divide this inequality by n, we obtain that $c - 1 \leq (m-1)/n < c$. Hence, $\lfloor (m-1)/n\rfloor = c - 1$ and we have the lemma.

c. Since $m = 0 \mod n$ and $n < m$, we have that $m = cn$ for some $c > 1$. Hence, any prime factor of c, which is also a prime factor of m, holds the lemma.

$\square$

Definition 3.2.17 *Let n be a positive integer. We define*

$$\mathcal{P}_n = \bigcup_{q \text{ is a prime factor of } n} q\mathbb{N}^{(\mathcal{I})}.$$

Lemma 3.2.18 *Let n, s be two positive integers such that $n \neq s$. Then, there do not exist $\alpha \in \mathbb{N}^{(\mathcal{I})} \setminus \mathcal{P}_n$ and $\eta \in \mathbb{N}^{(\mathcal{I})} \setminus \mathcal{P}_s$ such that $\alpha s = \eta n$.*

Proof. Suppose that there exist $\alpha \in \mathbb{N}^{(\mathcal{I})} \setminus \mathcal{P}_n$ and $\eta \in \mathbb{N}^{(\mathcal{I})} \setminus \mathcal{P}_s$ such that $\alpha s = \eta n$. If there were such a prime that divides n and s, then we could simplify it. So, we can assume that s and n do not have prime factors in common. Now, as s and n are not the same, one of them, we say s, has a prime factor q such that does not divide to the another one, in this case n. Since $\alpha s = \eta n$, we have that $\alpha_i s = \eta_i n$ for all $i \in \mathcal{I}$. So, q divide η_i for all $i \in \mathcal{I}$. Then $\eta = q\eta' \in \mathcal{P}_s$ and we have a contradiction.

$\square$

Fix $m > 1$ an integer and consider $m = q_1^{a_1} \cdots q_s^{a_s}$ its prime factorization, i.e. for all $j = 1, \ldots s$, q_j is a prime, $a_j > 0$ and $q_j \neq q_i$ if $i \neq j$. Let us consider $\beta \in \mathcal{P}_m$. Then, we can write $\beta = q_1^{b_1} \cdots q_s^{b_s}\eta$ where $b_j \geq 1$ for some $j \in \{1, \ldots, s\}$ and $\eta \in \mathbb{N}^{(\mathcal{I})}$ such that $q_j \nmid \eta$ for any $j = 1, \ldots, s$, i.e. for all j there exists η_{i_j} with $i_j \in \mathcal{I}$ such that $q_j \nmid \eta_{i_j}$. We can assume, without loss of generality, that there exists an integer l_β such that $0 \leq l_\beta \leq s$ and $a_j > b_j$ for all $j \leq l_\beta$ and $a_j \leq b_j$ for all $j > l_\beta$. Then, we define

$$n_\beta = \begin{cases} 1 & \text{if } l_\beta = 0 \\ q_1^{a_1 - b_1} \cdots q_{l_\beta}^{a_{l_\beta} - b_{l_\beta}} & \text{if } l_\beta \geq 1. \end{cases}$$

Lemma 3.2.19 *For each $\beta \in \mathcal{P}_m$, there exists a unique $n \in \mathbb{N}$ with $1 \leq n < m$ such that $m = 0 \mod n$ and $\beta n / m \notin \mathcal{P}_n$.*

Proof. We write $\beta = q_1^{b_1} \cdots q_s^{b_s}\eta$, where $\eta \in \mathbb{N}^{(\mathcal{I})}$ such that $q_j \nmid \eta$ for any $j = 1, \ldots, s$ and $b_j \geq 1$ for some $j \in \{1, \ldots, s\}$. We take $n = n_\beta$. It is obvious that n divides m and $1 \leq n < m$. We denote $l := l_\beta$ to simplify the notation. We put

$$\alpha := \frac{\beta n}{m} = \frac{\eta q_1^{b_1} \cdots q_s^{b_s} n}{q_1^{a_1} \cdots q_s^{a_s}}.$$

If $l = 0$, then $n = 1$ and $\mathcal{P}_1 = \emptyset$ so, $\alpha \notin \mathcal{P}_n$ (note that $\alpha \in \mathbb{N}^{(\mathcal{I})}$ because if $l = 0$, then $b_j \geq a_j$ for all $j = 1, \ldots, s$). If $l \geq 1$, then

$$\alpha = \frac{\eta q_1^{b_1} \cdots q_s^{b_s} q_1^{a_1 - b_1} \cdots q_l^{a_l - b_l}}{q_1^{a_1} \cdots q_s^{a_s}} = \frac{\eta q_1^{a_1} \cdots q_l^{a_l} q_{l+1}^{b_{l+1}} \cdots q_s^{b_s}}{q_1^{a_1} \cdots q_s^{a_s}} = q_{l+1}^{b_{l+1} - a_{l+1}} \cdots q_s^{b_s - a_s}\eta.$$

Note that the set of primes which divide n is $\{q_1,\ldots,q_l\}$. Hence, $q_j\nmid\alpha$ for all $j=1,\ldots,l$ (recall that $q_j\nmid\eta$). So, $\alpha\notin\mathcal{P}_n$.

Now, let us suppose that there exists another $n'\in\mathbb{N}$ holding the lemma, in particular $\alpha':=\beta n'/m\notin\mathcal{P}_{n'}$. Then, $\alpha n'=\alpha' n$ and we have a contradiction by Lemma 3.2.18.

□

Theorem 3.2.20 *Let $m\geq 1$ be an integer and $L=k[t_i\mid i\in\mathcal{I}]$ a polynomial ring. Let us consider $D\in\mathrm{HS}_L(R_L;m)$. Then, for all $n=1,\ldots,m$ there exists a finite subset L_n of $\mathbb{N}^{(\mathcal{I})}\setminus\mathcal{P}_n$ and an $N^{n,\alpha}\in\mathrm{HS}_k(R)$ for each $\alpha\in L_n$ such that*

$$D=\circ_{n=1}^{m}\left(\circ_{\alpha\in L_n}\left(\psi_\alpha^{n,m}\bullet\widetilde{N^{n,\alpha}}\right)\right)$$

where $\psi_\alpha^{n,m}:R_L[|\mu|]\to R_L[|\mu|]_m$ is the substitution map of constant coefficients given by $\psi_\alpha^{n,m}(\mu)=t^\alpha\mu^n$.

Proof. First, observe that, if $E\in\mathrm{HS}_L(R_L;m)$ then,

$$\psi_\alpha^{n,m}\bullet E=\tau_{\infty,m}\left(\left(t^\alpha\bullet E\right)[n]\right).$$

We prove this theorem by induction on m. Assume that $m=1$. Then, $D=(\mathrm{Id},D_1)\in\mathrm{HS}_L(R_L;1)$. Since L is free over k and $\{t^\alpha,\ \alpha\in\mathbb{N}^{(\mathcal{I})}\}$ is a k-basis of L, from Lemma 3.2.2, $D_1\in\mathrm{Der}_L(R_L)$ can be written as

$$D_1=\sum_{\alpha\in L_1}t^\alpha\widetilde{\delta_\alpha}$$

where L_1 is a finite subset of $\mathbb{N}^{(\mathcal{I})}$ and $\delta_\alpha\in\mathrm{Der}_k(R)$ for all $\alpha\in L_1$. Let us consider $N^{1,\alpha}$ an integral of δ_α for $\alpha\in L_1$. Then, $\widetilde{N^{1,\alpha}}\in\mathrm{HS}_L(R_L)$ is an integral of $\widetilde{\delta_\alpha}$. Hence,

$$D=\circ_{\alpha\in L_1}\left(t^\alpha\bullet\left(\mathrm{Id},\widetilde{\delta_\alpha}\right)\right)=\circ_{\alpha\in L_1}\left(\tau_{\infty,1}\left(t^\alpha\bullet\widetilde{N^{1,\alpha}}\right)\right)=\circ_{\alpha\in L_1}\left(\psi_\alpha^{1,1}\bullet\widetilde{N^{1,\alpha}}\right)$$

(note that the order of the composition in this equality is not important because $\mathrm{HS}_L(R_L;1)\equiv\mathrm{Der}_L(R)$ is a commutative group) and we have the result when $m=1$. Let us assume that the theorem is true for any HS-derivation of length $m-1$ and we will prove it for $D\in\mathrm{HS}_L(R_L;m)$. By induction hypothesis, for all $n=1,\ldots,m-1$, there exists a finite subset L'_n of $\mathbb{N}^{(\mathcal{I})}\setminus\mathcal{P}_n$ and an $N^{n,\alpha}\in\mathrm{HS}_k(R)$ for each $\alpha\in L'_n$ such that

$$\tau_{m,m-1}(D)=\circ_{n=1}^{m-1}\left(\circ_{\alpha\in L'_n}\left(\psi_\alpha^{n,m-1}\bullet\widetilde{N^{n,\alpha}}\right)\right).\tag{3.5}$$

We define

$$E:=\circ_{n=1}^{m-1}\left(\circ_{\alpha\in L'_n}\left(\psi_\alpha^{n,m}\bullet\widetilde{N^{n,\alpha}}\right)\right)$$

where the composition are in the same order that in (3.5). Note that $\psi_\alpha^{n,m-1}=\tau_{m,m-1}\circ\psi_\alpha^{n,m}$, and thanks to Lemma 1.4.7 and Corollary 1.4.8, we have that:

$$\begin{aligned}\tau_{m,m-1}(E)=&\ \circ_{n=1}^{m-1}\left(\circ_{\alpha\in L'_n}\left(\tau_{m,m-1}\bullet\left(\psi_\alpha^{n,m}\bullet\widetilde{N^{n,\alpha}}\right)\right)\right)\\ =&\ \circ_{n=1}^{m-1}\left(\circ_{\alpha\in L'_n}\left(\left(\tau_{m,m-1}\circ\psi_\alpha^{n,m}\right)\bullet\widetilde{N^{n,\alpha}}\right)\right)=\tau_{m,m-1}(D).\end{aligned}$$

Then, by Lemma 1.4.10, $D = E \circ (\mathrm{Id}, \delta)[m]$ where $\delta \in \mathrm{Der}_L(R_L)$. From Lemma 3.2.2, $\delta = \sum_{\beta \in \mathcal{J}} t^\beta \widetilde{\delta_\beta}$ where $\mathcal{J}$ is a finite subset of $\mathbb{N}^{(\mathcal{I})}$ and $\delta_\beta \in \mathrm{Der}_k(R)$ for all $\beta \in \mathcal{J}$. We denote $\Gamma = \{n \in \mathbb{N} \mid 1 \leq n \leq m-1, \ m = 0 \mod n\}$. For all $n \in \Gamma$, we define

$$\mathcal{J}_n := \{\beta \in \mathcal{J} \mid \beta = \alpha(m/n) \text{ for some } \alpha \in L'_n\}$$

and

$$\mathcal{L}_m = \mathcal{J} \setminus \mathcal{P}_m.$$

Claim 1. For $n, s \in \Gamma$ such that $n \neq s$, then $\mathcal{J}_n \cap \mathcal{J}_s = \emptyset$.

Let us suppose that there exists $\beta \in \mathcal{J}_n \cap \mathcal{J}_s$. In this case, there exist $\alpha \in L'_n \subseteq \mathbb{N}^{(\mathcal{I})} \setminus \mathcal{P}_n$ and $\eta \in L'_s \subseteq \mathbb{N}^{(\mathcal{I})} \setminus \mathcal{P}_s$ such that $\beta = \alpha(m/n) = \eta(m/s)$, i.e. $\alpha s = \eta n$ and this can not happen by Lemma 3.2.18.

Claim 2. $\mathcal{L}_m \cap \mathcal{J}_n = \emptyset$ for all $n \in \Gamma$.

By Lemma 3.2.16 c., there exists a prime factor, q, of m that divides m/n. Assume that $\beta \in \mathcal{J}_n$. Then, we have that $\beta = \alpha(m/n)$ for some $\alpha \in L'_n$. Then, $q|\beta$ so, $\beta \in \mathcal{P}_m$.

Let us write $\mathcal{J} = \sqcup_{n\in\Gamma} \mathcal{J}_n \sqcup \mathcal{L}_m \sqcup \overline{\mathcal{J}}$ where $\overline{\mathcal{J}} = \mathcal{J} \setminus (\sqcup_{n\in\Gamma} \mathcal{J}_n \sqcup \mathcal{L}_m)$. Observe that $\overline{\mathcal{J}} \subseteq \mathcal{P}_m$ so, from Lemma 3.2.19, for all $\beta \in \overline{\mathcal{J}}$, there exists a unique $n_\beta \in \Gamma$ such that $(\beta n_\beta)/m \notin \mathcal{P}_{n_\beta}$. Therefore, if we denote $\overline{\mathcal{J}}_n = \{\beta \in \overline{\mathcal{J}} \mid n_\beta = n\}$ for all $n \in \Gamma$, we can write

$$\mathcal{J} = \sqcup_{n\in\Gamma} \left(\mathcal{J}_n \sqcup \overline{\mathcal{J}}_n\right) \sqcup \mathcal{L}_m \qquad \text{and} \qquad \delta = \sum_{n\in\Gamma} \sum_{\beta \in \mathcal{J}_n \sqcup \overline{\mathcal{J}}_n} t^\beta \delta_\beta + \sum_{\alpha \in \mathcal{L}_m} t^\alpha \delta_\alpha.$$

Now, for each $n \in \Gamma$ we can define

$$\mathcal{L}'_n = \{\alpha \in L'_n | \ \alpha(m/n) \in \mathcal{J}_n\} \quad \text{ and } \quad \overline{\mathcal{L}}_n = \{\alpha \in \mathbb{N}^{(\mathcal{I})} \setminus L'_n | \ \alpha(m/n) \in \overline{\mathcal{J}}_n\} \nsubseteq \mathcal{P}_n.$$

Note that $\mathcal{L}'_n \cap \overline{\mathcal{L}}_n = \emptyset$. Let us denote $\mathcal{L}_n = \mathcal{L}'_n \cup \overline{\mathcal{L}}_n$. Hence, we can express

$$(\mathrm{Id}, \delta) = \circ_{n\in\Gamma} \left(\circ_{\alpha \in \mathcal{L}'_n} \left(\mathrm{Id}, t^{\alpha(m/n)} \widetilde{\delta_{\alpha(m/n)}} \right) \circ_{\alpha \in \overline{\mathcal{L}}_n} \left(\mathrm{Id}, t^{\alpha(m/n)} \widetilde{\delta_{\alpha(m/n)}} \right) \right) \circ \left(\circ_{\alpha \in \mathcal{L}_m} \left(\mathrm{Id}, t^\alpha \widetilde{\delta_\alpha} \right) \right).$$

By Corollary 1.4.8 and Lemma 1.1.6, for each $n \in \Gamma \cup \{m\}$ and $\alpha \in \mathcal{L}_n$, we have that:

$$\begin{aligned} \left(\mathrm{Id}, t^{\alpha(m/n)} \widetilde{\delta_{\alpha(m/n)}} \right)[m] &= \left(\left(t^{\alpha(m/n)} \bullet \left(\mathrm{Id}, \widetilde{\delta_{\alpha(m/n)}} \right) \right)[m/n] \right)[n] \\ &= \left(t^\alpha \bullet \left(\left(\mathrm{Id}, \widetilde{\delta_{\alpha(m/n)}} \right)[m/n] \right) \right)[n]. \end{aligned}$$

For each $n \in \Gamma \cup \{m\}$ and $\alpha \in \mathcal{L}_n$, let us consider $M^{n,\alpha} \in \mathrm{HS}_k(R)$ an integral of $\delta_{\alpha(m/n)}$. We know that $\widetilde{M^{n,\alpha}}$ is an integral of $\widetilde{\delta_{\alpha(m/n)}}$, so $\widetilde{M^{n,\alpha}}[m/n]$ is an integral of $\left(\mathrm{Id}, \widetilde{\delta_{\alpha(m/n)}} \right)[m/n]$. Hence, by Lemma 1.1.6, we have that

$$\begin{aligned} \psi^{n,m}_\alpha \bullet \left(\widetilde{M^{n,\alpha}}[m/n] \right) &= \tau_{\infty,m} \left(\left(t^\alpha \bullet \left(\widetilde{M^{n,\alpha}}[m/n] \right) \right)[n] \right) = \left(\tau_{\infty,m/n} \left(t^\alpha \bullet \left(\widetilde{M^{n,\alpha}}[m/n] \right) \right) \right)[n] \\ &= \left(t^\alpha \bullet \tau_{\infty,m/n} \left(\widetilde{M^{n,\alpha}}[m/n] \right) \right)[n] = \left(t^\alpha \bullet \left(\left(\mathrm{Id}, \widetilde{\delta_{\alpha(m/n)}} \right)[m/n] \right) \right)[n] \\ &= \left(\mathrm{Id}, t^{\alpha(m/n)} \widetilde{\delta_{\alpha(m/n)}} \right)[m]. \end{aligned}$$

To simplify the following expression, we put $\overline{\mathcal{L}}_n = \mathcal{L}'_n = \emptyset$ for all $n \in \{1, \ldots, m-1\} \setminus \Gamma$. Moreover, for all $n \in \{1, \ldots, m-1\}$, if $\alpha \in L'_n \setminus \mathcal{L}'_n$ then we consider $\delta_{\alpha(m/n)} = 0$ and $M^{n,\alpha} = \mathbb{I} \in \mathrm{HS}_k(R)$ an integral of $\delta_{\alpha(m/n)}$. Thanks to Lemmas 1.1.7 and 1.4.7 and the previous equation, we can write:

$$\begin{aligned}
D = \ & \circ_{n=1}^{m-1} \left(\circ_{\alpha \in L'_n} \left(\psi_\alpha^{n,m} \bullet \widetilde{N^{n,\alpha}} \right) \right) \circ \circ_{n=1}^{m-1} \left(\circ_{\alpha \in L'_n} \left(\left(\mathrm{Id}, t^{\alpha(m/n)} \widetilde{\delta_{\alpha(m/n)}} \right) [m] \right) \right. \\
& \left. \circ_{\alpha \in \overline{\mathcal{L}}_n} \left(\left(\mathrm{Id}, t^{\alpha(m/n)} \widetilde{\delta_{\alpha(m/n)}} \right) [m] \right) \right) \circ \left(\circ_{\alpha \in \mathcal{L}_m} \left(\mathrm{Id}, t^\alpha \widetilde{\delta_\alpha} \right) [m] \right) \\
= \ & \circ_{n=1}^{m-1} \left(\circ_{\alpha \in L'_n} \left(\psi_\alpha^{n,m} \bullet \widetilde{N^{n,\alpha}} \circ \left(\mathrm{Id}, t^{\alpha(m/n)} \widetilde{\delta_{\alpha(m/n)}} \right) [m] \right) \circ \left(\circ_{\alpha \in \overline{\mathcal{L}}_n} \left(\mathrm{Id}, t^{\alpha(m/n)} \widetilde{\delta_{\alpha(m/n)}} \right) [m] \right) \right) \\
& \circ \left(\circ_{\alpha \in \mathcal{L}_m} \left(\mathrm{Id}, t^\alpha \widetilde{\delta_\alpha} \right) [m] \right) \\
= \ & \circ_{n=1}^{m-1} \left(\circ_{\alpha \in L'_n} \left(\psi_\alpha^{n,m} \bullet \widetilde{N^{n,\alpha}} \circ \psi_\alpha^{n,m} \bullet (\widetilde{M^{n,\alpha}}[m/n]) \right) \circ_{\alpha \in \overline{\mathcal{L}}_n} \left(\psi_\alpha^{n,m} \bullet (\widetilde{M^{n,\alpha}}[m/n]) \right) \right) \\
& \circ \left(\circ_{\alpha \in \mathcal{L}_m} \left(\psi_\alpha^{m,m} \bullet \widetilde{M^{m,\alpha}} \right) \right) \\
= \ & \circ_{n=1}^{m-1} \left(\circ_{\alpha \in L'_n} \left(\psi_\alpha^{n,m} \bullet \left(\widetilde{N^{n,\alpha}} \circ \widetilde{M^{n,\alpha}}[m/n] \right) \right) \circ_{\alpha \in \overline{\mathcal{L}}_n} \left(\psi_\alpha^{n,m} \bullet (\widetilde{M^{n,\alpha}}[m/n]) \right) \right) \\
& \circ \left(\circ_{\alpha \in \mathcal{L}_m} \left(\psi_\alpha^{m,m} \bullet \widetilde{M^{m,\alpha}} \right) \right).
\end{aligned}$$

Thanks to Lemma 3.2.8 b., $\widetilde{M^{n,\alpha}}[m/n]$ is the extension of the HS-derivation $M^{n,\alpha}[m/n]$ and, by Lemma 3.2.8 a., $\widetilde{N^{n,\alpha}} \circ \widetilde{M^{n,\alpha}}[m/n]$ is the extension of $N^{n,\alpha} \circ M^{n,\alpha}[m/n]$. Therefore, if we denote $L_n = L'_n \cup \overline{\mathcal{L}}_n \subseteq \mathbb{N}^{\mathcal{I}} \setminus \mathcal{P}_n$ and $L_m = \mathcal{L}_m$, we have the theorem.

□

Theorem 3.2.21 *Let $m \geq 1$ be an integer, $L = k[t_i \mid i \in \mathcal{I}]$ a polynomial ring, $I \subseteq R$ an ideal and $D \in \mathrm{HS}_L(\log I^e; m)$. For all $n = 1, \ldots, m$, let L_n be a finite subset of $\mathbb{N}^{(\mathcal{I})} \setminus \mathcal{P}_n$ and $N^{n,\alpha} \in \mathrm{HS}_k(R)$ for each $\alpha \in L_n$ such that*

$$D = \circ_{n=1}^{m} \left(\circ_{\alpha \in L_n} \left(\psi_\alpha^{n,m} \bullet \widetilde{N^{n,\alpha}} \right) \right)$$

where $\psi_\alpha^{n,m} : R_L[|\mu|] \to R_L[|\mu|]_m$ is the substitution map given by $\psi_\alpha^{n,m}(\mu) = t^\alpha \mu^n$. Then, for all $n = 1, \ldots, m$ and $\alpha \in L_n$, $N^{n,\alpha} \in \mathrm{HS}_k(R)$ is an $\lfloor m/n \rfloor - I$-logarithmic HS-derivation.

Proof. We prove this result by induction on m. If $m = 1$, we have to prove that $N^{1,\alpha}$ is $1 - I$-logarithmic for all $\alpha \in L_1$, i.e. $N_1^{1,\alpha} \in \mathrm{Der}_k(\log I)$ for all $\alpha \in L_1$. In this case,

$$D = \circ_{\alpha \in L_1} \left(\psi_\alpha^{1,1} \bullet \widetilde{N^{1,\alpha}} \right) = \circ_{\alpha \in L_1} \left(\tau_{\infty,1} \left(t^\alpha \bullet \widetilde{N^{1,\alpha}} \right) \right) = \circ_{\alpha \in L_1} \left(\mathrm{Id}, t^\alpha \left(\widetilde{N^{1,\alpha}} \right)_1 \right).$$

Then,

$$D_1 = \sum_{\alpha \in L_1} t^\alpha \left(\widetilde{N^{1,\alpha}} \right)_1.$$

Since D_1 is I^e-logarithmic, doing the same process of Proposition 3.2.5, we have that $N^{n,\alpha}$ is $1 - I$-logarithmic. Assume that the theorem is true for all I^e-logarithmic HS-derivation of length $m-1$ and let us take $D \in \mathrm{HS}_L(\log I^e; m)$ such that

$$D = \circ_{n=1}^{m} \left(\circ_{\alpha \in L_n} \left(\psi_\alpha^{n,m} \bullet \widetilde{N^{n,\alpha}} \right) \right)$$

where $L_n \subseteq \mathbb{N}^{(\mathcal{I})} \setminus \mathcal{P}_n$ is a finite set and $N^{n,\alpha} \in \mathrm{HS}_k(R)$ for all $\alpha \in L_n$ and $n = 1, \ldots, m$. By Corollary 1.4.8, we have that

$$\tau_{m,m-1}(D) = \circ_{n=1}^{m-1} \left(\circ_{\alpha \in L_n} \tau_{m,m-1} \bullet \left(\psi_\alpha^{n,m} \bullet \widetilde{N^{n,\alpha}} \right) \right) \circ \left(\circ_{\alpha \in L_m} \tau_{m,m-1} \bullet \left(\psi_\alpha^{m,m} \bullet \widetilde{N^{n,\alpha}} \right) \right).$$

From Lemma 1.4.7, for any $E \in \mathrm{HS}_L(R_L)$, $\tau_{m,m-1} \bullet (\psi_\alpha^{n,m} \bullet (E)) = (\tau_{m,m-1} \circ \psi_\alpha^{n,m}) \bullet E = \psi_\alpha^{n,m-1} \bullet E$. Moreover, $\psi_\alpha^{m,m-1} \bullet E = \mathbb{I}$. So,

$$\tau_{m,m-1}(D) = \circ_{n=1}^{m-1} \left(\circ_{\alpha \in L_n} \psi_\alpha^{n,m-1} \bullet \widetilde{N^{n,\alpha}} \right).$$

Hence, since $\tau_{m,m-1}(D) \in \mathrm{HS}_L(\log I^e; m-1)$, we can apply the induction hypothesis and we deduce that $N^{n,\alpha} \in \mathrm{HS}_k(R)$ is $\lfloor (m-1)/n \rfloor - I$-logarithmic for all $\alpha \in L_n$ and $n = 1, \ldots, m-1$. We define

$$E^n := \circ_{\alpha \in L_n} \left(\psi_\alpha^{n,m} \bullet \widetilde{N^{n,\alpha}} \right) \Rightarrow D = E^1 \circ \cdots \circ E^m$$

where the order of the composition in E^n is the same that in D.

Claim. *E^n is $(m-1) - I^e$-logarithmic.*

Since $N^{n,\alpha}$ is $\lfloor (m-1)/n \rfloor - I$-logarithmic, by Lemma 3.2.8 c., $\widetilde{N^{n,\alpha}}$ is $\lfloor (m-1)/n \rfloor - I^e$-logarithmic. Hence, $t^\alpha \bullet \widetilde{N^{n,\alpha}}$ is also $\lfloor (m-1)/n \rfloor - I^e$-logarithmic. From Lemma 1.1.18, $\left(t^\alpha \bullet \widetilde{N^{n,\alpha}} \right)[n]$ is $((\lfloor (m-1)/n \rfloor + 1)n - 1) - I^e$-logarithmic. By Lemma 3.2.16 a., $m - 1 < (\lfloor (m-1)/n \rfloor + 1)n - 1$, so $\psi_\alpha^{n,m} \bullet \widetilde{N^{n,\alpha}}$ is $(m-1) - I^e$-logarithmic because $\psi_\alpha^{n,m} \bullet * = \tau_{\infty,m}((t^\alpha \bullet *)[n])$. Hence, by Lemma 1.1.19, E^n is $(m-1) - I^e$-logarithmic for all n.

Let us consider $n \in \{1, \ldots, m\}$ such that $n \nmid m$. Then, by Corollary 1.4.8,

$$E^n = \circ_{\alpha \in L_n} \left(\psi_\alpha^{n,m} \bullet \widetilde{N^{n,\alpha}} \right) = \circ_{\alpha \in L_n} \tau_{\infty,m} \left(\left(t^\alpha \bullet \widetilde{N^{n,\alpha}} \right)[n] \right) = \tau_{\infty,m} \left(\left(\circ_{\alpha \in L_n} \left(t^\alpha \bullet \widetilde{N^{n,\alpha}} \right) \right)[n] \right).$$

Hence, $E^n_m = 0$ by definition of $(*)[n]$. Moreover, by Lemma 3.2.16 b., $\lfloor (m-1)/n \rfloor = \lfloor m/n \rfloor$, so $N^{n,\alpha}$ is $\lfloor m/n \rfloor - I$-logarithmic. Therefore, to prove the theorem we have to show that $N^{n,\alpha}$ is $(m/n) - I$-logarithmic for $n|m$. By Lemma 3.2.16 b., $m/n = \lfloor (m-1)/n \rfloor + 1$ and, since $N^{n,\alpha}$ is $\lfloor (m-1)/n \rfloor - I$-logarithmic, it is enough to prove that $N^{n,\alpha}_{m/n}(I) \subseteq I$. Note that

$$\left(\psi_\alpha^{n,m} \bullet \widetilde{N^{n,\alpha}} \right)_m = \left(\tau_{\infty,m} \left(\left(t^\alpha \bullet \widetilde{N^{n,\alpha}} \right)[n] \right) \right)_m = t^{\alpha(m/n)} \left(\widetilde{N^{n,\alpha}} \right)_{m/n}$$

where $\left(\widetilde{N^{n,\alpha}} \right)_{m/n|R} = N^{n,\alpha}_{m/n}$ by Remark 3.2.7. Therefore, by Lemma 1.1.19

$$E^n_m = \sum_{\alpha \in L_n} t^{\alpha(m/n)} (\widetilde{N^{n,\alpha}})_{m/n} + F_n$$

where F_n is an I^e-differential operator. Hence, again by Lemma 1.1.19,

$$D_m = \sum_{n=1}^{m} E^n_m + F = \sum_{n|m} \sum_{\alpha \in L_n} t^{\alpha(m/n)} (\widetilde{N^{n,\alpha}})_{m/n} + F_n + F$$

where F is an I^e-differential operator. Since D_m is also an I^e-differential operator, we have that

$$\sum_{n|m}\sum_{\alpha\in L_n} t^{\alpha(m/n)}(\widetilde{N^{n,\alpha}})_{m/n} \text{ is an } I^e\text{-differential operator.}$$

Observe that $\alpha(m/n)\neq\eta(m/s)$ for all $\alpha\in L_n$ and $\eta\in L_s$ because $L_n\subseteq\mathbb{N}^{\mathcal{I}}\setminus\mathcal{P}_n$ and $L_s\in\mathbb{N}^{\mathcal{I}}\setminus\mathcal{P}_s$. Otherwise, if $\alpha(m/n)=\eta(m/s)$, then $\alpha s=\eta n$ and this is a contradiction by Lemma 3.2.18. Doing the same process than in the proof of Proposition 3.2.5, we can deduce that $N^{n,\alpha}\in \mathrm{HS}_k(R)$ is $\lfloor m/n\rfloor - I$-logarithmic for all $\alpha\in L_n$ and $n=1,\ldots,m$.

□

Theorem 3.2.22 *Let $m\geq 1$ be an integer, $L=k[t_i\mid i\in\mathcal{I}]$ a polynomial ring, A a finitely generated k-algebra and $E\in\mathrm{HS}_L(A_L;m)$. Then, for all $n=1,\ldots,m$ there exists a finite subset $L_n\subseteq\mathbb{N}^{(\mathcal{I})}\setminus\mathcal{P}_n$ and an $M^{n,\alpha}\in\mathrm{HS}_k(A;\lfloor m/n\rfloor)$ for each $\alpha\in L_n$ such that*

$$E=\circ_{n=1}^m\left(\circ_{\alpha\in L_n}\left(\phi_\alpha^{n,m}\bullet\widetilde{M^{n,\alpha}}\right)\right)$$

where $\phi_\alpha^{n,m}:A_L[|\mu|]_{\lfloor m/n\rfloor}\to A_L[|\mu|]_m$ is the substitution map of constant coefficients given by $\phi_\alpha^{n,m}(\mu)=t^\alpha\mu^n$.

Proof. Since A is a finitely generated k-algebra, we can take $A=R/I$ where $R=k[x_1,\ldots,x_d]$ and $I\subseteq R$ an ideal. By Proposition 1.2.2, there exists $D\in\mathrm{HS}_k(\log I^e;m)$ such that $\Pi^{I^e}_{\mathrm{HS};m}(D)=E$. By theorems 3.2.20 and 3.2.21 for all $n=1,\ldots,m$, there exists a finite subset L_n of $\mathbb{N}^{(\mathcal{I})}\setminus\mathcal{P}_n$ and an $\lfloor m/n\rfloor - I$-logarithmic HS-derivation $N^{n,\alpha}\in\mathrm{HS}_k(R)$ for each $\alpha\in L_n$ such that

$$D=\circ_{n=1}^m\left(\circ_{\alpha\in L_n}\left(\psi_\alpha^{n,m}\bullet\widetilde{N^{n,\alpha}}\right)\right)$$

where $\psi_\alpha^{n,m}:R_L[\mu|]\to R_L[|\mu|]_m$ is the substitution map given by $\psi_\alpha^{n,m}(\mu)=t^\alpha\mu^n$.

Let us consider $\theta_\alpha^{n,m}:R_L[|\mu|]_{\lfloor m/n\rfloor}\to R_L[|\mu|]_m$ the substitution map given by $\theta_\alpha^{n,m}(\mu)=t^\alpha\mu^n$. Then, $\psi_\alpha^{n,m}=\theta_\alpha^{n,m}\circ\tau_{\infty,\lfloor m/n\rfloor}$. So, let us rewrite $N^{n,\alpha}=\tau_{\infty,\lfloor m/n\rfloor}(N^{n,\alpha})\in\mathrm{HS}_k(\log I;\lfloor m/n\rfloor)$ and we have that

$$D=\circ_{n=1}^m\left(\circ_{\alpha\in L_n}\left(\theta_\alpha^{n,m}\bullet\widetilde{N^{n,\alpha}}\right)\right)$$

(recall that $\widetilde{\tau_{\infty s}(N)}=\tau_{\infty s}(\widetilde{N})$ for any $N\in\mathrm{HS}_k(R;m)$ and $s\geq 1$ by Lemma 3.2.8 b.). Moreover $\phi_\alpha^{n,m}$ is the induced map by $\theta_\alpha^{n,m}$ in A_L. Therefore, by Lemmas 1.4.12 and 3.2.9,

$$E=\Pi^{I^e}_{\mathrm{HS},m}(D)=\circ_{n=1}^m\left(\circ_{\alpha\in L_n}\left(\Pi^{I^e}_{\mathrm{HS},m}\left(\theta_\alpha^{n,m}\bullet\widetilde{N^{n,\alpha}}\right)\right)\right)$$
$$=\circ_{n=1}^m\left(\circ_{\alpha\in L_n}\left(\phi_\alpha^{n,m}\bullet\left(\Pi^{I^e}_{\mathrm{HS},\lfloor m/n\rfloor}(\widetilde{N^{n,\alpha}})\right)\right)\right)=\circ_{n=1}^m\left(\circ_{\alpha\in L_n}\left(\phi_\alpha^{n,m}\bullet\left(\widetilde{M^{n,\alpha}}\right)\right)\right)$$

where $\widetilde{M^{n,\alpha}}\in\mathrm{HS}_L(A_L;m)$ is the extension of $\Pi^{I}_{\mathrm{HS},\lfloor m/n\rfloor}(N^{n,\alpha})\in\mathrm{HS}_k(A;\lfloor m/n\rfloor)$ and the theorem is proved.

□

Corollary 3.2.23 *Let k be a ring, $L = k[t_i \mid i \in \mathcal{I}]$ and A a finitely generated k-algebra. We denote $A_L = L \otimes_k A$. Then, $\Phi_m^{L,A} : L \otimes_k \operatorname{IDer}_k(A; m) \to \operatorname{IDer}_L(A_L; m)$ is an isomorphism of A_L-modules for all $m \in \mathbb{N}$. Moreover, $\operatorname{Leaps}_k(A) = \operatorname{Leaps}_L(A_L)$.*

Proof. Since L is flat over k, from Lemma 3.2.11, $\Phi_m^{L,A}$ is injective. To prove the surjectivity, we take $\delta \in \operatorname{IDer}_L(A_L; m)$. By definition of integrability, there exists $E \in \operatorname{HS}_L(A_L; m)$ such that $E_1 = \delta$. By the previous theorem, we can write E as

$$E = \circ_{n=1}^{m} \left(\circ_{\alpha \in L_n} \left(\phi_\alpha^{n,m} \bullet \widetilde{M^{n,\alpha}} \right) \right)$$

where, for all $n = 1, \dots, m$, L_n is a finite subset of $\mathbb{N}^{(\mathcal{I})}$, for all $\alpha \in L_n$, $M^{n,\alpha} \in \operatorname{HS}_k(A; \lfloor m/n \rfloor)$ and $\phi_\alpha^{n,m} : A_L[|\mu|]_{\lfloor m/n \rfloor} \to A_L[|\mu|]_m$ is the substitution map given by $\phi_\alpha^{n,m}(\mu) = t^\alpha \mu^n$. If $n > 1$, then $\ell\left(\phi_\alpha^{n,m} \bullet N\right) > 1$ for all $N \in \operatorname{HS}_L(A_L; m)$ and if $n = 1$, then $M_1^{1,\alpha} \in \operatorname{IDer}_k(A; m)$. So,

$$\delta = E_1 = \left(\circ_{\alpha \in L_1} \left(\phi_\alpha^{1,m} \bullet \widetilde{M^{n,\alpha}} \right) \right)_1 = \sum_{\alpha \in L_1} t^\alpha \left(\widetilde{M^{n,\alpha}} \right)_1 = \Phi_m^{L,A} \left(\sum_{\alpha \in L_1} (t^\alpha \otimes M_1^{n,\alpha}) \right).$$

So, $\Phi_m^{L,A}$ is surjective. Moreover, since L is faithfully flat over k, $\operatorname{Leaps}_k(A) = \operatorname{Leaps}_L(A_L)$ by Lemma 3.2.13.

□

Let us assume that $k \to L$ is a pure transcendental field extension. Then, we can express $L = T^{-1}L'$ where $L' = k[t_i \mid i \in \mathcal{I}]$ and $T = L' \setminus \{0\}$. Hence, for any finitely generated k-algebra A, we have that

$$L \otimes_k \operatorname{IDer}_k(A; m) \cong T^{-1}L' \otimes_{L'} L' \otimes_k \operatorname{IDer}_k(A; m) \cong T^{-1}L' \otimes_{L'} \operatorname{IDer}_{L'}(A_{L'}; m). \tag{3.6}$$

Now, let us recall the following proposition:

Proposition 3.2.24 [Na2, **Corollary 2.3.5**] *Let C be a commutative ring, B a finitely presented C-algebra and $T \subseteq B$ a multiplicative set. Then, for any integer $m \geq 1$, the canonical map*

$$T^{-1} \operatorname{IDer}_C(B; m) \to \operatorname{IDer}_C(T^{-1}B; m)$$

is an isomorphism of $(T^{-1}B)$-modules.

Hence, if A is a finitely presented k-algebra,

$$T^{-1}L' \otimes_{L'} \operatorname{IDer}_{L'}(A_{L'}; m) \cong \operatorname{IDer}_{L'}(T^{-1}L' \otimes_L' A_{L'}; m) = \operatorname{IDer}_{L'}(A_L; m) \tag{3.7}$$

Moreover, it is easy to prove that if $T \subseteq L'$, then any HS-derivation over L' is $T^{-1}L'$-linear, so $\operatorname{HS}_{L'}(A_L; m) = \operatorname{HS}_{T^{-1}L'}(A_L; m)$ and $\operatorname{IDer}_{L'}(A_L; m) = \operatorname{IDer}_L(A_L; m)$. Therefore, thanks to the bijections (3.6) and (3.7), we have that

$$L \otimes_k \operatorname{IDer}_k(A; m) \cong T^{-1}L' \otimes_{L'} \operatorname{IDer}_{L'}(A_{L'}; m) \cong \operatorname{IDer}_L(A_L; m)$$

and we have proved the following corollary:

Corollary 3.2.25 *Let k be a field and L a pure transcendental field extension of k. Assume that A is a finitely presented k-algebra. Then, $\Phi_m^{L,A} : L \otimes_k \operatorname{IDer}_k(A; m) \to \operatorname{IDer}_L(A_L; m)$ is an isomorphism of A_L-modules for all $m \in \mathbb{N}$. Moreover, $\operatorname{Leaps}_k(A) = \operatorname{Leaps}_L(A_L)$.*

3.2.2.3 Separable extensions

Let us consider k a field of characteristic $p > 0$ and L a k-algebra containing k. Remember that L is separable over k if $L_K := K \otimes_k L$ is reduced for every possible extension K of k. In this section we prove that $\Phi_m^{L,A} : L \otimes_k \mathrm{IDer}_k(A; m) \to \mathrm{IDer}_L(A_L; m)$ is an isomorphism when L is a separable algebra over k and A a finitely generated k-algebra.

Hypothesis 3.2.26 *Let k be a ring of characteristic $p > 0$ and $k \to L$ a free ring extension. Then, we assume that the following conditions hold.*

1. *For every k-linearly independent subset $\{a_i,\ i \in \mathcal{I}\}$ of L, the subset $\{a_i^p,\ i \in \mathcal{I}\}$ of L continues to be k-linearly independent.*
2. *For every k-basis $\{a_i,\ i \in \mathcal{I}\}$ of L and every k-linearly independent set $\{b_1, \ldots, b_s\}$ of L, there exists $\mathcal{L} \subseteq \mathcal{I}$ such that $\{b_1, \ldots, b_s\} \cup \{a_i,\ i \in \mathcal{L}\}$ is a k-basis of L.*

In the rest of this chapter, we put $R = k[x_1, \ldots, x_d]$.

Hypothesis 3.2.27 *Let $l \geq 1$ be an integer. We say that the ideal $I \subseteq R$ satisfies S_l if $\Phi_m^{L,R,I}$ is surjective of all $m < p^l$.*

Note that if $k \to L$ is a flat ring extension where k is a ring of characteristic $p > 0$, S_1 is satisfied for all $I \subseteq R$ thanks to $\Phi_1^{L,R,I}$ is bijective and leaps only occur at powers of p.

Lemma 3.2.28 *Let $l \geq 1$ be an integer and k a ring of characteristic $p > 0$. Assume that $k \to L$ is a free ring extension and $I \subseteq R$ satisfies S_l. Let us consider a $\left(p^l - 1\right) - I$-logarithmic HS-derivation $D \in \mathrm{HS}_L\left(R_L; p^l\right)$. Then, for each k-basis $\{a_i,\ i \in \mathcal{I}\}$ of L, there exists a finite subset $\mathcal{I}_0 \subseteq \mathcal{I}$ and a $\left(p^l - 1\right) - I$-logarithmic HS-derivation $N^i \in \mathrm{HS}_k\left(R; p^l\right)$ for each $i \in \mathcal{I}_0$ such that if*

$$E = \circ_{i \in \mathcal{I}_0}\left(a_i \bullet \widetilde{N^i}\right)$$

(where we choose any order of composition) there exists a $\left(p^{l-1} - 1\right) - I^e$-logarithmic HS-derivation $T \in \mathrm{HS}_L\left(R_L; p^{l-1}\right)$ and an I^e-logarithmic HS-derivation $F \in \mathrm{HS}_L\left(\log I^e; p^l\right)$ with $\ell(F) > 1$ such that

$$D = E \circ T[p] \circ F.$$

Proof. Since $\Phi_{p^l-1}^{L,R,I} : L \otimes_k \mathrm{IDer}_k(\log I; p^l - 1) \to \mathrm{IDer}_L(\log I^e; p^l - 1)$ is surjective and $D_1 \in \mathrm{IDer}_L(\log I^e; p^l - 1)$, there exists a subset $\mathcal{I}_0 \subset \mathcal{I}$ and a $\delta_i \in \mathrm{IDer}_k(\log I; p^l - 1)$ for each $i \in \mathcal{I}_0$ such that

$$\Phi_{p^l-1}^{L,R,I}\left(\sum_{i \in \mathcal{I}_0} a_i \otimes \delta_i\right) = \sum_{i \in \mathcal{I}_0} a_i \widetilde{\delta_i} = D_1.$$

Let us consider a $(p^l - 1) - I$-logarithmic integral $N^i \in \mathrm{HS}_k(R; p^l)$ of δ_i for all $i \in \mathcal{I}_0$. Then,

$$E := \circ_{i \in \mathcal{I}_0}\left(a_i \bullet \widetilde{N^i}\right)$$

is a p^l-integral of D_1 (note that the order of the composition is not important, E is always an integral of D_1). Since N^i is $(p^l-1)-I$-logarithmic for all $i \in \mathcal{I}_0$, we have that $\widetilde{N^i}$ is $(p^l-1)-I^e$-logarithmic (see Lemma 3.2.8 c.). Hence, by Lemmas 1.1.19 and 1.4.11, E^* is a $(p^l-1)-I^e$-logarithmic integral of $-D_1$. Therefore, $E^* \circ D \in \mathrm{HS}_L(R_L;p^l)$ is a $(p^l-1)-I^e$-logarithmic HS-derivation such that $\ell(E^* \circ D) > 1$. So, we can apply Corollary 3.1.5 to this HS-derivation. Then, there exists a $(p^{l-1}-1)-I^e$-logarithmic HS-derivation $T \in \mathrm{HS}_L(R_L;p^{l-1})$ and $F \in \mathrm{HS}_L(\log I^e;p^l)$ with $\ell(F) > 1$ such that

$$E^* \circ D = T[p] \circ F \Rightarrow D = E \circ T[p] \circ F$$

and the result is proved. □

Theorem 3.2.29 *Let $l \geq 1$ be an integer and assume that $k \to L$ satisfies Hypothesis 3.2.26 and the ideal $I \subseteq R$ satisfies S_l. Let us consider a $\left(p^l-1\right)-I^e$-logarithmic HS-derivation $D \in \mathrm{HS}_L(R_L;p^l)$. Then, for every k-basis $\{a_i,\ i \in \mathcal{I}\}$ of L, there exists, for all $j = 0,\ldots,l$,*

- *a finite subset $\mathcal{I}_j$ of $\mathcal{I}$ and*
- *a $\left(p^{l-j}-1\right)-I$-logarithmic HS-derivation $N^{j,n,i,j-n} \in \mathrm{HS}_k\left(R;p^{l-j}\right)$ for each $i \in \mathcal{I}_{j-n}$, $0 \leq n \leq j$*

such that for all $j = 0,\ldots,l$

$$\bigcup_{m=0}^{j} \left\{a_i^{p^{j-m}},\ i \in \mathcal{I}_m\right\} \text{ is a } k\text{-linearly independent set of } L$$

and, if we take

$$E^j = \circ_{i\in\mathcal{I}_0}\left(a_i^{p^j} \bullet \widetilde{N^{j,j,i,0}}\right) \circ \circ_{i\in\mathcal{I}_1}\left(a_i^{p^{j-1}} \bullet \widetilde{N^{j,j-1,i,1}}\right) \circ \cdots \circ \circ_{i\in\mathcal{I}_j}\left(a_i \bullet \widetilde{N^{j,0,i,j}}\right)$$

for all $j = 0,\ldots,l$ then, there exists $F \in \mathrm{HS}_L(\log I^e;p^l)$ with $\ell(F) > 1$ such that

$$D = E^0 \circ E^1[p] \circ \cdots \circ E^l\left[p^l\right] \circ F.$$

Proof. By Lemma 3.2.28, there exists a finite subset $\mathcal{I}_0 \subseteq \mathcal{I}$ and a $(p^l-1)-I$-logarithmic HS-derivation $N^{0,0,i,0} \in \mathrm{HS}_k(R;p^l)$ for each $i \in \mathcal{I}_0$ such that, if we take $E^0 = \circ_{i\in\mathcal{I}_0}(a_i \bullet N^{0,0,i,0})$, there exists a $(p^{l-1}-1)-I^e$-logarithmic HS-derivation $T^1 \in \mathrm{HS}_L(R_L;p^{l-1})$ and $F \in \mathrm{HS}_L(\log I^e;p^l)$ with $\ell(F) > 1$ such that

$$D = E^0 \circ T^1[p] \circ F.$$

Observe that the set $\mathcal{C}_0 := \{a_i,\ i \in \mathcal{I}_0\}$ of L is k-linearly independent so, by Hypothesis 3.2.26 1., we have that the set $\mathcal{C}_0^p := \{a_i^p,\ i \in \mathcal{I}_0\}$ of L is also k-linearly independent and from the point 2 in Hypothesis 3.2.26 (taking $\{a_i,\ i \in \mathcal{I}\}$ as k-basis) we obtain a subset $\mathcal{L}_1 \subseteq \mathcal{I}$ such that $\mathcal{B}_1 = \mathcal{C}_0^p \cup \{a_i,\ i \in \mathcal{L}_1\}$ is a k-basis of L. Note that if $l \neq 1$, we can apply the previous lemma to T^1 using the k-basis $\mathcal{B}_1$ of L.

Assumption. Let us suppose that doing this process recursively we obtain that, for some integer j such that $0 \leq j \leq l$, there exists for all $s = 0,\ldots,j-1$,

- a finite subset $\mathcal{I}_s$ of $\mathcal{I}$,
- a $(p^{l-s}-1) - I$-logarithmic HS-derivation $N^{s,n,i,s-n} \in \mathrm{HS}_k(R;p^{l-s})$ for all $i \in \mathcal{I}_{s-n}$ and $0 \leq n \leq s$

such that for all $s = 0, \ldots, j-1$,

$$\mathcal{C}_s = \bigcup_{m=0}^{s} \left\{ a_i^{p^{s-m}},\ i \in \mathcal{I}_m \right\} \text{ is } k\text{-linearly independent set of } L$$

and if we take

$$E^s = \circ_{i\in\mathcal{I}_0} \left(a_i^{p^s} \bullet \widetilde{N^{s,s,i,0}} \right) \circ \circ_{i\in\mathcal{I}_1} \left(a_i^{p^{s-1}} \bullet \widetilde{N^{s,s-1,i,1}} \right) \circ \cdots \circ \circ_{i\in\mathcal{I}_s} \left(a_i \bullet \widetilde{N^{s,0,i,s}} \right)$$

for all $s = 0, \ldots, j-1$ (where we choose any order in $\mathcal{I}_*$) then, there exists

- $F \in \mathrm{HS}_L(\log I^e; p^l)$ with $\ell(F) > 1$ and
- a $(p^{l-j}-1) - I^e$-logarithmic HS-derivation $T^j \in \mathrm{HS}_L(R_L; p^{l-j})$

such that,

$$D = E^0 \circ E^1[p] \circ \cdots \circ E^{j-1}\left[p^{j-1}\right] \circ T^j \left[p^j\right] \circ F. \tag{3.8}$$

$\diamond$

Observe that, since $\mathcal{C}_{j-1}$ is k-linearly independent, then $\mathcal{C}_{j-1}^p = \bigcup_{m=0}^{j-1} \left\{ a_i^{p^{j-m}},\ i \in \mathcal{I}_m \right\}$ is also a k-linearly independent finite set of L. So, there exists a subset $\mathcal{L}_j \subseteq \mathcal{I}$ such that $\mathcal{B}_j := \mathcal{C}_{j-1}^p \cup \{a_i,\ i \in \mathcal{L}_j\}$ is a k-basis of L (see Hypothesis 3.2.26 2.).

Let us suppose that $j \neq l$, i.e. $l - j \geq 1$. Then, we can apply Lemma 3.2.28 to T^j using the k-basis $\mathcal{B}_j$ of L. Hence, there exists a finite subsets $\mathcal{I}'_m$ of $\mathcal{I}_m$ for all $m = 0, \ldots, j-1$, a finite set $\mathcal{I}'_j$ of $\mathcal{L}_j$ and a $(p^{l-j}-1) - I$-logarithmic HS-derivation $N^{j,n,i,j-n} \in \mathrm{HS}_k(R;p^{l-j})$ for each $0 \leq n \leq j$ and $i \in \mathcal{I}'_{j-n}$ such that, if we take

$$E^j = \circ_{i\in\mathcal{I}'_0} \left(a_i^{p^j} \bullet \widetilde{N^{j,j,i,0}} \right) \circ \circ_{i\in\mathcal{I}'_1} \left(a_i^{p^{j-1}} \bullet \widetilde{N^{j,j-1,i,1}} \right) \circ \cdots \circ \circ_{i\in\mathcal{I}'_j} \left(a_i \bullet \widetilde{N^{j,0,i,j}} \right)$$

then, there exists $F' \in \mathrm{HS}_L\left(\log I^e; p^{l-j}\right)$ with $\ell(F') > 1$ and a $(p^{l-(j+1)}-1) - I^e$-logarithmic HS-derivation $T^{j+1} \in \mathrm{HS}_L\left(R_L; p^{l-(j+1)}\right)$ such that

$$T^j = E^j \circ T^{j+1}[p] \circ F'.$$

Note that we can take $\mathcal{I}'_m = \mathcal{I}_m$ for all $0 \leq n \leq j-1$ (it is enough to take $N^{j,n,i,j-n} = \mathbb{I}$ for all $i \in \mathcal{I}_m \backslash \mathcal{I}'_m$) and let us rewrite $\mathcal{I}_j := \mathcal{I}'_j$. Moreover, the subset $C_j = \bigcup_{m=0}^{j} \left\{ a_i^{p^{j-m}},\ i \in \mathcal{I}_m \right\} \subseteq \mathcal{B}_j$ of L is k-linearly independent and, if we replace T^j in (3.8), we obtain that

$$D = E^0 \circ \cdots \circ E^{j-1}\left[p^{j-1}\right] \circ E^j[p^j] \circ T^{j+1}[p^{j+1}] \circ F'[p^j] \circ F.$$

Observe that $F[p^j] \in \mathrm{HS}_L(\log I^e; p^l)$ so, $F := F'[p^j] \circ F \in \mathrm{HS}_L(\log I^e; p^l)$ with $\ell(F) > 1$. Therefore, we have the same condition that **Assumption** for $j+1$. So that, we can apply this process until $j = l$.

Let us suppose that $j = l$ in **Assumption**. Then, $T^l \in \mathrm{HS}_L(R_L; 1) \equiv \mathrm{Der}_L(R_L)$ and, by Lemma 3.2.2 with the k-basis $\mathcal{B}_j = \mathcal{B}_l$, there exists a finite subset $\mathcal{I}_l \subseteq \mathcal{L}_l \subseteq \mathcal{I}$ such that

$$T^l = \circ_{i\in\mathcal{I}_0}\left(a_i^{p^j} \bullet \widetilde{N^{l,l,i,0}}\right) \circ \circ_{i\in\mathcal{I}_1}\left(a_i^{p^{j-1}} \bullet \widetilde{N^{l,l-1,i,1}}\right) \circ \cdots \circ \left(\circ_{i\in\mathcal{I}_l} a_i \bullet \widetilde{N^{l,0,i,l}}\right)$$

where $N^{l,n,i,l-n} \in \mathrm{HS}_k(R; 1)$ for each $i \in \mathcal{I}_{l-n}$ and $0 \le n \le l$. It is obvious that

$$\bigcup_{m=0}^{l} \left\{a_i^{p^{j-m}},\ i \in \mathcal{I}_{j-m}\right\}$$

is a k-linearly independent set of L and since $D = E_0 \circ E_1[p] \circ \cdots \circ E^{l-1}[p^{l-1}] \circ T^l[p^l] \circ F$, we have the result.

□

Theorem 3.2.30 *Let $k \to L$ be a ring extension satisfying Hypothesis 3.2.26 and A a commutative finitely generated k-algebra. Then, $\Phi_m^{L,A} : L \otimes_k \mathrm{IDer}_k(A; m) \to \mathrm{IDer}_L(A_L; m)$ is an isomorphism of A_L-modules for all $m \in \mathbb{N}$. Moreover, $\mathrm{Leaps}_k(A) = \mathrm{Leaps}_L(A_L)$.*

Proof. If $\Phi_m^{L,A}$ is bijective, since L is faithfully flat over k, we have that $\mathrm{Leaps}_k(A) = \mathrm{Leaps}_L(A_L)$ by Lemma 3.2.13. Moveover, by Lemma 3.2.11 1., $\Phi_m^{L,A}$ is injective for all $m \in \mathbb{N}$. So, we only need to prove that $\Phi_m^{L,A}$ is surjective.

Recall that we consider $A = R/I$ where $R = k[x_1, \ldots, x_d]$ is a polynomial ring in a finite number of variable and $I \subseteq R$ an ideal. Then, by Lemma 3.2.11 2., $\Phi_m^{L,A}$ is surjective if and only if $\Phi_m^{L,R,I} : L \otimes_k \mathrm{IDer}_k(\log I; m) \to \mathrm{IDer}_L(\log I^e; m)$ is surjective. So, we will prove that $\Phi_m^{L,R,I}$ is surjective for all $m \in \mathbb{N}$. Moreover, since leaps only occur at powers of p (Theorem 2.5.1), it is enough to see that $\Phi_m^{L,R,I}$ is surjective when $m = p^l$ for $l \ge 0$. We proceed by induction on $l \ge 0$.

The case $l = 0$ is Proposition 3.2.5. Now, let us assume that $\Phi_m^{L,R,I}$ is surjective for all $m < p^l$ with $l \ge 1$, i.e. I satisfies S_l, and we prove the theorem for $\Phi_{p^l}^{L,R,I}$ with $l \ge 1$.

Let $\delta \in \mathrm{IDer}_L(\log I^e, p^l)$ be an L-derivation of R_L, then there exists $D \in \mathrm{HS}_k(\log I^e; p^l)$ an integral of δ. In particular, D is $(p^l - 1) - I^e$-logarithmic and we can apply Theorem 3.2.29 to D. Let us consider a k-basis $\{a_i,\ i \in \mathcal{I}\}$ of L. Then, for all $j = 0, \ldots, l$, there exists

- a finite subset $\mathcal{I}_j$ of $\mathcal{I}$ and
- a $(p^{l-j} - 1) - I$-logarithmic HS-derivation $N^{j,n,i,j-n} \in \mathrm{HS}_k(R; p^{l-j})$ for each $i \in \mathcal{I}_{j-n}$ and $0 \le n \le j$

such that, for all $j = 0, \ldots, l$ the subset

$$\bigcup_{m=0}^{j} \left\{a_i^{p^{j-m}},\ i \in \mathcal{I}_m\right\} \text{ of } L \text{ is } k\text{-linearly independent}$$

and, if we take

$$E^j = \left(\circ_{i\in\mathcal{I}_0} a_i^{p^j} \bullet \widetilde{N^{j,j,i,0}}\right) \circ \cdots \circ \left(\circ_{i\in\mathcal{I}_j} a_i \bullet \widetilde{N^{j,0,i,j}}\right)$$

for all $j = 0, \ldots, l$, there exists $F \in \mathrm{HS}_L(\log I^e; p^l)$ with $\ell(F) > 1$ such that

$$D = E^0 \circ E^1[p] \circ \cdots \circ E^l\left[p^l\right] \circ F.$$

For each $j = 0, \ldots, l$, $N^{j,n,i,j-n}$ is $\left(p^{l-j}-1\right) - I$-logarithmic for all $0 \le n \le j$ and $i \in \mathcal{I}_{j-n}$. So, $\widetilde{N^{j,n,i,j-n}}$ is $\left(p^{l-j}-1\right) - I^e$-logarithmic for all $0 \le n \le j$ and $i \in \mathcal{I}_{j-n}$ (see Lemma 3.2.8 c.). Therefore, by Lemma 1.1.19, $E^j \in \mathrm{HS}_L(R_L; p^{l-j})$ is $\left(p^{l-j}-1\right) - I^e$-logarithmic and

$$E^j_{p^{l-j}} = \sum_{i\in\mathcal{I}_0} \left(a_i^{p^j}\right)^{p^{l-j}} \widetilde{N^{j,j,i,0}_{p^{l-j}}} + \cdots + \sum_{i\in\mathcal{I}_j} a_i^{p^{l-j}} \widetilde{N^{j,0,i,j}_{p^{l-j}}} + \textit{some } I^e\textit{-diff. op.}$$

Hence, from Lemma 1.1.18, $E^j\left[p^j\right] \in \mathrm{HS}_L\left(R_L; p^l\right)$ is $\left(p^l-1\right) - I^e$-logarithmic for all j and

$$E^j[p^j]_{p^l} = E^j_{p^{l-j}} = \sum_{k=0}^{j}\sum_{i\in\mathcal{I}_k} a_i^{p^{l-k}} \widetilde{N^{j,j-k,i,k}_{p^{l-j}}} + \textit{some } I^e\textit{-diff. op.}$$

So, by Lemma 1.1.19,

$$D_{p^l} = \sum_{j=0}^{l} E^j[p^j]_{p^l} + \textit{some } I^e\textit{-diff. op.} = \sum_{j=0}^{l}\sum_{k=0}^{j}\sum_{i\in\mathcal{I}_k} a_i^{p^{l-k}} \widetilde{N^{j,j-k,i,k}_{p^{l-j}}} + \textit{some } I^e\textit{-diff. op.}$$

Since D_{p^l} is an I^e-differential operator,

$$\sum_{j=0}^{l}\sum_{k=0}^{j}\sum_{i\in\mathcal{I}_k} a_i^{p^{l-k}} \widetilde{N^{j,j-k,i,k}_{p^{l-j}}} = \sum_{i\in\mathcal{I}_0} a_i^{p^l}\left(\sum_{j=0}^{l} \widetilde{N^{j,j,i,0}_{p^{l-j}}}\right) + \sum_{i\in\mathcal{I}_1} a_i^{p^{l-1}}\left(\sum_{j=1}^{l} \widetilde{N^{j,j-1,i,1}_{p^{l-j}}}\right) + \cdots + \sum_{i\in\mathcal{I}_l} a_i \widetilde{N_1^{l,0,i,l}}$$

is an I^e-differential operator.

Since $\mathcal{C} := \bigcup_{k=0}^{l}\left\{a_i^{p^{l-k}},\ i\in\mathcal{I}_k\right\}$ is a k-linearly independent finite set of L and $\{a_i,\ i\in\mathcal{I}\}$ is a k-basis of L, by Hypothesis 3.2.26, there exists $\mathcal{L} \subseteq \mathcal{I}$ such that $\mathcal{C} \cup \{a_i,\ i\in\mathcal{L}\}$ is a k-basis of L. Hence, we can deduce, in the same way that in the proof of Proposition 3.2.5, that

$$\sum_{j=0}^{l} N^{j,j,i,0}_{p^{l-j}} \text{ is an } I\text{-differential operator for all } i\in\mathcal{I}_0$$

(recall that, by Remark 3.2.7, $\widetilde{N^{j,j,i,0}_{p^{l-j}}}_{|R} = N^{j,j,i,0}_{p^{l-j}}$).

For all $i \in \mathcal{I}_0$, let us consider $D^i = N^{0,0,i,0} \circ N^{1,1,i,0}[p] \circ \cdots \circ N^{l,l,i,0}\left[p^l\right] \in \mathrm{HS}_k(R; p^l)$ an integral of $N_1^{0,0,i,0}$. Since $N^{j,j,i,0} \in \mathrm{HS}_k(R; p^{l-j})$ is $(p^{l-j}-1) - I$-logarithmic for all $j = 0, \ldots, l$,

$N^{j,j,i,0}[p^j] \in \mathrm{HS}_k(R;p^l)$ is $(p^l-1)-I$-logarithmic (Lemma 1.1.18) and by Lemma 1.1.19, $D^i \in \mathrm{HS}_k(R;p^l)$ is $(p^l-1)-I$-logarithmic and

$$D^i_{p^l} = \sum_{j=0}^{l} N^{j,j,i,0}_{p^{l-j}} + \textit{some } I\textit{-differential operator.}$$

So, $D^i \in \mathrm{HS}_k\left(\log I;p^l\right)$ and we can deduce that $N_1^{0,0,i,0} \in \mathrm{IDer}_k\left(\log I;p^l\right)$. On the other hand, we recall that

$$D = E^0 \circ E^1[p] \circ \cdots \circ E^l\left[p^l\right] \circ F$$

where $\ell(F) > 1$. Then, $D_1 = E^0_1$ and, since $E^0 = \circ_{i\in\mathcal{I}_0}\left(a_i \bullet \widetilde{N^{0,0,i,0}}\right)$, we have that

$$D_1 = \sum_{i\in\mathcal{I}_0} a_i\widetilde{N_1^{0,0,i,0}} = \Phi^{L,R,I}_{p^l}\left(\sum_{i\in\mathcal{I}_0}\left(a_i \otimes N_1^{0,0,i,0}\right)\right)$$

Therefore, $\Phi^{L,R,I}_{p^l}$ is bijective.

□

Remark 3.2.31 *If we change the condition 2. in Hypothesis 3.2.26 for*

2'. There exists $\{a_i,\ i\in\mathcal{I}\}$ a k-basis of L such that $\left\{a^{p^r},\ i\in\mathcal{I}\right\} \subseteq \{a_i,\ i\in\mathcal{I}\}$ for all $r\geq 1$.

then, Theorems 3.2.29 and 3.2.30 are true for that basis. For example, if we take $L = k[t_i\ |i \in \mathcal{I}]$, we can apply these theorems and we obtain that $\Phi^{L,A}_m$ is an isomorphism.

As we said at the beginning of this section, we want to prove that $\Phi^{L,A}_m : L\otimes_k \mathrm{IDer}_k(A:m) \to \mathrm{IDer}_L(A_L;m)$ is an isomorphism when L is a separable extension over a field k of characteristic $p>0$. Let us recall a characterization for such type of extensions that appears in [Bo]:

Theorem 3.2.32 [Bo, **§15.4. Th. 2**] *Let k be a field of characteristic $p>0$, $k^{p^{-\infty}}$ a perfect closure of k and L a commutative k-algebra. The following properties are equivalent:*

1. *L is separable.*
2. *There exists an extension k' of k such that k' is perfect and $k'\otimes_k L$ is reduced.*
3. *The ring $k^{p^{-\infty}} \otimes_k L$ is reduced.*
4. *The ring $k'\otimes_k L$ is reduced for every extension k' of k which is of finite degree and p-radical of height ≤ 1.*
5. *For every family $\{a_i\}$ of elements of L linearly independent over k, the family $\{a_i^p\}$ is linearly independent over k.*
6. *There exists a basis $\{a_i\}$ of the vector k-space L such that the family $\{a_i^p\}$ is linearly independent over k.*

Note that if k is a field, then the second condition of Hypothesis 3.2.26 always holds and the first one is equivalent to L being a separable k-algebra thanks to the previous theorem. Then, if L is separable over k, L satisfies Hypothesis 3.2.26 and we have as a straightforward consequence of Theorem 3.2.30 the following result.

Corollary 3.2.33 *Let k be a field of characteristic $p > 0$, $k \to L$ a separable ring extension and A a commutative finitely generated k-algebra. Then, $\Phi_m^{L,A} : L \otimes_k \mathrm{IDer}_k(A; m) \to \mathrm{IDer}_k(A_L; m)$ is an isomorphism of A_L-modules for all $m \geq 1$. Moreover, $\mathrm{Leaps}_k(A) = \mathrm{Leaps}_L(A_L)$.*

Chapter 4

Integrable derivations for some plane curves

Although there is an algorithm to decide whether a given HS-derivation of length $m-1$ can be extended to a HS-derivation of length m or not, at present we do not know any real algorithm to decide if a given derivation is m-integrable or not, the main difficulty is the fact that a derivation can be m-integrable, but not necessarily any previously known $(m-1)$-integral can be extended to an m-integral. So the effective computation of a system of generators of the modules of m-integrable derivations remains a difficult problem.

In this chapter we want to show how to calculate the modules of m-integrable derivations, for $m \in \overline{\mathbb{N}}$, of quotient of the polynomial ring $k[x,y]$ in two variables over an ideal generated by certain plane curves where k will be a reduced ring of positive characteristic (i.e. $\mathbb{F}_p \subseteq k$).

4.1 Integrable derivations for $x^n - y^q$

Let $R = k[x,y]$ be the polynomial ring in two variables over a reduced ring k of characteristic $p > 0$ and $h = x^n - y^q \in R$ with $n, q \neq 0$. In this section we will study the modules of m-integrable k-derivations of $A = R/\langle h \rangle$ of length $m \in \overline{\mathbb{N}}$.

In this section we will use the following notations: Let $\alpha := \mathrm{val}_p(n)$ be the p-adic valuation of n and $s = n/p^\alpha$. We will denote by m the remainder of the division of q by p and $\beta := \mathrm{val}_p(q-m)$. We write

$$\gamma := \min\{i \in \mathbb{N} \mid ip^\alpha \geq q-1\} = \lceil (q-1)/p^\alpha \rceil.$$

Proposition 4.1.1 *Let k be a commutative reduced ring of characteristic $p > 0$ and $R = k[x,y]$ the polynomial ring over k. We set $A = R/\langle h \rangle$ where $h = x^n - y^q$ with $n, q \neq 0$. For $\delta \in \mathrm{Der}_k(\log h)$, we denote $\overline{\delta} = \Pi_n^{\langle h \rangle}(\delta)$ (Corollary 1.2.3).*

- *If $n, q \neq 0 \mod p$, then $\mathrm{IDer}_k(A) = \mathrm{Der}_k(A) = \langle \overline{\delta_1}, \overline{\delta_2} \rangle$ where $\delta_1 = qx\partial_x + ny\partial_y$ and $\delta_2 = qy^{q-1}\partial_x + nx^{n-1}\partial_y$.*

- *If $n = 0 \mod p$ and $q = 1$, then $\mathrm{IDer}_k(A) = \mathrm{Der}_k(A) = \langle \overline{\partial_x} \rangle$.*

- *If $\alpha, m \geq 1$ and $q \geq 2$, then*

$$\operatorname{IDer}_k(A;i) = \begin{cases} \begin{cases} \langle \overline{\partial_x} \rangle & \text{if } 1 \leq i < p^\alpha \\ \langle \overline{x\partial_x}, \overline{y^\gamma \partial_x} \rangle & \text{if } p^\alpha \leq i < p^{\alpha+\beta} \\ \left\langle \overline{x\partial_x}, \overline{y^{\gamma+1}\partial_x} \right\rangle & \text{if } i \geq p^{\alpha+\beta} \text{ or } i = \infty \end{cases} & \text{if } s = 1,\ \alpha \leq \beta,\ m = 1 \\ \begin{cases} \langle \overline{\partial_x} \rangle & \text{if } 1 \leq i < p^\alpha \\ \langle \overline{x\partial_x}, \overline{y^\gamma \partial_x} \rangle & \text{if } i \geq p^\alpha \text{ or } i = \infty \end{cases} & \text{otherwise.} \end{cases}$$

Proof. Let $\delta = u\partial_x + v\partial_y$ be a k-derivation of R. To prove this result it is enough to show which derivations are h-logarithmically i-integrable for $i \in \overline{\mathbb{N}}$ (Corollary 1.2.3).

- $n, q \not\equiv 0 \mod p$.

We have to find the pairs $(u,v) \in R^2$ such that $\delta(h) = nux^{n-1} - qvy^{q-1} \in \langle h \rangle$, i.e. the pairs $(u,v) \in R^2$ such that there exists $F \in R$ holding the equation $nx^{n-1}u - qy^{q-1}v = F(x^n - y^q)$. Then, $x^{n-1}(nu - Fx) = y^{q-1}(qv - Fy)$. Hence,

$$\begin{cases} nu - Fx = Gy^{q-1} \\ qv - Fy = Gx^{n-1} \end{cases} \Rightarrow \begin{cases} u = G(1/n)y^{q-1} + F(1/n)x \\ v = G(1/q)x^{n-1} + F(1/q)y. \end{cases}$$

Therefore, $\operatorname{Der}_k(\log h) = \langle \delta_1, \delta_2 \rangle$ where $\delta_1 = qx\partial_x + ny\partial_y$ and $\delta_2 = qy^{q-1}\partial_x + nx^{n-1}\partial_y$. Note that h is a quasi-homogenous polynomial with respect to the weights $w(x) = q$ and $w(y) = n$. By Theorem 1.2.8, the Euler vector field, δ_1, is h-logarithmically ∞-integrable. On the other hand, the gradient of h is $J^0 = \langle x^{n-1}, y^{q-1} \rangle$, so $\delta_2 \in J^0 \operatorname{Der}_k(R)$ and from Proposition 1.2.7 we know that δ_2 is h-logarithmically ∞-integrable too. So, $\operatorname{IDer}_k(A) = \operatorname{Der}_k(A) = \langle \overline{\delta_1}, \overline{\delta_2} \rangle$.

- $n = 0 \mod p$ *and* $q = 1$.

The condition for δ to be h-logarithmic is that $v \in \langle h \rangle$, so $\operatorname{Der}_k(\log h) = \langle \partial_x, h\partial_y \rangle$. In this case $J^0 = \langle 1 \rangle$. Hence, any $\langle h \rangle$-logarithmic derivation is integrable (Proposition 1.2.7). Then, $\operatorname{IDer}_k(A) = \operatorname{Der}_k(A) = \langle \overline{\partial}_x \rangle$.

- $\alpha, m \geq 1$ *and* $q \geq 2$.

Note that $n = sp^\alpha$. In order for δ to be h-logarithmic, $qvy^{q-1} \in \langle h \rangle$, i.e. $qvy^{q-1} = F(x^n - y^q)$ for some $F \in R$. So, $(qv + Fy)y^{q-1} = Fx^n$. Hence,

$$\begin{cases} qv + Fy = Gx^n \\ F = Gy^{q-1} \end{cases} \Rightarrow v = (1/q)G(x^n - y^q) \text{ for some } G \in R.$$

Therefore, $\operatorname{Der}_k(\log h) = \langle \partial_x, h\partial_y \rangle$. Since $h\partial_y$ is the zero derivation on A, we can focus on the h-logarithmically integrability of $\delta = u\partial_x$ with $u \in R$. Let $u_x \in R$ and $u_y \in k[y]$ such that

$$u = u_x(x,y)x + u_y(y) \Rightarrow \delta = u\partial_x = u_x x\partial_x + u_y\partial_x.$$

Since h is a quasi-homogeneous polynomial with respect to the weights $w(x) = q$ and $w(y) = sp^\alpha$, the Euler vector field, $\chi = qx\partial_x$, is h-logarithmically integrable, and hence also $u_x x\partial_x$ are. Thanks to this and since $\mathrm{IDer}_k(\log h; i)$ is an R-module for all i,

$$\delta \in \mathrm{IDer}_k(\log h; i) \Leftrightarrow u_y\partial_x \in \mathrm{IDer}_k(\log h; i).$$

So, we will see the integrability of $\delta = u\partial_x$ with $u \in k[y]$. Let us consider $\varphi : R \to R[|\mu|]$ a generic integral of $u\partial_x$:

$$\begin{array}{rccl} \varphi : & R & \longrightarrow & R[|\mu|] \\ & x & \longmapsto & x + u\mu + u_2\mu^2 + \cdots \\ & y & \longmapsto & y \quad + \quad v_2\mu^2 + \cdots \end{array}$$

To show that δ is i-integrable it is enough to prove that there exist u_j, v_j for $2 \leq j \leq i$ such that the coefficients of μ^j in $\varphi(h)$ belong to $\langle h \rangle$ for all $j \leq i$. We will denote by μ_j the coefficient of μ^j in the equation

$$\varphi(h) = \left(x^{p^\alpha} + u^{p^\alpha}\mu^{p^\alpha} + u_2^{p^\alpha}\mu^{2p^\alpha} + \cdots\right)^s - \left(y + v_2\mu^2 + v_3\mu^3 + \cdots\right)^q. \tag{4.1}$$

Suppose that there exists i such that $2 \leq i < p^\alpha$. Then, $\mu_2 = -qy^{q-1}v_2$ has to belong to $\langle h \rangle$. As we saw before, that implies that $v_2 \in \langle h \rangle$, so we can put $v_2 = 0$. Let us assume that $v_l = 0$ for all $2 \leq l < i < p^\alpha$. In this case, $\mu_i = -qy^{q-1}v_i$ and, as the same before, we can put $v_i = 0$. Then, for all $i < p^\alpha$,

$$\mathrm{IDer}_k(A; i) = \mathrm{Der}_k(A) = \left\langle \overline{\partial_x} \right\rangle$$

and we can write the equation (4.1) as:

$$\left(x^{p^\alpha} + u^{p^\alpha}\mu^{p^\alpha} + u_2^{p^\alpha}\mu^{2p^\alpha} + \cdots\right)^s - \left(y + v_{p^\alpha}\mu^{p^\alpha} + v_{p^\alpha+1}\mu^{p^\alpha+1} + \cdots\right)^q \in \langle h \rangle \tag{4.2}$$

Now, let us consider

$$\mu_{p^\alpha} = sx^{p^\alpha(s-1)}u^{p^\alpha} - qy^{q-1}v_{p^\alpha}. \tag{4.3}$$

We have to see that if there is $v_{p^\alpha} \in R$ such that $\mu_{p^\alpha} \in \langle h \rangle$. Let $F \in R$ such that $sx^{p^\alpha(s-1)}u^{p^\alpha} - qy^{q-1}v_{p^\alpha} = F(x^n - y^q)$. Then,

$$x^{p^\alpha(s-1)}\left(su^{p^\alpha} - Fx^{p^\alpha}\right) = y^{q-1}\left(mv_{p^\alpha} - Fy\right) \Rightarrow su^{p^\alpha} - Fx^{p^\alpha} = Gy^{q-1}$$

for some $G \in R$. Since $u \in k[y]$, we can write $u = \sum u_i y^i$ where $u_i \in k$ and the previous expression implies that $u_i^{p^\alpha} = 0$ for all i such that $ip^\alpha < q - 1$. So that $u_i = 0$ because k is reduced. Hence, we can write $u = w(y)y^\gamma$ where $\gamma = \min\{i \in \mathbb{N} \mid ip^\alpha \geq q - 1\}$ and $w(y) \in k[y]$. Substituting the expression of u on (4.3), we can deduce that

$$sx^{p^\alpha(s-1)}w^{p^\alpha}y^{\gamma p^\alpha - (q-1)} - qv_{p^\alpha} \in \langle h \rangle \;\Rightarrow\; v_{p^\alpha} \in (s/q)x^{p^\alpha(s-1)}w^{p^\alpha}y^{\gamma p^\alpha - (q-1)} + \langle h \rangle \tag{4.4}$$

Therefore, A has a leap at p^α and

$$\mathrm{IDer}_k(A; p^\alpha) = \langle \overline{x\partial_x}, \overline{y^\gamma\partial_x} \rangle \text{ where } \gamma = \min\{i \in \mathbb{N} \mid ip^\alpha \geq q - 1\}.$$

Let us write $q = tp^\beta + m$. Note that the only case where $\gamma p^\alpha = q - 1$ is $q = tp^\beta + 1$ and $\alpha \leq \beta$. To see that we have to show when the equality $ip^\alpha = q - 1$ holds. If we substitute q, we obtain that $ip^\alpha = tp^\beta + m - 1$, then $ip^\alpha - tp^\beta = m - 1$. Hence, $m-1$ has to be a multiple of p and, since $m < p$, we have that $m - 1 = 0$. So, $ip^\alpha = tp^\beta$. Now, if $\beta < \alpha$, then $ip^{\alpha-\beta} = t \neq 0$ mod p giving us a contradiction. So, $\beta \geq \alpha$, $q = tp^\beta + 1$ and $i = tp^{\beta-\alpha}$.

Let us focus on the case when $s = 1$ and $q = tp^\beta + 1$ with $\alpha \leq \beta$.

- *Case $q = tp^\beta + 1$, $\alpha \leq \beta$ and $s = 1$.* Observe that $t \neq 0$ because $q \geq 2$ and we have that $\gamma = tp^{\beta-\alpha}$. We will study the integrability of $w(y)y^\gamma\partial_x$ in this particular case.

  Substituting the values of q and s in the equation (4.2) and (4.4) we obtain:

$$\left(x^{p^\alpha} + u^{p^\alpha}\mu^{p^\alpha} + u_2^{p^\alpha}\mu^{2p^\alpha} + \cdots\right) - \\ - \left(y^{p^\beta} + v_{p^\alpha}^{p^\beta}\mu^{p^{\alpha+\beta}} + v_{p^\alpha+1}^{p^\beta}\mu^{(p^\alpha+1)p^\beta} + \cdots\right)^t \left(y + v_{p^\alpha}\mu^{p^\alpha} + \cdots\right) \in \langle h \rangle$$

  and

$$v_{p^\alpha} \in cw^{p^\alpha} + \langle h \rangle$$

  for $c = s/q$. Let us consider i such that $p^\alpha < i < p^{\alpha+\beta}$. If $i = jp^\alpha$ for some $j \geq 2$, then $\mu_i = u_j^{p^\alpha} - y^{tp^\beta}v_i$. Otherwise, $\mu_i = -y^{tp^\beta}v_i$. So, $wy^\gamma\partial_x$ is h-logarithmically i-integrable for all $i < p^{\alpha+\beta}$ (it's enough to put $u_j = v_i = 0$, so that $\mu_i \in \langle h \rangle$). Now,

$$\mu_{p^{\alpha+\beta}} = u_{p^\beta}^{p^\alpha} - ty^{(t-1)p^\beta+1}v_{p^\alpha}^{p^\beta} - y^{tp^\beta}v_{p^{\alpha+\beta}}$$

  has to belong to $\langle h \rangle$. So, substituting the value of v_{p^α}, we have that

$$u_{p^\beta}^{p^\alpha} - ctw^{p^{\alpha+\beta}}y^{(t-1)p^\beta+1} - y^{tp^\beta}v_{p^{\alpha+\beta}} = G\left(x^{p^\alpha} - y^{tp^\beta+1}\right)$$

  for some $G \in R$. The coefficient of y^j with $j = (t-1)p^\beta + 1$ in this equality is $tcw_0^{p^\alpha} = 0$ where w_0 is the independent term of w. Since R is reduced, $w_0 = 0$. Hence, $y^\gamma\partial_x$ is not $p^{\alpha+\beta}$-integrable. However, if $w = w'y$ with $w' \in k[y]$, the previous equation is

$$u_{p^\beta}^{p^\alpha} - ctw'^{p^{\alpha+\beta}}y^{q+p^\beta(p^\alpha-1)} - y^{tp^\beta}v_{p^{\alpha+\beta}} = G\left(x^{p^\alpha} - y^{tp^\beta+1}\right).$$

  Then, there exists a solution, for instance $u_{p^\beta} = 0$ and $v_{p^{\alpha+\beta}} = -ctw'^{p^{\alpha+\beta}}y^{p^\beta(p^\alpha-1)+1}$. In conclusion, in this case A has a leap at $p^{\alpha+\beta}$ and

$$\mathrm{IDer}_k\left(A; p^{\alpha+\beta}\right) = \left\langle \overline{x\partial_x}, \overline{y^{\gamma+1}\partial_x} \right\rangle.$$

Until now we saw that, for all $q \geq 2$

$$\mathrm{IDer}_k\left(A; p^\alpha\right) = \left\langle \overline{x\partial_x}, \overline{y^\gamma\partial_x} \right\rangle \text{ where } \gamma = \min\{i \in \mathbb{N} \mid ip^\alpha \geq q - 1\}$$

and moreover, when $q = tp^\beta + 1$, $1 \leq \alpha \leq \beta$ and $s = 1$, $y^\gamma\partial_x$ is not h-logarithmically integrable but

$$\mathrm{IDer}_k\left(A; p^{\alpha+\beta}\right) = \left\langle \overline{x\partial_x}, \overline{y^{\gamma+1}\partial_x} \right\rangle.$$

Let us rewrite $\gamma := \gamma + 1$ in the latter case. We will see that $y^\gamma \partial_x$ is integrable on A for all $q \geq 2$. Consider

$$\begin{array}{rccl} \varphi : & A & \longrightarrow & A[|\mu|] \\ & x & \longmapsto & x + y^\gamma \mu \\ & y & \longmapsto & y + v_1 \mu^{p^\alpha} + v_2 \mu^{2p^\alpha} + \cdots \end{array}$$

where for all $i \geq 1$,

$$v_i = C_i x^{p^\alpha(s-\sigma)} y^{i\gamma p^\alpha - (\tau+1)q+1} \text{ for } i = \tau s + \sigma \text{ with } \tau \geq 0 \text{ and } \sigma = 1, \ldots, s,$$

$$C_i = \frac{1}{q}\left[\binom{s}{i} - \sum_{j \in I_i} D_j\right] \text{ where } \binom{s}{i} = 0 \text{ if } i > s,$$

$$I_i = \left\{ j = (j_0, j_1, \ldots, j_{i-1}) \in \mathbb{N}^i \mid j_r \geq 0 \ \forall r = 0, \ldots, i-1, \ |j| = q, \ \sum_{r=1}^{i-1} r j_r = i \right\}$$

and, for all $j = (j_0, j_1, \ldots, j_l)$ with $l \geq 1$,

$$D_j = \binom{q}{j} C_1^{j_1} \cdots C_l^{j_l} \text{ with } \binom{q}{j} = \frac{q!}{j_0! \cdots j_l!}.$$

We have to prove that φ is well-defined. First we will see that $i\gamma p^\alpha - (\tau+1)q + 1 \geq 0$, i.e. $(\tau s + \sigma)\gamma p^\alpha - \tau q \geq q - 1$.

- When $\gamma p^\alpha > q - 1$, then $\gamma p^\alpha \geq q$, but q is not multiple of p, so $\gamma p^\alpha \geq q + 1$ and therefore

  $$(\tau s + \sigma)\gamma p^\alpha - \tau q \geq (\tau s + \sigma)(q+1) - \tau q = (\tau(s-1) + \sigma)q + \tau s + \sigma \geq q - 1$$

  because $s - 1 \geq 0$ and $\sigma \geq 1$.

- Let us consider $\gamma p^\alpha = q - 1$. As we have seen before, the previous equality only holds if $q = tp^\beta + 1$ and $\alpha \leq \beta$. If $s = 1$, then we have considered $\gamma + 1$, so we are in the first point. Therefore, we just have to consider $s \geq 2$. In this case, we have to prove that $(\tau s + \sigma)\gamma p^\alpha - \tau q = (\tau s + \sigma)(q-1) - \tau q \geq q - 1$. Then

  $$(\tau s + \sigma)(q-1) - \tau q \geq (2\tau + \sigma)(q-1) - \tau q = (\tau + \sigma)q - (2\tau + \sigma).$$

  So,

  $$(\tau + \sigma)q - (2\tau + \sigma) \geq q - 1 \Leftrightarrow (\tau + \sigma - 1)q \geq 2\tau + \sigma - 1$$

  and this is true because $q \geq 2$ and $\tau + \sigma - 1 \geq 0$. Note that if $\tau + \sigma - 1 = 0$ then $\tau = 0$ and $\sigma = 1$, so $2\tau + \sigma - 1 = 0$ too.

Now, we have to show that $\varphi(h) = 0$ in $A[|\mu|]$. The equation is:

$$\varphi(h) = \left(x^{p^\alpha} + y^{\gamma p^\alpha} \mu^{p^\alpha}\right)^s - \left(y + v_1 \mu^{p^\alpha} + v_2 \mu^{2p^\alpha} + \cdots\right)^q.$$

Since all degrees of the monomial which appeared in this equation are multiple of p^α, let us denote μ_i to the coefficient of degree ip^α. Then

$$\mu_i = \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} - \widetilde{\mu_i}$$

where $\widetilde{\mu_i}$ is the coefficient of μ^{ip^α} of $\left(y + v_1\mu^{p^\alpha} + v_2\mu^{2p^\alpha} + \cdots\right)^q$. This coefficient can be found on

$$\left(y + v_1\mu^{p^\alpha} + \cdots + v_i\mu^{ip^\alpha}\right)^q = \sum_{|j|=q} \binom{q}{j} y^{j_0} v_1^{j_1} \cdots v_i^{j_i} \mu^{p^\alpha(j_1+\cdots+ij_i)}.$$

We just have to consider all j such that $j_1 + \cdots + ij_i = i$. Observe that there exists only one j holding this equation such that $j_i \neq 0$, This j is $(q-1, 0, \ldots, 0, 1)$ where 1 is in the position i. So, we can identify the set of all these j with $I_i \cup (q-1, 0, \ldots, 0, 1)$. Let us calculate a term of $\widetilde{\mu_i}$. Fixed j, we have

$$\binom{q}{j} y^{j_0} v_1^{j_1} \cdots v_i^{j_i} = \binom{q}{j} C_1^{j_1} \cdots C_i^{j_i} x^{ap^\alpha} y^b = D_j x^{ap^\alpha} y^b$$

where

$$a = \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}(s-\sigma) \geq 0 \quad \text{and} \quad b = j_0 + \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\left(\gamma p^\alpha(\tau s+\sigma) - (\tau+1)q + 1\right) \geq 0.$$

We are going to study these exponents.

$$a = s \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma} - \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\sigma = s(q-j_0) - \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\sigma.$$

On the other side, if we write $i = ls + r$ where $l \geq 0$ and $1 \leq r \leq s$, we have that

$$ls + r = i = \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}(\tau s+\sigma) = s \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\tau + \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\sigma.$$

Then, if we denote $T = \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\tau$ and we substitute on a, we have

$$a = s(q-j_0) - ((l-T)s + r) = s(q - j_0 - l + T) - r \geq 0$$

If $q - j_0 - l + T < 1$, then $a < 0$ so $q - j_0 - l + T \geq 1$ and we can write

$$a = (q - j_0 - l + T - 1)s + s - r.$$

Observe that $s - r \geq 0$ because $1 \leq r \leq s$. Now,

$$\begin{aligned} b = \; & j_0 + \gamma p^\alpha \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}(\tau s+\sigma) - q \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\tau - q \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma} + \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma} \\ = \; & j_0 + \gamma p^\alpha i - qT - q(q-j_0) + (q-j_0) = i\gamma p^\alpha - q(T + q - j_0 - 1). \end{aligned}$$

So,
$$\binom{q}{j} y^{j_0} v_1^{j_1} \cdots v_i^{j_i} = D_j x^{(q-j_0-l+T-1)sp^\alpha+(s-r)p^\alpha} y^{i\gamma p^\alpha - q(T+q-j_0-1)}.$$
Since $x^{sp^\alpha} = y^q$ in A,
$$\binom{q}{j} y^{j_0} v_1^{j_1} \cdots v_i^{j_i} = D_j x^{(s-r)p^\alpha} y^{i\gamma p^\alpha + q(q-j_0-l+T-1)-q(T+q-j_0-1)} = D_j x^{(s-r)p^\alpha} y^{i\gamma p^\alpha - lq}.$$
Recall that $i = ls + r$ where $l \geq 0$ and $1 \leq r \leq s$. Then, we have
$$\begin{aligned}
\widetilde{\mu_i} &= \sum_{\substack{|j|=q \\ j_1+\ldots+ij_i=i}} D_j x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} = \left(\sum_{j\in I_i} D_j + D_{(q-1,0,\ldots,0,1)} \right) x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} \\
&= \left(\sum_{j\in I_i} D_j + qC_i \right) x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} = \left(\sum_{j\in I_i} D_j + q(1/q) \left[\binom{s}{i} - \sum_{j\in I_i} D_j \right] \right) x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} \\
&= \binom{s}{i} x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq}.
\end{aligned}$$
So,
$$\mu_i = \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} - \binom{s}{i} x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq}.$$
If $i > s$, then $\binom{s}{i} = 0$, and hence $\mu_i = 0$. If $i \leq s$, then $i = 0 \cdot s + i$, i.e., $l = 0$ and $r = i$, so
$$\mu_i = \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} - \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} = 0.$$
Hence φ is well-defined and the proposition is proved.

□

Remark 4.1.2 *Let us consider $h = x^n - y^q \in R = k[x, y]$ with $n, q \neq 0$ and $A = R/\langle h\rangle$. Assume that $n \neq 0 \mod p$ and $q = 0 \mod p$. Then, we can compute $\mathrm{IDer}_k(A)$ thanks to Proposition 4.1.1. Observe that the map $f : A \to R/\langle x^q - y^n\rangle =: B$ given by $f(x) = y$ and $f(y) = x$ is an isomorphism of k-algebras and by Lemma 1.1.26, $\mathrm{IDer}_k(A; i) = \mathrm{IDer}_k(B; i)$. Hence, $\mathrm{Leaps}_k(A) = \mathrm{Leaps}_k(B)$.*

We recall the notations that we use: Let $\alpha := \mathrm{val}_p(n)$ be the p-adic valuation of n and $s = n/p^\alpha$. We will denote by m the remainder of the division of q by p and $\beta := \mathrm{val}_p(q - m)$.

As a straightforward consequence of Proposition 4.1.1 and Corollary 1.2.16, we have the following result.

Corollary 4.1.3 *Let k be a UFD of characteristic $p > 0$ and $h = x^n - y^q \in k[x, y]$ with $n, q \neq 0$. We denote $A = k[x, y]/\langle h\rangle$. Suppose $m = 0$, $\alpha \geq 1$ and $\beta = \mathrm{val}_p(q) \geq 1$. We write $\tau = \min\{\alpha, \beta\} \geq 1$, $n' = n/p^\tau$ and $q' = q/p^\tau$. Then, $\mathrm{Der}_k(A) = \langle \overline{\partial}_x, \overline{\partial}_y\rangle$ and for all $i \geq 0$,*
$$\mathrm{IDer}_k(A; p^{\tau+i}) = \left\{ \overline{\delta} \mid \delta \in \mathrm{IDer}_k \left(\log \left\langle x^{n'} - y^{q'} \right\rangle, p^i \right) \right\}.$$

Corollary 4.1.4 *Let k be a commutative reduced ring of characteristic $p > 0$ and $A = k[x,y]/\langle h\rangle$ where $h = x^n - y^q$ with $n, q \neq 0$. Then, we have the following properties.*

1. *If $n, q \neq 0 \mod p$ then, $\mathrm{Leaps}_k(A) = \emptyset$.*
2. *If $n = 0 \mod p$ and $q = 1$ then, $\mathrm{Leaps}_k(A) = \emptyset$.*
3. *If $\alpha, m \geq 1$ and $q \geq 2$, then*
$$\mathrm{Leaps}_k(A) = \begin{cases} \{p^\alpha, p^{\alpha+\beta}\} & \textit{if } s = 1,\ \alpha \leq \beta,\ m = 1 \\ \{p^\alpha\} & \textit{otherwise.} \end{cases}$$
4. *If $\alpha = 0$ (i.e. $n \neq 0 \mod p$) and $m = 0$ (i.e. $q = 0 \mod p$) then, $\mathrm{Leaps}_k(A) = \mathrm{Leaps}_k(A')$ where $A' = k[x,y]/\langle x^q - y^n\rangle$.*

Moreover, if k is a unique factorization domain, $m = 0$, $\alpha, \beta \geq 1$ and we denote $\tau = \min\{\alpha, \beta\} \geq 1$, $n' = n/p^\tau$ and $q' = q/p^\tau$, we have that

$$\mathrm{Leaps}_k(A) = \{p^\tau\} \cup \{ip^\tau \mid i \in \mathrm{Leaps}_k(B)\} \ \textit{where } B = k[x,y]/\left\langle x^{n'} - y^{q'}\right\rangle.$$

Proof. This corollary is a consequence of Proposition 4.1.1 and Proposition 1.2.17. □

Corollary 4.1.5 *Let k be a commutative reduced ring of characteristic $p > 0$ and $A = k[x,y]/\langle h\rangle$ where $h = x^n - y^q$ such that $n, q \neq 0$, $\alpha, m \geq 1$ and $q \geq 2$. We denote*

$$B_i := \mathrm{Ann}_A\left(\mathrm{IDer}_k(A; i-1)/\,\mathrm{IDer}_k(A; i)\right)$$

for $i > 1$. Then,

$$B_i = \begin{cases} \langle x, y^\gamma\rangle & \textit{if } i = p^\alpha \\ \langle y\rangle & \textit{if } i = p^{\alpha+\beta},\ s = 1,\ \alpha \leq \beta \textit{ and } m = 1. \end{cases}$$

Moreover, $B_i \supseteq J^0 = \langle y^{q-1}\rangle$ where J^0 is the gradient ideal of h defined in Proposition 1.2.7.

Proof. Let us start with $i = p^\alpha$. From Proposition 4.1.1, we can deduce that

$$\mathrm{IDer}_k\left(A; p^\alpha - 1\right)/\,\mathrm{IDer}_k\left(A; p^\alpha\right) = \langle\partial_x\rangle/\langle x\partial_x, y^\gamma\partial_x\rangle$$

where $\partial_x \in \mathrm{Der}_k(A)$. By definition, $a \in B_i$ if $a\partial_x = 0 \mod \langle x\partial_x, y^\gamma\partial_x\rangle$, i.e. if there exist $F, G \in A$ such that $a\partial_x = Fx\partial_x + Gy^\gamma\partial_x$. Applying this derivation to x, we have that $a \in \langle x, y^\gamma\rangle$.

Now, when $\alpha \leq \beta$, $s = m = 1$ and $i = p^{\alpha+\beta}$, from Proposition 4.1.1,

$$\mathrm{IDer}_k\left(A; p^{\alpha+\beta} - 1\right)/\,\mathrm{IDer}_k\left(A; p^{\alpha+\beta}\right) = \langle x\partial_x, y^\gamma\partial_x\rangle/\langle x\partial_x, y^{\gamma+1}\partial_x\rangle = \langle y^\gamma\partial_x\rangle/\langle y^{\gamma+1}\partial_x\rangle.$$

In this case, $a \in B_{p^\alpha+\beta}$ if and only if $ay^\gamma\partial_x \in \langle y^{\gamma+1}\partial_x\rangle$, i.e. if $(a - Fy)y^\gamma\partial_x = 0$ for some $F \in A$. This implies that $a \in \langle y\rangle$ and we have proved the corollary.

□

Examples 4.1.6 *Let us consider k a reduced ring of characteristic $p=3$ and $h=x^3-y^5\in k[x,y]$, then $\gamma=2$. According with Proposition 4.1.1 and Corollary 4.1.4,* $\mathrm{Leaps}_k(k[x,y]/\langle h\rangle)=\{3\}$ *and*

$$\mathrm{IDer}_k(k[x,y]/\langle h\rangle;i)=\begin{cases}\langle\overline{\partial}_x\rangle & 1\leq i<3\\ \langle\overline{x\partial_x},\overline{y^2\partial_x}\rangle & i\geq 3.\end{cases}$$

Now, if we consider $f=x^3-y^4\in k[x,y]$, then $\gamma=1$, $\mathrm{Leaps}_k(k[x,y]/\langle f\rangle)=\{3,9\}$ *and*

$$\mathrm{IDer}_k(k[x,y]/\langle f\rangle;i)=\begin{cases}\langle\overline{\partial}_x\rangle & 1\leq i<3\\ \langle\overline{x\partial_x},\overline{y\partial_x}\rangle & 3\leq i<9\\ \langle\overline{x\partial_x},\overline{y^2\partial_x}\rangle & i\geq 9.\end{cases}$$

Moreover, if we assume that k is a UFD, thanks to Corollary 4.1.3 and Corollary 4.1.4, we have that $\mathrm{Leaps}_k(A)=\{3,9,27\}$ *and*

$$\mathrm{IDer}_k\left(k[x,y]/\left\langle f^3\right\rangle\right)=\begin{cases}\langle\overline{\partial}_x,\overline{\partial}_y\rangle & 1\leq i<3\\ \langle\overline{\partial}_x\rangle & 3\leq i<9\\ \langle\overline{x\partial_x},\overline{y\partial_x}\rangle & 9\leq i<27\\ \langle\overline{x\partial_x},\overline{y^2\partial_x}\rangle & i\geq 27.\end{cases}$$

Remark 4.1.7 *Note that if k is not reduced, Proposition 4.1.1 is not true. For example, if $k=\mathbb{F}_3[t]/\langle t^3\rangle$ and $h=x^3-y^5$, then $\overline{t\partial_x}\in\mathrm{IDer}_k(A)$ with the integral*

$$\begin{array}{ccc}A & \longrightarrow & A[|\mu|]\\ x & \longmapsto & x+t\mu\\ y & \longmapsto & y\end{array}$$

$\square$

4.2 Other plane curves

In this section we calculate the modules of integrable derivations of the quotient of a polynomial ring over some non-binomial plane curves. These curves have been taken from [Gr].

Curve 1.

Let k be a domain of characteristic $p>0$ and $t\in k$. Let us consider $h=x^p+tx^{p+1}\in R=k[x]$ and $A=R/\langle h\rangle$. The module $\mathrm{Der}_k(\log h)$ is generated by $(1+tx)\partial_x$. From Theorem 2.5.1 and Corollary 1.2.4, $\mathrm{Der}_k(\log h)=\mathrm{IDer}_k(\log h;p-1)$. Hence, we have that $(1+tx)\partial_x$ is h-logarithmically $(p-1)$-integrable. So, let us consider $E\in\mathrm{HS}_k(\log h;p-1)$ an integral of $u(1+tx)\partial_x$ where $u\in R$. From Proposition 1.2.1, there exists $D\in\mathrm{HS}_k(R)$ an integral of E. In order for D to be h-logarithmic,

$$D_p(x^p+tx^{p+1})=D_1(x)^p+t(xD_1(x)^p+D_p(x)x^p)=u^p(1+tx)^{p+1}+tD_p(x)x^p\in\langle h\rangle$$

(to calculate this equality see Lemma 1.2.9). So, $u\in\langle x\rangle$ and $\mathrm{IDer}_k(\log h;p)=\langle x(1+tx)\partial_x\rangle$. Observe that this generator is ∞-integrable, for example the k-algebra homomorphism $R\to$

$R[|\mu|]$ given by $x \mapsto x + x(1+tx)\mu$ is a h-logarithmic integral of $x(1+tx)\partial_x$. In conclusion, $\mathrm{Leaps}_k(A) = \{p\}$ and

$$\mathrm{IDer}_k(A;i) = \begin{cases} \overline{\langle (1+tx)\partial_x \rangle} & \text{if } i < p \\ \overline{\langle x(1+tx)\partial_x \rangle} & \text{if } i \geq p. \end{cases}$$

Curve 2.

Let k be a domain of characteristic $p = 2$ and $h = x^4 + y^6 + y^7 \in R = k[x,y]$. Let $A = R/\langle h \rangle$. Let us consider $\delta = u\partial_x + v\partial_y$ for some $u, v \in R$. In order for δ to be h-logarithmic, $vy^6 \in \langle h \rangle$, that means that $vy^6 = F(x^4 + y^6 + y^7)$ for some $F \in R$. Hence, $(v - F)y^6 = F(x^4 + y^7)$. So, $F = Gy^6$ for some $G \in R$ and $v = G(x^4 + y^6 + y^7)$, i.e. $v \in \langle h \rangle$. Therefore,

$$\mathrm{Der}_k(\log h) = \langle \partial_x, h\partial_y \rangle.$$

Since $h\partial_y$ is h-logarithmically ∞-integrable, we can focus on the h-logarithmically integrability of $u\partial_x$ where $u \in R$. Let us suppose that $\delta = u\partial_x \in \mathrm{IDer}_k(\log I; 4(i-1))$ for some $i \geq 2$ and that there exists a $4(i-1) - \langle h \rangle$-logarithmic integral of δ of the form

$$\begin{array}{rccl} \varphi: & R & \longrightarrow & R[|\mu|] \\ & x & \longmapsto & x + u\mu \\ & y & \longmapsto & y + v_1\mu^4 + \cdots + v_{i-1}\mu^{4(i-1)} \end{array}$$

Then, for all $n \geq 4(i-1)$ such that $4 \nmid n$, the coefficient of μ^n in the equation $\varphi(h)$ is zero. Moreover, the coefficient of μ^{4i}, that we denote it by $\widetilde{\mu}_i$, is obtained from the expression

$$\begin{aligned} & \left(y + v_1\mu^4 + \cdots + v_{i-1}\mu^{4(i-1)}\right)^6 + \left(y + v_1\mu^4 + \cdots + v_i\mu^{4(i-1)}\right)^7 \\ = & \left(y^2 + v_1^2\mu^8 + \cdots + v_{i-1}^2\mu^{8(i-1)}\right)^3 \left(1 + y + v_1\mu^4 + \cdots + v_{i-1}\mu^{4(i-1)}\right) \\ = & \sum_{\substack{j=(j_0,\ldots,j_{i-1}) \\ |j|=3}} \binom{3}{j} y^{2j_0} v_1^{2j_1} \cdots v_{i-1}^{2j_{i-1}} \mu^{8(j_1+\cdots+(i-1)j_{i-1})} \left(1 + y + v_1\mu^4 + \cdots + v_{i-1}\mu^{4(i-1)}\right) \\ = & \sum_{l=0}^{i-1} \sum_{\substack{j=(j_0,\ldots,j_{i-1}) \\ |j|=3}} \binom{3}{j} v_l y^{2j_0} v_1^{2j_1} \cdots v_{i-1}^{2j_{i-1}} \mu^{8(j_1+\cdots+(i-1)j_{i-1})+4l} \end{aligned} \tag{4.5}$$

where $v_0 = 1 + y$. Then,

$$\widetilde{\mu}_i = \sum_{(j,l)\in\mathcal{I}_i} \mu_{i,j,l} \qquad \text{where} \qquad \mu_{i,j,l} = \sum_{(j,l)\in\mathcal{I}_i} \binom{3}{j} v_l y^{2j_0} v_1^{2j_1} \cdots v_{i-1}^{2j_{i-1}}$$

and

$$\mathcal{I}_i = \{(j,l) \in \mathbb{N}^i \times \mathbb{N} \mid 0 \leq l < i,\ j_s \geq 0\ \forall s = 0,\ldots,i-1,\ |j| = 3,\ 2(j_1 + \cdots + (i-1)j_{i-1}) = i - l\}.$$

Observe that $j_0 \leq 2$ for all $(j,l) \in \mathcal{I}_i$ because if $j_0 = 3$, then $j_s = 0$ for all $s = 1,\ldots,i-1$, so $i - l = 0$ and this is a contradiction. We have the following lemmas.

Lemma 4.2.1 *Let $i \geq 2$ be an integer and let us suppose that $v_n \in \langle y^2 \rangle$ for all $1 \leq n \leq i-1$. Then, $\widetilde{\mu}_i \in \langle y^8 \rangle$.*

Proof. Since $v_n \in \langle y^2 \rangle$,

$$\mu_{i,j,l} \in \left\langle y^{2j_0+4(j_1+\cdots+j_{i-1})} \right\rangle .$$

Since $j_0 + j_1 + \cdots + j_{i-1} = 3$ and $j_0 \leq 2$, we have $2j_0 + 4(j_1 + \cdots + j_{i-1}) = 2j_0 + 4(3 - j_0) = 4 - 2j_0 + 8$ where $4 - 2j_0 \geq 0$. Hence,

$$\mu_{i,j,l} \in \langle y^8 \rangle \text{ for all } (j,l) \in \mathcal{I}_i \Rightarrow \widetilde{\mu}_i \in \langle y^8 \rangle$$

and we have the result.

□

Lemma 4.2.2 *Let $i \geq 2$ be an integer and let us suppose that $v_n \in \langle x^4 \rangle$ for all $1 \leq n \leq i-1$. Then, $\widetilde{\mu}_i \in \langle x^8 \rangle$.*

Proof. Since $v_n \in \langle x^4 \rangle$, we have that

$$\mu_{i,j,l} \in \left\langle x^{8(j_1+\cdots+j_{i-1})} \right\rangle .$$

Since $j_0 \neq 3$, we have $8(j_1 + \cdots + j_{i-1}) = 8(3 - j_0) = 8(2 - j_0) + 8$ where $2 - j_0 \geq 0$. Hence,

$$\mu_{i,j,l} \in \langle x^8 \rangle \text{ for all } (j,l) \in \mathcal{I}_i \Rightarrow \widetilde{\mu}_i \in \langle x^8 \rangle .$$

□

Let us suppose that $\delta = u\partial_x \in \mathrm{IDer}_k(\log I; 4(i-1))$ for some $i \geq 2$ and it has a $4(i-1)-\langle h \rangle$-logarithmic integral

$$\begin{array}{rccl} \varphi : & R & \longrightarrow & R[|\mu|] \\ & x & \longmapsto & x + u\mu \\ & y & \longmapsto & y + v_1\mu^4 + \cdots + v_{i-1}\mu^{4(i-1)} \end{array}$$

where $v_n \in \langle y^2 \rangle$ (resp. $v_n \in \langle x^4 \rangle$) for all $n \geq 1$. By Lemma 4.2.1 (resp. Lemma 4.2.2), we have that $\widetilde{\mu}_i = Fy^8$ (resp. $\widetilde{\mu}_i = Fx^8$) for some $F \in R$. We put $v_i = Fy^2$ (resp. $v_i = F(1+y)x^4$) and we define an integral of δ

$$\begin{array}{rccl} \varphi' : & R & \longrightarrow & R[|\mu|] \\ & x & \longmapsto & x + u\mu \\ & y & \longmapsto & y + v_1\mu^4 + \cdots + v_{i-1}\mu^{4(i-1)} + v_i\mu^{4i} \end{array}$$

Then, φ' is $4i-\langle h \rangle$-logarithmic. It is clear that φ' is $(4i-1)-\langle h \rangle$-logarithmic and the coefficient of μ^{4i} in $\varphi'(h)$ is $y^6 v_i + \widetilde{\mu}_i \in \langle h \rangle$. Therefore, if $\delta = u\partial_x \in \mathrm{IDer}_k(\log I; 4)$ and there exists $v \in \langle y^2 \rangle$ or $v \in \langle x^4 \rangle$ such that

$$\begin{array}{rccl} \varphi : & R & \longrightarrow & R[|\mu|] \\ & x & \longmapsto & x + u\mu \\ & y & \longmapsto & y + v\mu^4 \end{array}$$

is a $4 - \langle h \rangle$-logarithmic integral of δ, then $\delta \in \mathrm{IDer}_k(\log I; m)$ for all $m \geq 1$. Thanks to this, we will calculate modules of integrable k-derivations.

Let us consider $\delta = u\partial_x$ where $u \in R$ and a generic integral $\varphi : R \to R[|\mu|]$ of $\delta = u\partial_x$:

$$\begin{array}{rccl} \varphi : & R & \longrightarrow & R[|\mu|] \\ & x & \longmapsto & x + u\mu + u_2\mu^2 + \cdots \\ & y & \longmapsto & y \quad + \quad v_2\mu^2 + \cdots \end{array}$$

The coefficient of μ^i for $i = 2, 3$ in $\varphi(h)$ is $y^6 v_i$ and they have to belong to $\langle h \rangle$. As we have seen before, that implies that $v_i \in \langle h \rangle$, so we can put $v_i = 0$ for $i = 2, 3$ and we can write:

$$\varphi(h) = (x + u\mu + u_2\mu^2 + \cdots)^4 + (y^2 + v_4^2\mu^8 + v_5^2\mu^{10} + \cdots)^3(1 + y + v_4\mu^4 + v_5\mu^5 + \cdots). \quad (4.6)$$

The coefficient of μ^4 in this equation is $u^4 + y^6 v_4$ and has to belong to $\langle h \rangle$. Let us suppose that $F \in R$ satisfies the equation

$$u^4 + y^6 v_4 = F\left(x^4 + y^6 + y^7\right). \quad (4.7)$$

Let us write $u = \sum u_i(x)y^i \in (k[x])[y]$ and $F = \sum F_i y^i \in (k[x])[y]$. Then,

$$\sum u_i^4 y^{4i} = \sum F_i x^4 y^i + (v_4 + (1 + y)F)y^6.$$

If we consider the independent term and the coefficient of y^4 in this equation, we have that $u_0^4 = F_0 x^4$ and $u_1^4 = F_1 x^4$. Hence, we can deduce that $u_0 = u_x x$ and $u_1 = u_{xy} x$ for $u_x, u_{xy} \in k[x]$. Therefore u and δ can be written as

$$u = u_x x + u_{xy} xy + u_y y^2 \Rightarrow \delta = u_x x\partial_x + u_{xy} xy\partial_x + u_y y^2\partial_x$$

where $u_y \in R$. Substituting the expression of u on (4.7), we have that

$$u_x^4 x^4 + u_{xy}^4 x^4 y^4 + u_y^4 y^8 + y^6 v_4 = F\left(x^4 + y^6 + y^7\right) \Rightarrow \left(u_x^4 + u_{xy}^4 y^4 + F\right)x^4 = y^6\left(v_4 + F(1 + y) + u_y^4 y^2\right).$$

Hence, there exists $G \in R$ such that

$$\left\{\begin{array}{l} u_x^4 x^4 + u_{xy}^4 y^4 + F = G y^6 \\ v_4 + F(1 + y) + u_y^4 y^2 = G x^4 \end{array}\right. \Rightarrow \quad F = G y^6 + u_x^4 x^4 + u_{xy}^4 y^4$$

Substituting F in the second equation, we have that

$$v_4 = G(x^4 + y^6 + y^7) + u_y^4 y^2 + (u_x^4 + u_{xy}^4 y^4)(1 + y).$$

Therefore,

$$\operatorname{IDer}_k(\log I; 4) = \langle x\partial_x, xy\partial_x, y^2\partial_x, h\partial_y \rangle = \langle x\partial_x, y^2\partial_x, h\partial_y \rangle.$$

Thanks to the previous computation we can see that

$$\begin{array}{ccl} R & \longrightarrow & R[|\mu|] \\ x & \longmapsto & x + xy\mu \\ y & \longmapsto & y + (1 + y)y^4\mu^4 \end{array} \quad \text{and} \quad \begin{array}{ccl} R & \longrightarrow & R[|\mu|] \\ x & \longmapsto & x + y^2\mu \\ y & \longmapsto & y + y^2\mu^4 \end{array}$$

are $4 - \langle h \rangle$-logarithmic integrals of $xy\partial_x$ and $y^2\partial_x$ respectively. So, both derivations are $\langle h \rangle$-logarithmically m-integrable for all $m \geq 1$.

If $\delta \in \mathrm{IDer}_k(\log h; 4)$, we have that $\delta = u_x x\partial_x + u_{xy} xy\partial_x + u_y y^2\partial_x$ for some $u_x \in k[x]$ and $u_{xy}, u_y \in R$. Since $\mathrm{IDer}_k(\log h; m)$ is an R-module and $xy\partial_x$ and $y^2\partial_x$ are h-logarithmically m-integrable, $\delta \in \mathrm{IDer}_k(\log h; m)$ if and only if $u_x x\partial_x \in \mathrm{IDer}_k(\log h; m)$. Therefore, we need to see the h-logarithmically integrability of $ux\partial_x$ where $u \in k[x]$.

Let us consider a $4 - \langle h\rangle$-logarithmic integral $\varphi : R \to R[|\mu|]$ of $ux\partial_x$:

$$\begin{array}{rcl} \varphi: \; R & \to & R[|\mu|] \\ x & \mapsto & x + ux\mu + u_2\mu^2 + \cdots \\ y & \mapsto & y + v_4\mu^4 + v_5\mu^5 + \cdots \end{array}$$

Then, $v_4 \in (1+y)u^4 + \langle h\rangle$. Observe that the coefficient of μ^i for $i = 5, 6, 7$ in $\varphi(h)$ (see (4.6)) is $y^6 v_i$. Since we want φ to be h-logarithmic, $y^6 v_i \in \langle h\rangle$, so we can put $v_i = 0$. Now, the coefficient of μ^8 is

$$\mu_8 := u_2^4 + y^6 v_8 + v_4^2(1+y)y^4 = u_2^4 + y^6 v_8 + (1+y)^3 u^8 y^4.$$

In order for μ_8 to be in $\langle h\rangle$,

$$u_2^4 + y^6 v_8 + (1+y)^3 u^8 y^4 = F\left(x^4 + y^6 + y^7\right)$$

for some $F \in R$. Observe that the coefficient of y^5 in the previous equation is $u_0^8 = 0$ where u_0 is the independent term of u. Since R is a domain, $u_0 = 0$, so $u \in \langle x\rangle$ and we can write $u = wx$. Hence, $v_4 = (1+y)w^4x^4 \in \langle x^4\rangle$ and if we put

$$v_8 = w^8(1+y)^3(1+y)x^4y^4 \in \langle x^4\rangle \text{ and } u_2 = 0$$

then $\mu_8 = 0 \mod \langle h\rangle$. Therefore,

$$\mathrm{IDer}_k(\log h; 8) = \langle x^2\partial_x, xy\partial_x, y^2\partial_x, h\partial_y\rangle$$

and

$$\begin{array}{rcl} R & \longrightarrow & R[|\mu|] \\ x & \longmapsto & x + x^2\mu \\ y & \longmapsto & y + (1+y)x^4\mu^4 \end{array}$$

is a $4 - \langle h\rangle$-logarithmic integral of $x^2\partial_x$, so $x^2\partial_x$ is h-logarithmically m-integrable for all $m \geq 1$. In conclusion, $\mathrm{Leaps}_k(A) = \{4, 8\}$ and

$$\mathrm{IDer}_k(A; i) = \begin{cases} \langle\overline{\partial_x}\rangle & \text{if } 1 \leq i < 4 \\ \langle\overline{x\partial_x}, \overline{y^2\partial_x}\rangle & \text{if } 4 \leq i < 8 \\ \langle\overline{x^2\partial_x}, \overline{xy\partial_x}, \overline{y^2\partial_x}\rangle & \text{if } i \geq 8. \end{cases}$$

Curve 3.

Let k be a domain of characteristic $p = 3$ and $h = x^3 + y^5 + x^2y^2 \in R = k[x, y]$. Let $A = R/\langle h\rangle$. Let us consider $\delta = u\partial_x + v\partial_y$. In order for δ to be h-logarithmic, $\delta(h) = 2uxy^2 + 2vx^2y + 2vy^4 \in \langle h\rangle$, i.e. $y(uxy + vx^2 + vy^3) = F(x^3 + x^2y^2 + y^5)$ for some $F \in R$. Since y is not a factor of h, we have that

$$uxy + vx^2 + vy^3 = F(x^3 + x^2y^2 + y^5) \Rightarrow (uy - Fx^2)x = (Fy^2 - v)(x^2 + y^3)$$

for some $F \in R$. Hence,

$$\left\{\begin{array}{l} Fy^2 - v = Gx \Rightarrow v = Fy^2 - Gx \\ uy - Fx^2 = G(x^2+y^3) \Rightarrow (u-Gy^2)y = (F+G)x^2 \Rightarrow \end{array}\right. \left\{\begin{array}{l} u - Gy^2 = Hx^2 \Rightarrow u = Gy^2 + Hx^2 \\ F+G = Hy \Rightarrow F = Hy - G \end{array}\right.$$

for some $G, H \in R$. Then,

$$\left\{\begin{array}{l} u = Gy^2 + Hx^2 \\ v = -G(x+y^2) + Hy^3. \end{array}\right.$$

Let us denote $\delta_1 := x^2\partial_x + y^3\partial_y$ and $\delta_2 := 2y^2\partial_x + (x+y^2)\partial_y$. Then,

$$\mathrm{Der}_k(\log h) = \langle \delta_1, \delta_2 \rangle.$$

These two derivations are h-logarithmically m-integrable for all $m \geq 1$. To verify this claim, let us consider a k-algebra homomorphism $\varphi : R \to R[|\mu|]$ given by

$$\begin{array}{rccl} \varphi : & R & \longrightarrow & R[|\mu|] \\ & x & \longmapsto & x + u_1\mu + u_2\mu^2 + \cdots \\ & y & \longmapsto & y + v_1\mu + v_2\mu^2 + \cdots \end{array}$$

We start to calculating a generic coefficient of $\varphi(h)$:

$$\begin{array}{rl} \varphi(h) = & (x^3 + u_1^3\mu^3 + u_2^3\mu^6 + \cdots) + \\ & (y + v_1\mu + v_2\mu^2 + \cdots)^2 \left[(x + u_1\mu + u_2\mu^2 + \cdots)^2 + y^3 + v_1^3\mu^3 + v_2^3\mu^6 + \cdots \right]. \end{array}$$

The coefficient of μ^i in $(x + u_1\mu + u_2\mu^2 + \cdots)^2$ is the coefficient of μ^i of

$$(x + u_1\mu + u_2\mu^2 + \cdots + u_i\mu^i)^2 = \sum_{|l|=2} \binom{2}{l} x^{l_0} u_1^{l_1} \cdots u_i^{l_i} \mu^{l_1 + \cdots + il_i}.$$

Let us denote $L_i = \left\{ l = (l_0, l_1, \ldots, l_i) \in \mathbb{N}^{i+1} \mid |l| = 2,\ \sum_{s=1}^{i} sl_s = i \right\}$ and

$$\nu_{i3} = \left\{\begin{array}{ll} 1 & \text{if } i = 0 \mod 3 \\ 0 & \text{otherwise.} \end{array}\right.$$

Then, the coefficient of μ^i in the term $(x + u_1\mu + u_2\mu^2 + \cdots)^2 + y^3 + v_1^3\mu^3 + v_2^3\mu^6 + \cdots$ is

$$\widetilde{\mu}_i = \sum_{l \in L_i} \binom{2}{l} x^{l_0} u_1^{l_1} \cdots u_i^{l_i} + \nu_{i3} v_{i/3}^3.$$

Now the coefficient of μ^i in $(y + v_1\mu + v_2\mu^2 + \cdots)^2$ is

$$\mu_i' = \sum_{j \in L_i} \binom{2}{j} y^{j_0} v_1^{j_1} \cdots v_i^{j_i}.$$

Hence, the coefficient of μ^i in $\varphi(h)$ is

$$\mu_i = \nu_{i3} u_{i/3}^3 + \sum_{n=0}^{i} \mu_n' \widetilde{\mu}_{i-n}.$$

Observe that if $n \neq 0, i$, then μ'_n and $\widetilde{\mu}_{i-n}$ do not have u_i or v_i as a factor of any term. Moreover, $\mu'_0 = y^2$ and $\widetilde{\mu}_0 = x^2 + y^3$. Let $j \in L_i$, then $j_1 + \cdots + ij_i = i$, so there exists only one j such that $j_i \neq 0$, namely $j = (1, 0, \ldots, 0, 1)$. Let us denote $\mathcal{L}_i = L_i \setminus \{(1, 0, \ldots, 0, 1)\}$. Then,

$$\begin{aligned} \mu_i = \; & \nu_{i3} u_{i/3}^3 + 2\left(x^2 + y^3\right) y v_i + 2xy^2 u_i + \left(x^2 + y^3\right) \left(\sum_{j \in \mathcal{L}_i} \binom{2}{j} y^{j_0} v_1^{j_1} \cdots v_{i-1}^{j_{i-1}} \right) \\ + \; & y^2 \left(\sum_{l \in \mathcal{L}_i} \binom{2}{j} x^{l_0} u_1^{l_1} \cdots u_{i-1}^{l_{i-1}} + \nu_{i3} v_{i/3}^3 \right) + \sum_{n=1}^{i-1} \mu'_n \widetilde{\mu}_{i-n}. \end{aligned}$$

Let us consider the following lemma:

Lemma 4.2.3 *Let $u_1 = x^2$ and $v_1 = y^3$ and $i \geq 2$ an integer. Suppose that $v_j = 0$ for all $j \geq 2$ and $u_n \in \langle x^2 \rangle$ for all $n < i$. Then, there exists $u_i \in \langle x^2 \rangle$ such that μ_i belongs to $\langle h \rangle$.*

Proof. Note that $L_1 = \{(1, 1)\}$ and $L_2 = \{(0, 2, 0), (1, 0, 1)\}$. Then,

$$\begin{aligned} \mu_2 = \; & 2xy^2 u_2 + (x^2 + y^3) v_1^2 + y^2 u_1^2 + 2y v_1 2x u_1 = 2xy^2 u_2 + x^2 y^6 + y^9 + x^4 y^2 + x^3 y^4 \\ = \; & 2xy^2 u_2 + x^4 y^2 + y^4 (x^3 + x^2 y^2 + y^5) = 2xy^2 u_2 + x^4 y^2 \mod \langle h \rangle. \end{aligned}$$

If we put $u_2 = x^3$, we have that $\mu_2 \in \langle h \rangle$ and the lemma is true for $i = 2$.

Let us assume that $i \geq 3$. Let $j \in L_n$ with $n \geq 3$, then if $j_s > 0$ for some $s \in \{2, \ldots, n\}$, the term associated with$j$ in μ'_n is zero. Hence, we can assume that $j_s = 0$ for all $s \geq 2$. In this case, $j_1 = n \geq 3$ but $j_0 + j_1 = 2$ and we have a contradiction. Therefore, $\mu'_n = 0$ for all $n \geq 3$. Observe that a similar argument can be applied to $j \in \mathcal{L}_i$ in the term $\left(\sum_{j \in \mathcal{L}_i} \binom{2}{j} y^{j_0} v_1^{j_1} \cdots v_{i-1}^{j_{i-1}} \right)$. So,

$$\mu_i = \nu_{i3} u_{i/3}^3 + 2xy^2 u_i + y^2 \left(\sum_{l \in \mathcal{L}_i} \binom{2}{l} x^{l_0} u_1^{l_1} \cdots u_{i-1}^{l_{i-1}} + \nu_{i3} v_{i/3}^3 \right) + \mu'_1 \widetilde{\mu}_{i-1} + \mu'_2 \widetilde{\mu}_{i-1}.$$

Observe that $\mu'_1 = 2y v_1$ and $\mu'_2 = v_1^2$, so

$$\mu_i = \begin{cases} \nu_{i3} u_{i/3}^3 + 2xy^2 u_i + y^2 \sum_{l \in \mathcal{L}_i} \binom{2}{l} x^{l_0} u_1^{l_1} \cdots u_{i-1}^{l_{i-1}} + 2y^4 \widetilde{\mu}_{i-1} + y^6 \widetilde{\mu}_{i-2} + y^{11} & \text{if } i = 3 \\ \nu_{i3} u_{i/3}^3 + 2xy^2 u_i + y^2 \sum_{l \in \mathcal{L}_i} \binom{2}{l} x^{l_0} u_1^{l_1} \cdots u_{i-1}^{l_{i-1}} + 2y^4 \widetilde{\mu}_{i-1} + y^6 \widetilde{\mu}_{i-2} & \text{otherwise.} \end{cases}$$

Let $l \in L_n$ with $n \geq 1$, then $l_1 + \cdots + l_n \geq 1$ because $l_1 + \cdots + nl_n = n$, so $l_0 \leq 1$. Moreover, since $u_s \in \langle x^2 \rangle$ for all $1 \leq s < i$, we have that

$$\binom{2}{l} x^{l_0} u_1^{l_1} \cdots u_n^{l_n} \in \left\langle x^{2(l_1 + \cdots + l_n) + l_0} = x^{4 - l_0} \right\rangle \subset \left\langle x^3 \right\rangle.$$

The same occurs when $l \in \mathcal{L}_i$. Then, $\widetilde{\mu_{i-1}} \in \nu_{(i-1)3} v_{(i-1)/3}^3 + \langle x^3 \rangle$ and $\widetilde{\mu_{i-2}} \in \mu_{(i-2)3} v_{(i-2)/3}^3 + \langle x^3 \rangle$. Hence,

$$y^2 \sum_{l \in \mathcal{L}_i} \binom{2}{l} x^{l_0} u_1^{l_1} \cdots u_{i-1}^{j_{i-1}} + 2y^4 \widetilde{\mu}_{i-1} + y^6 \widetilde{\mu}_{i-2} = Fx^3 y^2 + 2y^4 \nu_{(i-1)3} v_{(i-1)/3}^3 + y^6 \nu_{(i-2)3} v_{(i-2)/3}^3$$

for some $F \in R$. If we put $u_n = F_n x^2$ for $1 \leq n < i$, we have that

$$\mu_i = \begin{cases} \nu_{i3} F_{i/3}^3 x^6 + 2xy^2 u_i + Fx^3y^2 + y^{11} & \text{if } i = 3 \\ \nu_{i3} F_{i/3}^3 x^6 + 2xy^2 u_i + Fx^3y^2 + 2y^{13} & \text{if } i = 4 \\ \nu_{i3} F_{i/3}^3 x^6 + 2xy^2 u_i + Fx^3y^2 + y^{15} & \text{if } i = 5 \\ \nu_{i3} F_{i/3}^3 x^6 + 2xy^2 u_i + Fx^3y^2 & \text{otherwise.} \end{cases}$$

Then, if we take

$$u_i = \begin{cases} 2\nu_{i3} F_{i/3}^3 x^2(x^2+y^3) + Fx^2 + 2x^2y^4 + x^3y(x+y^2) & \text{if } i = 3 \\ 2\nu_{i3} F_{i/3}^3 x^2(x^2+y^3) + Fx^2 + 2x^3y(x+y^2)^2 & \text{if } i = 4 \\ 2\nu_{i3} F_{i/3}^3 x^2(x^2+y^3) + Fx^2 + x^3y^3(x+y^2)^2 & \text{if } i = 5 \\ 2\nu_{i3} F_{i/3}^3 x^2(x^2+y^3) + Fx^2 & \text{otherwise} \end{cases}$$

we have the result.

□

Let us consider $\delta_1 = x^2\partial_x + y^3\partial_y$. Then, by Lemma 4.2.3, there exists $u_2 \in \langle x^2 \rangle$ such that

$$\begin{array}{rcl} R & \to & R[|\mu|]_2 \\ x & \mapsto & x + x^2\mu + u_2\mu^2 \\ y & \mapsto & y + y^3\mu \end{array}$$

is a h-logarithmic 2-integral of δ_1. Doing this process recursively we can deduce that δ_1 is h-logarithmically m-integrable for all $m \geq 1$. To see the h-logarithmically integrability of δ_2, we consider the following lemma:

Lemma 4.2.4 *Let $u_1 = 2y^2$ and $v_1 = x + y^2$ and $i \geq 2$ an integer. Suppose $u_n \in \langle xy, y^3 \rangle$ and $v_n \in \langle y^2 \rangle$ for all $2 \leq n < i$. Then, there exist $u_i \in \langle xy, y^3 \rangle$ and $v_i \in \langle y^2 \rangle$ such that μ_i belongs to $\langle h \rangle$.*

Proof. We will start to calculate for $i = 2$:

$$\begin{aligned} \mu_2 &= 2(x^2+y^3)yv_2 + 2xy^2u_2 + (x^2+y^3)v_1^2 + y^2u_1^2 + xyv_1u_1 \\ &= 2(x^2+y^3)yv_2 + 2xy^2u_2 + (x^2+y^3)(x+y^2)^2 + y^6 + 2xy^3(x+y^2) \\ &= 2(x^2+y^3)yv_2 + 2xy^2u_2 + y^6 + (x+y^2)\left[(x^2+y^3)y^2 + x^3 + xy^3 + 2xy^3\right] \\ &= 2(x^2+y^3)yv_2 + 2xy^2u_2 + y^6 \mod \langle h \rangle. \end{aligned}$$

If we put $u_2 = 2xy$ and $v_2 = y^2$ we have the result for $i = 2$. Let us consider $i \geq 3$. We will study each component of μ_i.

- For $1 \leq n \leq i-1$, we have that

$$\mu_n' = \sum_{j \in L_n} \binom{2}{j} y^{j_0} v_1^{j_1} \cdots v_n^{j_n} = \sum_{j \in L_n} \binom{2}{j} y^{j_0} (x+y^2)^{j_1} v_2^{j_2} \cdots v_n^{j_n}.$$

  Observe that $j_0 \leq 1$ because $n > 0$ and if $j_0 = 2$, then $j_s = 0$ for all $s \geq 1$ and $0 = \sum_{s=1}^n s j_s = n$!!!. Moreover, since $v_s \in \langle y^2 \rangle$ for all $2 \leq s \leq i-1$, if $j \in L_n$, then

$$y^{j_0}(x+y^2)^{j_1} v_2^{j_2} \cdots v_n^{j_n} \in \left\langle (x+y^2)^{j_1} y^{j_0 + 2(j_2 + \cdots + j_n)} \right\rangle.$$

  We fix $j \in L_n$:

- If $l_1 = 0$, then the term associated with l belongs to $\langle y^3 \rangle$ because $j_0+2(j_2+\cdots+j_n) = 4 - j_0 \geq 3$.
- If $l_1 = 1$, then there exists $s \in \{0, 2, \ldots, s\}$ such that $j_s = 1$ and $v_s \in \langle y \rangle$. So, the term associated with j belongs to $\langle xy, y^3 \rangle$.
- If $j_1 = 2$, then $j_0 = j_2 = \cdots = j_n = 0$ and the term is $y^4 + 2xy^2 + x^2$.

Then,

$$\mu'_n \in \left\langle y^3, xy, x^2 \right\rangle .$$

Now, we denote

$$\widetilde{\mu}'_n = \sum_{l \in L_n} \binom{2}{l} x^{l_0} u_1^{l_1} \cdots u_n^{l_n} = \sum_{l \in L_n} \binom{2}{l} x^{l_0} (2y^2)^{l_1} u_2^{l_2} \cdots u_n^{l_n}.$$

Again, $l_0 \leq 1$ for all $l \in L_n$. We fix $l \in L_n$ and recall that $u_s \in \langle xy, y^3 \rangle$ for all $2 \leq s \leq i-1$.

- If $l_1 = 0$, then $l_0 = 1$ or $l_0 = 0$. In the first case, there exists $l_s = 1$ with $s \geq 2$, so the term associated with l belongs to $\langle x^2y, xy^3 \rangle$. Now, if $l_0 = 0$, then the term associated with l is in $\langle xy, y^3 \rangle^2 \subseteq \langle x^2y^2, xy^4, y^6 \rangle$.
- If $l_1 = 1$, then if $l_0 = 1$, the term associated with l belongs to $\langle xy^2 \rangle$. Otherwise, if $l_0 = 0$, then there exists $l_s = 1$ for some $s \geq 2$ and the term associated with l is in $\langle xy^3, y^5 \rangle$.
- If $l_1 = 2$, then $l_i = 0$ for all $i = 0, 2, \ldots, n$ and the term associated with l belongs to $\langle y^4 \rangle$.

Moreover, $v_{i/3}^3 \in \langle y^6, x^3 \rangle$, so

$$\widetilde{\mu}_n \in \left\langle x^2y, xy^2, y^4, x^3 \right\rangle .$$

Hence,

$$\sum_{n=1}^{i-1} \mu'_n \widetilde{\mu}_{i-n} \in \left\langle y^7, x^5, x^3y, x^2y^3, xy^5 \right\rangle .$$

- For $i \geq 3$ we denote by η_i the term

$$\eta_i := \sum_{l \in \mathcal{L}_i} \binom{2}{l} x^{l_0} u_1^{l_1} u_2^{l_2} \cdots u_{i-1}^{l_{i-1}} = \sum_{l \in \mathcal{L}_i} \binom{2}{l} x^{l_0} (2y^2)^{l_1} u_2^{l_2} \cdots u_{i-1}^{l_{i-1}}.$$

Let $l \in \mathcal{L}_i$, then $l_0 \leq 1$ because $i \geq 1$. If $l_0 = 1$, then there is only one $l_s = 1$ and $sl_s = i$, so $s = i$. That means that $l = (1, 0, \ldots, 0, 1) \notin \mathcal{L}_i$. Moreover, if $l_1 = 2$, then $2 < i = \sum_s sl_s = l_1 = 2$!!!. Hence, $l_0 = 0$ for all $l \in \mathcal{L}_i$ and $l_1 \leq 1$. We fix $l \in \mathcal{L}_i$. Since $u_s \in \langle xy, y^3 \rangle$ for all $2 \leq s \leq i-1$ we have that:

- If $l_1 = 1$, then there exists $l_s = 1$ for $s \geq 2$ and the term associated with l is in $\langle xy^3, y^5 \rangle$.
- If $l_1 = 0$, then the term associated with l belongs to $\langle xy, y^3 \rangle^2 \subseteq \langle x^2y^2, xy^4, y^6 \rangle$.

Then,

$$\eta_i \in \left\langle xy^3, y^5, x^2y \right\rangle.$$

Since $v_{i/3}^3 \in \langle y^6, x^3 \rangle$, we have that

$$y^2(\eta_i + \nu_{i3} v_{i/3}^3) \in \left\langle xy^5, y^7, x^2y^3, x^3y^2 \right\rangle.$$

- For $i \geq 3$, we denote by η_i' the following term

$$\eta_i' := \sum_{j \in \mathcal{L}_i} \binom{2}{j} y^{j_0} v_1^{l_1} v_2^{l_2} \cdots v_{i-1}^{l_{i-1}} = \sum_{j \in \mathcal{L}_i} \binom{2}{j} y^{j_0} (x+y^2)^{l_1} v_2^{l_2} \cdots v_{i-1}^{l_{i-1}}.$$

Note that $j_0 = 0$ and $j_1 \leq 1$ for same reason that in the previous point. We fix $j \in \mathcal{L}_i$. Since $v_s \in \langle y^2 \rangle$ for all $2 \leq s \leq i-1$, we have that:

  - If $j_1 = 1$ then there exists $j_s = 1$ for some $s \geq 2$ and the term associated with j is in $\langle xy^2, y^4 \rangle$.
  - If $j_1 = 0$ then there exists $j_s = 2$ or $j_s = j_t = 1$ for $s, t \geq 2$, so the term associated with j is in $\langle y^4 \rangle$.

Then

$$\eta_i' \in \left\langle xy^2, y^4 \right\rangle$$

and

$$(x^2+y^3)\eta_i' \in \left\langle (x^2+y^3)y^4, x^3y^2, xy^5 \right\rangle.$$

- We have that $u_{i/3}^3 \in \langle y^6, x^3y^3 \rangle$.

To sum up,

$$\mu_i \in 2(x^2+y^3)yv_i + 2xy^2u_i + \left\langle y^6, x^5, x^3y, x^2y^3, xy^5, (x^2+y^3)y^4 \right\rangle.$$

So, there exists a $\alpha_i \in k[x,y]$ for each $i = 1, \ldots, 6$ such that

$$\mu_i = 2(x^2+y^3)yv_i + 2xy^2u_i + \alpha_1 y^6 + \alpha_2 x^5 + \alpha_3 x^3 y + \alpha_4 x^2y^3 + \alpha_5 xy^5 + \alpha_6 (x^2+y^3)y^4$$

and we want to find $u_i \in \langle xy, y^3 \rangle$, $v_i \in \langle y^2 \rangle$ such that $\mu_i \in \langle h \rangle$. Then, if we put

$$\begin{aligned} u_i &= 2\alpha_1 xy + 2\alpha_2 xy^3 + \alpha_4 xy + \alpha_5 y^3 \\ v_i &= \alpha_1 y^2 + \alpha_2 xy^3 + 2\alpha_3 y^2 + \alpha_6 y^3 \end{aligned}$$

we have the result.

□

Since $\delta_2 = 2y^2\partial_x + (x+y^2)\partial_y$ is h-logarithmic, we can apply Lemma 4.2.4, to obtain $u_2 \in \langle xy, y^3 \rangle$ and $v_2 \in \langle y^2 \rangle$ such that the k-algebra homomorphism $\varphi : R \to R[|\mu|]_2$ defined by $\varphi(x) = x + 2y^2\mu + u_2\mu^2$ and $\varphi(y) = y + (x+y^2)\mu + v_2\mu^2$ is h-logarithmic. Applying Lemma 4.2.4 repeatedly, we can deduce that δ_2 is h-logarithmically m-integrable for all $m \geq 1$. Therefore, $\mathrm{Leaps}_k(A) = \emptyset$ and for all $m \geq 1$,

$$\mathrm{IDer}_k(A; m) = \langle \overline{\delta_1}, \overline{\delta_2} \rangle \text{ where } \delta_1 = x^2\partial_x + y^3\partial_y \text{ and } \delta_2 = 2y^2\partial_x + (x+y^2)\partial_y. \tag{4.8}$$

4.3 Leaps, semigroup of curves and integral closure of ideals

In this section we give two results that were suggested by Professor H. Mourtada. The first one tells us that leaps of an irreducible algebroid plane curve over a algebraically closed field is not determined by the semigroup of the curve. The second one tells us that leaps of a commutative algebra over a commutative ring are not preserved by integral closure of ideals.

Let us consider an irreducible plane algebroid curve A over the algebraically closed field k and we denote by F its quotient field. Let us consider the integral closure $\overline{A}$ of A in F. Then, we have the following theorem.

Theorem 4.3.1 [Ca, **Th. 1.3.1**] *$\overline{A}$ is a complete discrete valuation ring of F. If $\overline{\mathfrak{m}}$ is the maximal ideal of $\overline{A}$, $t \in \overline{\mathfrak{m}} \setminus \overline{\mathfrak{m}}^2$, and T is an indeterminate over k, the homomorphism given by $T \in k[|T|] \mapsto t \in \overline{A}$ is an isomorphism of k-algebras.*

For such a t, we write $\overline{A} = k[|t|]$ and $F = k((t))$. Let $\underline{v} : F \to \mathbb{Z}$ be the normalized natural valuation of $k((t))$. If $z \in A \subseteq \overline{A}$, then $z = s(t)$ with $s(T) \in k[|T|]$ and we have $\underline{v}(z) = \operatorname{ord}(s(T))$ (see [Ca, **§1.3**]).

Definition 4.3.2 *The semigroup $S(A) = \underline{v}(A \setminus \{0\}) \subseteq \mathbb{Z}_+$ will be called semigroup of values of A.*

Proposition 4.3.3 *Leaps of irreducible algebroid plane curve over an algebraically closed field are not determined by the semigroup of the curve.*

Proof. Let k be an algebraically closed field of characteristic 3 and $R = k[|x, y|]$ the formal power series ring in 2 variables over k. Let us consider $h = x^3 - y^5$ and $g = x^3 - y^5 + x^2y^2$ two polynomials in R. Let us denote $A = R/\langle h\rangle$ and $B = R/\langle g\rangle$. These two rings are irreducible algebroid plane curves with the same semigroup, $(3, 5)$ (see Ch. 4.3 of [Ca]). However, they do not have the same leaps. Note that the calculations made in Proposition 4.1.1 and Curve 3 in the previous section are valid for R. So, by Example 4.1.6, $\operatorname{Leaps}_k(A) = \{3\}$.

On the other hand, we have that the map

$$\begin{array}{cccc} f: & B & \to & R/\langle x^3 + y^5 + x^2y^2\rangle \\ & x & \mapsto & x \\ & y & \mapsto & -y \end{array}$$

is an isomorphism of k-algebra. Hence, by Lemma 1.1.26 and Curve 3. in the previous section (see (4.8)), we obtain that

$$\operatorname{Der}_k(B) = \operatorname{IDer}_k(B; n) = \langle \overline{\delta}_1, \overline{\delta}_2\rangle$$

where $\overline{\delta}_i = \Pi_n^{\langle g\rangle}(\delta_i)$ and $\delta_1 = x^2\partial_x + y^3\partial_y$ and $\delta_2 = y^2\partial_x + (x + y^2)\partial_y$. Therefore, $\operatorname{Leaps}_k(B) = \emptyset$. □

Let us consider k a commutative ring and A a commutative k-algebra. Remember that the integral closure of an ideal I of A is the ideal that consists of all elements of A that are integral over I, and is denoted $\overline{I}$.

Lemma 4.3.4 *Under the above condition, leaps of A/I are not the same that leaps of $A/\overline{I}$, i.e. leaps are not the same up to integral closure of ideals.*

Proof. Let us assume that k is a reduced ring of characteristic 2 and let us consider the ideal $I = \langle x^2, y^2\rangle \subseteq R = k[x,y]$. Then, its integral closure is $\overline{I} = \langle x^2, xy, y^2\rangle$. We will calculate modules of integrable k-derivations for $A = R/I$ and $\overline{A} = R/\overline{I}$ and will see that leaps of these two rings are different. We start with I-logarithmic k-derivations.

Let $\delta = u\partial_x + v\partial_y$ be a k-derivation of R. Then, $\delta(x^2) = \delta(y^2) = 0$. So, $\operatorname{Der}_k(\log I) = \langle \partial_x, \partial_y\rangle$. Let us consider $D = (\mathrm{Id}, \delta, D_2) \in \mathrm{HS}_k(R;2)$. Then, by Lemma 1.2.9, $D_2(x^2) = D_1(x)^2 = u^2 \in \langle x^2, y^2\rangle$. If we write $u = \sum u_{ij}x^iy^j$, then $u_{00}^2 = 0$. Since k is reduced $u_{00} = 0$ and $u \in \langle x, y\rangle$. Analogously, $D_2(y^2) = v^2 \Rightarrow v \in \langle x, y\rangle$. So,

$$\mathrm{IDer}_k(\log I; 2) = \langle x\partial_x, y\partial_y, x\partial_y, y\partial_y\rangle$$

It is easy to see that all these derivations are (∞-)integrable, it is enough to consider the k-algebra homomorphisms:

$$\begin{array}{rccl} \varphi_{xx}: & R & \to & R[|\mu|] \\ & x & \mapsto & x + x\mu \\ & y & \mapsto & y \end{array} \qquad \begin{array}{rccl} \varphi_{yx}: & R & \to & R[|\mu|] \\ & x & \mapsto & x + y\mu \\ & y & \mapsto & y \end{array}$$

$$\begin{array}{rccl} \varphi_{xy}: & R & \to & R[|\mu|] \\ & x & \mapsto & x \\ & y & \mapsto & y + x\mu \end{array} \qquad \begin{array}{rccl} \varphi_{yy}: & R & \to & R[|\mu|] \\ & x & \mapsto & x \\ & y & \mapsto & y + y\mu \end{array}$$

where φ_{ab} is an I-logarithmic integral of $a\partial_b$ for $a, b \in \{x, y\}$. In conclusion, $\mathrm{Leaps}_k(A) = \{2\}$ and

$$\mathrm{IDer}_k(A; n) = \begin{cases} \langle \partial_x, \partial_y\rangle & \text{if } n = 1 \\ \langle x\partial_x, y\partial_x, x\partial_y, y\partial_y\rangle & \text{if } n \geq 2 \text{ or } n = \infty. \end{cases}$$

Now, we calculate modules of integrable k-derivations of $\overline{A}$. Let us consider $\delta = u\partial_x + v\partial_y \in \operatorname{Der}_k(R)$. Then $\delta(x^2) = \delta(y^2) = 0$ and

$$\delta(xy) = uy + vx = Fx^2 + Gxy + Hy^2 \Rightarrow (u - Gx - Hy)y = (Fx - v)x$$

Then,

$$\begin{cases} u = Gx + Hy + Lx \\ v = Fx + Ly. \end{cases}$$

So,

$$\operatorname{Der}_k\left(\log \overline{I}\right) = \langle x\partial_x, y\partial_x, x\partial_y, y\partial_y\rangle.$$

Observe that φ_{ab} is an $\overline{I}$-logarithmic integral of $a\partial_b$ for all $a, b \in \{x, y\}$. Therefore, $\mathrm{Leaps}_k\left(\overline{A}\right) = \emptyset$ and

$$\mathrm{IDer}_k\left(\overline{A}; n\right) = \langle x\partial_x, y\partial_x, x\partial_y, y\partial_y\rangle \text{ if } n \geq 1 \text{ or } n = \infty$$

and the lemma is proved.

□

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