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INVARIANT FUNCTIONS AND CONTRACTIONS OF ALGEBRAS

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INVARIANT FUNCTIONS AND CONTRACTIONS OF ALGEBRAS

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This Ph. Doctoral Thesis in Mathematics, written in English and exposed in both Spanish and English languages, is dedicated to my wife and my children for supporting me in this stage, for which I have needed a lot of time and they have not had any problem in giving me theirs.

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Introduction

This manuscript constitutes the Ph. Doctoral Thesis presented by the author to apply for the Doctor in Mathematics' degree from the University of Seville. It deals with invariant functions and contractions of certain types of algebras, as Lie , Heisenberg, , Malcev or Kinematical algebras, for instance.

The most cited algebras in the manuscript are filiform Lie algebras. The motivation for dealing with these algebras is due that they are the most structured algebras within the nilpotent Lie algebras, which allows us to use and study them easier than other Lie algebras (an historical evolution of Lie algebras in general can be checked in [10]). With respect to the classification of these algebras can be consulted [9]).

Filiform Lie algebras were introduced by Vergne, in the late 60's of the past century [69], although before that, Blackburn [5] had already studied the analogous class of finite p-groups and used the term *maximal class* to call them, which is also now used for Lie algebras. Vergne showed that, within the variety of nilpotent Lie multiplications on a fixed vector space, the non-filiform ones can be relegated to small-dimensional components. On some occasions throughout this manuscript, especially in the page headings, filiform Lie algebras will be designated by the acronym FLA. Also, the word "dimension" will be denoted by dim. In any case, as it has been just mentioned, other different types of algebras, as Heisenberg , Malcev or Kinematical algebras, for instance, are also dealt with in this manuscript.

The main novel aspects and results obtained in this research which are shown in this manuscript are the following.

- The author has computed the invariant functions ψ and φ, introduced in 2007 by Hrivnák and Novotný [55], for the case of the filiform Lie algebras of lower dimensions (Chapter 3).

  In this respect, it is convenient to say that the reason for dealing with algebras of lower dimensions is because there are no complete classifications of Lie algebras in general of dimensions greater than 5 (for the filiform case, the maximal dimension that has been classified is 12 [9]) and, besides, few invariant and invariant functions for them are known. Therefore, the use of these invariant functions can be considered as a tool to get advances in the knowledge of these algebras, particularly in the study of contractions.

Indeed, the author has computed the *values of the invariant functions* ψ and φ for the filiform Lie algebras of dimensions 3, 4 and 5, for the Heisenberg algebra of dimension 3 and for other different types of algebras, all of them in the lower dimension (Chapter 3).

- As a relevant aspect of the research, we have introduced in the manuscript *three new invariant functions* for algebras, the one-parameter invariant function $\upsilon_{\mathfrak{g}}$, the two-parameter invariant function $\bar{\psi}_{\mathfrak{g}}$ and the two-parameter invariant function $\bar{\phi}_{\mathfrak{g}}$. We have obtained the main properties, some results and several applications of such functions to the study of contractions of algebras. These three new functions have allowed to give a step forward towards the knowledge of this topic and to make easier the computations needed for it (Chapter 4).

- With respect to the study of the *proper contractions* of filiform Lie algebras of lower dimensions, we have obtained two main results (Chapter 5).

  The first one is that there exists a proper contraction from the filiform Lie algebra $\mathfrak{f}_3$ both to the algebra $3\mathfrak{g}_1$ and the algebra $\mathfrak{g}_{3,2}$, whereas it exists no proper contraction either between $\mathfrak{f}_3$ and $\mathfrak{g}_{3,1}$, or $\mathfrak{f}_3$ and $\mathfrak{g}_{3,3}$.

  The second result is that there is no proper contraction from a filiform Lie algebra to a Heisenberg algebra. It implies that filiform Lie algebras cannot appear as a classical limit from the contraction of a quantum mechanical model built upon a Heisenberg algebra because in that case there would be a contraction from the Heisenberg algebra to the filiform Lie algebra of the same dimension.

- Finally, as application of the introduced new invariant function $\upsilon_{\mathfrak{g}}$, Kinematical Lie algebras are widely dealt with in this manuscript (Chapter 6).

  The main result obtained by the author on this subject has been the computation of the values of such a function for the eight kinematical Lie algebras studied by Tolar [65] (see Chapter 6). It also suppose a step forward in the knowledge of the study of contractions of algebras.

Therefore, according to the previous paragraphs, the structure of the manuscript is the following

In Chapter 1, we expose a brief historical evolution on invariant functions and contractions of certain types of algebras, with the objective of framing the problem under study and offering a historical overview of the evolution that these topics have followed so far.

Chapter 2 consists of those already known basic concepts and results on different types of algebras, gradings, invariant functions of them and contractions that we use throughout the manuscript.

In Chapter 3 we compute the invariant functions ψ and φ, introduced by Hrivnák and Novotný [55], in the particular case of model filiform Lie algebras of lower dimensions, particularly dimensions 3 and 4. Our intention is to deal with those of greater dimensions in a similar way in future work.

In Chapter 4 we firstly introduce a new two-parameter invariant function $\bar{\psi}_{\mathfrak{g}}$ and compute its value for different types of algebras: Malcev , Lie and other algebras. Secondly, two other invariant functions for algebras are also introduced: the one-parameter invariant function $\upsilon_{\mathfrak{g}}$ and the two-parameter invariant function $\bar{\phi}_{\mathfrak{g}}$.

Chapter 5 is devoted to the study of contractions of algebras, particularly proper contractions of algebras. We show several examples of proper contractions between different types of algebras.

Kinematical algebras are dealt with in Chapter 6. We consider the kinematical Lie algebras of four-dimensional spacetime, introduced by Tolar in [65], and compute the one-parameter invariant function υ introduced in the previous chapter for the eight kinematical Lie algebras given by Tolar .

Finally, a last chapter devoted to pose and analyze some open problems coming from this research has been also included.

At the end of the manuscript, after the bibliographic references, we expose an index with the names of researchers appearing in the manuscript.

Finally, let us remark that this manuscript is largely based upon the following papers

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, On contractions of Lie algebras, *Mathematics in Computer Science (Math. Comput. Sci.)* **10**:3 (2016), 353-364. DOI 10.1007/s11786-016-0266-0 [23]. Indicios de calidad: Índice de impacto 0.151

en índice SJR-2007 Applied Mathematics, posición 390 de 430, **Cuartil 4**. Tercil 3.

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, A new one-parameter invariant function for algebras, *Mathematics in Computer Science (Math. Comput. Sci* **12** (2018), 143-150 [25]. Indicios de calidad: Índice de impacto 0.151 en índice SJR-2007 Applied Mathematics, posición 390 de 430, **Cuartil 4**. Tercil 3.

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, Invariant functions and contractions of certain types of Lie algebras of lower dimensions, *Journal of Nonlinear Mathematical Physics* **25**:3 (2018), 358-374 [26]. Impact Factor 1.438 in JCR-17 Math App., 65/252, **Cuartil 1**.

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, Graded contractions of filiform Lie algebras, *Mathematical Methods in the Applied Sciences*, **41**:17 (2018), 7195-7201 [27]. Impact Factor 1.180 in JCR-17 Math App., 91/252, **Cuartil 2**.

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, Kinematical Lie algebras and invariant functions of algebras, *International Journal for Computational Methods in Engineering Science & Mechanics*. In press, 2018 [28]. Indicios de calidad: Revista Scopus. Índice de impacto 0.359 en SJR-2007 Computational Mathematics, posición 79 de 123, **Cuartil 3**. Tercil 2.

The following paper has been also submitted recently to one international journal included in the Journal Citation Reports and, at present, it is pending of acceptance

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, The invariant two-parameter function $\bar{\psi}$. Submitted, November 2018 [29].

And currently and with the intention of sending it for publication, the following paper is already in the final phase of its preparation

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, The two-parameter invariant function for algebras $\bar{\phi}$ [37].

Further, the results here exposed have already been pointed out in the following national and international conferences

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, Invariant functions of filiform Lie algebras. Poster in the *Congreso de la RSME 2015*, Granada, 2 al 6 de Febrero de 2015 [30].

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, Contractions of particular types of nilpotent Lie algebras of lower dimensions, *2nd International Conference on Numerical and Symbolic Computation Developments and Applications (SYMCOMP 2015)*, Faro, Algarve, 26-27 March 2015 [31].

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, Contractions of certain Lie algebras, *XI Encuentro Andaluz de Geometría.* Sevilla, 15 de mayo de 2015 [32].

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, Applying invariants to the study of contractions of algebras, *Third International School on Computer Algebra and Applications (EACAS)*, Sevilla, 18 a 21 de enero 2016 [33].

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, A new one-parameter invariant function for algebras, *3rd International Conference on Numerical and Symbolic Computation: Developments and Applications (SYMCOMP-2017)*, Guimares, Portugal, 6-7 April 2017 [34].

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, Graded contractions of filiform Lie algebras, *17th International Conference Computational and Mathematical Methods in Science and Engineering, CMMSE 2017*, 4-8 July 2017, Rota, Cadiz, Spain [35].

- J.M. Escobar, J. Núñez and P. Pérez-Fernández, Applications of the two-parameter invariant function $\bar{\psi}^0_{\mathfrak{g}}(\alpha, \beta)$ to filiform Lie algebras, *XXVII International Fall Workshop on Geometry and Physics*, Sevilla, 3 - 7 September 2018 [36].

Finally, we would like to say that without taking in consideration the own ones, the references that we have most used in our research have been [44, 55, 56] and [65] and somewhat less, although also frequently, [1, 4, 54] and [71].

Chapter 1

An historical evolution on the topics of the manuscript

As we indicated in the Introduction, we are dealing in this manuscript with invariant functions and contractions of certain types of algebras, as Lie, Heisenberg, Malcev or Kinematical algebras, for instance.

In this respect, it is convenient to note that many modern physical theories from the last times have a powerful mathematical apparatus underlying, in which different types of algebras play a very important role. As an example, it can be highlighted the importance that the Virasoro and Kac-Moody algebras have, among others, in the development of the Superstring Theory.

Specifically, Lie algebras (so called after Sophus Lie, Norwegian mathematician, 1842 - 1899) constitute certainly one of the most useful tools in order to describe symmetries underlying in these physical theories. Also, Lie algebras are essential in Conformal Field Theory, in which the semi-simple Lie algebras guarantee the existence of the non-degenerate Killing form, which in turn allows to build the *vertex algebras* so important in this theory (see the paper by Frenkel and Ben-Zvi [39]). But it is possible that the better known example is the application of Lie algebras to the celebrated Standard Model of particle physics, which describes the internal symmetries of the unitary product group $SU(3) \times SU(2) \times SU(1)$.

Regarding Heisenberg algebras, it is convenient to recall in the first place that the Heisenberg group, named after the German physicist Werner Karl Heisenberg (Würzburg, 1901 - Munich, 1976, known above all for formulating the *uncertainty principle*, a fundamental contribution to the development of Quantum Theory), is the group of 3×3 upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

under the operation of matrix multiplication. The elements a, b and c can be taken from any commutative ring with identity, often taken to be the ring of real numbers (resulting in the *continuous Heisenberg group*) or the ring of integers (resulting in the *discrete Heisenberg group*). The continuous Heisenberg group arises in the description of one-dimensional quantum mechanical systems, especially in the context of the Stone-von Neumann theorem. More generally, one can consider Heisenberg groups associated to n-dimensional systems, and most generally, to any symplectic vector space.

The study of other physical concepts has significantly been enhanced by the *limit processes* which allow us to relate Lie algebras between themselves. These processes were first investigated by Segal [63] in 1951, although, as it has been just noted, they are currently being widely used. Two are the better known examples of these processes. The first of them involves the connection between classical mechanic and relativistic mechanic, with their respective Poincaré symmetry group and Galilean symmetry group. The second one is the limit process by which Quantum Mechanics is contracted to classical mechanic, when $\hbar \to 0$, which actually corresponds to a contraction of the Heisenberg algebra to the abelian algebra of the same dimension. Remember also that classical mechanics is a limiting case of relativistic mechanics. Hence the group of the former, the Galilean group, must be in some sense a limiting case of the relativistic mechanics' group. This means that the representations of the Galilean symmetry group must be limiting cases of the representations of the Poincaré symmetry group (see [54] for further general information and [48] for complementary information).

Regarding Malcev algebras, they constitute a generalization of Lie algebras. It is well-known that themselves and Lie algebras are not disjoint sets. Indeed, every Lie algebra is a Malcev algebra, but the converse is not true. Therefore, we can distinguish between type Lie Malcev algebras and type non-Lie Malcev algebras. Obviously, those Malcev algebras which are of the type Lie verify both identities: Jacobi and Malcev.

If the Jacobi identity does not hold, then the Malcev algebra is said to have a *Jacobi anomaly*. In Quantum Mechanics, the existence of Jacobi anomalies in the underlying non-associative algebraic structure related to the coordinates and momenta of a quantum non-Hamiltonian dissipative system was already claimed by Dirac in [19] in the process of taking Poisson brackets. In String Theory, for instance, one such an anomaly is involved by the non-associative algebraic structure that is defined by coordinates (x) and velocities or momenta (v) of an electron moving in the field of a constant magnetic charge distribution, at the position of the location of the magnetic monopole [49]. In particular, $J(v_1, v_2, v_3) = -\overrightarrow{\nabla} \circ \overrightarrow{B}(x)$, where $\overrightarrow{\nabla} \circ \overrightarrow{B}(x)$ denotes the divergence of the magnetic field $\overrightarrow{B}(x)$. The underlying algebraic structure constitutes a non-Lie Malcev algebra [40], with the commutation relations $[x_a, x_b] = 0$, $[x_a, v_b] = i\,\delta_{ab}$ and $[v_a, v_b] = i\,\varepsilon_{abc}\,B_c(x)$, where $a, b, c \in \{1, 2, 3\}$, δ_{ab} denotes the Kronecker Delta and ε_{abc} denotes the Levi-Civita symbol. If the magnetic field is proportional to the coordinates, the latter can be normalized and $B_c(x)$ can then be supposed to coincide with x_c. The resulting algebra is then called magnetic. A generalization to electric charges has recently been considered in [41] by Günaydin and Minic, by defining the products $[x_a, x_b] = -i\,\varepsilon_{abc}\,E_c(x, v)$, where the electric field E as well as the magnetic field B must depend not only on coordinates but also on velocities. This latter algebra is called electric algebra [41, 14]. Remark that both magnetic and electric algebras constitute partial-magma algebras (this term "magma" is due to the group of French mathematicians who call themselves Nicholas Bourbaki. See [8] for these algebras).

Regarding the concept of *contractions* of algebras, there exist many works on it that address both the physical aspect of the applications of these contractions and the mathematical aspect focused in the study of the algebraic properties of these contractions. Indeed, after Segal, the concept of limit process between physical theories in terms of contractions of their associated symmetry groups was formulated by Erdal Inönü and Eugene Wigner [44, 45], who introduced the so-called *Inönü-Wigner contractions or IW- contractions*. Later, Saletan [62] studied a more general class of *one-parameter contractions*, for which the elements of the corresponding matrices are one-degree polynomials with respect the contraction parameter (in fact, WI-contractions are a subclass of Saletan contractions). Other extensions of the IW-contractions are, for instance, the *generalized Inönü-Wigner contractions*, introduced by Doebner and Melsheimer [20], the *parametric degenerations*, by Burde and Steinhoff [13, 11, 12, 64], very used in the Algebraic Invariants Theory, and the *singular contractions* [45].

One of the most relevant and useful results in Physics is the so called *Correspondence principle*, which colloquial and basically sets that a new theory should coincide with the old one in predictions for phenomena where these conditions are satisfied.

Indeed, this principle, which was formulated by Niels Bohr in 1920 [7], though he had previously made use of it as early as 1913 in developing his model of the atom, states that the behavior of systems described by the theory of Quantum Mechanics (or by the old quantum theory) reproduces classical physics in the limit of large quantum numbers. In other words, it says that for large orbits and for large energies, quantum calculations must agree with classical calculations. Therefore, the term codifies the idea that a new theory should reproduce under some conditions the results of older well-established theories in those domains where the old theories work (note, however, that this concept is somewhat different from the requirement of a formal limit under which the new theory reduces to the older, thanks to the existence of a deformation parameter).

The mathematical formulation of this principle for relativistic mechanics was given by Inön*ü* and Wigner [44] when introducing their IW-contractions.

The algebraical approach to contractions is called *graded contractions*, which were originally introduced in [52] as a generalization of IW-contractions. There are two types of graded contractions: continuous graded contractions, which correspond to IW-contractions and discrete graded contractions, which possess no equivalent in continuous contractions. The general solution of the graded contractions, considering only so called generic case, was achieved in [72, 73]. Since this solution depends solely on the grading group (the structure of the Lie algebra does not matter at all), it is obtained simultaneously for all Lie algebras which allow the given grading. However, this approach is in a certain sense too general. It motivates our study.

Indeed, graded contraction of several types of Lie algebras have been already dealt in previous papers. For instance, Novotny studied in deep graded contractions of the simple Lie algebra $sl(3,\mathbb{C})$ [55]. He showed 4 gradings for this algebra. Later, Novotny himself obtained the contractions for each grading. Bahturin, Goze and Remm [4] classified, up to isomorphism, gradings by abelian groups on nilpotent Lie algebras of nonzero rank and, in the case of rank 0, they described conditions to obtain non trivial Z_k-gradings.

Apart of the previously cited papers, there are lots of papers in the literature dealing with IW-contractions. Among them, [50, 70, 71, 15] can be checked. Indeed, Hegerfeldt dealt particularly with this topic in [42] and Popovych D.R. and Popovych R.O. presented in [59] a simple and rigorous proof of the claim by Weimar-Woods posed in [71] on that any diagonal contraction (e.g., a generalized Inön*ü*Wigner contraction) is equivalent to a generalized Inön*ü*-Wigner contraction with integer parameter powers. Moreover, a quite complete study on the theory of Inönü-Wigner contractions can be consulted in [71].

So, by continuing with this study, which we began to deal with in [23], we show in the paper the graded contractions of the model filiform Lie algebras of dimension 3 and 4. These ones, together with those of dimensions 5 and 6, which we have also include in this paper, us have allowed to study the general case of the contractions of n-dimensional model filiform Lie algebras, which is dealt with in Section 5. Moreover, with the objective of comparing the model case with the non-model one, the graded contractions of a non-model 6-dimensional filiform Lie algebra has been also obtained. It is convenient to say that our motivation for dealing with this type of algebras is due to the fact of that these algebras, which were introduced by Vergne in 1966, in her Ph. D. Thesis, later published in 1979 [69], constitute the most structured subset of nilpotent Lie algebras.

To finish this historical evolution of the main concepts dealt with in this work, we are going to write on Kinematics.

Since the middle of the last century, Lie algebras and groups have been very used as a tool in the study of several topics in Kinematics. A first example of this assertion is due to Bacry and Lévy-Leblond, who dealt with *kinematical or relativity groups* in 1968 (see [2] and [65] and the references therein). These groups include, besides space-time translations and spatial rotations, inertial transformations connecting different inertial frames of reference. When parity and time-reversal are required to be automorphisms of the groups and a weak hypothesis on causality is made, the only possible groups are found to consist of the de Sitter groups and their rotation-invariant contractions. Besides the de Sitter, Poincaré, and Galilei groups, two other types of groups are found to present some interest. The first one is the *static group*, which applies to the static models, with infinitely massive particles, and the second one, which is halfway between the de Sitter and the Galilei groups, contains two non-relativistic cosmological groups describing a non-relativistic curved space-time.

All these groups have a great importance in Physics, since through these symmetry groups of space-time the basic invariance of the laws of that discipline can be implemented, particularly in the special theory of relativity, with the ten-parameter Poincaré group containing (as transformation group of the four-dimensional Minkowski space-time) the time and space translations, space rotations, and boosts (inertial transformations). In 1986, Bacry himself and Nuyts classify in reference [3], under certain natural physical assumptions, all the abstract ten-dimensional real Lie algebras that contain as a subalgebra the algebra of the three-dimensional rotation group (generators J) and decompose under the rotation group into three-vector representation spaces (J itself, K, and P) and a scalar (generator H), showing the existence of a homogeneous space of dimension 4 in all cases.

Another example of the application of Lie algebras and groups in Kinematics is found later, in reference [57], where Park and Ravani generalize the concept of Bézier curves to curved spaces, and illustrate this generalization with an application in Kinematics. Indeed, they show how De Casteljau's algorithm for constructing Bézier curves can be extended in a natural way to a special class of Riemannian manifold, the Lie groups, since that these groups, due to their own group structure, admit an elegant and efficient recursive algorithm for constructing those curves. Spatial displacements of a rigid body also form a Lie group, and can therefore be interpolated (in the Bézier sense) using the recursive algorithm which they construct and apply to the kinematic problem of trajectory generation or motion interpolation for a moving rigid body. The orientation trajectory of motions generated in this way have the important property of being invariant with respect to choices of inertial and body-fixed reference frames.

In the same sense, another example of the use of Lie algebras and groups in Kinematics is supplied by Rico, Gallardo and Ravani, who have dealt with an instantaneous form of the mobility criterion based on the theory of subspaces and subalgebras of the Lie Algebra of the Euclidean group and their possible intersections. They show that certain results on mobility of over-constraint linkages derived previously using screw theory are not complete and accurate and their theory provides for a computational approach that allows efficient automation of the new group-theoretic mobility criterion [60].

A different example of the application of Lie algebras and groups in Kinematics is shown by Chevallier in [17]. He develops systematic coordinate-free exposition of the different algebraic operations in the set of infinitesimal displacements (screws) and their relations with finite displacements is developed. He introduces six basic operations which generate several algebraic structures, in particular Lie algebra and module over the dual number ring endowed with a dual valued inner product.

As the last example which we point out of the use of Lie algebras in Kinematics, we cite a paper by Khrushchev and Leznov, [47] in which they consider a deformation of the canonical algebra for kinematic observables of quantum field theory in Minkowski space-time under the condition of Lorentz invariance. The relativistically invariant algebra which they obtain depends on additional fundamental constants M, L and H with the dimensions of mass, length and action, respectively. They observe that in some limiting cases the algebra goes over into the well-known Snyder or Yang algebras, whereas in the general case, the algebra represents a class of Lie algebras, which consists of both simple algebras and semidirect sums of simple and integrable algebras, some of which are not invariant under the T and C transformations.

There are several papers in the literature regarding the classification of kinematical Lie algebras. Among them, Campoamor and Rausch dealt with and classified kinematical algebras which appear in the framework of Lie superalgebras or Lie algebras of order three. All these algebras are related through generalised In*ö*n*ü*-Wigner contractions from either the orthosymplectic superalgebra or the de Sitter Lie algebra of order three [16] and recently, Figueroa has obtained partial classifications in them (see [38] and the references therein). All of this gives an idea of the importance that these algebras are currently acquiring.

Finally, let us recall again that a contraction of an algebra is a procedure for which, starting from an algebra one can obtain a new algebra, satisfying certain conditions, which is not isomorphic to the initial one. At present, contractions of algebras play a relevant role in Theoretical Physics since Wigner and Inönü discussed the possibility to obtain from a given Lie group a different (non-isomorphic) Lie group by a group contraction with respect to a continuous subgroup of it [44].

Let us now recall brief preliminaries on this subject.

Chapter 2

Preliminaries

2.1 Preliminaries on Lie algebras

In this section we show some preliminaries on Lie algebras in general and on filiform Lie algebras in particular. For a further review on these topics, the reader can consult [43, 68] for Lie algebras, for instance, and [69] for filiform Lie algebras.

An n-dimensional *Lie algebra* $\mathfrak{g}$ over a field K is an n-dimensional vector space over K endowed with a second inner law, named *bracket product*, which is bilinear and anti-commutative and satisfies the following expression, named *Jacobi identity*

$$J(u,v,w) = [u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0, \text{ for all } u,v,w \in \mathfrak{g}. \tag{2.1}$$

The *center* of $\mathfrak{g}$ is the set $Z(\mathfrak{g}) = \{u \in \mathfrak{g} \mid [u,v] = 0, \text{ for all } v \in \mathfrak{g}\}$. The Lie algebra is said to be *abelian* if $[u,v] = 0$, for all $u,v \in \mathfrak{g}$.

A *derivation* on $\mathfrak{g}$ is a linear map $d : \mathfrak{g} \to \mathfrak{g}$ verifying the following condition: $d([u,v]) = [d(u),v] + [u,d(v)]$, for all $u,v \in \mathfrak{g}$.

Two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are *isomorphic* if there exists a vector space isomorphism f between them such that

$$f([u,v]) = [f(u),f(v)], \text{ for all } u,v \in \mathfrak{g}. \tag{2.2}$$

The existence of such a map is denoted as $\mathfrak{g} \cong \mathfrak{h}$.

2.2 Preliminaries on filiform Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. The *lower central series* of $\mathfrak{g}$ is defined as

$$\mathfrak{g}^1 = \mathfrak{g},\ \mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}],\ \ldots,\ \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}],\ \ldots \tag{2.3}$$

If there exists $m \in \mathbb{N}$ such that $\mathfrak{g}^m \equiv 0$, then $\mathfrak{g}$ is called *nilpotent*. The *nilpotency class* of $\mathfrak{g}$ if the smallest natural c such that $\mathfrak{g}^{c+1} \equiv 0$.

An n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is said to be *filiform* if it is verified that

$$\dim\, \mathfrak{g}^k = n - k,\ \text{ for all } k \in \{2, \ldots, n\}. \tag{2.4}$$

The only n-dimensional filiform Lie algebra for $n < 3$ is the abelian. For $n \geq 3$, it is always possible to find which is called *adapted basis* $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$ such that

$[e_1, e_{j-1}] = e_j$, for all $j \in \{3, \ldots, n\}$, where the other brackets are null.

If $n \geq 4$, then the following two integers are invariants by isomorphism [22].

$$z_1 = \min\{i \geq 4 | [e_i, e_n] \neq 0\}, \quad z_2 = \min\{i \geq 4 | [e_i, e_{i+1}] \neq 0\}. \tag{2.5}$$

From the condition of filiformity and the Jacobi identity (2.1), the bracket product of $\mathfrak{g}$ is determined by the previous brackets and the new ones

$$[e_i, e_j] = \sum_{k=2}^{min\{i-1, n-2\}} c_{ij}^k e_k, \quad \text{for} \quad 4 \leq i < j \leq n, \tag{2.6}$$

where $c_{i,j}^k \in K$ are called *structure constants* of $\mathfrak{g}$. If all of them are zeros, then the filiform Lie algebra $\mathfrak{g}$ is called *model*. From the invariants (2.5), the model algebra is not isomorphic to any other algebra of the same dimension. From the original products and (2.6), every n-dimensional filiform Lie algebra $\mathfrak{g}$ having an adapted basis $\{e_1, \ldots, e_n\}$ verifies that

$$\mathfrak{g}^2 = \langle e_2, \ldots, e_{n-1} \rangle,\ \mathfrak{g}^3 = \langle e_2, \ldots, e_{n-2} \rangle, \ldots, \mathfrak{g}^{n-1} = \langle e_2 \rangle,\ \mathfrak{g}^n = 0. \tag{2.7}$$

2.3 Preliminaries on Heisenberg Lie algebras

Let n be a nonnegative integer or infinity. The *n-th Heisenberg algebra* is the Lie algebra with basis $\mathcal{B} = \{p_1, \ldots, p_n, q_1, \ldots, q_n, z\}$ with the following relations, known as *canonical commutation relations*

1. $[p_i, q_j] = c_{ij}\, z, \quad 1 \leq i, j \leq n.$

2. $[p_i, z] = [q_i, z] = [p_i, p_j] = [q_i, q_j] = 0, \quad 1 \leq i, j \leq n.$

Note that the dimension of an n-th Heisenberg algebra is not n, but $2n+1$. In fact, the n in the above definition is called the *rank* of the Heisenberg algebra, although it is not, however, a rank in any of the usual meanings that this word has in the theory of Lie algebras. Due to it, this Lie algebra is also known as the *Heisenberg algebra of rank n*.

2.4 Preliminaries on Malcev algebras

These algebras, so-called after the Russian mathematician Anatolii Ivánovich Mátsev, constitute a generalization of Lie algebras. We now recall some preliminary concepts on them, taking into account that a general overview on them can be consulted in [51, 61]. From here on, we are only considering finite-dimensional algebras over the complex number field $\mathbb{C}$.

A *Malcev algebra* $\mathcal{M}$ is a vector space with a second bilinear inner composition law $([\cdot,\cdot])$ called the *bracket product* or *commutator*, which satisfies the two following conditions

1. $[u, v] = -[v, u]$, $\forall u, v \in \mathcal{M}$.

2. $[[u, v], [u, w]] = [[[u, v], w], u] + [[[v, w], u], u] + [[[w, u], u], v]$, $\forall u, v, w \in \mathcal{M}$.

The second condition is named the *Malcev identity* and we use the notation $M(u, v, w) = [[u, v], [u, w]] - [[[u, v], w], u] - [[[v, w], u], u] - [[[w, u], u], v]$.

Given a basis $\{e_i\}_{i=1}^n$ of a n-dimensional Malcev algebra $\mathcal{M}$, the *structure constants* $c_{i,j}^h$ are defined as $[e_i, e_j] = \sum_{h=1}^n c_{i,j}^h e_h$, for $1 \leq i, j \leq n$.

It is immediate to see that Malcev algebras and Lie algebras are not disjoint sets. Indeed, every Lie algebra is a Malcev algebra, but the converse is not true. Therefore, as we previously said we can distinguish between type Lie Malcev algebras and type non-Lie Malcev algebras. Obviously, those Malcev algebras which are of the type Lie verify both identities: Jacobi and Malcev .

2.5 Gradings

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra over $\mathbb{C}$. A decomposition $\Gamma : \mathfrak{g} = \oplus_{i \in I} \mathfrak{g}_i$ of the vector space $\mathfrak{g}$ into a direct sum of vector subspaces $\mathfrak{g}_i \neq 0$, $i \in I$, is called a *grading* of $\mathfrak{g}$ if for any pair of indices $i, j \in I$, there exists $k \in I$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_k$. Vector subspaces $\mathfrak{g}_i$ are called *grading subspaces.* The number of grading subspaces is equal to the cardinality $|I|$ of the index set I.

A grading $\Gamma : \mathfrak{g} = \oplus_{i \in I} \mathfrak{g}_i$ is called *group grading* (respectively, *semigroup grading*) if there exist an abelian group (resp. semigroup) G and an injective mapping $f : I \to G$ such that for any pair of indices $i, j \in I$, the equality $f(i \circ j) = f(i) + f(j)$ holds, where $+$ denotes the binary operation in G. The group (resp. semigroup) G is called *grading group (resp. semigroup).*

The *universal group* is the Abelian finitely generated group U which contains the set of indices J of a grading group $J \subset U$.

Let $\Gamma : \mathfrak{g} = \oplus_{i \in I} \mathfrak{g}_i$ be a grading of the Lie algebra $\mathfrak{g}$, with $|I| = m \in \mathbb{N}$ grading subspaces. A complex Lie algebra $\mathfrak{g}^{\varepsilon}$ endowed with a Lie bracket $[,]_{\varepsilon}$ and satisfying the two conditions: *i)* the underlying vector space of $\mathfrak{g}^{\epsilon}$ is the underlying vector space of $\mathfrak{g}$, i.e. $\mathfrak{g}^{\epsilon} = \oplus_{i \in I} \mathfrak{g}_i$, and *ii)* for all $i, j \in I$, there exists $\varepsilon_{ij} \in \mathbb{C}$ such that $[x, y]_{\varepsilon} = \varepsilon_{ij}[x, y]$, for all $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$, is called Γ-*graded contraction* of the Lie algebra $\mathfrak{g}$.

We define the *contraction matrix* ε of the Lie algebra $\mathfrak{g}^{\varepsilon}$ as a matrix whose elements are ε_{ij}. This matrix determines the Lie algebra $\mathfrak{g}^{\varepsilon}$.

The elements ε_{ij} associated with Lie brackets verifying $[\mathfrak{g}_i, \mathfrak{g}_j] \neq 0$ will be called *relevant* elements. On the contrary, those elements verifying $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ will be called *non-relevant* elements and they will be considered null. The null relevant elements will be called *singular* elements. The set of the pairs (i, j) such that ε_{ij} is a relevant element of the contraction matrix ε will be denoted by $\mathcal{I}$.

Let Π_n denote the symmetric group of the set $\{1, 2, ..., n\}$. We define an equivalence relation on $I^n = I \times I \times \ldots \times I (n \text{ times})$ as follows: two n-tuples $(x_1, ..., x_n), (y_1, ..., y_n) \in I^n$ are equivalent if and only if there exists $\sigma \in \Pi_n$ such that $x_i = y_{\sigma(i)}$, for all $i = 1, ..., n$. The classes $(x_1\, x_2 \ldots x_n) = \{(x_{\sigma 1}, ..., x_{\sigma n}) | \sigma \in \Pi_n\}$ defined by this relation are called *unordered n-tuples* and the set of all unordered n-tuples with entries in I is denoted by I^n_u.

According to Jin Tolar [65], to apply the new theory of graded contractions of Lie algebras in Physics, it is desirable to define the gradings on the basis of some physical principles. For instance, the gradings can be defined by the automorphisms of the given Lie algebra induced by some discrete Abelian invariance group. In his physical applications [53, 66] the gradings are defined by involutive Lie algebra automorphisms induced by discrete transformations of space inversion and time reversal, being both of them preseved. In order to preserve physical types of Lie algebra generators under contraction, Trávnícek and Tolar impose in addition that their transformation properties under space rotations are preserved. In [67] the transformation of space inversion is trivial on the Lie algebra and is replaced by a discrete canonical transformation exchanging positions and momenta. Such additional assumptions then lead to a substantially restricted but physically well justified classification of contractions.

For further information on gradings, the reader can consult [58].

2.6 Invariant functions of Lie algebras

In this subsection we recall the definitions and main properties of invariant functions ψ and φ, obtained by Hrivnák and Novotný [55] in 2007. One of the objectives of this work is to introduce new invariant functions of algebras starting from these two last functions.

2.6.1 The invariant function ψ

Definition 2.6.1. *Let $\mathfrak{g}$ be a Lie algebra. An endomorphism d of $\mathfrak{g}$ is said to be a (α,β,γ)-*derivation *of $\mathfrak{g}$ if there exist $\alpha,\beta,\gamma\in\mathbb{C}$ such that*

$$\alpha\, d[X,Y]=\beta\,[dX,Y]+\gamma\,[X,dY],\quad \forall X,Y\in\mathfrak{g}.$$

The set of $(\alpha,\beta,\gamma)-$*derivations* of $\mathfrak{g}$ will be denoted by $Der_{(\alpha,\beta,\gamma)}\mathfrak{g}$.

Note that this definition is the extension of the usual definition of *derivation* of a Lie algebra, which is the case in which $\alpha=\beta=\gamma=1$.

Theorem 2.6.2. *Let $f:\mathfrak{g}\mapsto\widetilde{\mathfrak{g}}$ be an isomorphism between two complex Lie algebras $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$. Then, the mapping $\rho: End\,\mathfrak{g}\mapsto End\,\widetilde{\mathfrak{g}}$ defined as $\rho(d)=f\circ d\circ f^{-1}$ is an isomorphism between the corresponding vector spaces $Der_{(\alpha,\beta,\gamma)}\mathfrak{g}$ and $Der_{(\alpha,\beta,\gamma)}\widetilde{\mathfrak{g}}$, $\forall\alpha,\beta,\gamma\in\mathbb{C}$.*

Corollary 2.6.3. *The dimension of the vector space $Der_{(\alpha,\beta,\gamma)}\mathfrak{g}$ is an invariant of the Lie algebra $\mathfrak{g}$, $\forall\alpha,\beta,\gamma\in\mathbb{C}$.*

Definition 2.6.4. *The functions $\psi_{\mathfrak{g}},\psi^0_{\mathfrak{g}}:\mathbb{C}\mapsto\{0,1,2,...,(\dim\mathfrak{g})^2\}$ defined as*

$$(\psi_{\mathfrak{g}})(\alpha)=\dim Der_{(\alpha,1,1)}\mathfrak{g}, \tag{2.8}$$

$$(\psi_{\mathfrak{g}}^0)(\alpha) = \dim Der_{(\alpha,1,0)}\mathfrak{g}, \tag{2.9}$$

are called $\psi_{\mathfrak{g}}$ *and* $\psi_{\mathfrak{g}}^0$ invariant functions *corresponding to the* (α,β,γ)*-derivations of* $\mathfrak{g}$.

Theorem 2.6.5. *Two 3-dimensional complex Lie algebras* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are isomorphic if and only if* $\psi_{\mathfrak{g}_1} = \psi_{\mathfrak{g}_2}$.

2.6.2 The invariant function φ

Definition 2.6.6. *Let* (V, f) *be a representation of the Lie algebra* $\mathfrak{g}$*, where* V *is a complex vector space. A* V-cochain *of dimension* q *is a* q*-linear mapping* $c : \underbrace{\mathfrak{g}\times\ldots\times\mathfrak{g}}_{q-times} \mapsto V$*, such that* $c(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_q) + c(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_q) = 0$*, for all indices* i,j*, with* $1 \le i < j \le q$.

The vector space of all V-cochains of dimension q with $q \in \mathbb{N}$ will be denoted by $C^q(\mathfrak{g}, V)$ and $C^0(\mathfrak{g}, V) = V$. Now, we define the mapping $d : C^q(\mathfrak{g}, V) \mapsto C^{q+1}(\mathfrak{g}, V)$, with $q = 0, 1, 2, \ldots$ as

$$dc(x) = f(x)\,c, \text{ with } c \in C^0(\mathfrak{g}, V),$$

$$dc(x_1,\ldots,x_{q+1}) = \sum_{i=1}^{q+1}(-1)^{i+1} f(x_i)\, c(x_1,\ldots,\hat{x}_i,\ldots,x_{q+1})$$

$$+\sum_{\substack{i,j=1\\ i<j}}^{q+1}(-1)^{i+j}\, c([x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{q+1}).$$

where $\hat{x}_i$ means that x_i has been omitted.

Under the same conditions as before, let $\kappa = (\kappa_{ij})$ be a symmetric complex matrix of dimension $(q+1)\times(q+1)$.

Definition 2.6.7. *A* κ*-twisted cocycle (or simply* κ*-cocycle) is any* $c \in C^q(\mathfrak{g}, V)$*, with* $q \in \mathbb{N}$*, verifying*

$$0 = \sum_{i=1}^{q+1}(-1)^{i+1}\kappa_{ii} f(x_i)c(x_1,\ldots,\hat{x}_i,\ldots,x_{q+1})$$
$$+\sum_{\substack{i,j=1\\ i<j}}^{q+1}(-1)^{i+j}\kappa_{ij}c([x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{q+1}).$$

If the vector space V is identified with the algebra $\mathfrak{g}$, then the adjoint representation can be used as an action (see [43]). So, the equality of the definition 2.6.7 can be written as

$$0=\sum_{i=1}^{q+1}(-1)^{i+1}\kappa_{ii}\ [x_i,c(x_1,\ldots,\hat{x}_i,\ldots,x_{q+1})]$$

$$+\sum_{\substack{i,j=1\\ i<j}}^{q+1}(-1)^{i+j}\kappa_{ij}\,c([x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{q+1}).$$

The set of all κ-cocycles of dimension q will be denoted by $Z^q(\mathfrak{g},f,\kappa)$. Clearly, it is a vector subspace of $C^q(\mathfrak{g},V)$.

If the following notation is considered

$$coc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\mathfrak{g}=Z^2\left(\mathfrak{g},ad_{\mathfrak{g}},\begin{pmatrix}\beta_1&\alpha_2&\alpha_3\\ \alpha_2&\beta_3&\alpha_1\\ \alpha_3&\alpha_1&\beta_2\end{pmatrix}\right),$$

it is easy to see that the vector space $coc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)$ is constituted by the $B\in C^2(\mathfrak{g},\mathfrak{g})$ such that $\forall X,Y,Z\in\mathfrak{g}$,

$$\begin{aligned}0&=\alpha_1B(X,[Y,Z])+\alpha_2B(Z,[X,Y])+\alpha_3B(Y,[Z,X])\\ &+\beta_1[X,B(Y,Z)]+\beta_2[Z,B(X,Y)]+\beta_3[Y,B(Z,X)].\end{aligned}\tag{2.10}$$

The two following results will be used in next sections of the manuscript.

Theorem 2.6.8. *[56] Let $g:\mathfrak{g}\mapsto\widetilde{\mathfrak{g}}$ be an isomorphism between Lie algebras $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$. Then, the mapping $\rho:C^q(\mathfrak{g},\mathfrak{g})\mapsto C^q(\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}})$, for $q\in\mathbb{N}$, defined by $(\rho c)(x_1,\ldots,x_q)=gc(g^{-1}x_1,\ldots,g^{-1}x_q)$, $\forall c\in C^q(\mathfrak{g},\mathfrak{g})$ and $\forall x_1,\ldots,x_q\in\widetilde{\mathfrak{g}}$, is an isomorphism between the vector spaces $C^q(\mathfrak{g},\mathfrak{g})$ and $C^q(\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}})$.*

Corollary 2.6.9. *[56] The dimension of the vector space $Z^q(\mathfrak{g},ad_{\mathfrak{g}},\kappa)$ is an invariant of the Lie algebra $\mathfrak{g}$, for any $q\in\mathbb{N}$ and any complex $(q+1)$-square symmetric matrix κ.*

Definition 2.6.10. *The invariant functions φ and φ^0 corresponding to the n-dimensional Lie algebra $\mathfrak{g}$ are defined as*

$$\varphi:\mathbb{C}\mapsto\{0,1,\ldots,\frac{n^2(n-1)}{2}\},$$

$$(\varphi\mathfrak{g})(\alpha)=\dim coc_{(1,1,1,\alpha,\alpha,\alpha)}\mathfrak{g},\tag{2.11}$$

and

$$\varphi^0:\mathbb{C}\mapsto\{0,1,\ldots,\frac{n^2(n-1)}{2}\},$$

$$(\varphi^0\mathfrak{g})(\alpha)=\dim coc_{(0,1,1,\alpha,1,1)}\mathfrak{g},\tag{2.12}$$

respectively.

2.7 Contractions of Lie algebras

Let $\mathfrak{g} = (V, [,])$ be an n-dimensional *Lie algebra* and $U : (0,1] \mapsto \mathfrak{gl}(V)$ be an one-parameter mapping. If the limit

$$[X,Y]_0 = \lim_{\varepsilon \to 0^+} U^{-1}(\varepsilon)\,[U(\varepsilon)X, U(\varepsilon)Y]$$

exists for all $X, Y \in \mathfrak{g}$, we say that $\mathfrak{g}_0 = (V, [,]_0)$ is an one-parameter contraction of the algebra $\mathfrak{g}$ and we write $\mathfrak{g} \mapsto \mathfrak{g}_0$.

A contraction $\mathfrak{g} \mapsto \mathfrak{g}_0$ is said to be *proper* if $\mathfrak{g}$ is not isomorphic to $\mathfrak{g}_0$.

The following results were shown in [56]

Theorem 2.7.1. *If $\mathfrak{g}_0$ is a proper contraction of the complex Lie algebra $\mathfrak{g}$, then*

1. $\dim Der(\mathfrak{g}) < \dim Der(\mathfrak{g}_0)$.
2. $\psi\mathfrak{g} \leq \psi\mathfrak{g}_0$ *and* $\psi\mathfrak{g}(1) < \psi\mathfrak{g}_0(1)$.
3. $\varphi\mathfrak{g} \leq \varphi\mathfrak{g}_0$ *and* $\varphi^0\mathfrak{g} \leq \varphi^0\mathfrak{g}_0$.

Moreover, it happens that, in dimension 3, Condition 2 is a characterization of proper contractions of $\mathfrak{g}$.

Chapter 3

New results on the ψ and φ invariant functions in the case of filiform Lie algebras

In this chapter we deal with the invariant functions ψ and φ introduced by Hrivnák and Novotný [55] in the particular case of model filiform Lie algebras of lower dimensions (those of greater dimensions will be tackled in a similar way in future work).

3.1 ψ and φ functions for the 3-dim. model filiform Lie algebra

Let $\mathfrak{f}_3$ be the model filiform Lie algebra of dimension 3 defined by the law $[e_1, e_2] = e_3$ (remember that, by agreement, all possible brackets not appearing in the expression of the law are considered null, that is, in this case, $[e_1, e_3] = [e_2, e_3] = 0$).

3.1.1 The ψ invariant function for the 3-dimensional model filiform Lie algebras

Let $\alpha \in \mathbb{C}$ and consider $d \in Der_{(\alpha,1,1)}\mathfrak{f}_3$. Then,

$$\alpha d\big([X, Y]\big) = [d(X), Y] + [X, d(Y)] \quad \forall X, Y \in \mathfrak{f}_3. \tag{3.1}$$

Also, let

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be the matrix associated with the endomorphism d.

We wish to obtain the elements of this matrix. To do this, for the pair of generators (e_1, e_2), condition (3.1) is now $\alpha d\big([e_1, e_2]\big) = [d(e_1), e_2] + [e_1, d(e_2)]$, and $d(e_i) = \sum_{h=1}^{3} a_{ih}\, e_h$.

We can get the first condition to be fulfilled for the elements of this endomorphism starting from $\alpha\, d\big([e_1, e_2]\big) = \alpha\, d(e_3)$. Indeed, $\alpha\, a_{31}e_1 + \alpha\, a_{32}e_2 + \alpha\, a_{33}e_3 = [a_{11}e_1 + a_{12}e_2 + a_{13}e_3, e_2] + [e_1, a_{21}e_1 + a_{22}e_2 + a_{23}e_3] = a_{11}e_3 + a_{22}e_3$.

Therefore, $\alpha\, a_{31}e_1 + \alpha\, a_{32}e_2 + \alpha\, a_{33}e_3 = a_{11}e_3 + a_{22}e_3$ and thus, taking into consideration the linear dependence, the following conditions are obtained: $\alpha\, a_{31} = 0$, $\alpha\, a_{32} = 0$, and $\alpha\, a_{33} = a_{11} + a_{22}$.

Proceeding in the same way with the following pair (e_1, e_3), we deduce, by taking into account the law of the algebra, that $[e_1, a_{31}e_1 + a_{32}e_2 + a_{33}e_3] = a_{32}e_3 = 0$, which implies $a_{32} = 0$.

Finally, by proceeding in the same way with the last pair (e_2, e_3), from $\alpha d\big([e_2, e_3]\big) = [d(e_2), e_3] + [e_2, d(e_3)]$, we obtain $0 = [a_{21}e_1 + a_{22}e_2 + a_{23}e_3, e_3] + [e_2, a_{31}e_1 + a_{32}e_2 + a_{33}e_3]$, and therefore $-a_{31}e_3 = 0$. As a result, we find that $a_{31} = 0$.

Summarizing, we have obtained the following conditions

From pair (e_i, e_j)	Conditions
(e_1, e_2)	$\alpha\, a_{31} = 0, \quad \alpha\, a_{32} = 0, \quad \alpha\, a_{33} = a_{11} + a_{22}.$
(e_1, e_3)	$a_{32} = 0.$
(e_2, e_3)	$a_{31} = 0.$

which allow us to determine the vector space $Der_{(\alpha,1,1)}\mathfrak{f}_3$. Indeed

For $\alpha \neq 0$:

$$Der_{(\alpha,1,1)}\mathfrak{f}_3 = \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left.\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&0&0\\0&0&\frac{1}{\alpha}\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\0&0&\frac{1}{\alpha}\end{pmatrix}\right\}.$$

For $\alpha = 0$:

$$Der_{(0,1,1)}\mathfrak{f}_3 = \text{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}.$$

Therefore, $\psi_{\mathfrak{f}_3}(\alpha) = 6$, $\forall \alpha \in \mathbb{C}$. According to the notation used in [55], it is expressed in the following way

α	$\forall\,\alpha \in \mathbb{C}$
$\psi_{\mathfrak{f}_3}(\alpha)$	6

Note that this dimension is always the same independently of the value of α.

3.1.2 The φ invariant function for the 3-dimensional model filiform Lie algebras

The $\mathfrak{f}_3$-cochains $B_i \in C^2(\mathfrak{f}_3,\mathfrak{f}_3)$, $\forall i \in \{0, 1, ..., \frac{n^2(n-1)}{2}\}$, are defined starting from their non-null commutativity relations $B_r(e_s, e_t) = ke_u$, $s < t$ which verify Equation 2.10. The $\mathfrak{f}_3$-cochains which constitute a basis of the vector space $coc_{(1,1,1,\lambda,\lambda,\lambda)}\mathfrak{f}_3$ whose dimension determines the invariant function $\varphi_{\mathfrak{f}_3}$ are the following

For $\lambda = 0$:

B_1: $B_1(e_1, e_2) = e_1$. B_2: $B_2(e_1, e_2) = e_2$. B_3: $B_3(e_1, e_2) = e_3$.
B_4: $B_4(e_2, e_3) = e_1$. B_5: $B_5(e_2, e_3) = e_2$. B_6: $B_6(e_2, e_3) = e_3$.
B_7: $B_7(e_1, e_3) = e_1$. B_8: $B_8(e_1, e_3) = e_2$. B_9: $B_9(e_1, e_3) = e_3$.

For $\lambda \neq 0$:

C_1: $C_1(e_1, e_2) = e_1$, $C_1(e_2, e_3) = e_3$. C_2: $C_2(e_2, e_3) = e_2$.
C_3: $C_3(e_2, e_3) = e_1$. C_4: $C_4(e_1, e_2) = e_2$.
C_5: $C_5(e_1, e_2) = e_3$. C_6: $C_6(e_1, e_3) = e_1$.
C_7: $C_7(e_1, e_3) = e_2$. C_8: $C_8(e_1, e_3) = e_3$.

Therefore, we have the following vector spaces

$$\begin{aligned} coc_{(1,1,1,0,0,0)}\mathfrak{f}_3 &= span_{\mathbb{C}}\{B_i, 1 \leq i \leq 9\}, \\ coc_{(1,1,1,\lambda,\lambda,\lambda)}\mathfrak{f}_3 &= span_{\mathbb{C}}\{C_i, 1 \leq i \leq 8\}, \forall \lambda \neq 0. \end{aligned}$$

So, we have as a result

λ	0	$\forall \lambda \in \mathbb{C} \setminus \{0\}$
$\varphi_{\mathfrak{f}_3}(\lambda)$	9	8

Let us now consider these computations for the 4 and 5-dimensional filiform Lie algebras.

3.2 The ψ and φ invariant functions for the 4-dimensional model filiform Lie algebras

We deal now in this section with the cases of dimension 4.

3.2.1 The ψ and φ functions for the 4-dim. model filiform Lie algebra

Let $\mathfrak{f}_4$ be the model filiform Lie algebra of dimension 4 defined by the brackets $[e_1, e_2] = e_3$ and $[e_1, e_3] = e_4$.

In the same way as before, we wish to obtain a basis of the vector space $Der_{(\alpha,1,1)}\mathfrak{f}_4$, for all $\alpha \in \mathbb{C}$. To do this, let us consider $d \in Der_{(\alpha,1,1)}\mathfrak{f}_4$. Then, $\alpha\, d\big([X, Y]\big) = [d(X), Y] + [X, d(Y)]$, for all $X, Y \in \mathfrak{f}_4$.

Let

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

be the associated matrix with the endomorphism d. From the previous expression, we have in the first place that $\alpha d\big([e_1, e_2]\big) = [d(e_1), e_2] + [e_1, d(e_2)]$, which allows us to obtain the first conditions for the elements of the previous matrix. Indeed, as $[a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4, e_2] + [e_1, a_{21}e_1 + a_{22}e_2 + a_{23}e_3 + a_{24}e_4] = \alpha\, a_{31}e_1 + \alpha\, a_{32}e_2 + \alpha\, a_{33}e_3 + \alpha\, a_{34}e_4$, we have that $\alpha\, a_{31}e_1 + \alpha\, a_{32}e_2 + \alpha\, a_{33}e_3 + \alpha\, a_{34}e_4 = a_{11}e_3 + a_{22}e_3 + a_{23}e_4$. It implies that $\alpha\, a_{31} = 0$, $\alpha\, a_{32} = 0$, $\alpha\, a_{33} = a_{11} + a_{22}$ and $\alpha\, a_{34} = a_{23}$.

Similarly, we obtain the rest of conditions for the elements of that matrix in the same way as we did in Subsection 3.1.1. These conditions will allow us to obtain the vector space $Der_{(\alpha,1,1)}\mathfrak{f}_4$.

From pair (e_i, e_j)	Conditions
(e_1, e_2)	$\alpha\, a_{31} = 0,\ \alpha\, a_{32} = 0,$ $\alpha\, a_{33} = a_{11} + a_{22},\ \alpha\, a_{34} = a_{23}.$
(e_1, e_3)	$\alpha\, a_{41} = 0,\ \alpha\, a_{42} = 0,$ $\alpha\, a_{43} = a_{32},\ \alpha\, a_{44} = a_{11} + a_{33}.$
(e_1, e_4)	$a_{42} = 0,\ a_{43} = 0.$
(e_2, e_3)	$a_{31} = a_{21} = 0.$
(e_3, e_4)	$a_{41} = 0.$

Note that no conditions are obtained from some pairs of generators, for instance, from the pair (e_2, e_4) in this case.

Now, starting from these results we obtain the vector space $Der_{(\alpha,1,1)}\mathfrak{f}_4$, being $\alpha \neq 0$, and $Der_{(0,1,1)}\mathfrak{f}_4$.

For $\alpha \neq 0$:

$$Der_{(\alpha,1,1)}\mathfrak{f}_4 = \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&\alpha&0&0\\0&0&1&0\\0&1&0&\frac{1}{\alpha} \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&\frac{1}{\alpha}\\0&0&0&0 \end{pmatrix}\Bigg\}.$$

For $\alpha = 0$:

$$Der_{(0,1,1)}\mathfrak{f}_4 = \text{span}_{\mathbb{C}}\Bigg\{\begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}\Bigg\}.$$

Therefore, after proceeding in the same way as we did in the previous subsection, we obtain that, according to the notation in [55], the invariant function $\psi_{\mathfrak{f}_4}$ is

α	$\forall\,\alpha \in \mathbb{C}$
$\psi_{\mathfrak{f}_4}(\alpha)$	7

Regarding the φ invariant function for the 4-dimensional model filiform Lie algebra, we are going to prove that

λ	0	$\forall\lambda \in \mathbb{C}\setminus\{0\}$
$\varphi_{\mathfrak{f}_4}(\lambda)$	16	18

Indeed, as we see in the case of dimension 3, the $\mathfrak{f}_4$-cochains $B_i \in C^2(\mathfrak{f}_4,\mathfrak{f}_4)$, for all $i \in \{0, 1, ..., \frac{n^2(n-1)}{2}\}$ are defined starting from their non-null commutativity relations $B_r(e_s, e_t) = ke_u$, $s < t$, which verify the equality given by Equation 2.10. The $\mathfrak{f}_4$-cochains which constitute a basis of the vector space $coc_{(1,1,1,\lambda,\lambda,\lambda)}\mathfrak{f}_4$, whose dimension determines the invariant function $\varphi_{\mathfrak{f}_4}$, are the following, for $\lambda = 0$

For $\lambda \neq 0$:

B_1:	$B_1(e_1,e_2)=e_1.$	B_2:	$B_2(e_1,e_2)=e_2.$	B_3:	$B_3(e_1,e_2)=e_3.$
B_4:	$B_4(e_1,e_2)=e_4.$	B_5:	$B_5(e_1,e_4)=e_1.$	B_6:	$B_6(e_1,e_4)=e_2.$
B_7:	$B_7(e_1,e_4)=e_3$	B_8:	$B_8(e_1,e_4)=e_4.$	B_9:	$B_9(e_1,e_3)=e_1.$
B_{10}:	$B_{10}(e_1,e_3)=e_2$	B_{11}:	$B_{11}(e_1,e_3)=e_3.$	B_{12}:	$B_{12}(e_1,e_3)=e_4.$
B_{13}:	$B_{13}(e_1,e_4)=e_1.$	B_{14}:	$B_{14}(e_1,e_4)=e_2.$	B_{15}:	$B_{15}(e_1,e_4)=e_3.$
B_{16}:	$B_{16}(e_1,e_4)=e_4.$				

C_1: $C_1(e_1,e_2)=e_2.$ $\quad$ C_2: $C_2(e_1,e_2)=e_3.$ $\quad$ C_3: $C_3(e_1,e_2)=e_4.$

C_4: $C_4(e_1,e_4)=e_2.$ $\quad$ C_5: $C_5(e_1,e_4)=e_3.$ $\quad$ C_6: $C_6(e_1,e_4)=e_4.$

C_7: $C_7(e_1,e_3)=e_2.$ $\quad$ C_8: $C_8(e_1,e_3)=e_3.$ $\quad$ C_9: $C_9(e_1,e_3)=e_4.$

C_{10}: $C_{10}(e_3,e_4)=e_1.$ $\quad$ C_{11}: $C_{11}(e_3,e_4)=e_2.$

C_{12}: $C_{12}(e_2,e_4)=\frac{1}{\lambda}e_3,$ $\quad$ $C_{12}(e_2,e_3)=e_2,$ $\quad$ $C_{12}(e_1,e_3)=-\frac{1}{\lambda^2}e_1.$

C_{13}: $C_{13}(e_2,e_4)=\frac{1}{\lambda}e_3,$ $\quad$ $C_{13}(e_2,e_3)=e_2,$
$C_{13}(e_1,e_3)=(\frac{-1+\lambda^2}{\lambda^2})e_1,$ $\quad$ $C_{13}(e_3,e_4)=e_4.$

C_{14}: $C_{14}(e_2,e_4)=\frac{1}{\lambda}e_3,$ $\quad$ $C_{14}(e_2,e_3)=e_2,$ $\quad$ $C_{14}(e_1,e_3)=\frac{-1}{\lambda^2}e_1,$
$C_{14}(e_1,e_4)=e_1,$ $\quad$ $C_{14}(e_3,e_4)=-e_3.$

C_{15}: $C_{15}(e_2,e_4)=\frac{1}{\lambda}e_3,$ $\quad$ $C_{15}(e_2,e_3)=e_2,$ $\quad$ $C_{15}(e_3,e_4)=\frac{1}{\lambda^2}e_4,$

C_{16}: $C_{16}(e_2,e_3)=e_3,$ $\quad$ $C_{16}(e_2,e_4)=\lambda e_2,$ $\quad$ $C_{16}(e_1,e_2)=\frac{-1}{\lambda}e_1,$
$C_{16}(e_1,e_3)=e_1,$ $\quad$ $C_{16}(e_1,e_4)=e_1,$ $\quad$ $C_{16}(e_3,e_4)=e_4.$

C_{17}: $C_{17}(e_2,e_3)=e_3,$ $\quad$ $C_{17}(e_2,e_4)=\lambda e_2,$ $\quad$ $C_{17}(e_1,e_2)=\frac{-1}{\lambda}e_1,$
$C_{17}(e_3,e_4)=e_3.$

C_{18}: $C_{18}(e_2,e_3)=e_3,$ $\quad$ $C_{18}(e_2,e_4)=\lambda e_2,$ $\quad$ $C_{18}(e_1,e_2)=\frac{-1}{\lambda}e_1,$
$C_{18}(e_1,e_4)=e_1.$

Therefore, we have the following vector spaces

$$coc_{(1,1,1,0,0,0)}\mathfrak{f}_4 = span_{\mathbb{C}}\{B_i \mid 1\le i\le 16\}.$$

$$coc_{(1,1,1,\lambda,\lambda,\lambda)}\mathfrak{f}_4 = span_{\mathbb{C}}\{C_i \mid 1 \leq i \leq 18\}, \forall \lambda \neq 0.$$

It concludes the proof.

Note that unlike the previous case, these dimensions are now dependent on the value of λ.

3.3 The ψ invariant function for the model filiform Lie algebra of dimension 5

With the next objective of proving that there is no proper contraction from a Heisenberg algebra to a 5-dimensional filiform Lie algebra, we have also obtained the invariant function $\psi_{\mathfrak{f}_5}$, where $\mathfrak{f}_5$ is the 5-dimensional model filiform Lie algebra with non-null brackets $[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_5$.

Indeed, by proceeding in the same way as in the previous dimensions (computations are not included for reasons of length), we obtain that

α	$\forall \alpha \in \mathbb{C}$
$\psi_{\mathfrak{f}_5}(\alpha)$	9

This chapter concludes with these computations, although the intention of the author in future work is to start from them to obtain the values of the invariant functions ψ and φ in the general case of the n-dimensional model filiform Lie algebra.

Chapter 4

Introducing new invariant functions

In this chapter we introduce three new invariant functions for algebras: the two-parameter function $\bar{\psi}$, the one-parameter function υ and the two-parameter function $\bar{\phi}$.

4.1 Introducing the two-parameter invariant function $\bar{\psi}$

In this section we introduce in the first place the two-parameter function $\bar{\psi}$. To do this and in order to generalize this study as much as possible, we firstly prove some results regarding Malcev algebras.

Let us first recall that if $\mathfrak{g}$ is a Lie type Malcev algebra, then

$$[[x,y],[x,z]] = [[[x,y],z],x] + [[[z,x],x],y] + [[[y,z],x],x], \quad \forall x,y,z \in \mathfrak{g}, \tag{4.1}$$

and that if $d \in der\mathfrak{g}$ is a derivation of $\mathfrak{g}$, then

$$d[[x,y],[x,z]] = [d[x,y],[x,z]] + [[x,y],d[x,z]], \quad \forall x,y,z \in \mathfrak{g}. \tag{4.2}$$

Now, from $J(x,y,[x,z]) = 0$, it is to say, from the Jacobi identity applied to x, y and $[x,z]$, we have

$$[[x,y],[x,z]] = [[[x,z],y],x] + [[[z,x],x],y], \quad \forall x,y,z \in \mathfrak{g}. \tag{4.3}$$

Then, starting from (4.2) and the result of applying d in (4.3), we have

$$[d[x,y],[x,z]] + [[x,y],d[x,z]] = d[[[x,z],y],x] + d[[[z,x],x],y] \quad \forall x,y,z \in \mathfrak{g}. \tag{4.4}$$

Now, the generalization of this last expression produces the following definitions

Definition 4.1.1. *Let $\mathfrak{g}$ a type Lie Malcev algebra. The set*

$$\{d \in End(\mathfrak{g}) : \alpha[d[x,y],[x,z]] + \beta[[x,y],d[x,z]] = \gamma d[[[x,z],y],x] + \tau d[[[z,x],x],y]\},$$

for all $(\alpha,\beta,\gamma,\tau) \in \mathbb{C}^4$, is called the set of the $(\alpha,\beta,\gamma,\tau)$-derivations *of the algebra $\mathfrak{g}$. It will be denoted by $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g}$.*

Proposition 4.1.2. *Let $\mathfrak{g}$ be a type Lie Malcev algebra. Then, $dim_{(1,1,1,1)}\mathfrak{g}$ is an algebraic invariant of $\mathfrak{g}$.*

Proof. Let us consider $d \in Der_{(1,1,1,1)}\mathfrak{g}$. Then, it is satisfied that

$$[d[x,y],[x,z]] + [[x,y],d[x,z]] = d[[[x,z],y],x] + d[[[z,x],x],y].$$

Now, by appliying the Jacobi identity $J(x,y,[x,z]) = 0$, we have

$$[[x,y],[x,z]] = [[[x,z],y],x] + [[[z,x],x],y] \quad \forall x,y,z \in \mathfrak{g}.$$

This expression and the previous one imply that $[d[x,y],[x,z]] + [[x,y],d[x,z]] = d[[x,y],[x,z]]$ if and only if $d \in Der\mathfrak{g}$, that is to say, $Der_{(1,1,1,1)}\mathfrak{g} = Der\mathfrak{g}$. It implies that $\dim\big(Der_{(1,1,1,1)}\mathfrak{g}\big) = \dim\big(Der\mathfrak{g}\big)$.

Since $\dim\big(Der\mathfrak{g}\big)$ is an invariant of $\mathfrak{g}$, it follows that $\dim\big(Der_{(1,1,1,1)}\mathfrak{g}\big)$ is an invariant of $\mathfrak{g}$.

□

Theorem 4.1.3. *Let $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ be two Malcev algebras Lie and let $f : \mathfrak{g} \to \bar{\mathfrak{g}}$ be an isomorphism. Then, the mapping $\rho : End\ \mathfrak{g} \to End\ \bar{\mathfrak{g}}$, defined by $d \longrightarrow fdf^{-1}$ is an isomorphism between the vector spaces $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g}$ and $Der_{(\alpha,\beta,\gamma,\tau)}\bar{\mathfrak{g}}$, $\forall(\alpha,\beta,\gamma,\tau) \in \mathbb{C}^4$.*

Proof. Let $\mathfrak{g} = (V, \cdot)$ and $\bar{\mathfrak{g}} = (\bar{V}, *)$ be two Malcev algebras Lie and let us consider $d \in Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g}$, for any $(\alpha,\beta,\gamma,\tau) \in \mathbb{C}^4$ and for all $x, y, z \in \bar{\mathfrak{g}}$. Then,

$$\alpha d\Big(f^{-1}(x) \cdot f^{-1}(y)\Big) \cdot \Big(f^{-1}(x) \cdot f^{-1}(z)\Big) + \beta\Big(f^{-1}(x) \cdot f^{-1}(y)\Big) \cdot d\Big(f^{-1}(x) \cdot f^{-1}(z)\Big) =$$

$$\gamma d\Bigg(\bigg(\Big(f^{-1}(x) \cdot f^{-1}(z)\Big) \cdot f^{-1}(y)\bigg) \cdot f^{-1}(x)\Bigg) + \tau d\Bigg(\bigg(\Big(f^{-1}(z) \cdot f^{-1}(x)\Big) \cdot f^{-1}(x)\bigg) \cdot f^{-1}(y)\Bigg).$$

From this expression, it is deduced that

$$\gamma d\Bigg(\bigg(\Big(f^{-1}(x) \cdot f^{-1}(z)\Big) \cdot f^{-1}(y)\bigg) \cdot f^{-1}(x)\Bigg) = \gamma d\bigg(\Big(f^{-1}(x * z) \cdot f^{-1}(y)\Big) \cdot f^{-1}(x)\bigg) =$$

$$\gamma df^{-1}((x * z) * y) \cdot f^{-1}(x)) = \gamma df^{-1}(((x * z) * y) * x),$$

and similarly,

$$\tau d\Bigg(\bigg(\Big(f^{-1}(z) \cdot f^{-1}(x)\Big) \cdot f^{-1}(x)\bigg) \cdot f^{-1}(y)\Bigg) = \tau df^{-1}(((z * x) * x) * y),$$

$$\alpha d\Big(f^{-1}(x) \cdot f^{-1}(y)\Big) \cdot \Big(f^{-1}(x) \cdot f^{-1}(z)\Big) = \alpha df^{-1}(x * y) \cdot f^{-1}(x * z),$$

$$\beta\Big(f^{-1}(x) \cdot f^{-1}(y)\Big) \cdot d\Big(f^{-1}(x) \cdot f^{-1}(z)\Big) = \beta f^{-1}(x * y) \cdot df^{-1}(x * z).$$

So, $\alpha Df^{-1}(x * y) \cdot f^{-1}(x * z) + \beta f^{-1}(x * y) \cdot Df^{-1}(x * z) = \gamma Df^{-1}(((x * z) * y) * x) + \tau Df^{-1}(((z * x) * x) * y)$.

Then, the result of applying f to the previous expression is

$$\alpha\Big(fdf^{-1}\Big)(x * y) * (x * z) + \beta(x * y) * \Big(fdf^{-1}\Big)(x * z) =$$

$$\gamma\Big(fdf^{-1}\Big)((x * z) * y) * x + \tau\Big(fdf^{-1}\Big)((z * x) * x) * y.$$

Therefore, $fdf^{-1} \in Der_{(\alpha,\beta,\gamma,\tau)}\bar{\mathfrak{g}}$, which concludes the proof. □

An immediate consequence of this result is the following

Corollary 4.1.4. *Let $\mathfrak{g}$ be a type Lie Malcev algebra. The dimension of the vector space $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g}$ is an invariant of the algebra, for all $(\alpha,\beta,\gamma,\tau) \in \mathbb{C}^4$.*

Lemma 4.1.5. *(Technical Lemma) Under the above conditions, the following expressions are verified*

1. *d[[[z,x],x],y]=d[[x,y],[x,z]]-d[[[y,z],x],x].*
2. *d[[[y,x],x]z]=d[[x,z],[x,y]]-d[[[z,y],x],x].*
3. *d[[[x,z],y],x]=d[[x,y],[x,z]]-d[[[z,x],x],y].*
4. *d[[[x,y],z],x]=d[[x,z],[x,y]]-d[[[y,x],x],z].*

Proof. All expressions are immediate consequences of the properties of the derivations (see Preliminaries). □

Lemma 4.1.6. *Let $\mathfrak{g} = (V, [,])$ be a type Lie Malcev algebra. Then,*

$$Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(\alpha+\beta,\alpha+\beta,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(\alpha-\beta,\beta-\alpha,0,0)}\mathfrak{g}.$$

Proof. Let suppose $d \in Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g}$. Then, for all $(x,y,z) \in \mathfrak{g}$ we have

$$\alpha[d[x,y],[x,z]] + \beta[[x,y],d[x,z]] = \gamma d[[[x,z],y],x] + \tau d[[[z,x],x],y].$$

Intercharging now y and z between themselves, we have

$$\alpha[d[x,z],[x,y]] + \beta[[x,z],d[x,y]] = \gamma d[[[x,y],z],x] + \tau d[[[y,x],x],z],$$

and by adding the two first expressions of Lemma 4.1.5 and taking the anti-skew property of the Lie bracket into consideration, we have that

$$(\alpha-\beta)[d[x,y],[x,z]] + (\beta-\alpha)[[x,y],d[x,z]] =$$
$$\gamma(d[[[x,z],y],x] + d[[[x,y],z],x]) + \tau(d[[[z,x],x],y] + d[[[y,x],x],z]).$$

Similarly, starting from the two last expressions of Lemma 4.1.5, we obtain that $d[[[z,x],x],y]+d[[[y,x],x],z] = 0$, and by repeating the same procedure we obtain $d[[x,z],y],x]+ d[[x,y],z],x] = 0$.

Now, from both expressions we have $(\alpha-\beta)[d[x,y],[x,z]]+(\beta-\alpha)[[x,y],d[x,z]]=0$. It implies that $d \in Der_{(\alpha-\beta,\beta-\alpha,0,0)}\mathfrak{g}$.

Now, by subtracting the two first expressions of the proof and taking into account the anti-skew property, we have $(\alpha+\beta)[d[x,y],[x,z]]+(\beta+\alpha)[[x,y],d[x,z]] = \gamma(d[[[x,z],y],x]-d[[[x,y],z],x]) + \tau(d[[[z,x],x],y]-d[[[y,x],x],z])$.

We will use now in the previous equality the following two expressions, which were respectively obtained from previous expressions: $d[[[y,x],x],z] = -d[[[z,x],x],y]$ and $d[[x,y],z],x] = -d[[x,z],y],x]$.

Then, we have that $(\alpha+\beta)[d[x,y],[x,z]]+(\alpha+\beta)[[x,y],d[x,z]] = 2\gamma d[[[x,z],y],x]+2\tau d[[[z,x],x],y]$. It involves that $d \in Der_{(\alpha+\beta,\alpha+\beta,2\gamma,2\tau)}\mathfrak{g}$. Therefore, it is verified that $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} \subset Der_{(\alpha+\beta,\alpha+\beta,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(\alpha-\beta,\beta-\alpha,0,0)}\mathfrak{g}$.

If $d \in Der_{(\alpha+\beta,\alpha+\beta,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(\alpha-\beta,\beta-\alpha,0,0)}\mathfrak{g}$, then d has to verify both equations: $(\alpha+\beta)[d[x,y],[x,z]]+(\alpha+\beta)[[x,y],d[x,z]] = 2\gamma d[[[x,z],y],x]+2\tau d[[[z,x],x],y]$ and $(\alpha-\beta)[d[x,y],[x,z]]+(\beta-\alpha)[[x,y],d[x,z]]=0$.

Then, by adding these last equations and simplifying, we observe that d verifies

$$\alpha[d[x,y],[x,z]]+\beta[[x,y],d[x,z]] = \gamma d[[[x,z],y],x]+\tau d[[[z,x],x],y].$$

So $d \in Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(\alpha+\beta,\alpha+\beta,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(\alpha-\beta,\beta-\alpha,0,0)}\mathfrak{g}$. It completes the proof. $\square$

Theorem 4.1.7. *Let $\mathfrak{g}$ be a type Lie Malcev algebra. Then, for all $(\alpha,\beta,\gamma,\tau) \in \mathbb{C}^4$, it exists $(\lambda_1,\lambda_2) \subset \mathbb{C}^2$ such that $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g}$ is one of the following four sets: $Der_{(0,0,\lambda_1,\lambda_2)}\mathfrak{g}$, $Der_{(1,-1,\lambda_1,\lambda_2)}\mathfrak{g}$, $Der_{(1,0,\lambda_1,\lambda_2)}\mathfrak{g}$ or $Der_{(1,1,\lambda_1,\lambda_2)}\mathfrak{g}$.*

Proof. Let consider $(\alpha,\beta,\gamma,\tau) \in \mathbb{C}^4$. We distinguish the following cases

Case 1: If $\alpha+\beta=0$. We distinguish now the following two subcases

- Subcase 1.1: If $\alpha=\beta=0$. Then, $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(0,0,\gamma,\tau)}\mathfrak{g}$. Therefore, $\gamma=\lambda_1$ and $\lambda_2=\tau$.

- Subcase 1.2: If $\alpha=-\beta$. In this subcase, by Lemma 4.1.6 we have that $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(0,0,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(-2\beta,2\beta,0,0)}\mathfrak{g} = Der_{(0,0,\gamma,\tau)}\mathfrak{g} \cap Der_{(-1,1,0,0)}\mathfrak{g}$. Apart from that, it is also verified that $Der_{(-1,1,\gamma,\tau)}\mathfrak{g} = Der_{(0,0,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(-2,2,0,0)}\mathfrak{g} = Der_{(0,0,\gamma,\tau)}\mathfrak{g} \cap Der_{(-1,1,0,0)}\mathfrak{g}$. Therefore, $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(-1,1,\gamma,\tau)}\mathfrak{g}$. It involves that $\lambda_1=\gamma$ and $\lambda_2=\tau$.

Case 2: If $\alpha+\beta \neq 0$. Two subcases are also considered

- Subcase 2.1: If $\alpha \neq \beta$.

By Lemma 4.1.6 we have $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(1,1,\frac{2\gamma}{\alpha+\beta},\frac{2\tau}{\alpha+\beta})}\mathfrak{g} \cap Der_{(1,-1,0,0)}\mathfrak{g}$. Since $Der_{(1,0,\frac{\gamma}{\alpha+\beta},\frac{\tau}{\alpha+\beta})}\mathfrak{g} = Der_{(1,1,\frac{2\gamma}{\alpha+\beta},\frac{2\tau}{\alpha+\beta})}\mathfrak{g} \cap Der_{(1,-1,0,0)}\mathfrak{g}$, it is deduced that $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(1,0,\frac{\gamma}{\alpha+\beta},\frac{\tau}{\alpha+\beta})}\mathfrak{g}$. It involves that $\lambda_1 = \frac{\gamma}{\alpha+\beta}$ and $\lambda_2 = \frac{\tau}{\alpha+\beta}$.

- Subcase 2.2: If $\alpha = \beta \neq 0$.

In this subcase, $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(1,1,\frac{\gamma}{\alpha},\frac{\tau}{\alpha})}\mathfrak{g}$. Therefore, $\lambda_1 = \frac{\gamma}{\alpha}$ and $\lambda_2 = \frac{\tau}{\alpha}$.

□

These two two-parameter sets $Der_{(1,0,\lambda_1,\lambda_2)}\mathfrak{g}$ and $Der_{(1,1,\lambda_1,\lambda_2)}\mathfrak{g}$ previously defined allow us to define the following invariant two-parameter functions of Lie algebras.

Definition 4.1.8. *The two-parameter functions* $\bar{\psi}_\mathfrak{g}, \bar{\psi}^0_\mathfrak{g} : \mathbb{C}^2 \mapsto \mathbb{N}$ *defined, respectively, as* $(\bar{\psi}_\mathfrak{g})(\alpha,\beta) = \dim Der_{(1,1,\alpha,\beta)}\mathfrak{g}$ *and* $(\bar{\psi}^0_\mathfrak{g})(\alpha,\beta) = \dim Der_{(1,0,\alpha,\beta)}\mathfrak{g}$ *are called* $\bar{\psi}_\mathfrak{g}$ *and* $\bar{\psi}^0_\mathfrak{g}$ invariant functions *corresponding to the* $(1,1,\alpha,\beta)$*-derivations and* $(1,0,\alpha,\beta)$*-derivations of* $\mathfrak{g}$*, respectively.*

Corollary 4.1.9. *If two Malcev algebras Lie* $\mathfrak{g}$ *and* $\mathfrak{f}$ *are isormorphic, then* $\bar{\psi}_\mathfrak{g} = \bar{\psi}_\mathfrak{f}$ *and* $\bar{\psi}^0_\mathfrak{g} = \bar{\psi}^0_\mathfrak{f}$.

4.2 Introducing the one-parameter invariant function υ

In this section we introduce a new invariant function starting from the just studied two-parameter invariant function $\bar{\psi}$.

Definition 4.2.1. *Let* $\mathfrak{g}$ *be an algebra. The one-parameter function* $\upsilon_\mathfrak{g} : \mathbb{C} \mapsto \mathbb{N}$ *defined as* $\upsilon_\mathfrak{g}(\lambda) = \bar{\psi}_\mathfrak{g}(1,\lambda) = \dim Der_{(1,1,1,\lambda)}\mathfrak{g}$ *is called the* $\upsilon_\mathfrak{g}$ invariant function *corresponding to the* $(1,1,1,\lambda)$*-derivations of* $\mathfrak{g}$.

A natural question which arises is to ask ourselves if the invariant functions $\bar{\psi}$ and ψ are really different. To see it we are going to compare the new invariant function υ just introduced with the invariant function ψ introduced by Hrivnák and Novotný in 2007 [55] to prove that although both functions are very similar, they are really different. To do this, we will compute both functions for a same Lie algebra, in the particular case of being $\alpha = 1$. Concretely, we will use the Lie algebra induced by the Lorentz group $SO(3,1)$, which we denote by $\mathfrak{g}_6$.

Computing $\psi_{\mathfrak{g}_6}$ for $\lambda = 1$

Let us recall that Minkowski defined the spacetime as a four-dimensional manifold with the metric $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. We introduce the metric tensor

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we rename $(ct, x, y, z) \to (x^0, x^1, x^2, x^3)$, then the expression ds^2 can be written as $ds^2 = \eta_{\mu\gamma}dx^\mu dx^\gamma$ (summed over μ and γ). Recall that this distance is invariant under the following type of transformations $x^\mu \to \lambda^\mu_\gamma x^\gamma$ such that the coefficients λ^μ_γ are the elements of an matrix Λ (which is called *Lorentz transformations*) that satisfies $\Lambda^t\eta\Lambda = \eta$. Since the metric in the three-dimensional Euclidean space corresponds to the identity matrix, if R is the matrix of a rotation then $R^t 1 R = 1$ and comparing this expression with $\Lambda^t\eta\Lambda = \eta$ it is possible to say that the Lorentz transformations are rotations in the Minkowski space. These transformations form a group called the *Lorentz group* which we denote by $SO(3,1)$.

Now we focus our study on the *infinitesimal* Lorentz transformations. A Lorentz transformation matrix can be written as $\Lambda^\mu_\gamma = \delta^\mu_\gamma + \omega^\mu_\gamma$, where the parameters ω^μ_γ are infinitesimal and verify that $\omega^\mu_\gamma = -\omega^\gamma_\mu$ so that the Lorentz transformation is valid. The action of this transformation on the coordinates x^μ in the Minkowski space can be written as $\delta x^\mu = \Lambda^\mu_\gamma\, x^\gamma$.

If we define $A_{\rho\sigma}$ such that $\Lambda^\mu_\gamma = \dfrac{1}{2}\lambda^{\rho\sigma}(A_{\rho\sigma})^\mu_\gamma$, we can write the above action as $\delta x^\mu = \dfrac{1}{2}\lambda^{\rho\sigma}(A_{\rho\sigma})^\mu_\gamma x^\gamma$. Then, it is easily proved that $(A_{\rho\sigma})^\mu_\gamma = \delta^\mu_\rho\eta_{\sigma\gamma} - \delta^\mu_\sigma\eta_{\rho\gamma}$.

Explicitly

$$A_{10} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad A_{20} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$A_{30} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$A_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad A_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Now, by defining the Lie product as the usual commutator product

$$[A_{ij}, A_{hk}] = A_{ij} \cdot A_{hk} - A_{hk} \cdot A_{ij},$$

it is easy to see that matrices $A_{10}, A_{20}, A_{30}, A_{12}, A_{23}$ and A_{31} generate a Lie algebra which we will denote by $\mathfrak{g}_6$.

Let us consider $D \in Der_{(1,1,1)}\mathfrak{g}_6$ and let $A = (a_{ij})$, $1 \leq i, j \leq 6$ be the 6×6 square matrix associated with the endomorphism d.

To obtain the elements of this matrix, for the pair of generators (e_i, e_j), with $i < j$, the derivation d satisfies $d\big([e_i, e_j]\big) = [d(e_i), e_j] + [e_i, d(e_j)]$ and $d(e_i) = \sum_{h=1}^{6} a_{ih}\, e_h$. In this way the following conditions, indicated in the table in the next page, are obtained

From pair (e_i, e_j)	Conditions
(e_1, e_2)	$a_{41} = a_{14}, \quad a_{42} = a_{24}, \quad a_{43} = -a_{15} - a_{26},$ $a_{44} = a_{11} + a_{22}, \quad a_{45} = -a_{13}, \quad a_{46} = -a_{23}.$
(e_1, e_3)	$a_{61} = a_{16}, \quad a_{62} = -a_{15} - a_{34}, \quad a_{63} = a_{36},$ $a_{64} = -a_{32}, \quad a_{65} = -a_{12}, \quad a_{66} = a_{33} + a_{11}.$
(e_1, e_4)	$a_{21} = -a_{12}, \quad a_{22} = a_{11} + a_{44}, \quad a_{23} = -a_{46},$ $a_{24} = a_{42}, \quad a_{25} = -a_{16}, \quad a_{26} = a_{15} - a_{43}.$
(e_1, e_5)	$a_{13} = 0, \quad a_{54} = 0, \quad a_{12} - a_{56} = 0,$ $a_{16} + a_{52} = 0, \quad a_{14} + a_{53} = 0.$
(e_1, e_6)	$a_{31} = -a_{13}, \quad a_{32} = -a_{64}, \quad a_{33} = a_{11} + a_{66},$ $a_{34} = a_{15} - a_{62}, \quad a_{35} = -a_{14}, \quad a_{36} = a_{63}.$
(e_2, e_3)	$a_{51} = -a_{26} - a_{34}, \quad a_{52} = a_{25}, \quad a_{53} = a_{35},$ $a_{54} = -a_{31}, \quad a_{55} = a_{22} + a_{33}, \quad a_{56} = -a_{21}.$
(e_2, e_4)	$a_{11} = a_{22} + a_{44}, \quad a_{12} = -a_{21}, \quad a_{13} = -a_{45},$ $a_{14} = a_{41}, \quad a_{15} = a_{26} - a_{43}, \quad a_{16} = -a_{25}.$
(e_2, e_5)	$a_{31} = -a_{54}, \quad a_{32} = -a_{23}, \quad a_{33} = a_{22} + a_{55},$ $a_{34} = a_{26} - a_{51}, \quad a_{35} = a_{53}, \quad a_{36} = -a_{24}.$
(e_2, e_6)	$a_{23} - a_{64} = 0, \quad -a_{21} + a_{65} = 0, \quad a_{25} + a_{61} = 0,$ $a_{24} + a_{63} = 0.$
(e_3, e_5)	$a_{21} = -a_{56}, \quad a_{22} = a_{33} + a_{55}, \quad a_{23} = -a_{32},$ $a_{24} = -a_{36}, \quad a_{25} = a_{52}, \quad a_{26} = a_{34} - a_{51}.$
(e_3, e_6)	$a_{11} = a_{33} + a_{66}, \quad a_{12} = -a_{65}, \quad a_{13} = -a_{31},$ $a_{14} = -a_{35} + a_{34}, \quad a_{15} = -a_{62}, \quad a_{16} = a_{61}.$
(e_4, e_5)	$a_{61} = -a_{52}, \quad a_{62} = a_{43} + a_{51}, \quad a_{63} = -a_{42},$ $a_{64} = -a_{46}, \quad a_{65} = -a_{56}, \quad a_{66} = a_{44} + a_{55}.$
(e_4, e_6)	$a_{51} = a_{43} + a_{62}, \quad a_{52} = -a_{61}, \quad a_{53} = -a_{41},$ $a_{54} = -a_{45}, \quad a_{55} = a_{44} + a_{66}, \quad a_{56} = -a_{65}.$
(e_5, e_6)	$a_{41} = -a_{53}, \quad a_{42} = -a_{63}, \quad a_{43} = a_{51} + a_{62},$ $a_{44} = a_{55} + a_{66}, \quad a_{45} = -a_{54}, \quad a_{46} = -a_{64}.$
(e_3, e_4)	$-a_{32} + a_{46} = 0, \quad a_{31} - a_{45} = 0, \quad a_{36} + a_{42} = 0,$ $a_{35} + a_{41} = 0.$

It follows from these conditions on a_{ij} that,

$a_{11} = a_{55}$,	$a_{12} = -a_{65}$,	$a_{13} = 0$,	$a_{14} = a_{41}$,	$a_{15} = 0$,	$a_{16} = a_{61}$.
$a_{21} = a_{65}$,	$a_{22} = a_{66}$,	$a_{23} = -a_{46}$,	$a_{24} = a_{42}$,	$a_{25} = -a_{61}$,	$a_{26} = 0$.
$a_{31} = 0$,	$a_{32} = a_{46}$,	$a_{33} = a_{44}$,	$a_{34} = 0$,	$a_{35} = -a_{41}$,	$a_{36} = -a_{42}$.
		$a_{43} = 0$,		$a_{45} = 0$.	
$a_{51} = 0$,	$a_{52} = -a_{61}$,	$a_{53} = -a_{41}$,	$a_{54} = 0$,		$a_{56} = -a_{65}$.
	$a_{62} = 0$,	$a_{63} = -a_{42}$,	$a_{64} = -a_{46}$.		

where a_{41}, a_{42}, a_{44}, a_{46}, a_{55}, a_{61}, a_{65}, $a_{66} \in \mathbb{C}$. This implies that

$$\psi_{\mathfrak{g}_6}(1) = dim\big(der_{(1,1,1)}\mathfrak{g}_6\big) = 8.$$

Computing $\upsilon_{\mathfrak{g}_6}$ for $\lambda = 1$

Let us consider $D \in Der_{(1,1,1,1)}\mathfrak{g}_6$. Then $[D[u,v],[u,w]]+[[u,v],D[u,w]] = D[[[u,w],v],u]+D[[[w,u],u],v] \quad \forall u,v,w \in \mathfrak{g}_6$.

To obtain the elements a_{ij} of the corresponding 5×5 square matrix associated with D, we see that for each triplets of generators (e_i, e_j, e_k) of the algebra, the previous expression is written as

$$[D[e_i,e_j],[e_i,e_k]] + [[e_i,e_j],D[e_i,e_k]] = D[[[e_i,e_k],e_j],e_i] + D[[[e_k,e_i],e_i],e_j].$$

Starting from it, we obtain the following conditions

From triplet (e_i, e_j, e_k)	Conditions
(e_1, e_2, e_3)	$a_{51} = a_{43} + a_{62}$, $a_{52} = -a_{61}$, $a_{53} = -a_{41}$, $a_{54} = -a_{45}$, $a_{55} = a_{66} + a_{44}$, $a_{56} = -a_{65}$.
(e_1, e_2, e_4)	$a_{11} = a_{22} + a_{44}$, $a_{12} = -a_{21}$, $a_{13} = -a_{45}$, $a_{14} = -a_{41}$, $a_{15} = a_{26} - a_{43}$, $a_{16} = -a_{25}$.
(e_1, e_2, e_5)	No conditions.
(e_1, e_2, e_6)	$a_{32} + a_{46} = 0$, $e_{31} + a_{45} = 0$, $a_{36} + a_{42} = 0$, $a_{35} + a_{41} = 0$.
(e_1, e_3, e_4)	$-a_{23} + a_{64} = 0$, $a_{21} - a_{65} = 0$, $a_{25} + a_{61} = 0$. $a_{24} + a_{63} = 0$.
(e_1, e_3, e_5)	No conditions.
(e_1, e_3, e_6)	$a_{33} + a_{66} = a_{11}$, $a_{65} = -a_{12}$, $a_{31} = -a_{13}$, $a_{35} = -a_{14}$, $a_{34} - a_{62} = a_{15}$, $a_{61} = a_{16}$.
(e_1, e_4, e_5)	No conditions.
(e_1, e_4, e_6)	$a_{51} = -a_{26} - a_{34}$, $a_{52} = a_{25}$, $a_{53} = a_{35}$, $a_{54} = -a_{31}$, $a_{55} = a_{22} + a_{33}$, $a_{56} = -a_{21}$.
(e_1, e_5, e_6)	No conditions.
(e_2, e_3, e_4)	$-a_{13} + a_{54} = 0$, $a_{12} - a_{56} = 0$, $a_{16} + a_{52} = 0$, $a_{14} + a_{53} = 0$.
(e_2, e_3, e_5)	$a_{21} = -a_{56}$, $a_{22} = a_{33} + a_{55}$, $a_{23} = -a_{32}$, $a_{24} = -a_{36}$, $a_{25} = a_{52}$, $a_{26} = a_{34} - a_{51}$.
(e_2, e_3, e_6)	No conditions.
(e_2, e_4, e_5)	$a_{61} = -a_{16}$, $a_{62} = a_{15} + a_{34}$, $a_{63} = -a_{36}$, $a_{64} = a_{32}$, $a_{65} = a_{12}$, $a_{66} = -a_{11} - a_{33}$.
(e_2, e_4, e_6)	No conditions.
(e_2, e_5, e_6)	No conditions.
(e_3, e_4, e_5)	No conditions.
(e_3, e_4, e_6)	No conditions.
(e_4, e_5, e_6)	$a_{41} = -a_{53}$, $a_{42} = -a_{63}$, $a_{43} = a_{51} + a_{62}$, $a_{44} = a_{55} + a_{66}$, $a_{45} = -a_{54}$, $a_{46} = -a_{64}$.
(e_3, e_5, e_6)	$a_{41} = a_{14}$, $a_{42} = a_{24}$, $a_{43} = -a_{15} - a_{26}$, $a_{44} = a_{11} + a_{22}$, $a_{45} = -a_{13}$, $a_{46} = -a_{23}$.

It follows from these conditions that coefficients $a_{ij} = 0$, for all $1 \leq i, j \leq 6$. This implies that $\upsilon_{\mathfrak{g}_6}(1) = dim\big(Der_{(1,1,1,1)}\mathfrak{g}_6\big) = 0$, which proves that $\psi \neq \upsilon$ in general.

Now, in the following section, we prove some properties of the invariant function υ.

4.2.1 Properties of the invariant function υ

Let us see some properties of the invariant function υ.

Let us first consider the family $\mathfrak{g}_\varepsilon$ of Lie algebras defined as follows

$$[e_1, e_{p_2}] = \varepsilon_{p_2} e_{p_3}, \ [e_1, e_{p_3}] = \varepsilon_{p_3} e_{p_4}, \ \ldots, \ [e_1, e_{p_m}] = \varepsilon_{p_m} e_{p_{m+1}},$$

for all $m \in \mathbb{N}$ such that $2 \leq m \leq n-1$ and for all subindices $(p_2, \ldots, p_m) \in \mathbb{Z}^{m-1}$, with $1 \leq p_1 < \ldots < p_m \leq n-1$, where ε_{p_r} are non-zero real constants, for all $r \in \{1, \ldots, m\}$.

Proposition 4.2.2. *The invariant function υ is constant for any n-dimensional Lie algebra belonging to the family $\mathfrak{g}_\varepsilon$.*

Proof

In the first place, let us suppose that $m \geq 4$. Let $D = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ be the matrix of the endomorphism of the definition of υ.

As e_s is in the center of the Lie algebra, for all $s \in \{2, \ldots, n\} \setminus \{p_2, \ldots, p_m\}$, the only triples of vectors which can supply restrictions to the matrix D are (e_1, e_{p_i}, e_{p_j}), for all $i, j \in \mathbb{N}$ such that $2 \leq i < j \leq m$.

Let us first suppose that $2 \leq i < j < m$. Then, according to the definition of υ we have

$$[D[e_1, e_{p_i}], [e_1, e_{p_j}]] + [[e_1, e_{p_i}], D[e_1, e_{p_j}]] = 0. \tag{4.5}$$

Therefore, for the triples (e_1, e_{p_i}, e_{p_j}), we have

$$[\varepsilon_{p_i} D(e_{p_{(i+1)}}), \varepsilon_{p_j} e_{p_{(j+1)}}] + [\varepsilon_{p_i} e_{p_{(i+1)}}, \varepsilon_{p_j} D(e_{p_{(j+1)}})] = \varepsilon_{p_i}\varepsilon_{p_j}[a_{p_{(i+1)}1} e_1 + \ldots +$$
$$a_{p_{(i+1)}n} e_n, e_{p_{(j+1)}}] + \varepsilon_{p_i}\varepsilon_{p_j}[e_{p_{(i+1)}}, a_{p_{(j+1)}1} e_1 + \ldots + a_{p_{(j+1)}n} e_n] =$$
$$\varepsilon_{p_i}\varepsilon_{p_j}[a_{p_{(i+1)}1} e_1, e_{p_{(j+1)}}] + \varepsilon_{p_i}\varepsilon_{p_j}[e_{p_{(i+1)}}, a_{p_{(j+1)}1} e_1] =$$
$$\varepsilon_{p_i}\varepsilon_{p_j}\varepsilon_{p_{(j+1)}} a_{p_{(i+1)}1} e_{p_{(j+2)}} - \varepsilon_{p_i}\varepsilon_{p_j}\varepsilon_{p_{(i+1)}} a_{p_{(j+1)}1} e_{p_{(i+2)}} = 0,$$

and from this expression it is deduced that $a_{p_3 1} = 0, \ldots, a_{p_m 1} = 0$.

Secondly, by proceeding in an analogue way with $2 \leq i < j \leq m$, we also obtain the expression

$$\varepsilon_{p_i}\varepsilon_{p_m}[a_{p_{(i+1)}1} e_1 + \ldots + a_{p_{(i+1)}n} e_n, e_{p_{(m+1)}}] + \varepsilon_{p_i}\varepsilon_{p_m}[e_{p_{(i+1)}}, a_{p_{(m+1)}1} e_1 + \ldots + a_{p_{(m+1)}n} e_n] = 0$$

from which we deduce that the first summand is zero because of $e_{p_{(m+1)}}$ belongs to the center of the algebra. So,

$$\varepsilon_{p_i}\varepsilon_{p_m}\varepsilon_{p_{(m+1)}}a_{p_{(m+1)}1} = 0.$$

Therefore, the following restriction for the elements of D must be added to those previously obtained: $a_{p_{(m+1)}1} = 0$.

Therefore,

$$\dim\left(Der_{(1,1,1,\lambda)}\mathfrak{g}_\varepsilon\right) = n^2 - (m+1-3) - 1 = n^2 - (m-2),$$

and thus, $\upsilon_{\mathfrak{g}_\varepsilon}(\lambda) = n^2 - (m-2)$, for all $\lambda \in \mathbb{C}$.

Finally, we now consider the cases $m = 2$ and $m = 3$.

If $m = 2$, the expression 4.5 consists of the triple (e_1, e_{p_2}, e_{p_3}), which is zero because e_{p_3} belongs to the center of the algebra. So, no restrictions on the elements of the matrix D are obtained in this case and $\upsilon_{\mathfrak{g}_\varepsilon}(\lambda) = n^2$, for all $\lambda \in \mathbb{C}$.

If $m = 3$, the unique triple which might supply restrictions to the elements of D is (e_1, e_{p_2}, e_{p_3}). Therefore, it is the same case as when $2 \leq i < j = 3$, and thus, the only restriction on D is $a_{p_4 1} = 0$. It implies that $\upsilon_{\mathfrak{g}_\varepsilon}(\lambda) = n^2 - 1$, for all $\lambda \in \mathbb{C}$.

Note that in the previous result $m-1$ coincides with the number of non-zero constants and that the n-dimensional model filiform Lie algebra is the algebra $\mathfrak{g}_\varepsilon$ when $m = n-1$, $p_i = i$, for all $i \in 2, \ldots, n-1$ and $\varepsilon_i = 1$, for all $i \in 2, \ldots, n-1$. Therefore, $\upsilon_{f_n}(\lambda) = n^2 - (m-2) = n^2 - (n-3)$.

So, as a consequence of the previous result, the following assertion is proved

Theorem 4.2.3. *For $n \geq 5$, if f_n denotes the n-dimensional model filiform Lie algebra, then $\upsilon_{f_n}(\lambda) = n^2 - (n-3)$, for all $\lambda \in \mathbb{C}$.*

A new concept that will be used next to obtain other properties of this function υ is the following

Definition 4.2.4. *Let $\mathfrak{g}_0$ be a graded contraction of a Lie algebra $\mathfrak{g}$. The graded contraction $\mathfrak{g}_\varepsilon$ obtained from $\mathfrak{g}_0$ by making tend to zero some of its relevant parameters is called an* ε-contraction.

Now, we show new results on the invariant function υ, which will be checked in the next section on the 8 kinematical algebras previously indicated. The first of them is a necessary condition for the existence of contractions of graded contractions.

Proposition 4.2.5. *If* $\mathfrak{g}_0$ *is a graded contraction of a Lie algebra* $\mathfrak{g}$ *and* $\mathfrak{g}_\varepsilon$ *is an* ε-*contraction of* $\mathfrak{g}_0$, *then* $\upsilon_{\mathfrak{g}_0} \leq \upsilon_{\mathfrak{g}_\varepsilon}$.

Proof. Let $\mathfrak{g}_\varepsilon$ be a ε-contraction of $\mathfrak{g}_0$. The commutation relation in $\mathfrak{g}_\varepsilon$ is given by $[x,y]_{\mathfrak{g}_\varepsilon} = \varepsilon_{x,y}[x,y]$, where some $\varepsilon_{x,y}$ tends to zero. Observe that

If $d_0 \in Der_{(1,1,1,\lambda)}\mathfrak{g}_0$, then $[d_0[x,y]_{\mathfrak{g}_0}, [x,z]_{\mathfrak{g}_0}]_{\mathfrak{g}_0} + [[x,y]_{\mathfrak{g}_0}, d_0[x,z]_{\mathfrak{g}_0}]_{\mathfrak{g}_0} =$

$d_0[[[x,z]_{\mathfrak{g}_0}, y]_{\mathfrak{g}_0}, x]_{\mathfrak{g}_0} + \lambda d_0[[[z,x]_{\mathfrak{g}_0}, x]_{\mathfrak{g}_0}, y]_{\mathfrak{g}_0} \forall (x,y,z) \in \mathfrak{g}_0$.

If $d_\varepsilon \in Der_{(1,1,1,\lambda)}\mathfrak{g}_\varepsilon$, then $[d_\varepsilon[x,y]_{\mathfrak{g}_\varepsilon}, [x,z]_{\mathfrak{g}_\varepsilon}]_{\mathfrak{g}_\varepsilon} + [[x,y]_{\mathfrak{g}_\varepsilon}, d_\varepsilon[x,z]_{\mathfrak{g}_\varepsilon}]_{\mathfrak{g}_\varepsilon} =$

$d_\varepsilon[[[x,z]_{\mathfrak{g}_\varepsilon}, y]_{\mathfrak{g}_\varepsilon}, x]_{\mathfrak{g}_\varepsilon} + \lambda d_\varepsilon[[[z,x]_{\mathfrak{g}_\varepsilon}, x]_{\mathfrak{g}_\varepsilon}, y]_{\mathfrak{g}_\varepsilon}$, for all $(x,y,z) \in \mathfrak{g}_\varepsilon$.

So, $\kappa_1[d_\varepsilon[x,y],[x,z]] + \kappa_2[[x,y], d_\varepsilon[x,z]] = \kappa_3 d_\varepsilon[[[x,z],y],x] + \lambda\ \kappa_4 d_\varepsilon[[[z,x],x],y]$, for all $(x,y,z) \in \mathfrak{g}_\varepsilon$, where κ_k are constants obtained as the product of constants of contractions of the form $\varepsilon_{xy} \cdot \varepsilon_{uv}$.

It implies that the equations which are deduced when imposing the restrictions on the elements of the matrix of the endomorphism D_ε are the same as those deduced when obtaining the restrictions on the elements of the matrix associated to the endomorphism D_0, except that each term of the new system is multiplied by a constant κ. It implies in turn that when some of these contraction constants ε_{xy} tend to zero, the equation systems used to obtain the restrictions on the elements of the matrices associated to D_ε and D_0 are the same, although some variables do not appear in the last system due that they vanish for being multiplied by zero. These variables become independent and thus they increase the $\dim(Der_{(1,1,1,\lambda)}\mathfrak{g}_\varepsilon)$ with respect to $\dim(Der_{(1,1,1,\lambda)}\mathfrak{g}_0)$. It involves that $\dim(Der_{(1,1,1,\lambda)}\mathfrak{g}_0) \leq \dim(Der_{(1,1,1,\lambda)}\mathfrak{g}_\varepsilon)$ and thus, $\upsilon_{\mathfrak{g}_o} \leq \upsilon_{\mathfrak{g}_\varepsilon}$. □

4.2.2 The invariant function υ in the case of model filiform Lie algebras of lower dimensions

We now compute the values of the invariant function υ for the case of model filiform Lie algebras of lower dimensions. We show here the computations related with these algebras of dimension 3, 4 and 5. The results obtained allow us to give a general expression for the value of this function in this type of algebras.

The 3-dimensional filiform Lie algebra $\mathfrak{f}_3$

Let $\mathfrak{f}_3$ be the model filiform Lie algebra of dimension 3 defined by the bracket $[e_1, e_2] = e_3$.

Let us consider $d \in Der_{(1,1,1,\lambda)}\mathfrak{f}_3$, with $\lambda \in \mathbb{C}$ and let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be the matrix associated with the endomorphism d.

In this case, only one triple (e_1, e_2, e_3) can be taken into consideration.

Taking into consideration that the following expression is satisfied: $\alpha[d[x,y],[x,z]] + \beta[[x,y],d[x,z]] = \gamma d[[[x,z],y],x] + \tau d[[[z,x],x],y]$, if we set $\alpha = \beta = \gamma = 1$, we have

$$[d[x,y],[x,z]] + [[x,y],d[x,z]] = d[[[x,z],y],x] + \tau d[[[z,x],x],y]. \tag{4.6}$$

Since $[e_2, e_3] = [e_1, e_3] = 0$, we see that all the terms in Equation 4.6 are zero. This implies that there are no restrictions for the elements a_{ij} of the matrix A. Consequently,

$$\upsilon_{\mathfrak{f}_3}(\lambda) = \dim\left(Der_{(1,1,1,\lambda)}\mathfrak{f}_3\right) = 9, \quad \forall \lambda \in \mathbb{C}.$$

The 4-dimensional filiform Lie algebra $\mathfrak{f}_4$

Let $\mathfrak{f}_4$ be the 4-dimensional model filiform Lie algebra defined by the brackets $[e_1, e_2] = e_3, [e_1, e_3] = e_4$.

Let us consider $d \in Der_{(1,1,1,\lambda)}\mathfrak{f}_4$, with $\lambda \in \mathbb{C}$, and let $A = (a_{ij}),\ 1 \leq i, j \leq 4$, be the matrix associated with the endomorphism d.

We observe that the only triple producing non-zero results when applying equation Equation 4.6 is (e_1, e_2, e_3). Indeed, that equation for that triple is

$$[d[e_1,e_2],[e_1,e_3]] + [[e_1,e_2],d[e_1,e_3]] = d[[[e_1,e_3],e_2],e_1] + \lambda d[[[e_3,e_1],e_1],e_2]$$

and thus, the only non-null term is $[[e_1, e_2], d[e_1, e_3]]$. That term is $[[e_1, e_2], d[e_1, e_3]] = [e_3, d(e_4)] = [e_3, a_{41}e_1 + a_{42}e_2 + a_{43}e_3 + a_{44}e_4] = a_{41}e_4$. Therefore, $a_{41}e_2 = 0$ if and only if $a_{41} = 0$.

This implies that the equation system which allows us to obtain the elements of the endomorphism d has one equation and 16 variables. Therefore,

$$\upsilon_{\mathfrak{f}_4}(\lambda) = \dim\left(Der_{(1,1,1,\lambda)}\mathfrak{f}_4\right) = 16 - 1 = 15, \quad \forall \lambda \in \mathbb{C}.$$

The 5-dimensional filiform Lie algebra $\mathfrak{f}_5$

Let $\mathfrak{f}_5$ be the model filiform Lie algebra of dimension 5 defined by the brackets $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$.

Let us consider $d \in Der_{(1,1,1,\lambda)}\mathfrak{f}_5$, with $\lambda \in \mathbb{C}$, and let $A = (a_{ij}),\ 1 \leq i, j \leq 5$, be the matrix associated with it.

Now, applying Equation 4.6 to all the possible triples among the generators of the algebra, we obtain expressions involving the elements of the matrix A. Indeed, from the triple (e_1, e_2, e_3) we have $a_{31} = a_{41} = 0$ and from the triple (e_1, e_2, e_4) we obtain that $a_{51} = 0$. Moreover, this same result is obtained starting from the triple (e_1, e_3, e_4), whereas we obtain $a_{51} = a_{41} = 0$. Triples containing e_5 do not supply any condition on elements of A because in such cases Equation 4.6 is identically null.

So, the equation system which allows us to obtain a basis of the vector space $Der_{(1,1,1,\lambda)}\mathfrak{f}_5$ is constituted by 3 equations and 25 variables. This implies that

$$\upsilon_{\mathfrak{f}_5}(\lambda) = \dim\left(Der_{(1,1,1,\lambda)}\mathfrak{f}_5\right) = 22, \quad \forall \lambda \in \mathbb{C}.$$

The n-dimensional filiform Lie algebras $\mathfrak{f}_\mathfrak{n}$

Proceeding in the same way, it is easy to obtain the following assertion (see also Theorem 4.2.3)

Theorem 4.2.6. *Let $\mathfrak{f}_\mathfrak{n}$ be the the model n-dimensional filiform Lie algebra. The function $\upsilon_{\mathfrak{f}_\mathfrak{n}}$ verifies*

$$\upsilon_{\mathfrak{f}_\mathfrak{n}}(\lambda) = \dim\left(Der_{(1,1,1,\lambda)}\mathfrak{f}_\mathfrak{n}\right) = \begin{cases} 9, & \textit{if} \quad n = 3, \\ 15, & \textit{if} \quad n = 4, \\ n^2 - (n-3), & \textit{if} \quad n \geq 5. \end{cases}$$

for all $\lambda \in \mathbb{C}$.

4.2.3 The invariant function υ in the case of other algebras of lower dimensions

In this section we deal with the invariant function υ in the case of different types of algebras in lower dimensions.

The $\mathfrak{so}(3)$ algebra

Let $\mathfrak{so}(3)$ be the 3-dimensional Lie algebra defined by the brackets $[e_1, e_2] = -e_3$, $[e_2, e_3] = -e_1$ and $[e_3, e_1] = -e_2$

Let us consider $d \in Der_{(1,1,1,\lambda)}\mathfrak{so}(3)$, with $\lambda \in \mathbb{C}$, and let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be the matrix associated with the endomorphism d.

The only possible triple (e_1, e_2, e_3), we have

$$\begin{aligned} [d[e_1, e_2], [e_1, e_3]] &= -a_{33}e_1 + a_{31}e_3, \\ [[e_1, e_2], d[e_1, e_3]] &= -a_{22}e_1 + a_{21}e_2, \\ d[[[e_1, e_3], e_2], e_1] &= 0, \\ \lambda d[[[e_3, e_1], e_1], e_2] &= -\lambda a_{11}e_1 - \lambda a_{12}e_2 - \lambda a_{13}e_3. \end{aligned}$$

Therefore, $(-a_{33} - a_{22})e_1 + a_{21}e_2 + a_{31}e_3 = -\lambda a_{11}e_1 - \lambda a_{12}e_2 - \lambda a_{13}e_3$, which implies $a_{33} + a_{22} = \lambda a_{11}$, $a_{21} = -\lambda a_{12}$ and $a_{31} = -\lambda a_{13}$. Then, we have

$$\upsilon_{so(3)}(\lambda) = \dim\left(Der_{(1,1,1,\lambda)}\mathfrak{so}(3)\right) = 9 - 3 = 6, \ \forall \lambda \in \mathbb{C}.$$

The 3-dimensional Malcev algebra M_3

Let M_3 the Malcev algebra of dimension 3 defined by the law $[e_1, e_2] = e_1 + e_3$, $[e_2, e_3] = e_1 + e_2 + e_3$ and $[e_3, e_1] = -e_1$.

Let us consider $d \in Der_{(1,1,1,\lambda)}M_3$, with $\lambda \in \mathbb{C}$ and let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be the matrix associated with the endomorphism d.

In this case, similarly to what happened in Subsection 4.2.2, only one triple (e_1, e_2, e_3) can be taken into consideration. By applying Equation 4.6 we have $[d(e_1+e_3), -e_1]+[e_1+e_3, d[(-e_1)] = d[[-e_1, e_2], e_1]$. The first member of this expression is: $[d(e_1)+d(e_3), -e_1]]+[e_1+e_3, d(-e_1)] = (-a_{33} - a_{11} + a_{32} + a_{12})e_1 + a_{12}e_2 + (a_{32} + a_{12})e_3$ and the second one is: $d[[-e_1, e_2], e_1] = -a_{11}e_1 - a_{12}e_2 - a_{13}e_3$. So, we obtain three conditions on the elements of the endomorphism matrix given by $-a_{33} - a_{11} + a_{32} + a_{12} = -a_{11},\ a_{12} = 0$ and $a_{32} + a_{12} = -a_{13}$.

Therefore, we have

$$\upsilon_{M_3}(\lambda) = \dim\left(Der_{(1,1,1,\lambda)}M_3\right) = 9 - 3 = 6,\ \forall \lambda \in \mathbb{C}.$$

The 3-dimensional abelian Lie algebra $\mathfrak{g}_3$ and the algebra $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$

Let $\mathfrak{g}_3$ be the 3-dimensional abelian Lie algebra, given by $[e_i, e_j] = 0, 1 \leq i, j \leq 3$, and $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$ be the 3-dimensional algebra defined by the only bracket $[e_1, e_2] = e_2$.

By using the previously procedure with these two algebras, of the same dimension, we observe that no restrictions can be obtained, due to all the terms of the Equation 4.6 are identically null, $\forall \lambda \in \mathbb{C}$. So,

$$\upsilon_{\mathfrak{g}_3}(\lambda) = \upsilon_{\mathfrak{g}_{2,1}\oplus\mathfrak{g}_1}(\lambda) = 9.$$

4.2.4 A quantum-mechanical model based on a 5-th Heisenberg algebra

In this section and by using the invariant function previously introduced υ, we prove the following result

Theorem 4.2.7. Main Theorem

A 5-dimensional classical-mechanical model built upon certain types of 5-dimensional Lie algebras cannot be obtained as a limit process of a quantum-mechanical model based on a 5-th Heisenberg algebra.

Proof. Let $\mathbb{H}_5$ be the 5-th Heisenberg algebra, defined by the brackets $[e_1, e_3] = e_5$ and $[e_2, e_4] = e_5$.

Let us consider $d \in Der_{(1,1,1,1)}\mathbb{H}_5$. Then $[d[u,v],[u,w]]+[[u,v],d[u,w]] = d[[[u,w],v],u]+d[[[w,u],u],v]$, $\forall u, v, w \in \mathbb{H}_5$.

To obtain the elements a_{ij} of the corresponding 5×5 square matrix associated with d, we see that for each triplet of generators (e_i, e_j, e_k) of the algebra, the previous expression is written as

$$[d[e_i, e_j], [e_i, e_k]] + [[e_i, e_j], d[e_i, e_k]] = d[[[e_i, e_k], e_j], e_i] + d[[[e_k, e_i], e_i], e_j].$$

Note that, in this case, there is no restriction on the elements of the matrix associated with d and thus, $\upsilon\mathbb{H}_5(1,1) = dim\big(Der_{(1,1,1,1)}\mathbb{H}_5\big) = 25$.

For another part, if $\mathfrak{f}_5$ is the 5-dimensional filiform Lie algebra, defined by $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$, we have already computed (see Subsection 4.2.2) that $\upsilon_{\mathfrak{f}_5}(1) = 22$.

Next, to continue with the proof, we make use of the highly non-trivial result, which was originally proved by Borel in [6]: *If $\mathfrak{g}_0$ is a proper contraction of a complex Lie algebra $\mathfrak{g}$ then it holds:* $\dim\big(Der\mathfrak{g}\big) < \dim\big(Der\mathfrak{g}_0\big)$.

Indeed, by using the proposition 4.1.2 we have obtained that

$$\upsilon_{\mathbb{H}_5}(1) = \dim\big(Der_{(1,1,1,1)}\mathbb{H}_5\big) = \dim\big(Der\big(\mathbb{H}_5\big)\big) = 25$$

and

$$\upsilon_{\mathfrak{f}_5}(1) = \dim\big(Der_{(1,1,1,1)}\mathfrak{f}_5\big) = \dim\big(Der\big(\mathfrak{f}_5\big)\big) = 22.$$

It implies that there not exists any proper contraction transforming the 5-th Heisemberg algebra $\mathbb{H}_5$ into the filiform Lie algebra $\mathfrak{f}_5$. So, since both algebras are not isomorphic, the 5-dimensional classical-mechanical model built upon a 5-dimensional filiform Lie algebra cannot be obtained as a limit process of a quantum-mechanical model based on a 5-th Heisenberg algebra. □

4.3 Introducing the two-parameter invariant function $\bar{\phi}$

In the following definition we introduce a novel twisted cocycle that we will use to construct a new two-parameter invariant function $\bar{\phi}$. To generalize as much as possible this study, we deal with Malcev algebras of Lie type.

Definition 4.3.1. *Let $\mathfrak{g}$ be a Malcev algebra of Lie type and let us consider $B \in C^2(\mathfrak{g}, \mathfrak{g})$. B is a new twisted cocycle if*

$$\alpha_1 \, B([[x,z],y],x) \; + \; \alpha_2 \, B([[y,x],z],x) \; + \; \alpha_3 \, B([[z,y],x],x) \; + \; \beta_1 \, [B([x,z],y),x] +$$
$$\beta_2 \, [B([y,x],z),x] \; + \; \beta_3 \, [B([z,y],x),x] \; = 0, \tag{1}$$

for all $(x, y, z) \in \mathfrak{g}$.

It is easy to see that the set of these new twisted cocycles can be endowed with a vector space structure. It will be denoted by $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g}$.

Some properties related to this concept are next proved.

Proposition 4.3.2. *Let $\mathfrak{g}$ be a Malcev algebra of Lie type. Then,*

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(\alpha_2, \alpha_1, \alpha_3, \beta_2, \beta_1, \beta_3)\mathfrak{g},$$

for all $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \in \mathbb{C}^4$.

Proof. If $B \in cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g}$, with $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \in \mathbb{C}^4$, changing each other y and z in (1) we have $\alpha_1 \, B([[x,y],z],x) + \alpha_2 \, B([[z,x],y],x) + \alpha_3 \, B([[y,z],x],x) + \beta_1 \, [B([x,y],z),x] + \beta_2 \, [B([z,x],y),x] + \beta_3 \, [B([y,z],x),x] = 0$.

Now, by taking opposite signs and reordering, it is obtained

$$\alpha_2 \, B([[x,z],y],x) + \alpha_1 \, B([[y,x],z],x) + \; \alpha_3 \, B([[z,y],x],x) \; + \; \beta_2 \, [B([x,z],y),x] +$$
$$\beta_1 \, [B([y,x],z),x] + \beta_3 \, [B([z,y],x),x] = 0.$$

Therefore, $B \in cc(\alpha_2, \alpha_1, \alpha_3, \beta_2, \beta_1, \beta_3)\mathfrak{g}$.

In the other sense, the proof is similar. □

Proposition 4.3.3. *Let $\mathfrak{g}$ be a Malcev algebra of Lie type. Then,*

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} \; = \; cc(\alpha_1 \; + \; \alpha_2, \alpha_1 \; + \; \alpha_2, 2\,\alpha_3, \beta_1 \; + \; \beta_2, \beta_1 \; + \; \beta_2, 2\,\beta_3)\mathfrak{g}$$
$$\cap \; cc(\alpha_1 \; - \; \alpha_2, \alpha_2 \; - \alpha_1, 0, \beta_1 \; - \; \beta_2, \beta_2 \; - \; \beta_1, 0)\mathfrak{g},$$

for all $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \in \mathbb{C}^4$.

Proof. If $B \in cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g}$, taking previous result into consideration we have $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(\alpha_2, \alpha_1, \alpha_3, \beta_2, \beta_1, \beta_3)\mathfrak{g}$.

Then,

$$\alpha_1 B([[x,y],z],x) + \alpha_2 B([[z,x],y],x) + \alpha_3 B([[y,z],x],x) +$$
$$\beta_1[B([x,y],z),x] + \beta_2[B([z,x],y),x] + \beta_3[B([y,z],x),x] = 0$$

and

$$\alpha_2 B([[x,z],y],x) + \alpha_1 B([[y,x],z],x) + \alpha_3 B([[z,y],x],x) +$$
$$\beta_2[B([x,z],y),x] + \beta_1[B([y,x],z),x] + \beta_3[B([z,y],x),x] = 0.$$

By now adding in the first place both expressions, we have

$$(\alpha_1 + \alpha_2) B([[x,y],z],x) + (\alpha_1 + \alpha_2) B([[z,x],y],x) + 2\alpha_3 B([[y,z],x],x) +$$
$$(\beta_1 + \beta_2)[B([x,y],z),x] + (\beta_1 + \beta_2)[B([z,x],y),x] + 2\beta_3[B([y,z],x),x] = 0,$$

from which we deduce that $B \in cc(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, 2\alpha_3, \beta_1 + \beta_2, \beta_1 + \beta_2, 2\beta_3)\mathfrak{g}$.

And by subtracting them secondly, we obtain

$$(\alpha_1 - \alpha_2) B([[x,y],z],x) + (\alpha_2 - \alpha_1) B([[z,x],y],x) +$$
$$\beta_1 - \beta_2)[B([x,y],z),x] + (\beta_2 - \beta_1)[B([z,x],y),x] = 0.$$

So, $B \in cc(\alpha_1 - \alpha_2, \alpha_2 - \alpha_1, 0, \beta_1 - \beta_2, \beta_2 - \beta_1, 0)\mathfrak{g}$.

Both results imply that

$$B \in cc(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, 2\alpha_3, \beta_1 + \beta_2, \beta_1 + \beta_2, 2\beta_3)\mathfrak{g} \cap cc(\alpha_1 - \alpha_2, \alpha_2 - \alpha_1, 0, \beta_1 - \beta_2, \beta_2 - \beta_1, 0)\mathfrak{g}$$

and thus

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} \subset cc(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, 2\alpha_3, \beta_1 + \beta_2, \beta_1 + \beta_2, 2\beta_3)\mathfrak{g}$$
$$\cap\ cc(\alpha_1 - \alpha_2, \alpha_2 - \alpha_1, 0, \beta_1 - \beta_2, \beta_2 - \beta_1, 0)\mathfrak{g}.$$

In the other sense, the proof is similar. $\square$

Theorem 4.3.4. *Let $\mathfrak{g}$ be a Malcev algebra of Lie type. Then, there exists a quatern $(\alpha, \beta, \gamma, \tau) \in \mathbb{C}^4$ such that $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g}$ is one of the following vextor spaces*

1. $cc(0, 0, \alpha, 0, 0, \beta)\mathfrak{g}, cc(0, 0, \alpha, -1, 1, \beta)\mathfrak{g}$
   $cc(-1, 1, \alpha, 0, 0, \beta)\mathfrak{g}, cc(\alpha, -\alpha, \beta, 1, -1, \gamma)\mathfrak{g}$
2. $cc(0, 0, \alpha, 1, 0, \beta)\mathfrak{g}, cc(0, 0, \alpha, 1, 1, \beta)\mathfrak{g}$
   $cc(\alpha, -\alpha, \beta, 1, 0, \gamma)\mathfrak{g}, cc(1, -1, \alpha, 1, 1, \beta)\mathfrak{g}$
3. $cc(1, 0, \alpha, 0, 0, \beta)\mathfrak{g}, cc(1, 1, \alpha, 0, 0, \beta)\mathfrak{g}$
   $cc(1, 0, \alpha, \beta, -\beta, \gamma)\mathfrak{g}, cc(1, 1, \alpha, 1, -1, \beta)\mathfrak{g}$
4. $cc(1, 0, \alpha, \beta + \gamma, \beta - \gamma, \tau)\mathfrak{g}, cc(\alpha + 1, \alpha - 1, \beta, 1, 1, \gamma)\mathfrak{g}$
   $cc(1, 1, \alpha, \beta + 1, \beta - 1, \gamma)\mathfrak{g}, cc(1, 1, \alpha, \beta, \beta, \gamma)\mathfrak{g}$

Proof. We distinguish the following cases

Case 1 If $\alpha_1 + \alpha_2 = 0$ and $\beta_1 + \beta_2 = 0$.

In this case, we consider the following subcases

- **Subcase 1.1** If $\alpha_1 = -\alpha_2 = 0$ and $\beta_1 = -\beta_2 = 0$.

  In this subcase, $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(0, 0, \alpha_3, 0, 0, \beta_3)\mathfrak{g}$. Then, taking $\alpha = \alpha_3$ and $\beta = \beta_3$ we obtain that

  $$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} \; = \; cc(0, 0, \alpha, 0, 0, \beta)\mathfrak{g}.$$

- **Subcase 1.2** If $\alpha_1 = -\alpha_2 = 0$ and $\beta_1 = -\beta_2 \neq 0$.

  According to Proposition 4.3.3 we have
  $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(0, 0, 2\alpha_3, 0, 0, 2\beta_3)\mathfrak{g} \cap$
  $cc(0, 0, 0, -2\beta_2, 2\beta_2, 0)\mathfrak{g} = cc(0, 0, \alpha_3, 0, 0, \beta_3)\mathfrak{g} \; \cap \; cc(0, 0, 0, -1, 1, 0)\mathfrak{g}$.

  Apart from that, we also have that $cc(0, 0, \alpha_3, -1, 1, \beta_3)\mathfrak{g} =$
  $cc(0, 0, 2\alpha_3, 0, 0, 2\beta_3)\mathfrak{g} \cap cc(0, 0, 0, -2, 2, 0)\mathfrak{g} = cc(0, 0, \alpha_3, 0, 0, \beta_3)\mathfrak{g} \cap$
  $cc(0, 0, 0, -1, 1, 0)\mathfrak{g}$.

  Then, taking $\alpha = \alpha_3$ and $\beta = \beta_3$, we obtain that

  $$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(0, 0, \alpha, -1, 1, \beta)\mathfrak{g}.$$

- **Subcase 1.3** If $\alpha_1 = -\alpha_2 \neq 0$ and $\beta_1 = -\beta_2 = 0$.

  In a similar way as before, according to Proposition 4.3.3 we have
  $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(0, 0, 2\alpha_3, 0, 0, 2\beta_3)\mathfrak{g} \cap cc(-2\alpha_2, 2\alpha_2, 0, 0, 0, 0)\mathfrak{g}$
  $= cc(0, 0, \alpha_3, 0, 0, \beta_3)\mathfrak{g} \cap cc(-1, 1, 0, 0, 0, 0)\mathfrak{g}$.

  Besides, $cc(-1, 1, \alpha_3, 0, 0, \beta_3)\mathfrak{g} = cc(0, 0, 2\alpha_3, 0, 0, 2\beta_3)\mathfrak{g} \cap cc(-2, 2, 0, 0, 0, 0)\mathfrak{g}$
  $= cc(0, 0, \alpha_3, 0, 0, \beta_3)\mathfrak{g} \cap cc(-1, 1, 0, 0, 0, 0)\mathfrak{g}$.

  Taking $\alpha = \alpha_3$ and $\beta = \beta_3$, we obtain that

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(-1, 1, \alpha, 0, 0, \beta)\mathfrak{g}.$$

- **Subcase 1.4** If $\alpha_1 = -\alpha_2 \neq 0$ and $\beta_1 = -\beta_2 \neq 0$.

  In a similar way as before, according to Proposition 4.3.3 we have
  $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(0, 0, 2\alpha_3, 0, 0, 2\beta_3)\mathfrak{g} \cap$
  $cc(\alpha_1 - \alpha_2, \alpha_2 - \alpha_1, 0, \beta_1 - \beta_2, \beta_2 - \beta_1, 0)\mathfrak{g} = cc(0, 0, 2\alpha_3, 0, 0, 2\beta_3)\mathfrak{g} \cap$
  $cc(2\frac{\alpha_1-\alpha_2}{\beta_1-\beta_2}, -2\frac{\alpha_1-\alpha_2}{\beta_1-\beta_2}, 0, 2, -2, 0)\mathfrak{g} = cc(\frac{\alpha_1-\alpha_2}{\beta_1-\beta_2}, -\frac{\alpha_1-\alpha_2}{\beta_1-\beta_2}, \alpha_3, 1, -1, \beta_3)\mathfrak{g}$.

Taking now $\alpha = \frac{\alpha_1-\alpha_2}{\beta_1-\beta_2}$, $\beta = \alpha_3$, $\gamma = \alpha_3$, we have

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(\alpha, -\alpha, \beta, 1, -1, \gamma)\mathfrak{g}.$$

Case 2 If $\alpha_1 + \alpha_2 = 0$ and $\beta_1 + \beta_2 \neq 0$.

In this case, we consider the following subcases

- **Subcase 2.1** $\alpha_1 = -\alpha_2 = 0$ and $\beta_1 - \beta_2 \neq 0$.

  In this subcase, we have
  $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(0, 0, 2\alpha_3, \beta_1 + \beta_2, \beta_2 + \beta_1, 2\beta_3)\mathfrak{g} \cap$
  $= cc(0, 0, 0, \beta_1 - \beta_2, \beta_2 - \beta_1, 0)\mathfrak{g} = cc(0, 0, \frac{2\alpha_3}{\beta_1+\beta_2}, 1, 1, \frac{2\beta_3}{\beta_1+\beta_2})\mathfrak{g} \cap$
  $cc(0, 0, 0, 1, -1, 0)\mathfrak{g} = cc(0, 0, \frac{\alpha_3}{\beta_1+\beta_2}, 1, 0, \frac{\beta_3}{\beta_1+\beta_2})\mathfrak{g}$.

  Then, if we now take $\alpha = \frac{\alpha_3}{\beta_1+\beta_2}$ and $\beta = \frac{\beta_3}{\beta_1+\beta_2}$, we obtain that

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(0, 0, \alpha, 1, 0, \beta)\mathfrak{g}.$$

- **Subcase 2.2** $\alpha_1 = -\alpha_2 = 0$ and $\beta_1 = \beta_2 \neq 0$.

  In this subcase, we have $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(0, 0, \frac{\alpha_3}{\beta_1}, \frac{\beta_1}{\beta_1}, \frac{\beta_1}{\beta_1}, \frac{\beta_3}{\beta_1})\mathfrak{g}$.

Then, taking $\alpha = \frac{\alpha_3}{\beta_1}$ and $\beta = \frac{\beta_3}{\beta_1}$ we obtain that

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(0, 0, \alpha, 1, 1, \beta)\mathfrak{g}.$$

- **Subcase 2.3** $\alpha_1 = -\alpha_2 \neq 0$ and $\beta_1 - \beta_2 \neq 0$.

  In this subcase we have $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} =$
  $cc(0, 0, 2\alpha_3, \beta_1+\beta_2, \beta_2+\beta_1, 2\beta_3)\mathfrak{g} \cap\ cc(\alpha_1-\alpha_2, \alpha_2-\alpha_1, 0, \beta_1-\beta_2, \beta_2-\beta_1, 0)\mathfrak{g} =$
  $cc(0, 0, \frac{2\alpha_3}{\beta_1+\beta_2}, 1, 1, \frac{2\beta_3}{\beta_1+\beta_2})\mathfrak{g} \cap cc(\frac{2\alpha_1}{\beta_1-\beta_2}, -\frac{2\alpha_1}{\beta_1-\beta_2}, 0, 1, -1, 0)\mathfrak{g} =$
  $cc(\frac{\alpha_1}{\beta_1-\beta_2}, -\frac{\alpha_1}{\beta_1-\beta_2}, \frac{\alpha_3}{\beta_1+\beta_2}, 1, 0, \frac{\beta_3}{\beta_1+\beta_2})\mathfrak{g}$.

  Then, taking $\alpha = \frac{\alpha_1}{\beta_1-\beta_2}$, $\beta = \frac{\alpha_3}{\beta_1+\beta_2}$ $\gamma = \frac{\beta_3}{\beta_1+\beta_2}$ we obtain that

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(\alpha, -\alpha, \beta, 1, 0, \gamma)\mathfrak{g}.$$

- **Subcase 2.4** $\alpha_1 = -\alpha_2 \neq 0$ and $\beta_1 = \beta_2 \neq 0$.

  In this subcase we have $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(0, 0, 2\alpha_3, \beta_1 + \beta_2, \beta_2 + \beta_1, 2\beta_3)\mathfrak{g} \cap cc(\alpha_1-\alpha_2, \alpha_2-\alpha_1, 0, \beta_1-\beta_2, \beta_2-\beta_1, 0)\mathfrak{g} = cc(0, 0, \frac{4\alpha_3}{\beta_1+\beta_2}, 2, 2, \frac{4\beta_3}{\beta_1+\beta_2})\mathfrak{g}$ $\cap\ cc(2, -2, 0, 0, 0, 0)\mathfrak{g} =\ cc(1, -1, \frac{2\alpha_3}{\beta_1+\beta_2}, 1, 1, \frac{2\beta_3}{\beta_1+\beta_2})\mathfrak{g}$.

  Then, taking $\alpha = \frac{2\alpha_3}{\beta_1+\beta_2}$ and $\beta = \frac{2\beta_3}{\beta_1+\beta_2}$ we obtain that

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(1, -1, \alpha, 1, 1, \beta)\mathfrak{g}.$$

Case 3 If $\alpha_1 + \alpha_2 \neq 0$ and $\beta_1 + \beta_2 = 0$.

In this case, we consider the following subcases

- **Subcase 3.1** $\alpha_1 - \alpha_2 \neq 0$ and $\beta_1 = -\beta_2 = 0$.
  In this subcase, we have
  $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(\alpha_1 + \alpha_2, \alpha_2 + \alpha_1, 2\alpha_3, 0, 0, 2\beta_3)\mathfrak{g} \cap$
  $cc(\alpha_1 - \alpha_2, \alpha_2 - \alpha_1, 0, 0, 0, 0,)\mathfrak{g} = cc(1, 1, \frac{2\alpha_3}{\alpha_1+\alpha_2}, 0, 0, \frac{2\beta_3}{\alpha_1+\alpha_2})\mathfrak{g} \cap$
  $cc(1, -1, 0, 0, 0, 0)\mathfrak{g} = cc(1, 0, \frac{\alpha_3}{\alpha_1+\alpha_2}, 0, 0, \frac{\beta_3}{\alpha_1+\alpha_2})\mathfrak{g}$.
  Then, if we now take $\alpha = \frac{\alpha_3}{\alpha_1+\alpha_2}$ and $\beta = \frac{\beta_3}{\alpha_1+\alpha_2}$, we obtain that

$$cc(\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2, \alpha_3)\mathfrak{g} = cc(1, 0, \alpha, 0, 0, \beta)\mathfrak{g}.$$

- **Subcase 3.2** $\alpha_1 = \alpha_2 \neq 0$ and $\beta_1 = -\beta_2 = 0$.
  In this subcase, we have $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} =$
  $cc(\frac{\alpha_1}{\alpha_1}, \frac{\alpha_1}{\alpha_1}, \frac{\alpha_3}{\alpha_1}, 0, 0, \frac{\beta_3}{\alpha_1})\mathfrak{g}$.

Then, taking $\alpha = \frac{\alpha_3}{\alpha_1}$ and $\beta = \frac{\beta_3}{\alpha_1}$ we obtain that

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(1, 1, \alpha, 0, 0, \beta)\mathfrak{g}.$$

- **Subcase 3.3** $\alpha_1 - \alpha_2 \neq 0$ and $\beta_1 = -\beta_2 \neq 0$.

  In this subcase we have $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} =$
  $cc(\alpha_1 + \alpha_2, \alpha_2 + \alpha_1, 2\alpha_3, 0, 0, 2\beta_3)\mathfrak{g} \cap$
  $cc(\alpha_1 - \alpha_2, \alpha_2 - \alpha_1, 0, \beta_1 - \beta_2, \beta_2 - \beta_1, 0)\mathfrak{g} =$
  $cc(1, 1, \frac{2\alpha_3}{\alpha_1+\alpha_2}, 0, 0, \frac{2\beta_3}{\alpha_1+\alpha_2})\mathfrak{g} \quad \cap cc(1, -1, 0, \frac{2\beta_1}{\alpha_1-\alpha_2}, -\frac{2\beta_1}{\alpha_1-\alpha_2}, 0)\mathfrak{g} =$
  $cc(1, 0, \frac{\alpha_3}{\alpha_1+\alpha_2}, \frac{\beta_1}{\alpha_1-\alpha_2}, -\frac{\beta_1}{\alpha_1-\alpha_2}, \frac{\beta_3}{\alpha_1+\alpha_2})\mathfrak{g}$.
  Then, taking $\alpha = \frac{\alpha_3}{\alpha_1+\alpha_2}$, $\beta = \frac{\beta_1}{\alpha_1-\alpha_2}$ and $\gamma = \frac{\beta_3}{\alpha_1+\alpha_2}$ and we obtain that

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(1, 0, \alpha, \beta, -\beta, \gamma)\mathfrak{g}.$$

- **Subcase 3.4** $\alpha_1 = \alpha_2 \neq 0$ and $\beta_1 = -\beta_2 \neq 0$

  In this subcase we have $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(\alpha_1 + \alpha_2, \alpha_2 + \alpha_1, 2\alpha_3, 0, 0, 2\beta_3)\mathfrak{g} \cap cc(\alpha_1 - \alpha_2, \alpha_2 - \alpha_1, 0, \beta_1 - \beta_2, \beta_2 - \beta_1, 0)\mathfrak{g}$
  $= cc(2, 2, \frac{4\alpha_3}{\alpha_1+\alpha_2}, 0, 0, \frac{4\beta_3}{\alpha_1+\alpha_2})\mathfrak{g} \cap cc(0, 0, 0, 2, -2, 0)\mathfrak{g} =$
  $cc(1, 1, \frac{2\alpha_3}{\alpha_1+\alpha_2}, 1, -1, \frac{2\beta_3}{\alpha_1+\alpha_2})\mathfrak{g}$.
  Then, taking $\alpha = \frac{2\alpha_3}{\alpha_1+\alpha_2}$ and $\beta = \frac{2\beta_3}{\alpha_1+\alpha_2}$ we obtain that

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(1, 1, \alpha, 1, -1, \beta)\mathfrak{g}.$$

Case 4 $\alpha_1 + \alpha_2 \neq 0$ and $\beta_1 + \beta_2 \neq 0$

In this case we consider the following subcases

- **Subcase 4.1** $\alpha_1 - \alpha_2 \neq 0$ and $\beta_1 - \beta_2 \neq 0$.

  In this subcase we have $cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} =$
  $cc(\alpha_1+\alpha_2, \alpha_1+\alpha_2, 2\alpha_3, \beta_1+\beta_2, \beta_2+\beta_1, 2\beta_3)\mathfrak{g} \cap cc(\alpha_1-\alpha_2, \alpha_2-\alpha_1, 0, \beta_1-\beta_2, \beta_2 - \beta_1, 0)\mathfrak{g} =$
  $cc(1, 1, \frac{2\alpha_3}{\alpha_1+\alpha_2}, \frac{\beta_1+\beta_2}{\alpha_1+\alpha_2}, \frac{\beta_1+\beta_2}{\alpha_1+\alpha_2}, \frac{2\beta_3}{\alpha_1+\alpha_2})\mathfrak{g} \cap cc(1, -1, 0, \frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}, -\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}, 0)\mathfrak{g} =$
  $cc(1, 0, \frac{\alpha_3}{\alpha_1+\alpha_2}, \frac{\beta_1+\beta_2}{2(\alpha_1+\alpha_2)} + \frac{\beta_1-\beta_2}{2(\alpha_1-\alpha_2)}, \frac{\beta_1+\beta_2}{2(\alpha_1+\alpha_2)} - \frac{\beta_1-\beta_2}{2(\alpha_1-\alpha_2)}, \frac{\beta_3}{\alpha_1+\alpha_2})\mathfrak{g}$.
  Taking now $\alpha = \frac{\alpha_3}{\alpha_1+\alpha_2}$, $\beta = \frac{\beta_1+\beta_2}{2(\alpha_1+\alpha_2)}$, $\gamma = \frac{\beta_1-\beta_2}{2(\alpha_1-\alpha_2)}$ and $\tau = \frac{\beta_3}{\alpha_1+\alpha_2}$ we obtain that

$$cc(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\mathfrak{g} = cc(1, 0, \alpha, \beta + \gamma, \beta - \gamma, \tau)\mathfrak{g}.$$

- **Subcase 4.2** $\alpha_1 - \alpha_2 \neq 0$ and $\beta_1 = \beta_2 \neq 0$.

  In this subcase we have

$cc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\mathfrak{g} = cc(\alpha_1+\alpha_2,\alpha_1+\alpha_2,2\alpha_3,2\beta_1,2\beta_1,2\beta_3)\mathfrak{g}\cap cc(\alpha_1-\alpha_2,\alpha_2-\alpha_1,0,0,0,0)\mathfrak{g} = cc(\frac{\alpha_1+\alpha_2}{\beta_1},\frac{\alpha_1+\alpha_2}{\beta_1},\frac{2\alpha_3}{\beta_1},2,2,\frac{2\beta_3}{\beta_1})\mathfrak{g}\ \cap$ $cc(2,-2,0,0,0,0)\mathfrak{g} = cc(\frac{\alpha_1+\alpha_2}{2\beta_1}+1,\frac{\alpha_1+\alpha_2}{2\beta_1}-1,\frac{\alpha_3}{\beta_1},1,1,\frac{\beta_3}{\beta_1})\mathfrak{g}$.

Then, taking $\alpha=\frac{\alpha_1+\alpha_2}{2\beta_1}$, $\beta=\frac{\alpha_3}{\beta_1}$ and $\gamma=\frac{\beta_3}{\beta_1}$ we obtain that

$$cc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\mathfrak{g} = cc(\alpha+1,\alpha-1,\beta,1,1,\gamma)\mathfrak{g}.$$

- **Subcase 4.3** $\alpha_1=\alpha_2\neq 0$ and $\beta_1-\beta_2\neq 0$.

  In this subcase we have
  $cc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\mathfrak{g} = cc(2,2,\frac{2\alpha_3}{\alpha_2},\frac{\beta_1+\beta_2}{\alpha_2},\frac{\beta_1+\beta_2}{\alpha_2},\frac{2\beta_3}{\alpha_2})\mathfrak{g}\cap$ $cc(0,0,0,2,-2,0)\mathfrak{g} = cc(1,1,\frac{\alpha_3}{\alpha_2},\frac{\beta_1+\beta_2}{2\alpha_2}+1,\frac{\beta_1+\beta_2}{2\alpha_2}-1,\frac{\beta_3}{\alpha_2})\mathfrak{g}$.

  Then, if we now take $\alpha=\frac{\alpha_3}{\alpha_2}$, $\beta=\frac{\beta_1+\beta_2}{2\alpha_2}$, and $\gamma=\frac{\beta_3}{\alpha_2}$ we obtain that

$$cc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\mathfrak{g} = cc(1,1,\alpha,\beta+1,\beta-1,\gamma)\mathfrak{g}.$$

- **Subcase 4.4** $\alpha_1=\alpha_2\neq 0$ and $\beta_1=\beta_2\neq 0$.

  In this subcase we have
  $cc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\mathfrak{g} = cc(\frac{\alpha_1}{\alpha_1},\frac{\alpha_1}{\alpha_1},\frac{\alpha_3}{\alpha_1},\frac{\beta_1}{\alpha_1},\frac{\beta_1}{\alpha_1},\frac{\beta_3}{\alpha_1})\mathfrak{g}$.

  If we now take $\alpha=\frac{\alpha_3}{\alpha_1}$, $\beta=\frac{\beta_1}{\alpha_1}$ and $\gamma=\frac{\beta_3}{\alpha_1}$ we obtain that

$$cc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\mathfrak{g} = cc(1,1,\alpha,\beta,\beta,\gamma)\mathfrak{g}.$$

□

Theorem 4.3.5. *Let $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ be two type Lie Malcev algebras and let $f:\mathfrak{g}\to\bar{\mathfrak{g}}$ be an isomorphism. Under these conditions, the mapping $\rho: End\ \mathfrak{g}\to End\ \bar{\mathfrak{g}}$, defined by $D\longrightarrow fDf^{-1}$ is an isomorphism between the vector spaces $cc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\mathfrak{g}$ and $cc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\bar{\mathfrak{g}}$, for all $(\alpha,\beta,\gamma,\tau)\in\mathbb{C}^4$.*

Proof. Let $\mathfrak{g}=\big(V,\cdot\big)$ and $\bar{\mathfrak{g}}=\big(\bar{V},*\big)$ be two Type Lie Malcev algebras and let us consider $B\in cc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\mathfrak{g}$, for any $(\alpha,\beta,\gamma,\tau)\in\mathbb{C}^4$ and for all $x,y,z\in\mathfrak{g}$. Then, we have

$$\alpha_1 B\big(((f^{-1}(x)\cdot f^{-1}(z))\cdot f^{-1}(y)),f^{-1}(x)\big)+\alpha_2 B\big(((f^{-1}(y)\cdot f^{-1}(x))\cdot f^{-1}(z)),f^{-1}(x)\big)+ \alpha_3 B\big(((f^{-1}(z)\cdot f^{-1}(y))\cdot f^{-1}(x)),f^{-1}(x)\big)+\beta_1(B\big((f^{-1}(x)\cdot f^{-1}(z)),f^{-1}(y)\big)\cdot f^{-1}(x))+ \beta_2(B\big((f^{-1}(y)\cdot f^{-1}(x)),f^{-1}(z)\big)\cdot f^{-1}(x))+\beta_3(B\big((f^{-1}(z)\cdot f^{-1}(y)),f^{-1}(x)\big)\cdot f^{-1}(x))=0$$

We also have that

$\alpha_1 B\big(((f^{-1}(x)\cdot f^{-1}(z))\cdot f^{-1}(y)), f^{-1}(x)\big) = \alpha_1 Bf^{-1}\big(((x*z)*y), x\big)$,

and similarly,

$\alpha_2 B\big(((f^{-1}(y)\cdot f^{-1}(x))\cdot f^{-1}(z)), f^{-1}(x)\big) = \alpha_2 Bf^{-1}\big(((y*x)*z), x\big)$,

$\alpha_3 B\big(((f^{-1}(z)\cdot f^{-1}(y))\cdot f^{-1}(x)), f^{-1}(x)\big) = \alpha_3 Bf^{-1}\big(((z*y)*x), x\big)$.

On the other hand, $\beta_1(B\big((f^{-1}(x)\cdot f^{-1}(z)), f^{-1}(y)\big)\cdot f^{-1}(x))$

$= \beta_1(Bf^{-1}\big((x*z), y\big)\cdot f^{-1}(x))$, and similarly,

$\beta_2(B\big((f^{-1}(y)\cdot f^{-1}(x)), f^{-1}(z)\big)\cdot f^{-1}(x)) = \beta_2(Bf^{-1}\big((y*x), z\big)\cdot f^{-1}(x))$ and

$\beta_3(B\big((f^{-1}(z)\cdot f^{-1}(y)), f^{-1}(x)\big)\cdot f^{-1}(x)) = \beta_3(Bf^{-1}\big((z*y), x\big)\cdot f^{-1}(x))$.

So,

$\alpha_1 Bf^{-1}\big(((x*z)*y), x\big) + \alpha_2 Bf^{-1}\big(((y*x)*z), x\big) + \alpha_3 Bf^{-1}\big(((z*y)*x), x\big) + \beta_1(Bf^{-1}\big((x*z), y\big)\cdot f^{-1}(x)) + \beta_2(Bf^{-1}\big((y*x), z\big)\cdot f^{-1}(x)) + \beta_3(Bf^{-1}\big((z*y), x\big)\cdot f^{-1}(x)) = 0$

Then, by applying f to the previous expression, we have

$\alpha_1 fBf^{-1}\big(((x*z)*y), x\big) + \alpha_2 fBf^{-1}\big(((y*x)*z), x\big) + \alpha_3 fBf^{-1}\big(((z*y)*x), x\big) + \beta_1(fBf^{-1}\big((x*z), y\big)*x) + \beta_2(fBf^{-1}\big((y*x), z\big)*x) + \beta_3(fBf^{-1}\big((z*y), x\big)*x) = 0$.

Therefore, $fBf^{-1} \in cc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\bar{\mathfrak{g}}$, which concludes the proof.

□

Corollary 4.3.6. *Let $\mathfrak{g}$ be a type Lie Malcev algebra. The dimension of the vector space $cc(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\mathfrak{g}$ is an invariant of the algebra, for all $(\alpha,\beta,\gamma,\tau)\in\mathbb{C}^4$.*

Next, taking into consideration the two following two-parameter sets $cc(0,0,\alpha,1,1,\beta)\mathfrak{g}$ and $cc(1,1,\alpha,1-1,\beta)\mathfrak{g}$ of the previous theorem, we introduce two new invariant functions for Lie type Malcev algebras as follows

Definition 4.3.7. *The two functions $\bar{\phi}^0_{\mathfrak{g}}, \bar{\phi}_{\mathfrak{g}} : \mathbb{C}^2 \mapsto \mathbb{Z}$ defined as*

$$\bar{\phi}^0_{\mathfrak{g}}(\alpha,\beta) = \dim cc(0,0,\alpha,-1,1,\beta)\mathfrak{g} \tag{4.7}$$

$$\bar{\phi}_{\mathfrak{g}}(\alpha,\beta) = \dim cc(1,1,\alpha,1,-1,\beta)\mathfrak{g} \tag{4.8}$$

are respectively called $\bar{\phi}^0_{\mathfrak{g}}$ and $\bar{\phi}_{\mathfrak{g}}$ invariant functions *corresponding to the 2-dimensional new twisted cocycles of the adjoint representation of a Lie algebra $\mathfrak{g}$.*

Taking now into consideration this definition, the concept of isomorphism of algebras and Theorem 4.3.5, it is easy to check that the following result is satisfied

Theorem 4.3.8. *If two Lie type Malcev algebras $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic, then*

1. $\bar{\phi}_{\mathfrak{g}} = \bar{\phi}_{\mathfrak{h}}$.
2. $\bar{\phi}^0_{\mathfrak{g}} = \bar{\phi}^0_{\mathfrak{h}}$.

Note that, proceeding in a similar way as we did, other invariant functions of algebras could be defined by using the rest of vector spaces that appear in Theorem 4.3.4. In the respective cases, they would be two, three or four-parameter.

Chapter 5

Some examples of proper contractions between different types of algebras

In this chapter we deal with proper contractions between different types of algebras.

5.1 Some examples of proper contractions between different types of algebras

In this section we study proper contractions from filiform Lie algebras of lower dimensions to different types of algebras. We will consider two cases: filiform Lie and Heisenberg algebras.

5.1.1 Proper contractions of 3-dimensional filiform Lie algebras

Theorem 2.7.1 allows us to know if there exists a proper contraction between 3-dimensional Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_{3,1}$, $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{3,2}$, $\mathfrak{g}_{3,3}$ and the 3-dimensional filiform Lie algebras already studied in this paper. By using the corresponding invariant functions $\psi_{3\mathfrak{g}_1}$, $\psi_{\mathfrak{g}_{3,1}}$, $\psi_{\mathfrak{g}_{2,1}\oplus\mathfrak{g}_1}$, $\psi_{\mathfrak{g}_{3,2}}$ and $\psi_{\mathfrak{g}_{3,3}}$, already calculated by Novotný and Hrivnák in [56], we obtain that

$\mathfrak{g}_1$: Abelian Lie algebra

α	$\forall\alpha\in\mathbb{C}$
$\psi_{\mathfrak{g}_1}(\alpha)$	9

$\mathfrak{g}_{3,1}$: $[e_2,e_3]=e_1$

α	$\forall\alpha\in\mathbb{C}$
$\psi_{\mathfrak{g}_{3,1}}(\alpha)$	6

$\mathfrak{g}_{2,1}\oplus\mathfrak{g}_1 : [e_1,e_2]=e_2$

α	$\forall\alpha\in\mathbb{C}\setminus\{0\}$	0
$\psi_{\mathfrak{g}_{2,1}\oplus\mathfrak{g}_1}(\alpha)$	4	6

$\mathfrak{g}_{3,2} : [e_1,e_3]=e_1, [e_2,e_3]=e_1+e_2$

α	1	$\forall\alpha\in\mathbb{C}\setminus\{1\}$
$\psi_{\mathfrak{g}_{3,2}}(\alpha)$	4	3

$\mathfrak{g}_{3,3} : [e_1,e_3]=e_1, [e_2,e_3]=e_2$

α	1	$\forall\alpha\in\mathbb{C}\setminus\{1\}$
$\psi_{\mathfrak{g}_{3,3}}(\alpha)$	6	3

As $\psi_{\mathfrak{f}_3}\le\psi_{\mathfrak{g}_1}$ and $\psi_{\mathfrak{f}_3}(1)<\psi_{\mathfrak{g}_1}(1)$, Theorem 2.7.1 assures the existence of a proper contraction from $\mathfrak{f}_3$ to $3\mathfrak{g}_1$. Analogously, the same occurs between $\mathfrak{g}_{3,2}$ and $\mathfrak{f}_3$ since $\psi_{\mathfrak{g}_{3,2}}\le\psi_{\mathfrak{f}_3}$ and $\psi_{\mathfrak{g}_{3,2}}(1)<\psi_{\mathfrak{f}_3}(1)$.

Moreover, note that $\psi_{\mathfrak{f}_3}(1)=6$ and $\psi_{\mathfrak{g}_{3,1}}(1)=6$. Therefore, the same theorem assures that there is no any proper contraction between $\mathfrak{g}_{3,1}$ and $\mathfrak{f}_3$. Similarly, the same occurs between $\mathfrak{g}_{3,3}$ and $\mathfrak{f}_3$ due to that $\psi_{\mathfrak{f}_3}(1)=6$ and $\psi_{\mathfrak{g}_{3,3}}(1)=6$.

Besides, according to Theorem 2.6.5, the algebras $\mathfrak{g}_{3,1}$ and $\mathfrak{f}_3$ are isomorphic. This implies that there is no proper contraction between themselves.

5.1.2 Proper contractions between Heisenberg algebras and filiform Lie algebras

We have just seen that there is no proper contraction between $\mathfrak{g}_{3,1}$ and $\mathfrak{f}_3$. Since $\mathfrak{g}_{3,1}$ is a 3-th Heisenberg algebra, we asked ourselves if there exists a proper contraction between a Heisenberg algebra and a filiform Lie algebra in the case of dimension five.

To deal with this question, let us consider the Heisenberg algebra of dimension 5, defined by the brackets $[e_1,e_3]=e_5$ and $[e_2,e_4]=e_5$.

We want to obtain a basis of $Der_{(\alpha,1,1)}\mathbb{H}_5$ $\forall\alpha\in\mathbb{C}$. To do this, let us consider $d\in$

$Der_{(\alpha,1,1)}\mathbb{H}_5$ and the associated matrix with the endomorphism d

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$$

By proceeding in the same way as in the subsection 3.1.1, we obtain the following conditions for the elements of the matrix

From pair (e_i, e_j)	Conditions
(e_1, e_2)	$a_{14} = a_{23}$.
(e_1, e_3)	$\alpha\, a_{51} = 0, \quad \alpha\, a_{52} = 0, \quad \alpha\, a_{53} = 0,$ $\alpha\, a_{54} = 0, \quad \alpha\, a_{55} = a_{11} + a_{33}$.
(e_1, e_4)	$a_{12} = -\, a_{43}$.
(e_1, e_5)	$a_{53} = 0$.
(e_2, e_3)	$a_{21} = -a_{34}$.
(e_2, e_4)	$a_{11} + a_{33} = a_{22} + a_{44}$.
(e_2, e_5)	$a_{54} = 0$.
(e_3, e_4)	$a_{32} = a_{41}$.
(e_3, e_5)	$a_{51} = 0$.
(e_4, e_5)	$a_{52} = 0$.

This implies that

$$dim(Der_{(\alpha,1,1)}\mathbb{H}_5) = 15, \forall \alpha \in \mathbb{C}$$

and thus

α	$\forall \alpha \in \mathbb{C}$
$\psi_{\mathbb{H}_5}(\alpha)$	15

So, since $\psi_{\mathbb{H}_5} > \psi_{\mathfrak{f}_5}$, Theorem 2.7.1 proves that there is no proper contraction between a Heisenberg algebra and a filiform Lie algebra, both of dimension 5.

5.2 Graded contractions of filiform Lie algebras

In this section we deal with graded contractions of the model filiform Lie algebras of lower dimensions. We study these contractions for the model filiform Lie algebras of dimensions $3, 4, 5$ and 6 and also for a two concrete non-model filiform Lie algebras of dimensions

6 and 8, respectively. The four first cases will allow us to deduce the expressions of the graded contractions in the general case of the the model n-dimensional filiform Lie algebra.

5.2.1 Graded contractions of the model filiform Lie algebra of dimension 3

Let $\mathfrak{f}_3 : [e_1, e_2] = e_3$ be the model filiform Lie algebra of dimension 3. A grading of $\mathfrak{f}_3$ is given by (see [4])

$$\Gamma : \mathfrak{f}_3 = \mathfrak{f}^3_{(1,0)} \oplus \mathfrak{f}^3_{(0,1)} \oplus \mathfrak{f}^3_{(1,1)}.$$

The universal group U of $\mathfrak{f}_3$ is $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ and $\mathfrak{f}^3_{(1,0)} = \langle e_1 \rangle$, $\mathfrak{f}^3_{(0,1)} = \langle e_2 \rangle$ and $\mathfrak{f}^3_{(1,1)} = \langle e_3 \rangle$.

Let us now denote

$$I = \{(1,0), (0,1), (1,1)\}$$

and let us consider a order relation on I

$$\mathcal{O} : \begin{cases} 1 & \rightarrow & (1,0), \\ 2 & \rightarrow & (0,1), \\ 3 & \rightarrow & (1,1). \end{cases}$$

Let us consider

$$\Pi_3 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \text{ with } a \in \mathbb{Z}_2,$$

and the set $G_{\Pi_3} = \{g \in Aut(\mathfrak{f}^3) \mid \exists \pi \in \Pi_3 \mid g(\mathfrak{f}^3_i) = \mathfrak{f}^3_{\pi(i)}, \text{ for all } i \in I\}$.

We define π as the mapping $I \mapsto I$, which maps an element $i \in I$ into the matrix product in I. Then, it is easy to see that G_{Π_3} is a subgroup of $Aut(\mathfrak{f}^3)$. Indeed, if $g_1, g_2 \in G_{\Pi_3}$, then there exist $\pi_1, \pi_2 \in \Pi_3$, such that $g_1(f^3_i) = f^3_{\pi_1(i)}$ and $g_2(f^3_i) = f^3_{\pi_2(i)}$, for all $i \in I$. Therefore, $(g_1 g_2^{-1}) f^3_i = g_1(g_2^{-1}(f^3_{\pi_2(\pi_2^{-1}(i))})) = g_1(f^3_{\pi_2^{-1}(i)}) = f^3_{\pi_1(\pi_2^{-1}(i))}$. Therefore, $g_1 g_2^{-1} \in G_{\Pi_3}$, and thus G_{Π_3} is a subgroup of de $Aut(\mathfrak{f}^3)$.

1 Orbits of I.

Let us recall that the concept of *orbit* is the following: If G is a group acting on a set I, the *orbit* of an element x in I is the set of elements in I to which x can be moved by the elements of G, that is, $G \cdot x = \{g \cdot x \mid g \in G\}$.

Now, let us see how we can obtain the orbits of I.

Let

$$\pi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \pi_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

be the generators of Π_3. Then

$$\pi_1((1,0)) = (1,0)\pi_1 = (1,0) \in I, \quad \pi_2((1,0)) = (1,0)\pi_2 = (1,0) \in I.$$

Therefore, $(1,0)$ represents a orbit which contains itself. Similarly, as

$$\pi_1((0,1)) = (0,1)\pi_1 = (0,1) \in I, \pi_2((0,1)) = (0,1)\pi_2 = (1,1) \in I,$$

$(0,1)$ represents a orbit containing itself and the index $(1,1)$. So, we have

Represented by the points	Orbits
$(1,0)$	$(1,0)$
$(0,1)$	$(0,1)$ and $(1,1)$

Similarly, we obtain and show in the following tables the rest of orbits.

2 Orbits of the 3 points of I_u^2

These orbits are obtained by the following definition: $\pi_i((p,q)(r,s)) = (\pi_i(p,q)\pi_i(r,s))$, for all $i \in \{1,2,3\}$. We obtain that

Orbits	Represented by the points
$((1,0)(0,1))$	$((1,0)(0,1))$ and $((1,0)(1,1))$
$((0,1)(1,1))$	$((0,1)(1,1))$

3 Orbit of the unique point of I_u^3

Similarly, these orbits are obtained by the following definition: $\pi_i((m,n)(p,q)(r,s)) = (\pi_i(m,n)\pi_i(p,q)\pi_i(r,s))$, for all $i \in \{1,2,3\}$. We have

Orbits	Represented by the points
$((1,0)(0,1)(1,1))$	$((1,0)(0,1)(1,1))$

Let us observe that this orbit contains an unique triple, which is the one of indices of the vectors which satisfy Jacobi Identity.

4 Orbit of the 3 points of $\mathcal{I}$:

The non-relevant elements of the contraction matrix ε which might be different from zero are $\varepsilon_{(1,0)(0,1)}$ and $\varepsilon_{(1,0)(1,1)}$. We obtain

Orbits	Represented by the points
$((1,0)(0,1))$	$((1,0)(0,1))$ and $((1,0)(1,1))$

However, $\varepsilon_{(1,0)(1,1)} = 0$, because $0 = [\mathfrak{f}^3_{(1,0)}, \mathfrak{f}^3_{(1,1)}]_\varepsilon = \varepsilon_{(1,0)(1,1)}\,[\mathfrak{f}^3_{(1,0)}, \mathfrak{f}^3_{(1,1)}] = \varepsilon_{(1,0)(1,1)}$ $[\mathfrak{f}^3_{\pi_2(1,0)}, \mathfrak{f}^3_{\pi_2(1,1)}] = \varepsilon_{(1,0)(1,1)} g_2\,[\mathfrak{f}^3_{(1,0)}, \mathfrak{f}^3_{(0,1)}]$, but the bracket $[\mathfrak{f}^3_{(1,0)}, \mathfrak{f}^3_{(0,1)}] \neq 0$. So, the element $\varepsilon_{(1,0)(1,1)}$ of the contraction matrix is singular.

The explicit form of the contraction matrix ε with respect to the chosen order $\mathcal{O}$ is

$$\begin{pmatrix} 0 & \varepsilon_{(1,0)(0,1)} & 0 \\ \varepsilon_{(1,0)(0,1)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we are trying to find if the relevant elements of the contraction matrix verify some particular conditions. To do this, the elements of this matrix have to satisfy the following conditions

By imposing the Jacobi identity $[X,[Y,Z]_\varepsilon]_\varepsilon + [Y,[Z,X]_\varepsilon]_\varepsilon + [Z,[X,Y]_\varepsilon]_\varepsilon = 0$, for all $X \in \mathfrak{f}^3_{(i,j)}$, $Y \in \mathfrak{f}^3_{(k,l)}$, $Z \in \mathfrak{f}^3_{(m,n)}$ and for all $((i,j)(k,l)(m,n)) \in I^3_u$, we obtain that

$$\varepsilon_{(i,j)(k+m,l+n)}\varepsilon_{(k,l)(m,n)}[X_{i,j},[X_{k,l},X_{m,n}]] + \varepsilon_{(k,l)(m+i,n+j)}\varepsilon_{(m,n)(i,j)}[X_{k,l},[X_{m,n},X_{i,j}]]$$

$$+\varepsilon_{(m,n)(i+k,j+l)}\varepsilon_{(i,j)(k,l)}[X_{m,n},[X_{i,j},X_{k,l}]] = 0, \forall((i,j)(k,l)(m,n)) \in I^3_u. \tag{5.1}$$

From this expression and for all $\pi \in \Pi_3$, it is deduced that

$\varepsilon_{\pi(i,j)\pi(k+m,l+n)}\varepsilon_{\pi(k,l)\pi(m,n)}[X_{\pi(i,j)},[X_{\pi(k,l)},X_{\pi(m,n)}]] + \varepsilon_{\pi(k,l)\pi(m+i,n+j)}\varepsilon_{\pi(m,n)\pi(i,j)}$ $[X_{\pi(k,l)},[X_{\pi(m,n)},X_{\pi(i,j)}]] + \varepsilon_{\pi(m,n)\pi(i+k,j+l)}\varepsilon_{\pi(i,j)\pi(k,l)}[X_{\pi(m,n)},[X_{\pi(i,j)},X_{\pi(k,l)}]] = 0$, for all $((i,j)(k,l)(m,n)) \in I^3_u$.

Let now $g \in G_{\Pi_3}$ be such that $g(X_{(k,l)}) = X_{\pi(k,l)}$, for all $(k,l) \in I$. We have the following restrictions for the elements of the contraction matrix

$\varepsilon_{\pi(i,j)\pi(k+m,l+n)}\varepsilon_{\pi(k,l)\pi(m,n)}\ g[X_{(i,j)},[X_{(k,l)},X_{(m,n)}]]+\varepsilon_{\pi(k,l)\pi(m+i,n+j)}\varepsilon_{\pi(m,n)\pi(i,j)}\ g[X_{(k,l)},$ $[X_{(m,n)},X_{(i,j)}]]+\varepsilon_{\pi(m,n)\pi(i+k,j+l)}\varepsilon_{(i,j)(k,l)}g[X_{(m,n)},[X_{(i,j)},X_{(k,l)}]]=0,\ \forall((i,j)(k,l)(m,n))$ $\in I_u^3$.

$g(\varepsilon_{\pi(i,j)\pi(k+m,l+n)}\varepsilon_{\pi(k,l)\pi(m,n)}\ [X_{(i,j)},[X_{(k,l)},X_{(m,n)}]]+\varepsilon_{\pi(k,l)\pi(m+i,n+j)}\varepsilon_{\pi(m,n)\pi(i,j)}\ [X_{(k,l)},$ $[X_{(m,n)},X_{(i,j)}]]\varepsilon_{\pi(m,n)\pi(i+k,j+l)}\varepsilon_{(i,j)(k,l)}[X_{(m,n)},[X_{(i,j)},X_{(k,l)}]])=0,\ \forall((i,j)(k,l)(m,n))\in$ I_u^3.

$\varepsilon_{\pi(i,j)\pi(k+m,l+n)}\varepsilon_{\pi(k,l)\pi(m,n)}\ [X_{(i,j)},[X_{(k,l)},X_{(m,n)}]]+\varepsilon_{\pi(k,l)\pi(m+i,n+j)}\varepsilon_{\pi(m,n)\pi(i,j)}[X_{(k,l)},$ $[X_{(m,n)},X_{(i,j)}]]+\varepsilon_{\pi(m,n)\pi(i+k,j+l)}\varepsilon_{(i,j)(k,l)}[X_{(m,n)},[X_{(i,j)},X_{(k,l)}]]=0,\ \forall((i,j)(k,l)(m,n))\in$ I_u^3.

These expressions allow us to obtain the restrictions which verify the rest of elements of the contraction matrix. It implies that the relevant elements $\varepsilon_{\pi(i,j)}$ are also elements of that matrix and satisfy the same conditions as $\varepsilon_{(i,j)}$.

Moreover, $[X_{i,j},[X_{k,l},X_{m,n}]]$, $[X_{m,n},[X_{i,j},X_{k,l}]]$ and $[X_{k,l},[X_{m,n},X_{i,j}]]$ are null for $\mathfrak{f}_3$. This implies that the element $\varepsilon_{(1,0)(0,1)}$ can take any complex value.

On the other hand, if $\varepsilon=(\varepsilon_{ij})$ is a contraction matrix, then we define $\tau=(\tau_{ij})$ such that $\tau_{ij}=\frac{1}{\varepsilon_{ij}}$, if $\varepsilon_{ij}\neq 0$ or $\tau_{ij}=0$, otherwise. Besides, $\varepsilon\diamond\tau$ ($\diamond$ means the Hadamard product, that is the binary operation that takes two matrices of the same dimensions, and produces another matrix where each element *pq* is the product of elements *pq* of the original two matrices) is a contraction matrix in which all non-null elements are 1. We call *normalized contraction matrix* of ε to the matrix $\varepsilon\diamond\tau$, and we denote by $N(\mathfrak{f}_3)$ to the set of all normalized contraction matrices of $\mathfrak{f}_3$. This set has 2 elements, which are the 3×3 null matrix and the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

5.2.2 Graded contractions of the model filiform Lie algebra of dimension 4

Let $\mathfrak{f}_4:[e_1,e_k]=e_{k+1}$, for $2\leq k\leq 3$ be the model filiform Lie algebra of dimension 4. A grading of $\mathfrak{f}_4$ is given by

$$\Gamma:\mathfrak{f}^4=\mathfrak{f}^4_{(1,0)}\oplus\mathfrak{f}^4_{(0,1)}\oplus\mathfrak{f}^4_{(1,1)}\oplus\mathfrak{f}^4_{(2,1)}.$$

The universal group of $\mathfrak{f}_4$ is $\mathbb{Z}_3\otimes\mathbb{Z}_2$ and $\mathfrak{f}^4_{(1,0)}=\langle e_1\rangle$, $\mathfrak{f}^4_{(0,1)}=\langle e_2\rangle$, $\mathfrak{f}^4_{(1,1)}=\langle e_3\rangle$ and $\mathfrak{f}^4_{(2,1)}=\langle e_4\rangle$.

Let us consider

$$\Pi_4 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \text{ with } a \in \mathbb{Z}_3 \text{ and } \mathrm{I} = \{(1,0),(0,1),(1,1),(2,1)\}.$$

We now consider the following order $\mathcal{O}$ on I : $1 \to (1,0)$, $2 \to (0,1)$, $3 \to (1,1)$ and $4 \to (2,1)$. By straightforward computations, we obtain that the elements of I constitute the following orbits

Represented by the points	Orbits
$(1,0)$	$(1,0)$
$(0,1)$	$(0,1),(1,1)$ and $(2,1)$

Similarly, we show in the following tables the following orbits

2 Orbits of the 6 points of I_u^2

Orbits	Represented by the points
$((1,0)(0,1))$	$((1,0)(0,1)),((1,0)(1,1))$ and $((1,0)(2,1))$
$((0,1)(1,1))$	$((0,1)(1,1)),((1,1)(2,1))$ and $((2,1)(0,1))$

3 Orbits of the 4 point of I_u^3

Orbits	Represented by the points
$((1,0)(0,1)(1,1))$	$((1,0)(0,1)(1,1)),((1,0)(1,1)(2,1))$ and $((1,0)(2,1)(0,1))$
$((0,1)(1,1)(2,1))$	$((0,1)(1,1)(2,1))$

Let us observe that these two orbits contain $\binom{4}{3} = 4$ triples, which correspond with the indices of the triples of vectors which must satisfy the Jacobi Identity.

4 Orbits of the 3 points of $\mathcal{I}$

Orbits	Represented by the points
$((1,0)(0,1))$	$((1,0)(0,1)),((1,0)(1,1))$ and $((1,0)(2,1))$

The non-relevant elements of the contraction matrix ε that might be different from zero are $\varepsilon_{(1,0)(0,1)}$, $\varepsilon_{(1,0)(1,1)}$ and $\varepsilon_{(1,0)(2,1)}$. However, a similar reasoning as in the previous case shows that the element $\varepsilon_{(1,0)(2,1)}$ is singular. So, the explicit form of the contraction matrix ε with respect to the chosen order $\mathcal{O}$ is

$$\begin{pmatrix} 0 & \varepsilon_{(1,0)(0,1)} & \varepsilon_{(1,0)(1,1)} & 0 \\ \varepsilon_{(1,0)(0,1)} & 0 & 0 & 0 \\ \varepsilon_{(1,0)(1,1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the non-null elements of this matrix have to satisfy Equation 5.1.

Moreover, from that equation is also deduced that the elements $\varepsilon_{(1,0)(0,1)}$ and $\varepsilon_{(1,0)(1,1)}$ can take any complex value, since that the three following brackets $[X_{(i,j)}, [X_{(k,l)}, X_{(m,n)}]]$, $[X_{(m,n)}, [X_{(i,j)}, X_{(k,l)}]]$ and $[X_{(k,l)}, [X_{(m,n)}, X_{(i,j)}]]$ are null for $\mathfrak{f}_4$.

By reasoning as we did in the previous dimension, we deduce that the set $N(\mathfrak{f}_4)$ of all normalized contraction matrices of $\mathfrak{f}_4$ has 4 elements, which are the following matrices

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{and the } 3 \times 3 \text{ null matrix.}$$

5.2.3 Graded contractions of the model filiform Lie algebra of dimension 5

Let $\mathfrak{f}_5 : [e_1, e_k] = e_{k+1}$, for $2 \leq k \leq 4$ be the model filiform Lie algebra of dimension 5. A grading of $\mathfrak{f}_5$ is given by

$$\Gamma : \mathfrak{f}_5 = \mathfrak{f}^5_{(1,0)} \oplus \mathfrak{f}^5_{(0,1)} \oplus \mathfrak{g}^5_{(1,1)} \oplus \mathfrak{f}^5_{(2,1)} \oplus \mathfrak{f}^5_{(3,1)}$$

By proceeding as in the previous cases we obtain the following results.

The universal group of $\mathfrak{f}_5$ is $\mathbb{Z}_4 \otimes \mathbb{Z}_2$ and $\mathfrak{f}^5_{(1,0)} = \langle e_1 \rangle$, $\mathfrak{f}^5_{(0,1)} = \langle e_2 \rangle$, $\mathfrak{f}^5_{(1,1)} = \langle e_3 \rangle$, $\mathfrak{f}^5_{(2,1)} = \langle e_4 \rangle$, $\mathfrak{f}^5_{(3,1)} = \langle e_5 \rangle$.

Let us consider

$$\Pi_5 = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, a \in \mathbb{Z}_4 \right\}, \text{ with } a \in \mathbb{Z}_3 \text{ and I} = \{(1,0), (0,1), (1,1), (2,1), (3,1)\}.$$

By considering the following order $\mathcal{O}$ on $I : \{1 \to (1,0);\ 2 \to (0,1);\ 3 \to (1,1);\ 4 \to (2,1);\ 5 \to (3,1)\}$, we obtain that the five points of I form the following orbits: $(1,0)$ represents a orbit formed by the point $(1,0)$, whereas $(0,1)$ represents a orbit formed by four points $(0,1)$, $(1,1)$, $(2,1)$, and $(3,1)$. Indeed,

Represented by the points	Orbits
$(1,0)$	$(1,0)$
$(0,1)$	$(0,1),(1,1),(2,1)$ and $(3,1)$

Similarly, we show in the following tables the following orbits

2 Orbits of the 10 points of I_u^2

Orbits	Represented by the points
$((1,0)(0,1))$	$((1,0)(0,1)),((1,0)(1,1)),((1,0)(2,1))$ and $((1,0)(3,1))$
$((0,1)(1,1))$	$((0,1)(1,1)),((1,1)(2,1)),((2,1)(3,1))$ and $((3,1)(0,1))$
$((0,1)(2,1))$	$((0,1)(2,1))),((1,1)(3,1))$

3 Orbits of the 10 points of I_u^3

Orbits	Represented by the points
$((1,0)(0,1)(1,1))$	$((1,0)(0,1)(1,1)),((1,0)(1,1)(2,1)),((1,0)(2,1)(3,1))$ and $((1,0)(3,1)(0,1))$
$((0,1)(1,1)(2,1))$	$((0,1)(1,1)(2,1)),((1,1)(2,1)(3,1)),((2,1)(3,1)(0,1))$ and $((3,1)(0,1)(1,1))$
$((1,0)(0,1)(2,1))$	$((1,0)(0,1)(2,1)),((1,0)(1,1)(3,1))$

Let us observe that these orbits contain $\binom{5}{3} = 10$ triples, which correspond with the indices of the triples of vectors which must satisfy the Jacobi Identity.

4 Orbits of the 4 points of $\mathcal{I}$:

Orbits	Represented by the points
$((1,0)(0,1))$	$((1,0)(0,1)),((1,0)(1,1)),((1,0)(2,1))$ and $((1,0)(3,1))$

The elements of the contraction matrix ε that might be different from zero are $\varepsilon_{(1,0)(0,1)}$, $\varepsilon_{(1,0)(1,1)}$, $\varepsilon_{(1,0)(2,1)}$ and $\varepsilon_{(1,0)(3,1)}$. However, a similar reasoning as in the previous cases shows that $\varepsilon_{(1,0)(3,1)} = 0$. Indeed, $0 = [\mathfrak{f}^5_{(1,0)}, \mathfrak{f}^5_{(3,1)}]_\varepsilon = \varepsilon_{(1,0)(3,1)}\, [\mathfrak{f}^5_{(1,0)}, \mathfrak{f}^5_{(3,1)}] = \varepsilon_{(1,0)(3,1)}$ $[\mathfrak{f}^5_{\pi_3(1,0)}, \mathfrak{f}^5_{\pi_3(1,1)}] = \varepsilon_{(1,0)(3,1)}\; G_3\, [\mathfrak{f}^5_{(1,0)}, \mathfrak{f}^5_{(1,1)}]$, and $[\mathfrak{f}^5_{(1,0)}, \mathfrak{f}^5_{(1,1)}] \neq 0$. It implies that $\varepsilon_{(1,0)(3,1)} = 0$. So, the explicit form of the contraction matrix ε with respect to the chosen order $\mathcal{O}$ is

$$\begin{pmatrix} 0 & \varepsilon_{(1,0)(0,1)} & \varepsilon_{(1,0)(1,1)} & \varepsilon_{(1,0)(2,1)} & 0 \\ \varepsilon_{(1,0)(0,1)} & 0 & 0 & 0 & 0 \\ \varepsilon_{(1,0)(1,1)} & 0 & 0 & 0 & 0 \\ \varepsilon_{(1,0)(2,1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the non-null elements of this matrix have to satisfy Equation 5.1.

Moreover, the three following brackets $[X_{(i,j)}, [X_{(k,l)}, X_{(m,n)}]]$, $[X_{(m,n)}, [X_{(i,j)}, X_{(k,l)}]]$ and $[X_{(k,l)}, [X_{(m,n)}, X_{(i,j)}]]$ are null for $\mathfrak{f}_5$. This implies that the elements $\varepsilon_{(1,0)(0,1)}$, $\varepsilon_{(1,0)(1,1)}$ and $\varepsilon_{(1,0)(2,1)}$ can take any complex value.

Then, by reasoning as we did in the previous dimension, we deduce that the set $N(\mathfrak{f}_5)$ of all normalized contraction matrices of $\mathfrak{f}_5$ has $2^3 = 8$ elements. These are the following

$$\begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&1&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&1&1&0\\0&0&0&0&0\\1&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&1&0&1&0\\1&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

5.2.4 Graded contractions of the model filiform Lie algebra of dimension 6

We finished the study of the graded contractions of model filiform Lie algebras with the 6-dimensional case. It will allow us to obtain some general conclusions in the next subsection.

Let $\mathfrak{f}_6 : [e_1, e_k] = e_{k+1}$, for $2 \leq k \leq 5$ be the model filiform Lie algebra of dimension 6. A grading of $\mathfrak{f}_6$ is given by

$$\Gamma : \mathfrak{f}_6 = \mathfrak{f}^6_{(1,0)} \oplus \mathfrak{f}^6_{(0,1)} \oplus \mathfrak{f}^6_{(1,1)} \oplus \mathfrak{f}^6_{(2,1)} \oplus \mathfrak{f}^6_{(3,1)} \oplus \mathfrak{f}^6_{(4,1)}$$

By proceeding as in the previous cases we obtain the following results.

The universal group of $\mathfrak{f}_6$ is $\mathbb{Z}_5 \otimes \mathbb{Z}_2$ and $\mathfrak{f}^6_{(1,0)} = \langle e_1 \rangle$, $\mathfrak{f}^6_{(0,1)} = \langle e_2 \rangle$, $\mathfrak{f}^6_{(1,1)} = \langle e_3 \rangle$, $\mathfrak{f}^6_{(2,1)} = \langle e_4 \rangle$, $\mathfrak{f}^6_{(3,1)} = \langle e_5 \rangle$, $\mathfrak{f}^6_{(4,1)} = \langle e_6 \rangle$.

Let us consider

$$\Pi_6 = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, a \in \mathbb{Z}_5 \right\}, \text{ with } a \in \mathbb{Z}_3 \text{ and I} = \{((1,0),(0,1),(1,1),(2,1),(3,1),(4,1)\}.$$

If we consider the order $\mathcal{O}$ on $I : \{1 \to (1,0);\ 2 \to (0,1);\ 3 \to (1,1);\ 4 \to (2,1);\ 5 \to (3,1);\ 6 \to (4,1)\}$, we obtain that the orbits formed by the 6 points of I are: $(1,0)$ represents a orbit formed by the point $(1,0)$, whereas $(0,1)$ represents a orbit formed by four points $(0,1)$, $(1,1)$, $(2,1)$, $(3,1)$ and $(4,1)$. The elements of I form the following orbits

Represented by the points	Orbits
$(1,0)$	$(1,0)$
$(0,1)$	$(0,1),(1,1),(2,1),(3,1)$ and $(4,1)$

Similarly, in the following tables are shown the following orbits

2 Orbits of the 15 points of I_u^2

Orbits	Represented by the points
$((1,0)(0,1))$	$((1,0)(0,1)),((1,0)(1,1)),((1,0)(2,1)),((1,0)(3,1))$ and $((1,0)(4,1))$
$((0,1)(1,1))$	$((0,1)(1,1)),((1,1)(2,1)),((2,1)(3,1)),((3,1)(4,1))$ and $((4,1)(0,1))$
$((0,1)(2,1))$	$((0,1)(2,1))),((1,1)(3,1)),((2,1)(4,1)),((3,1)(0,1))$ and $((4,1)(1,1))$

3 Orbits of the 20 points of I_u^3

Orbits	Represented by the points
$((1,0)(0,1)(1,1))$	$((1,0)(0,1)(1,1)),((1,0)(1,1)(2,1)),((1,0)(2,1)(3,1)),$ $((1,0)(3,1)(4,1))$ and $((1,0)(4,1)(0,1))$
$((1,0)(0,1)(2,1))$	$((1,0)(0,1)(2,1)),((1,0)(1,1)(3,1)),((1,0)(2,1)(4,1)),$ $((1,0)(3,1)(0,1))$ and $((1,0)(4,1)(1,1))$
$((0,1)(1,1)(2,1))$	$((0,1)(1,1)(2,1)),((1,1)(2,1)(3,1)),((2,1)(3,1)(4,1)),$ $((3,1)(4,1)(0,1))$ and $((4,1)(0,1)(1,1))$
$((0,1)(1,1)(3,1))$	$((0,1)(1,1)(3,1)),((1,1)(2,1)(4,1)),((2,1)(3,1)(0,1)),$ $((3,1)(4,1)(1,1))$ and $((4,1)(0,1)(2,1))$

Let us observe that these orbits contain $\binom{6}{3} = 20$ triples, which correspond with the indices of the triples of vectors which must satisfy the Jacobi Identity.

4 Orbits of the 5 points of $\mathcal{I}$:

Orbits	Represented by the points
$((1,0)(0,1))$	$((1,0)(0,1)),((1,0)(1,1)),((1,0)(2,1)),((1,0)(3,1))$ and $((1,0)(4,1))$

The elements of the contraction matrix ε that might be different from zero are $\varepsilon_{(1,0)(0,1)}$, $\varepsilon_{(1,0)(1,1)}$, $\varepsilon_{(1,0)(2,1)}$, $\varepsilon_{(1,0)(3,1)}$ and $\varepsilon_{(1,0)(4,1)}$. The resting elements are null. However, we have that $\varepsilon_{(1,0)(4,1)} = 0$, due to that $0 = [\mathfrak{g}^6_{(1,0)}, \mathfrak{g}^6_{(4,1)}]_\varepsilon = \varepsilon_{(1,0)(4,1)} \ [\mathfrak{g}^6_{(1,0)}, \mathfrak{g}^6_{(4,1)}]$

$\varepsilon_{(1,0)(4,1)}[\mathfrak{g}^6_{\pi_4(1,0)}, \mathfrak{g}^6_{\pi_4(1,1)}] = \varepsilon_{(1,0)(3,1)} G_4[\mathfrak{g}_{(1,0)^6}, \mathfrak{g}^6_{(1,1)}]$, but $[\mathfrak{g}^6_{(1,0)}, \mathfrak{g}^6_{(1,1)}] \neq 0$. So, the explicit form of the contraction matrix ε with respect to the chosen order $\mathcal{O}$ is

$$\begin{pmatrix} 0 & \varepsilon_{(1,0)(0,1)} & \varepsilon_{(1,0)(1,1)} & \varepsilon_{(1,0)(2,1)} & \varepsilon_{(1,0)(3,1)} & 0 \\ \varepsilon_{(1,0)(0,1)} & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{(1,0)(1,1)} & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{(1,0)(2,1)} & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{(1,0)(3,1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the non-null elements of this matrix have to satisfy Equation 5.1.

Moreover, $[X_{i,j}, [X_{k,l}, X_{m,n}]]$, $[X_{m,n}, [X_{i,j}, X_{k,l}]]$ and $[X_{k,l}, [X_{m,n}, X_{i,j}]]$ are null for $\mathfrak{f}_6$. This implies que the elements $\varepsilon_{(1,0)(0,1)}$, $\varepsilon_{(1,0)(0,1)}$, $\varepsilon_{(1,0)(1,1)}$, $\varepsilon_{(1,0)(2,1)}$ y $\varepsilon_{(1,0)(3,1)}$ can take any complex value.

Then, by reasoning as we did in the previous dimensions, we deduce that the set of all normalized contraction matrices of $\mathfrak{f}_6$ has $2^4 = 16$ elements. These matrices are the following

$$\begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&1&1&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&1&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&1&1&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1&1&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us observe that as a consequence of the results obtained in the previous sections, the general n-dimensional case can be also dealt with in a similar way. We do it in the following subsection.

5.2.5 Graded contractions of the n-dimensional filiform Lie algebra

Let $\mathfrak{f}_n : [e_1, e_k] = e_{k+1}$, for $2 \leq k \leq n-1$ be the model filiform Lie algebra of dimension n. A grading of $\mathfrak{f}_n$ is given by

$$\Gamma : \mathfrak{f}_n = \mathfrak{f}^n_{(1,0)} \oplus \mathfrak{f}^n_{(0,1)} \oplus \mathfrak{f}^n_{(1,1)} \oplus \mathfrak{f}^n_{(2,1)} \oplus \mathfrak{f}^n_{(3,1)} \oplus \mathfrak{f}^n_{(4,1)} \oplus \ldots \oplus \mathfrak{f}^n_{(n-2,1)}.$$

The universal group of $\mathfrak{f}_n$ is $\mathbb{Z}_{n-1} \otimes \mathbb{Z}_2$ and $\mathfrak{f}^n_{(1,0)} = \langle e_1 \rangle$, $\mathfrak{f}^n_{(0,1)} = \langle e_2 \rangle$, $\mathfrak{f}^n_{(1,1)} = \langle e_3 \rangle$, $\mathfrak{f}^n_{(2,1)} = \langle e_4 \rangle$, $\mathfrak{f}^n_{(3,1)} = \langle e_5 \rangle$, $\mathfrak{f}^n_{(4,1)} = \langle e_6 \rangle, \ldots, \mathfrak{f}^n_{(n-2,1)} = \langle e_n \rangle$.

Let us consider

$\Pi_n = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, with $a \in \mathbb{Z}_{n-1}$ and $\mathrm{I} = \{(1,0),(0,1),(1,1),(2,1),(3,1),\ldots,(n-2,1)\}$.

We now consider the following order $\mathcal{O}$ on I : $1 \to (1,0)$, $2 \to (0,1)$, $3 \to (1,1)$, $4 \to (2,1)$, $\ldots$, and $n \to (n-2,1)$ and starting from this point and by using any symbolic computation package for computations we proceed in the same way as the indicated in the previous particular cases. Indeed, we obtain the orbits of the points of I, I_u^2 and I_u^3 and consider the elements of the $n \times n$ contraction matrix ε that might be different from zero.

These elements are $\varepsilon_{(1,0)(0,1)}$, $\varepsilon_{(1,0)(1,1)}$, $\varepsilon_{(1,0)(2,1)}$, $\varepsilon_{(1,0)(3,1)}, \ldots$ and $\varepsilon_{(1,0)(n-2\ 1)}$. The resting elements are null. Moreover, we find that $\varepsilon_{(1,0)(n-2\ 1)} = 0$. It allows us to obtain the explicit form of the contraction matrix ε with respect to the chosen order $\mathcal{O}$.

Let us recall that for any n-dimensional filiform Lie algebra, the elements, the non-null ones have to verify the following conditions (Equation 5.1)

$$\varepsilon_{(i,j)(k+m,l+n)}\varepsilon_{(k,l)(m,n)}[X_{i,j},[X_{k,l},X_{m,n}]] + \varepsilon_{(k,l)(m+i,n+j)}\varepsilon_{(m,n)(i,j)}[X_{k,l},[X_{m,n},X_{i,j}]]$$
$$+ \varepsilon_{(m,n)(i+k,j+l)}\varepsilon_{(i,j)(k,l)}[X_{m,n},[X_{i,j},X_{k,l}]] = 0, \forall((i,j)(k,l)(m,n)) \in I_u^3.$$

Moreover, $[X_{i,j},[X_{k,l},X_{m,n}]]$, $[X_{m,n},[X_{i,j},X_{k,l}]]$ and $[X_{k,l},[X_{m,n},X_{i,j}]]$ are null for $\mathfrak{f}_n$. This implies that the elements $\varepsilon_{(1,0)(0,1)}$, $\varepsilon_{(1,0)(0,1)}$, $\varepsilon_{(1,0)(1,1)}$, $\varepsilon_{(1,0)(2,1)}$, $\ldots$, $\varepsilon_{(1,0)(n-3\,1)}$ can take any complex value, which allow us to obtain the conclusions in each dimension.

In any case, $\varepsilon \diamond \tau$ is the contraction matrix in which all the non-null elements are 1 and we find that the set $N(\mathfrak{g}^n)$ of all normalized contraction matrices of $\mathfrak{f}^n$ has 2^{n-2} elements.

Now, we are going to study the particular case of two non-model filiform Lie algebras.

5.2.6 Graded contractions of the 6-dimensional non-model filiform Lie algebra Q_6

Let $\mathcal{Q}_6$ be a 6-dimensional non model filiform Lie algebra defined by the law $[e_1,e_k] = e_{k+1}$, for $2 \leq k \leq 4$, $[e_2,e_5] = -e_6$ and $[e_3,e_4] = e_6$. A grading of $\mathcal{Q}_6$ is given by

$$\Gamma : \mathcal{Q}_6 = Q^6_{(1,0)} \oplus Q^6_{(0,1)} \oplus Q^6_{(1,1)} \oplus Q^6_{(2,1)} \oplus Q^6_{(3,1)} \oplus Q^6_{(3,2)}.$$

By proceeding in a similar way as in previous sections, we have that the universal group of $\mathcal{Q}_6$ is $\mathbb{Z}_4 \otimes \mathbb{Z}_3$. Besides, $\mathcal{Q}^6_{(1,0)} = \langle e_1 + e_2\rangle$, $Q_{(0,1)} = \langle e_2\rangle$, $\mathcal{Q}^6_{(1,1)} = \langle e_3\rangle$, $\mathcal{Q}^6_{(2,1)} = \langle e_4\rangle$,

$\mathcal{Q}^6_{(3,1)} = \langle e_5 \rangle$ and $\mathcal{Q}^6_{(3,2)} = \langle e_6 \rangle$.

Let now consider

$$H_{\Pi_6} = \begin{pmatrix} 1 & 0 \\ 2\,a & 1 \end{pmatrix}, \text{ with } a \in \mathbb{Z}_4 \text{ and } \mathrm{I} = \{(1,0),(0,1),(1,1),(2,1),(3,1),(3,2)\}.$$

The considered order $\mathcal{O}$ on I is $\{1 \to (1,0);\, 2 \to (0,1);\, 3 \to (1,1);\, 4 \to (2,1);\, 5 \to (3,1);\, 6 \to (3,2)\}$. The tables now obtained for the elements of I and the orbits of the points of I_u^2 and I_u^3 are the following

Represented by the points	Orbits
$(1,0)$	$(1,0)$
$(0,1)$	$(0,1),(2,1)$
$(1,1)$	$(1,1),(3,1)$
$(3,2)$	$(3,2)$

Orbits of the 15 points of I_u^2

Orbit	Represented by the points
$((1,0)(0,1))$	$((1,0)(0,1)),((1,0)(2,1))$
$((1,0)(1,1))$	$((1,0)(1,1)),((1,0)(3,1))$
$((1,0)(3,2))$	$((1,0)(3,2))$
$((0,1)(1,1))$	$((0,1)(1,1)),((2,1)(3,1))$
$((0,1)(2,1))$	$((0,1)(2,1))$
$((0,1)(3,1))$	$((0,1)(3,1)),((2,1)(1,1))$
$((0,1)(3,2))$	$((0,1)(3,2)),((2,1)(3,1))$
$((1,1)(3,2))$	$((1,1)(3,2)),((3,1)(3,2))$
$((1,1)(3,1))$	$((1,1)(3,1))$

Orbits of the 20 points of I_u^3

Orbit	Represented by the points
$((1,0)(0,1)(1,1))$	$((1,0)(0,1)(1,1)),((1,0)(2,1)(3,1))$
$((1,0)(0,1)(2,1))$	$((1,0)(0,1)(2,1))$
$((1,0)(0,1)(3,1))$	$((1,0)(0,1)(3,1)),((1,0)(2,1)(1,1))$
$((1,0)(1,1)(3,1))$	$((1,0)(1,1)(3,1))$
$((1,0)(1,1)(3,2))$	$((1,0)(1,1)(3,2)),((1,0)(3,1)(3,2))$
$((1,0)(0,1)(3,2))$	$((1,0)(0,1)(3,2)),((1,0)(2,1)(3,2))$
$((0,1)(1,1)(2,1))$	$((0,1)(1,1)(2,1)),((2,1)(3,1)(0,1))$
$((0,1)(1,1)(3,1))$	$((0,1)(1,1)(3,1)),((2,1)(3,1)(1,1))$
$((0,1)(1,1)(3,2))$	$((0,1)(1,1)(3,2)),((2,1)(3,1)(3,2))$
$((0,1)(3,1)(3,2))$	$((0,1)(3,1)(3,2)),((2,1)(1,1)(3,2))$
$((1,1)(3,1)(3,2))$	$((1,1)(3,1)(3,2))$
$((0,1)(2,1)(3,2))$	$((0,1)(2,1)(3,2))$

Orbits of the 6 points of $\mathcal{I}$:

Orbit	Represented by the points
$((1,0)(0,1))$	$((1,0)(0,1)),((1,0)(1,1)),((1,0)(2,1)),((1,0)(3,1))$ and $((2,1)(1,1))$

The explicit form of the contraction matrix ε with respect to chosen order $\mathcal{O}$ is

$$\begin{pmatrix} 0 & \varepsilon_{(1,0)(0,1)} & \varepsilon_{(1,0)(1,1)} & \varepsilon_{(1,0)(2,1)} & 0 & 0 \\ \varepsilon_{(1,0)(0,1)} & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{(1,0)(1,1)} & 0 & 0 & \varepsilon_{(1,1)(2,1)} & 0 & 0 \\ \varepsilon_{(1,0)(2,1)} & 0 & \varepsilon_{(1,1)(2,1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The conditions for the non-null elements of the contraction matrix are

$$\varepsilon_{(i,j)(k+m,l+n)}\varepsilon_{(k,l)(m,n)}[X_{ij},[X_{kl},X_{mn}]]+\varepsilon_{(k,l)(m+i,n+j)}\varepsilon_{(m,n)(i,j)}[X_{kl},[X_{mn},X_{ij}]]$$
$$+\varepsilon_{(m,n)(i+k,j+l)}\varepsilon_{(i,j)(k,l)}[X_{mn},[X_{ij},X_{kl}]]=0,\forall((i,j)(k,l)(m,n))\in I_u^3.$$

In this case, the following restriction $\varepsilon_{(1,0)(0,1)}\varepsilon_{(1,1)(2,1)}=0$ is obtained.

Moreover, the elements of any contraction matrix verify the following conditions

- If $\varepsilon_{(1,0)(0,1)} \neq 0$, then $\varepsilon_{(1,1)(2,1)} = 0$ and the parameters $\varepsilon_{(1,0)(1,1)}$ and $\varepsilon_{(1,0)(2,1)}$ could be null. So, there are 2^2 different types of contraction matrices with $\varepsilon_{(1,0)(0,1)} \neq 0$.

- If $\varepsilon_{(1,0)(0,1)} = 0$, then the parameters $\varepsilon_{(1,0)(1,1)}$, $\varepsilon_{(1,0)(2,1)}$ and $\varepsilon_{(1,1)(2,1)}$ could be null. So, there are 2^3 different types of contraction matrices with $\varepsilon_{(1,0)(0,1)} = 0$.

If $\varepsilon = (\varepsilon_{ij})$ is a contraction matrix, then $\tau = (\tau_{ij})$, such that $\tau_{ij} = \frac{1}{\varepsilon_{ij}}$ if $\varepsilon_{ij} \neq 0$ or $\tau_{ij} = 0$, if $\varepsilon_{ij} = 0$ is also a contraction matrix. Moreover, $\varepsilon \diamond \tau$ is the contraction matrix in which all the non-null elements are 1. That matrix is the normalized contraction matrix of ε. If we denote by $N(\mathcal{Q}_6)$ to the set of all normalized contraction matrices of $\mathcal{Q}_6$, we find that this set has 12 elements. They are the following

$$
\begin{pmatrix} 0&1&1&1&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&1&1&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&1&0&1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},
$$

$$
\begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&0&1&1&0&0\\ 0&0&0&0&0&0\\ 1&0&0&1&0&0\\ 1&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 1&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},
$$

$$
\begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&0&1&1&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},
$$

$$
\begin{pmatrix} 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}.
$$

5.2.7 Graded contractions of the 8-dimensional filiform Lie algebra Q_8

Let $\mathcal{Q}_8$ be a 8-dimensional non model filiform Lie algebra defined by the law $[e_1, e_k] = e_{k+1}$, for $2 \leq k \leq 6$, $[e_2, e_7] = -e_8$, $[e_3, e_6] = e_8$, and $[e_4, e_5] = -e_8$.

A grading of $\mathcal{Q}_8$ is given by

$$\Gamma = Q^8_{(1,0)} \oplus Q^8_{(0,1)} \oplus Q^8_{(1,1)} \oplus Q^8_{(2,1)} \oplus Q^8_{(3,1)} \oplus Q^8_{(4,1)} \oplus Q^8_{(5,1)} \oplus Q^8_{(5,2)}.$$

Now, by proceeding in a similar way as in the previous procedures, we have that the universal group of $\mathcal{Q}_8$ is $\mathbb{Z}_6 \otimes \mathbb{Z}_3$. Besides, $Q^8_{(1,0)} = \langle e_1 + e_2 \rangle$, $Q^8_{(0,1)} = \langle e_2 \rangle$, $Q^8_{(1,1)} = \langle e_3 \rangle$, $Q^8_{(2,1)} = \langle e_4 \rangle$, $Q^8_{(3,1)} = \langle e_5 \rangle$, $Q^8_{(4,1)} = \langle e_6 \rangle$, $Q^8_{(5,1)} = \langle e_7 \rangle$ and $Q^8_{(5,2)} = \langle e_8 \rangle$.

Let now consider

$$H_{\Pi_8} = \begin{pmatrix} 1 & 0 \\ 3a & 1 \end{pmatrix}, \text{ with } a \in \mathbb{Z}_2 \text{ and } \mathrm{I} = \{(1,0), (0,1), (1,1), (2,1), (3,1), (4,1), (5,1), (5,2)\}.$$

The considered order $\mathcal{O}$ on I is $\{1 \to (1,0), 2 \to (0,1), 3 \to (1,1), 4 \to (2,1), 5 \to (3,1), 6 \to (4,1), 7 \to (5,1), 8 \to (5,2).\}$

The tables now obtained for the elements of I and the orbits of the points of I^2_u and I^3_u are the following

Orbits of the elements of I	
Orbits	Points
$(1,0)$	$(1,0)$
$(0,1)$	$(0,1),(3,1)$
$(1,1)$	$(1,1),(4,1)$
$(2,1)$	$(2,1),(5,1)$
$(5,2)$	$(5,2)$

Orbits of the elements of I_u^2	
Orbits	Points
$((1,0)(0,1))$	$((1,0)(0,1)),((1,0)(3,1))$
$((1,0)(1,1))$	$((1,0)(1,1)),((1,0)(4,1))$
$((1,0)(2,1))$	$((1,0)(2,1)),((1,0)(5,1))$
$((1,0)(5,2))$	$((1,0)(5,2))$
$((1,1)(2,1))$	$((1,1)(2,1)),((4,1)(5,1))$
$((1,1)(3,1))$	$((1,1)(3,1)),((4,1)(0,1))$
$((1,1)(5,1))$	$((1,1)(5,1)),((4,1)(2,1))$
$((1,1)(5,2))$	$((1,1)(5,2)),((4,1)(5,2))$
$((0,1)(2,1))$	$((0,1)(2,1)),((3,1)(5,1))$
$((0,1)(3,1))$	$((3,1)(0,1))$
$((0,1)(5,1))$	$((0,1)(5,1)),((3,1)(2,1))$
$((0,1)(5,2))$	$((0,1)(5,2)),((3,1)(5,2))$
$((2,1)(5,1))$	$((2,1)(5,1)),((5,1)(2,1))$
$((2,1)(5,2))$	$((2,1)(5,2)),((5,1)(5,2))$
$((3,1)(4,1))$	$((3,1)(4,1)),((0,1)(1,1))$

Orbits of the elements of $\mathcal{I}$	
Orbits	Points
$((1,0)(3,1))$	$((1,0)(3,1)),((1,0)(0,1))$
$((0,1)(5,1))$	$((0,1)(5,1)),((3,1)(2,1))$
$((1,1)(4,1))$	$((1,1)(4,1))$
$((1,0)(1,1))$	$((1,0)(1,1)),((1,0)(4,1))$
$((1,1)(4,1))$	$((1,1)(4,1))$
$((1,0)(2,1))$	$((1,0)(2,1)),((1,0)(5,1))$

Orbits of the elements of I_u^3	
Orbits	Points
$((1,0)(0,1)(1,1))$	$((0,1)(1,0)(1,1)),((1,0)(3,1)(4,1))$
$((0,1)(1,0)(2,1))$	$((0,1)(1,0)(2,1)),((1,0)(3,1)(5,1))$
$((0,1)(1,0)(3,1))$	$((0,1)(1,0)(3,1))$
$((0,1)(1,0)(4,1))$	$((0,1)(1,0)(4,1)),((1,0)(1,1)(3,1))$
$((0,1)(1,0)(5,1))$	$((0,1)(1,0)(5,1)),((1,0)(2,1)(3,1))$
$((0,1)(1,0)(5,2))$	$((0,1)(1,0)(5,2)),((1,0)(3,1)(5,2))$
$((1,0)(1,1)(2,1))$	$((1,0)(1,1)(2,1)),((1,0)(4,1)(5,1))$
$((1,0)(1,1)(4,1))$	$((1,0)(1,1)(4,1))$
$((1,0)(1,1)(5,1))$	$((1,0)(1,1)(5,1)),((1,0)(2,1)(4,1))$
$((1,0)(1,1)(5,2))$	$((1,0)(1,1)(5,2)),((1,0)(4,1)(5,2))$
$((1,0)(2,1)(5,1))$	$((1,0)(2,1)(5,1))$
$((1,0)(2,1)(5,2))$	$((1,0)(2,1)(5,2)),((1,0)(5,1)(5,2))$
$((0,1)(1,1)(2,1))$	$((0,1)(1,1)(2,1)),((3,1)(4,1)(5,1))$
$((0,1)(1,1)(3,1))$	$((0,1)(1,1)(3,1)),((0,1)(3,1)(4,1))$
$((0,1)(1,1)(41))$	$((0,1)(1,1)(4,1)),((1,1)(3,1)(4,1))$
$((0,1)(1,1)(5,1))$	$((0,1)(1,1)(5,1)),((2,1)(3,1)(4,1))$
$((0,1)(1,1)(5,2))$	$((0,1)(1,1)(5,2)),((3,1)(4,1)(5,2))$
$((0,1)(2,1)(3,1))$	$((0,1)(2,1)(3,1)),((0,1)(3,1)(5,1))$
$((0,1)(2,1)(4,1))$	$((0,1)(2,1)(4,1)),((1,1)(3,1)(5,1))$
$((0,1)(2,1)(5,1))$	$((0,1)(2,1)(5,1)),((2,1)(3,1)(5,1))$
$((0,1)(2,1)(5,2))$	$((0,1)(2,1)(5,2)),((3,1)(5,1)(5,2))$
$((0,1)(3,1)(5,2))$	$((0,1)(3,1)(5,2))$
$((0,1)(4,1)(5,1))$	$((0,1)(4,1)(5,1)),((1,1)(3,1)(2,1))$
$((0,1)(4,1)(5,2))$	$((0,1)(4,1)(5,2)),((1,1)(3,1)(5,2))$
$((0,1)(5,1)(5,2))$	$((0,1)(5,1)(5,2)),((2,1)(3,1)(5,2))$
$((1,1)(2,1)(4,1))$	$((1,1)(2,1)(4,1)),((1,1)(4,1)(5,1))$
$((1,1)(2,1)(5,2))$	$((1,1)(2,1)(5,2)),((4,1)(5,1)(5,2))$
$((1,1)(4,1)(5,2))$	$((1,1)(4,1)(5,2))$
$((1,1)(5,1)(5,2))$	$((1,1)(5,1)(5,2))((2,1)(4,1)(5,2))$
$((2,1)(4,1)(5,1))$	$((2,1)(4,1)(5,1)),((1,1)(4,1)(5,2))$
$((2,1)(5,1)(5,2))$	$((2,1)(5,1)(5,2))$

Taking into account Equation 5.1 and that the following brackets $[X_{(i,j)},[X_{(k,l)},X_{(m,n)}]]$, $[X_{(m,n)},[X_{(i,j)},X_{(k,l)}]]$ and $[X_{(k,l)},[X_{(m,n)},X_{(i,j)}]]$ are null for Q_8, the elements which could be null are $\varepsilon_{(1,0)(3,1)}$, $\varepsilon_{(0,1)(5,1)}$, $\varepsilon_{(3,1)(2,1)}$, $\varepsilon_{(1,1)(4,1)}$, $\varepsilon_{(1,0)(1,1)}$, $\varepsilon_{(1,0)(0,1)}$ and $\varepsilon_{(1,1)(4,1)}$.

Moreover, if $\varepsilon_{(0,1)(5,1)}$ or $\varepsilon_{(3,1)(2,1)}$ are null, then they will be null simultaneously. And it also occurs with $\varepsilon_{(1,0)(3,1)}$ and $\varepsilon_{(1,0)(0,1)}$ and with $\varepsilon_{(1,0)(1,1)}$ and $\varepsilon_{(1,1)(4,1)}$. Apart from that, $[x_i, [[x_j, x_k]]]$, for all $(i, j, k) \in I_u^3$, which implies that (Equation 5.1) is satisfied.

Then, the explicit form of the contraction matrix ε with respect to the chosen order $\mathcal{O}$ is

$$\begin{pmatrix} 0 & \varepsilon_{(1,0)(0,1)} & \varepsilon_{(1,0)(1,1)} & 0 & \epsilon_{(1,0)(3,1)} & \varepsilon_{(1,0)(4,1)} & 0 & 0 \\ \varepsilon_{(1,0)(0,1)} & 0 & 0 & 0 & 0 & 0 & \varepsilon_{(0,1)(5,1)} & 0 \\ \varepsilon_{(1,0)(1,1)} & 0 & 0 & 0 & 0 & \epsilon_{(1,1)(4,1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{(2,1)(3,1)} & 0 & 0 & 0 \\ \varepsilon_{(1,0)(3,1)} & 0 & 0 & \varepsilon_{(2,1)(3,1)} & 0 & 0 & 0 & 0 \\ \varepsilon_{(1,0)(4,1)} & 0 & \varepsilon_{(1,1)(4,1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{(0,1)(5,1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Just like before, if $\varepsilon = (\varepsilon_{ij})$ is a contraction matrix, them $\tau = (\tau_{ij})$, such that $\tau_{ij} = \frac{1}{\varepsilon_{ij}}$ if $\varepsilon_{ij} \neq 0$ or $\tau_{ij} = 0$, if $\varepsilon_{ij} = 0$ is also contraction matrix. Moreover, $\varepsilon \diamond \tau$ is the contraction matric in which all the non-null elements are 1. That matrix is the normalized contraction matrix of ε, If we denote by $N(Q_8)$ to the set of all normalized contraction matrices of Q_8, we find that this set has $2^4 = 16$ elements. They are, written with a smaller letter than the normal one for reasons of not wasting space, the following

$$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0&1&1&0&0 \\ 1&0&0&0&0&0&1&0 \\ 1&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0 \\ 1&0&0&1&0&0&0&0 \\ 1&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&0&1&0&0&0 \\ 1&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 1&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&1&0&0&0 \\ 1&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0 \\ 1&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0&1&1&0&0 \\ 1&0&0&0&0&0&1&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 1&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&1&0&1&1&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 1&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0&1&1&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Obviously, it is not possible to generalize any result starting from only two particular cases treated, but it could be convenient to indicate that according to the results obtained for these two cases, it could be conjectured that the number of normalized contraction matrices of for the filiform Lie algebras belonging to the family Q_n, defined by the brackets $[e_1, e_k] = e_{k+1}$, for $2 \leq k \leq n-1$, $[e_2, e_{n-1}] = -e_n$, $[e_3, e_{n-2}] = e_n$, $[e_4, e_{n-3}] = -e_n$ and so on, could be $2n$.

Chapter 6

Kinematical Lie algebras

In this chapter we deal with the kinematical Lie algebras of four-dimensional spacetime, introduced by Tolar in [65]. Particularly, we compute the one-parameter invariant function υ already considered for the eight kinematical Lie algebras by Tolar .

6.1 Kinematical Lie algebras of four-dimensional spacetime

The concept of *kinematical (or relativity)* groups has a fundamental importance since through these symmetry groups of spacetimes the basic invariance of the laws of physics can be implemented. The paradigm is the special theory of relativity, with the ten-parameter Poincaré group containing (as transformation group of the four-dimensional Minkowski spacetime) the time and space translations, space rotations, and boosts (inertial transformations).

The possible Lie algebras L of kinematical groups were classified by Tolar (see [65]) under the following natural physical assumptions

1. L is a real ten-dimensional Lie algebra and its generators correspond to time translations (H), space translations (P_i), space rotations (J_i), and inertial transformations (K_i), $i = 1, 2, 3$.

2. Rotational invariance of space imposes special transformation properties of the generators under rotations.

3. Space inversion and time-reversal transformations are automorphisms of L.

4. Inertial transformations in any given direction form noncompact one-parameter subgroups of the kinematical group.

The ten "kinematical" generators of L satisfy the following commutation relations

$$\begin{array}{llll} [\mathrm{J},\mathrm{J}]=\text{-}\mathrm{J}, & [\mathrm{J},\mathrm{H}]=0, & [\mathrm{J},\mathrm{P}]=\text{-}\mathrm{P}, & [\mathrm{J},\mathrm{K}]=\text{-}\mathrm{K}, \\ & [\mathrm{H},\mathrm{H}]=0, & [\mathrm{H},\mathrm{P}]=\mathrm{K}, & [\mathrm{H},\mathrm{K}]=\text{-}\mathrm{P}, \\ & & [\mathrm{P},\mathrm{P}]=\text{-}\mathrm{J}, & [\mathrm{P},\mathrm{K}]=\mathrm{H}, \\ & & & [\mathrm{K},\mathrm{K}]=\text{-}\mathrm{J}, \end{array} \tag{6.1}$$

where the notation $[A, B] = C$, for $[A_i, B_j] = \varepsilon_{ijk}\, C_k$, $[A, B] = D$, for $[A_i, B_j] = \delta_{ij}\, D$ and $[A, D] = B$, for $[A_i, D] = B_i$ is used, where ε_{ijk} denotes the Levi-Civita symbol.

These generators joint with the commutation law constitute the Lie algebra B_2, which is dealt with as a graded contraction in [65].

The commuting involutive automorphisms of space-inversion Π and time reversal Θ induce two Z_2-gradings

$$\Pi : B_2 = \mathrm{span}\{J, H\} \oplus \mathrm{span}\{P, K\},$$

$$\Theta : B_2 = \mathrm{span}\{J, P\} \oplus \mathrm{span}\{H, K\}.$$

Taken both expressions simultaneously, Tolar and Trávnícek induced in [66] the $\Pi \times \Theta$-grading with the grading group $G = Z_2 \times Z_2$

$$B_2 = L_a \oplus L_b \oplus L_c \oplus L_d = \mathrm{span}\{J\} \oplus \mathrm{span}\{H\} \oplus \mathrm{span}\{P\} \oplus \mathrm{span}\{K\}.$$

Now, with the objective of obtaining contractions of the previously indicated commutation relations (6.1) they introduced the following modifications

$$\begin{array}{llll} [\mathrm{J},\mathrm{J}]=-\varepsilon_{a,a}\ \mathrm{J}, & [\mathrm{J},\mathrm{H}]=\varepsilon_{a,b}\, 0, & [\mathrm{J},\mathrm{P}]=-\varepsilon_{a,c}\, \mathrm{P}, & [\mathrm{J},\mathrm{K}]=-\varepsilon_{a,d}\, \mathrm{K}, \\ & [\mathrm{H},\mathrm{H}]=\varepsilon_{b,b}\, 0, & [\mathrm{H},\mathrm{P}]=\varepsilon_{b,c}\, \mathrm{K}, & [\mathrm{H},\mathrm{K}]=-\varepsilon_{b,d}\, \mathrm{P}, \\ & & [\mathrm{P},\mathrm{P}]=-\varepsilon_{c,c}\, \mathrm{J}, & [\mathrm{P},\mathrm{K}]=\varepsilon_{c,d}\, \mathrm{H}, \\ & & & [\mathrm{K},\mathrm{K}]=-\varepsilon_{d,d}\mathrm{J}. \end{array} \tag{6.2}$$

Real numbers $\varepsilon_{i,j}$ are the elements of a 4×4 symmetric matrix with rows and columns ordered a, b, c and d, the *contraction matrix*, which will be denoted by ε. Besides, the form of the matrices is the one used by Bacry and Lévy-Leblond in [2].

$$\varepsilon^{R1} = \begin{pmatrix} 1 & & 1 & 1 \\ & & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$ is the complex simple Lie algebra $so(5,\mathbb{C})$, denoted by B_2.

If $\varepsilon_{b,c}, \varepsilon_{c,c} \mapsto 0$

$$\varepsilon^{R2} = \begin{pmatrix} 1 & & 1 & 1 \\ & & 0 & 1 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix}$$ is complex Poincaré Lie algebra P^C.

If $\varepsilon_{b,d}, \varepsilon_{d,d} \mapsto 0$

$$\varepsilon^{R3} = \begin{pmatrix} 1 & & 1 & 1 \\ & & 1 & 0 \\ & & 1 & 1 \\ & & & 0 \end{pmatrix}$$ is the complex para-Poincaré Lie algebra P'^C.

If $\varepsilon_{b,c}, \varepsilon_{b,d}, \varepsilon_{c,c}, \varepsilon_{d,d} \mapsto 0$

$$\varepsilon^{R4} = \begin{pmatrix} 1 & & 1 & 1 \\ & & 0 & 0 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$ is the complex Carroll Lie algebra C^C.

If $\varepsilon_{c,c}, \varepsilon_{c,d}, \varepsilon_{d,d} \mapsto 0$

$$\varepsilon^{A1} = \begin{pmatrix} 1 & & 1 & 1 \\ & & 1 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$ is the complex Newton-Hooke Lie algebra N^C.

If $\varepsilon_{b,c}, \varepsilon_{c,c}, \varepsilon_{c,d}, \varepsilon_{d,d} \mapsto 0$

$$\varepsilon^{A2} = \begin{pmatrix} 1 & & 1 & 1 \\ & & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$ is the complex Galilei Lie algebra G^C.

If $\varepsilon_{b,d}, \varepsilon_{c,c}, \varepsilon_{c,d}, \varepsilon_{d,d} \mapsto 0$

$$\varepsilon^{A3} = \begin{pmatrix} 1 & & 1 & 1 \\ & & 1 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \qquad \text{is the complex para-Galilei Lie algebra } {G'}^{C}.$$

If $\varepsilon_{b,c}, \varepsilon_{b,d}, \varepsilon_{c,c}, \varepsilon_{c,d}, \varepsilon_{d,d} \mapsto 0$

$$\varepsilon^{A4} = \begin{pmatrix} 1 & & 1 & 1 \\ & & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \qquad \text{is the complex static Lie algebra } {S_t}^{C}.$$

In the next section we compute the values of the invariant function υ (see Section 4.2 in Chapter 4) and compute its value for the eight kinematical Lie algebras introduced by Tolar (see [65]).

6.2 The values of the invariant function υ for the 8 kinematical algebras introduced by Tolar

By applying an algorithm designed by ourselves which we describe at the end of the section, we have obtained the following results

For B_2 : $\upsilon(\lambda) = 0$, for all $\lambda \in \mathbb{C} \setminus \{0\}$,
$\upsilon(0) = 3$.

For P^C : $\upsilon(\lambda) = 24$, for all $\lambda \in \mathbb{C} \setminus \{0, 1\}$,
$\upsilon(0) = 31$,
$\upsilon(1) = 25$.

For ${P'}^C$: $\upsilon(\lambda) = 68$, for all $\lambda \in \mathbb{C}$.

For C^C : $\upsilon(\lambda) = 85$, for all $\lambda \in \mathbb{C}$.

For N^C : $\upsilon(\lambda) = 87$, for all $\lambda \in \mathbb{C}$.

For G^C : $\upsilon(\lambda) = 91$, for all $\lambda \in \mathbb{C}$.

For ${G'}^C$: $\upsilon(\lambda) = 94$, for all $\lambda \in \mathbb{C}$.

For ${S_t}^C$: $\upsilon(\lambda) = 94$, for all $\lambda \in \mathbb{C}$.

As we have previously indicated, to obtain these results, we have developed an algorithm, which can be implemented in any symbolic computation package (we have used SAGEMATH), which allows us to make easier the computations needed. The steps of that algorithm are the following

- Step 1: Defining the generators of a basis of the algebra (recall that in the case of the algebras which have been dealt with, the dimension of them is 10.

- Step 2: Constructing the generic matrix associated with the endomorphism D.

- Step 3: Introducing the brackets of the algebra.

- Step 4: Obtaining the equations of the linear system which allows us to determine the elements $a_{i,j}$ of the matrix D.

- Step 5: Solving the previous system.

- Step 6: Finally obtaining the value of the υ invariant function for the considered algebra.

For instance, we show next the application of the algorithm in the case of the kinematical Lie algebra B_2.

After defining the ten vectors according to

sage:var('e1,e2,e3,e4,e5,e6,e7,e8,e9,e10')

we construct the matrix D with the following instructions (we only indicate the three first rows of D)

sage:var('a11,a12,a13,a14,a15,a16,a17,a18,a19,a110')

sage:var('a21,a22,a23,a24,a25,a26,a27,a28,a29,a210')

sage:var('a31,a32,a33,a34,a35,a36,a37,a38,a39,a310')

Now, we introduce the brackets $[D(e_i), e_j]$ of the algebras with the instruction

*sage:deiej=aii*eiej+aij*ejej+ai3*e3ej+ai4*e4ej+ai5*e5ej+ai6*e6ej*

*+ai7*e7ej+ai8*e8ej+ai9*e9ej+aii0*ei0ej;deiej*

sage:ejdei=-deiej; para i,j=1,...,10

and we obtain (the first three rows of them are only indicated)

sage:e8e9=-e10

sage:e9e8=e10

sage:e8e10=e9

Now, we obtain the equations for the linear system which will allow us to determine the elements a_{ij} of the matrix D. In the case of B_2 this system has 298 equations, 100 unknown variables and it depends on one parameter. The equations of the system are (only the first three of them are indicated)

*sage:E1=(a53-a62==a101*l)*

*sage:E2=(a61==a102*l)*

*sage:E3=(-a51==a103*l)*

Then, by solving the system and suppressing the elements $a_{i,j}$ which are zero, the rest of such elements determine the values of $\dim Der_{(1,1,1,\lambda)}\mathfrak{g}$, which, in turn, allows us to obtain the invariant function υ.

Indeed, for this algebra B_2 we find that

1. If λ is non-null, then all elements a_{ij} in the matrix D of the endomorphism are null and thus $\upsilon(\lambda) = 0$.

2. If λ is null, then the only elements a_{ij} which can be non-null are $a_{2\ 10}, a_{3\ 10}$ and $a_{5\ 10}$. Therefore, $\upsilon(\lambda) = 3$.

Applying the algorithm to the rest of kinematical Lie algebras indicated, the previously mentioned results have been obtained.

Finally, a consequence of Proposition 4.2.5 on these kinematical Lie algebras is the following

Corollary 6.2.1. *If $\mathfrak{g}$ is any kinematical algebra, then* $0 \leq \upsilon_{\mathfrak{g}} \leq 94$. □

We think that this result joint other similar which can be obtained could suppose one step forward in the knowledge of the study of contractions of algebras. In this way, we will try to apply this computational technique based on that function υ to other different types of Lie and non-Lie algebras in future work. And another research on this topic could be consider the recently published book by Chevallier and Lerbet [18] and try to tackle the open problem cited in it.

Chapter 7

Some personal reflections, conclusions and further works

In the first place, the author would like to indicate that there exist several open problems related with all the research which he has presented in this manuscript.

Indeed, regarding the study of invariant functions for algebras made in Chapter 3 and 4, we have found that for the ψ invariant function the dimension is always the same independently of the value of α, which does not occur in the case of the φ invariant function. This feature occurs at least for lower dimensional filiform Lie algebras (recall that our motivation for dealing with this type of algebras has been explained in the Introduction of this manuscript). We think that it could be interesting to check whether this is a general characteristic irrespective of the dimension since that, at present, we do not know exactly which is the meaning of this fact. One reason could be that both invariant functions, ψ and φ are not comparable because they are defined in a totally different way, since the function ψ is defined as the dimension of a derivation whereas φ is defined as the dimension of a cocycle, which involves that their computations were completely different.

Indeed, although by using a different procedure, we have confirmed some results by those authors referred to both functions ψ and φ and have also obtained new results in the case of filiform Lie algebras of dimension 5. In our opinion this is an improvement since this dimension is not studied in detail in [56] (in fact, the word *filiform* is not even used by the authors in their paper).

Furthermore, in our study we have dealt with several examples that could be of potential interest for the computation of invariant functions for other nilpotent Lie algebras different to the filiform or Heisenberg ones. Note that we have also given the procedure and computed the invariant functions for other types of Lie algebras, as the ones built by means of direct sums of other Lie algebras, for instance.

Therefore, our intention is to deal with the invariant function φ for filiform Lie algebras of greater dimensions in future work. Indeed, we are now trying to model a *Bose-Hubbard-like model* for describing interacting spinless bosons on a lattice by means a filiform Lie algebra. That is a nonlinear problem that if we succeed, it could be mapped to a system with one-body interactions, being therefore linear. Studies of invariance groups in that kind of system and their connection with contractions it is also something worth exploring.

Regarding contractions (see Chapter 5), we have obtained the graded contractions of some lower-dimensional filiform Lie algebra, concretely, of the model filiform Lie algebras of dimensions $3, 4, 5$ and 6. Then, as a consequence of the results obtained, we have dealt with the general case n-dimensional for this type of algebras. Moreover, we have repeated this study for the dimensions 6 and 8 with a non-model filiform Lie algebra. All of it completes previous papers on this topic by different authors, like Inönü and Wigner [44] in 1953, Weimar-Woods [72] in 2006 or Bahturin, Goze and Remm [4] in 2013, for instance.

We have also calculated the invariant function ψ for the 5-th Heisenberg algebra and have proved that there is no proper contraction from a lower dimensional filiform Lie algebra to a Heisenberg algebra. Furthemore, because neither these algebras are isomorphic, then it can not exist a non proper contraction between them. Therefore, we can conclude that for a 5-dimensional classical-mechanical model built upon a 5-dimensional filiform Lie algebra can not be obtained as a limit process of a quantum-mechanical model based on a 5-th Heisenberg algebra.

However, we think that there exists the possibility of setting new theoretical results on it. Indeed, as consequences of our study, we find some question which could make us think of giving an answer to the following facts, thereby determining some conjectures. For instance: a) will the orbits of I_u^2, for the filiform Lie algebras $\mathfrak{f}_5$ and $\mathfrak{f}_6$, have the same representatives? Do these orbits have 4 points at most? b) do the contraction matrices of the filiform Lie algebras $\mathfrak{f}_n$ and Q_n have an unique null relevant contraction parameter? and c) which is the form of the symmetry groups of filiform Lie algebras $\mathfrak{f}_n$ and Q_n? Our intention is to give responses to these questions in future work.

Moving now on the chapter devoted to the study of Kinematical Lie algebras (see Chapter 6), we have computed the values of the invariant function $\upsilon_{\mathfrak{g}}$, introduced by ourselves in [25], for the eight kinematical Lie algebras by Tolar [65], with the goal of giving steps forward in the knowledge of the study of contractions of algebras and making easier the computations needed to get such a purpose. In this way, we will try to apply this computational technique based on that function to other different types of Lie and non-Lie algebras also in future work.

Apart from all what we have just commented, and although the main objective of this doctoral dissertation was to deepen and advance in the study of the invariant functions and contraction of algebras, we have always tried to keep in mind, as secondary motivation, trying to find some possible interesting physical applications for the filiform Lie algebras.

At this respect, we think that we have developed some novel mathematical tools which could be adequate for dealing with certain physical aspects, although it is true that we have not found yet a concrete and suitable physical problem to be tackled.

Nevertheless, in this chapter we show some open problems that have arisen in a natural way during the course of this research. These open problems are concerning to the application to the field of Physics the concepts and results obtained in the previous chapters. In the following, we mention some of them.

- One of the possible physical applications of the present topic is given by the possibility of describing a many-body system based on interacting spinless boson particles located in a lattice of n sites by means a filiform Lie algebra. This system could be a kind of Bose-Hubbard model, which is well known in the Condensed Matter community and widely studied. The Hamiltonian corresponding to that system can be described in terms of semisimple Lie algebras and is a quadratic model since it contains up to two-body operators. Therefore, we wonder if we could describe the same system employing filiform Lie algebras and if we could obtain new information using the tools developed in this doctoral dissertation.

  In order to perform this task, it is necessary to write the boson operators involved in the Hamiltonian in term of new ones that fullfiles the commutation relations for a given filiform Lie algebra. However, at that point, we find the difficulty that we should employ a tensorial product of two filiform Lie algebras in order to describe the system properly. That means, it should be exist an isomorphism between the semisimple Lie algebra of the original hamiltonian and the filiform Lie algebra proposed to describe the physical system. Fortunatelly, It seems that there is a theorem that can confirm that kind of isomorphism.

  Now, the advantage that we gain employing a filiform Lie algebra instead of a semisimple Lie algebra is that we could map a nonlinear problem such a the problem described by a system with up to two-body interactions onto a linear problem with just one-body interactions. On the other hand, once we have described the system in terms of the filiform Lie algebra, it is necessary to define the branching rules, that is to find the irreducible representations of an algebra $\mathfrak{g}'$ contained in a given representation of $\mathfrak{g}$. Since the representations are interpreted as quantum mechanical states, it is necessary to provide a complete set of quantum numbers (labels) to characterize uniquely the basis of the system. This is a nontrivial task that it may even lead to a new doctoral dissertation.

- Another possible physical applications of the present topic is to study phase spaces by using filiform Lie algebras as a tool.

  At this respect, Arzano and Nettel in their paper entitled *Deformed phase spaces with group valued momenta* in 2016 [1] introduced a general framework for describing deformed phase spaces with group valued momenta. Using techniques from the theory of Poisson-Lie groups and Lie bialgebras, they developed tools for constructing Poisson structures on the deformed phase space starting from the minimal input of the algebraic structure of the generators of the momentum Lie group. These tools developed are used to derive Poisson structures on examples of group momentum space much studied in the literature such as the n-dimensional generalization of the κ-deformed momentum space and the $SL(2, R)$ momentum space in three space-time dimensions. They also discussed classical momentum observables associated to multiparticle systems and argued that these combined according the usual four-vector addition despite the non-Abelian group structure of momentum space (see [1] for further information).

  In that paper, the authors work with a phase space $\Gamma = T \times G$, given by the Cartesian product of a n-dimensional Lie group configuration space T and a n-dimensional Lie group momentum space G. Since T and G are Lie groups, we can consider their associated Lie algebras $\mathfrak{t}$ and $\mathfrak{g}$ so that we can define a Lie -Poisson algebra which can endow a mathematical structure to the phase space Γ. Indeed, Arzano and Nettel consider a phase space Γ in which the component related to momentum is an n-dimensional Lie sub-group of the $(n + 2)$-dimensional Lorentz group $SO(n + 1, 1)$, denoted as $AN(n)$.

  Taking into consideration this paper, we have tried to construct a phase space similar to the one by those authors, although we have taken the $(n+2)$-dimensional Lorentz group $SO(n + 1, 2)$ as the Lie group related to momentum.

  We began our research on this subject considering the Lie group $SO(2, 2)$ and using the same procedure as Arzano and Nettel did. However, we realized that that attempt was going to be very complicated because of the great dimensions of the matrices involved (in the computations, a 49×49 $r-$matrix appeared).

  Therefore, the fact of finding a Poisson structure that hat allows us to endow the phase space $\Gamma = T \times SO(n+1, 2)$ with a mathematical structure is another problem, which we consider open. As it was the case with the previous problem, this might also even lead to a new doctoral dissertation.

- Quantum algebras.

  Finally, we would like to note that both the doctoral student and the advisors would like to tackle the study of quantum algebras, also named quantum affine algebras or affine quantum groups.

Although the deep study of these is not long ago, because they were introduced independently by Drinfeld and Jimbo, both in 1985 ([21] and [46], respectively), as a special case of their general construction of a quantum group from a Cartan matrix, these algebras, which are Hopf algebras that are a q-deformation of the universal enveloping algebra of an affine Lie algebra, constitute nowadays a new and growing field of Mathematics with vast potential for applications in Physics.

We think that the results obtained in this dissertation could be useful to give steps forward in the study of these algebras, quantum algebras, although we have not taken any significant steps in this respect yet.

APPENDIX: Using the symbolic computation package Sage

We have used the symbolic computation package SageMath for computations performed in the manuscript. This package is a free open-source mathematics software system licensed under the GPL. It was created to provide a viable free open source (this is one of its main characteristics) alternative to other private packages, as Magma, Maple, Mathematica and Matlab, for example. Sage builds on top of many existing open-source packages: NumPy, SciPy, matplotlib, Sympy, Maxima, GAP, FLINT, R and many more and access their combined power through a common, Python-based language or directly via interfaces or wrappers. From its reliability speaks the fact that in 2007 Sage awarded the first prize in the scientific software category at *Les Trophes du libre*, an international free software competition (see [74] for further information).

As we have already previously indicated, we have used throughout the entire manuscript the SAGE symbolic computation package for doing the computations needed to obtain our results. On the sake of example, we show here some of such computations.

• When computing the invariant function φ it is necessary to permute the indices of the basis vectors. A routine to do this with SAGE is the following

```
sage: from sage.combinat.permutation import from_cycles
sage: for n in range(1,6):
....: for p in Permutations(n):
....: if from_cycles(n, p.to_cycles()) != p:
....: print "There is a problem with ",p
....: break
sage: size = 10000
```

sage: sample = (Permutations(size).random_element() for i in range(5))

True

• To obtain the bracket products of the contracted algebra from the ones of the initial algebra, which we do by taking the limit when the variable appearing in the last brackets tends to 0, we have used the following pseudo-code

```
For i from 1 to 3 do
  ei= i-th vector of the canonical basis;
  Uei=U*ei
end do
```

```
For i from 1 to 2 do
For j from i+1 to 3 do
  [ei,ej]e=inverse(U)*(Uei*Uej-Uej*Uei);
end do
end do
```

• Finally, as an example, we show the way in which we find that the contracted Lie algebra from a certain initial fixed algebra (the Heisenberg algebra $\mathbb{H}_3$ in this case) is filiform and has dimension 3.

- We start from Heisenberg algebra $\mathbb{H}_3$ and define a parameter matrix U

```
sage: \varepsilon=var('\varepsilon');
sage: U = matrix([[0,\varepsilon,0],[\varepsilon,0,0],[0,0,1]])
```

- We define the vectors of the basis of the new vector space V.

```
sage: e1 = matrix([[1],[0],[0]]);
sage: e2 = matrix([[0],[1],[0]]);
sage: e3 = matrix([[0],[0],[1]]);
```

- We map the matrix U to these vectors.

```
sage: Ue1=U*e1;
sage: Ue2=U*e2;
sage: Ue3=U*e3;
```

- The resulting bracket products $[Ue_i, Ue_j]_{\mathbb{H}_3}$, $\forall i, j \in \{1, 2, 3\}$, are

```
sage: Ue1Ue2=matrix([[0],[0],[0]]);
sage: Ue1Ue2=matrix([[e],[0],[0]]);
sage: Ue2Ue3=matrix([[0],[0],[0]]);
```

- Finally, we obtain $U^{-1} * [Ue_i, Ue_j]_{\mathbb{H}_3}$, $\forall i, j \in \{1, 2, 3\}$, and these expressions define a family of brackets which, in principle, will depend on the parameter ε. When the parameter tends to 0, we obtain the bracket $[\,,\,]_0$, which endows V with a new algebra structure, which is the contracted algebra from the Heisenberg algebra $\mathbb{H}_3$.

```
sage: e1xe2=U^-1*Ue1xhUe2;
sage: e1xe3=U^-1*Ue1xhUe3;
sage: e2xe3=U^-1*Ue2xhUe3;
```

After running the programme we obtain

```
sage: e1xe2;
[0]
[0]
[0]
sage: e1xe3;
[0]
[1]
[0]
sage: e2xe3;
[0]
[0]
[0]
```

Therefore, since these brackets do not depend on the parameter ε, it implies that they are the same in the initial and in the contracted algebra (with the usual notation, they are: $[e_1, e_2] = e_3$, $[e_1, e_3] = [e_2, e_3] = 0$). We conclude that the algebra we have obtained is a 3-dimensional filiform Lie algebra.

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