Convex analysis applied to location theory

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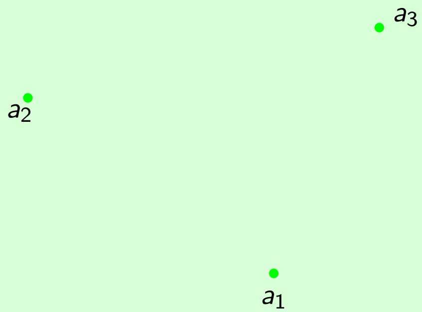

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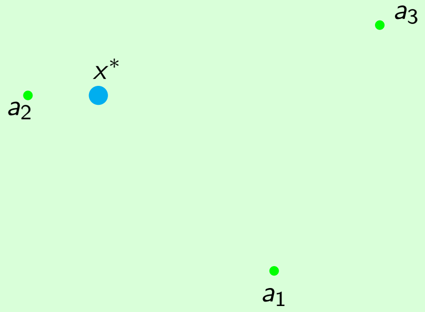

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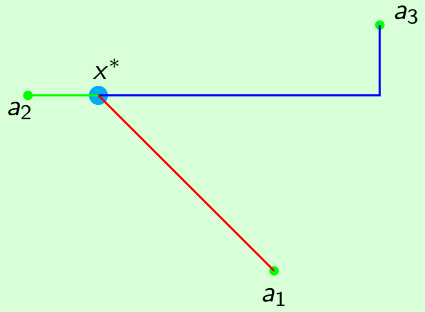

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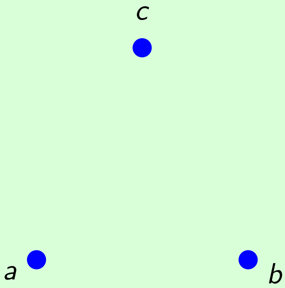

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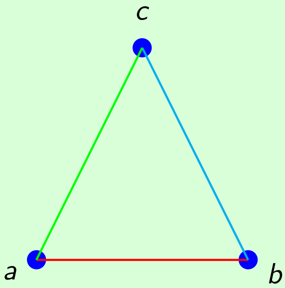

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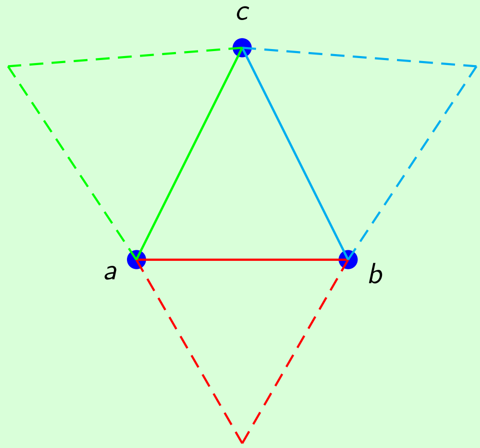

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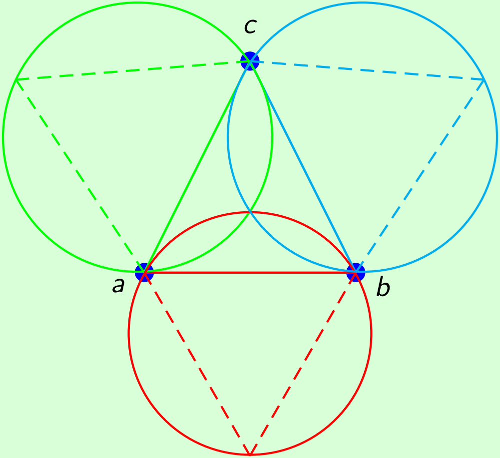

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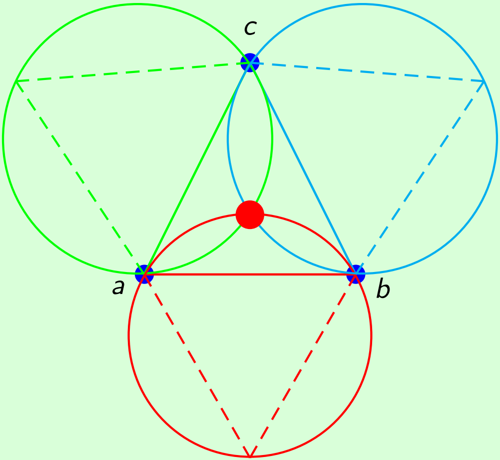

Elements of a location problem

Support space

- Continuous Spaces ($\mathbb{R}^n$).
  - Sphere
- Networks
- Discrete spaces

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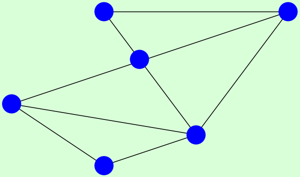

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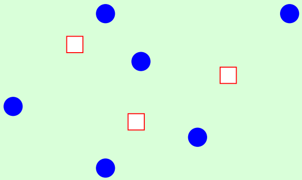

Objective Function

- Fermat-Weber Problem (Minisum)
- Center Problem (minimax)
- Cent-dian Problem
- k-centrum Problem
- Ordered Median Problem
- Multiobjective Problem

$$\min\{F_1(x), \ldots, F_Q(x)\}$$

Objective Function

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$$\sum_{a \in A} w_a d_a(x,a) \quad \text{con} \qquad i = 1, \ldots, M$$

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$$\max_{a \in A} u_a d_a(x, a) \quad \text{con} \quad i = 1, \ldots, M$$

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$$\min\{F_1(x), \ldots, F_k(x)\}$$

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$$\alpha \sum_{a \in A} w_a d_a(x,a) + (1-\alpha) \max_{a \in A} u_a d_a(x,a)$$

- k-centrum Problem
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$$\min\{F_1(x), \ldots, F_k(x)\}$$

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$$\sum_{j=M-k}^{M} w_j d_{\sigma_j}(x, a_{\sigma_j})$$

where $d_{\sigma_1}(x, a_{\sigma_1}) \leq \ldots \leq d_{\sigma_M}(x, a_{\sigma_M})$.

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$$\sum_{j=1}^{M} \lambda_j d_{\sigma_j}(x, a_{\sigma_j})$$

donde $d_{\sigma_1}(x, a_{\sigma_1}) \leq \ldots \leq d_{\sigma_M}(x, a_{\sigma_M})$.

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Introduction

- Facilities are represented by isolated points?
- Introducing sets instead of points introduces important differences in the mathematical analysis of these problems.
- Our approach: minimization of expected distances

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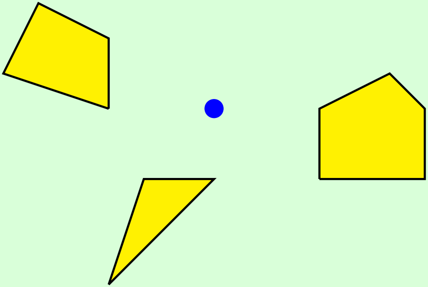

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- Application point of view:
  1. Case of the stationing of rescue helicopters [Ehrgott 02].
  2. Location of planes used to extinguish fires in reserves or natural parks.
  3. Locating a read/write head of a computer hard-disk to easily access the stored data. [Vickson, Gerchak and Rotem 95] and [Puerto and Rodríguez-Chía 03].

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- Our approach: minimization of expected distances

- **Our goal:** Geometrical characterization of the solution set for a single facility location model with sets as demand facilities using average distances. Networks: [Hakimi64, Hooker91]. Continuous location problems: [Durier and Michelot 85, Durier95, PF00, NPR03].
  - Basic model
  - General model
  - Discretization result.

Basic tools and definitions

- X is a real separable Banach space.
- γ norm with unit ball B.
- $f : X \to \mathbb{R} \cup \{+\infty\}$ convex function.

  $p \in \partial f(x) \subseteq X^*$ iff $f(y) \geq f(x) + \langle p, y - x \rangle \quad$ for each $y \in X$.
- Conjugate function:
  $f^*(p) = \sup\{\langle p, x\rangle - f(x) : x \in \text{dom } f\} \, \forall p \in X^*$.

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Result: For a closed and proper convex function, [Barbu, Precupanu 75]:

$$p \in \partial f(x),\ x \in X,\ p \in X^* \quad \text{iff} \quad x \in \partial f^*(p).$$

Remarks:

- $\partial \gamma(x) = \{p \in B^o \ : \ \gamma(x) = \langle p, x \rangle\}$.
- $\partial \gamma^*(p) = \{x \in X \ : \ \gamma(x) = \langle p, x \rangle\}$.

The basic model

$$\inf_{x\in X} \phi(x) := \int_T \varphi_t(x)\, d\mu(t), \qquad (P_\phi(T))$$

where $\varphi_t(x) := \gamma_t(x-t)$, μ a σ-finite, positive measure.

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Result: ϕ is convex on X.
Result: $F : G \longrightarrow L^1(X, \mathbb{R})$ such that $(F(x))(t) = \varphi_t(x)$.
If X is separable or T is countable, then ϕ is continuous at x_o and

$$\partial\phi(x_o) = \int_T \partial\varphi_t(x_o)\mu(dt) = \int_T \partial F(x_o)\mu(dt).$$

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Remark: μ is concentrated on a finite set of points, $P_\phi(T)$ reduces to the classical Fermat-Weber problem.

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Existence results: [Garkavi and Smatkov 74],

- X is finite dimension and φ_t are lower-semicontinuous (l.s.) in the t argument;
- X is reflexive and φ_t are sequentially l.s. in the t argument for the weak topology;
- X is the dual space to a separable space and φ_t are sequentially l.s. in the t argument for the weak topology
- X is a dual space, φ_t are l.s. in the t argument for the weak* topology and T is μ-separable.

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Uniqueness results: [Garkavi and Smatkov 74]

- $\mu(T) < +\infty$ and X is a strictly normed space. $P_\phi(T)$ has a unique solution iff T does not contain two nonintersecting subsets T_1 and T_2 such that $\mu(T_1) = \mu(T_2) = \mu(T)/2$, being T_1 and T_2 enclosed in nonintersecting rays ℓ_1 and ℓ_2, respectively and lying in the same straight line.
- $\mu(T)$ is not finite.
  Let dim$(X) \geq 2$. If γ_t are strict norms and μ is absolutely continuous with respect to any measure that assigns null measure to any subspace of dimension less than or equal to 1 then the considered problem has a unique optimal solution.

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Result: $\varphi_t(x) = \gamma(x-t)\ \forall t \in T$. The closure of $co(T)$ contains at least an optimal solution of Problem $P_\phi(T)$ if $dim(X) = 2$ or γ is a norm derived from an inner product when $dim(X) \geq 3$.

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Remark: These results extend some results proved in [Carrizosa et al. 95] for 2-dimensional spaces and in [Durier85] for finite set of points in $\mathbb{R}^n$.

Optimality Conditions:

Result: Let X be a separable Banach space, then:

1. If $M_\phi(T) \neq \emptyset$, $\exists q \in L^1(X, X^*)$ such that $\int_{T'} q(t)\mu(dt) = 0$ and $M_\phi(T) = \bigcap_{t \in T} \partial\varphi_t^*(q(t)) = \bigcap_{t \in T}(t + N_t(q(t))) := \mathcal{C}_q(T)$.
2. If $\exists q \in L^1(X, X^*)$, such that $\int_T q(t)\mu(dt) = 0$ and $\bigcap_{t \in T'} \partial\varphi_t^*(q(t)) \neq \emptyset$ then $M_\phi(T) = \bigcap_{t \in T} \partial\varphi_t^*(q(t)) = \bigcap_{t \in T}(t + N_t(q(t)))$.

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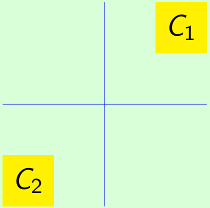
Example: $X = \mathbb{R}^2$, ℓ_1-norm and $f(t) = 1/2\delta_{C_1}(t) + 1/2\delta_{C_2}(t)$,



$$\min_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \|x - t\|_1 f(t)dt.$$

For

$$q(t) = \begin{cases} (-1,-1) & \text{if} \quad t \in C_1 \\ (1,1) & \text{if} \quad t \in C_2 \\ (0,0) & \text{if} \quad t \notin C_1 \cup C_2 \end{cases} . \quad \int_{\mathbb{R}^2} q(t)f(t)dt = (0,0).$$

Moreover, $t + N_{B^0}(q(t)) = \begin{cases} t - \mathbb{R}^2_+ & \text{if} \quad t \in C_1 \\ t + \mathbb{R}^2_+ & \text{if} \quad t \in C_2 \end{cases}$; and thus

$$\bigcap_{t \in C_1 \cup C_2} (t + N_{B^0}(q(t)) = conv\{(-1,-1),(-1,1),(1,-1),(1,1)\}.$$

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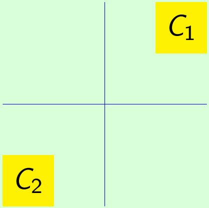

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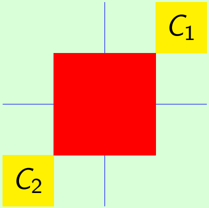

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Extended model

- $\Phi(\cdot)$ which is a monotone norm on $\mathbb{R}^M$.
- μ_i σ-finite, positive measures and $T \subseteq X$.
- $\bar{d}_i(x) := \int_T \varphi_t(x)\mu_i(dt)$, where $\varphi_t(x) = \gamma_t(x - t)$.

$$\inf_{x \in X} F(x) := \Phi((\bar{d}_1(x), \ldots, \bar{d}_M(x))), \qquad (P_\Phi(\Upsilon))$$

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Properties:

- Particular instances: center, cent-dian, k-centrum, etc.
- $F = \Phi \circ \overline{D}$ is convex on $\mathbb{R}^M$.
- Existence and uniqueness results are still valid.

Result: Let $x \in X$ be such that $\overline{D}(x) \neq 0 \in \mathbb{R}^M$.

$\beta \in \partial F(x)$ iff $\exists p_i \in \partial \bar{d}_i(x), \forall i$ and $\delta \in \partial \Phi(\overline{D}(x))$, such that, $\beta = \sum_{i=1}^{M} \delta_i p_i$

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Definition: $p = (p_1, \ldots, p_M) \in (X^*)^M$ and $I \subseteq \{1, \ldots, M\}$.

$$\overline{C}_I(p) := \bigcap_{i \in I} \partial \bar{d}_i^*(p_i).$$

For any $\delta = (\delta_1, \ldots, \delta_M) \geq 0$

$$\overline{D}_I(\delta) := \{x \; : \; \sum_{i \in I} \delta_i \bar{d}_i(x) = F(x)\}.$$

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Definition: (I, δ, p) is a suitable triplet if

1. $I \neq \emptyset$, $I \subseteq \{1, \ldots, M\}$,
2. $\delta = (\delta_1, \ldots, \delta_M)$; $\delta_i > 0 \ (i \in I)$, and $\delta_i = 0 \ (i \notin I)$; $\Phi^o(\delta) = 1$
3. $p = (p_1, \ldots, p_M)$ such that $p_i \in \partial \bar{d}_i(x) \ \forall i$ and for some $x \in X$, with $\sum_{i=1}^{M} \delta_i p_i = 0$

Result:

1. If $M_\Phi(\Upsilon) \neq \emptyset$, $\exists (I, \delta, p)$ such that $M_\Phi(\Upsilon) = \overline{C}_I(p) \cap \overline{D}_I(\delta)$.
2. (I, δ, p) s.t. with $\overline{C}_I(p) \cap \overline{D}_I(\delta) \neq \emptyset$, $M_\Phi(\Upsilon) = \overline{C}_I(p) \cap \overline{D}_I(\delta)$.

Remark:

- We only need to find a suitable triplet (I, δ, p) such that $\overline{C}_I(p) \cap \overline{D}_I(\delta) \neq \emptyset$.
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The polyhedral planar case

Result: There exists a finite partition of $\mathbb{R}^2$ such that $\bar{d}(x, T)$ has a common closed form expression on each element of the partition (linear or quadratic).

Example:

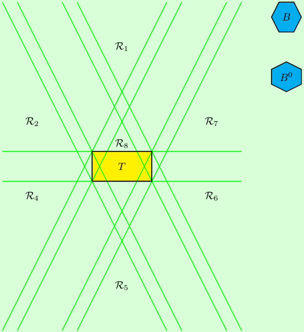
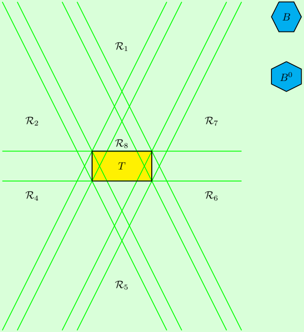
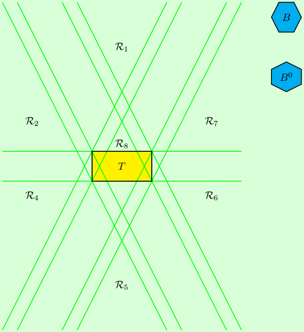


Figura: Partition of $\mathbb{R}^2$ generated by the norm γ.



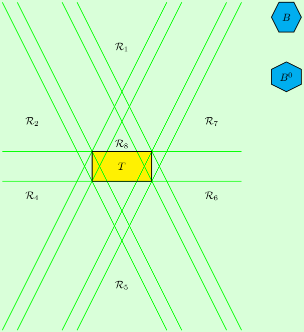
μ is a uniform probability density on T.

$$\bar{d}(x, T) = \int_T \gamma(x - t)\mu(dt) = \frac{1}{\mu(T)} \int_T \gamma(x - t)\, dt.$$



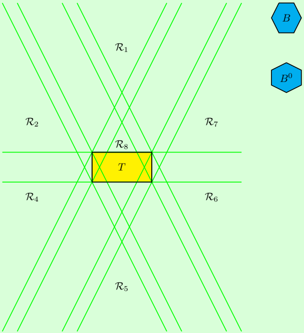
$x \in \mathcal{R}_1$: $\gamma(x-t) = \langle (0,-1), (t_1 - x_1, t_2 - x_2) \rangle$.

$$\bar{d}(x,T) = \frac{1}{8} \int_T \gamma(x-t)\, dt = x_2.$$



$x \in \mathcal{R}_2$: $\gamma(x-t) = \langle (1, -0{,}5), (t_1 - x_1, t_2 - x_2) \rangle$.

$$\bar{d}(x, T) = \frac{1}{8} \int_T \gamma(x-t)\, dt = -x_1 + \frac{x_2}{2}.$$



$x \in \mathcal{R}_8$: $\gamma(x-t) =$

$$\begin{cases} \langle (1, 0{,}5), (x_1 - t_1, x_2 - t_2) \rangle & \text{if } t \in T \text{ and } t_1 \leq \frac{t_2 + 2x_1 - x_2}{2} \\ \langle (0, 1), (x_1 - t_1, x_2 - t_2) \rangle & \text{if } t \in T \text{ and } \frac{t_2 + 2x_1 - x_2}{2} \leq t_1 \leq \frac{-t_2 + 2x_1 + x_2}{2} \\ \langle (-1, 0{,}5), (x_1 - t_1, x_2 - t_2) \rangle & \text{if } t \in T \text{ and } t_1 \geq \frac{-t_2 + 2x_1 + x_2}{2}. \end{cases}$$

Thus,

$$\bar{d}(x, T) = \frac{1}{8} \int_T \gamma(x - t)\, dt = 2\left(x_1 + \frac{x_2}{2}\right)\left(x_1 - \frac{x_2}{2} - 2\right) - \frac{15}{2}.$$

Complexity analysis:

- $O(Mgk_{\max})$ lines, the overall complexity of finding the arrangementis $O((Mgk_{\max})^2)$. [Eldel87].
- $\mathcal{R}_{j_0}$ with $j_0 \in J$, the upper envelope defining Φ has a complexity of at most $O(\lambda_4(r^0))$.
- Subpartition $\{\mathcal{R}'_j\}_{j \in J'} \cap \mathcal{R}_{j_0}$ has $O(\lambda_4(r^0))$ elements.
- The number of elements in the partition induced by the family $\overline{C}_I(p) \cap \overline{D}_I(\delta)$ is $O((Mgk_{\max})^2 \lambda_4(r^0))$. Moreover, it can be computed in $O((Mgk_{\max})^2 \lambda_4(r^0) \log(r^0))$.

Remark:

- $\lambda_s(n)$ is the maximum length of a Davenport-Schinzel sequence of order s on n symbols. [Sharir and Agarwal].
- $\lambda_1(n) = O(n)$, $\lambda_2(n) = O(n)$, $\lambda_3(n) = \theta(n\alpha(n))$, and $\lambda_4(n) = \theta(n2^{\alpha(n)})$, where $\alpha(n)$ is the inverse of the Ackermann function.

Example:

$$\Phi_1(\overline{D}(x)) = \sum_{i=1}^{4} \bar{d}_i(x)$$

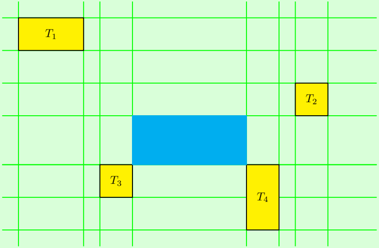
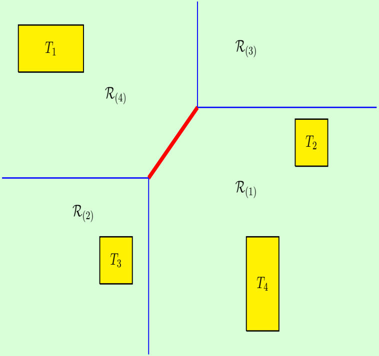


Figura: Minisum problem.

- Taking $I = \{1,2,3,4\}$, $\delta = (1,1,1,1)$, $p_1 = (1,-1)$, $p_2 = (-1,-1)$, $p_3 = (1,1)$, $p_4 = (-1,1)$, (I, δ, p) is a suitable triplet. $\overline{D}_I(\delta) = \mathbb{R}^2$.

Example:

$$\Phi_3(\overline{D}(x)) = \max_{i=1,\ldots,4} \bar{d}_i(x)$$

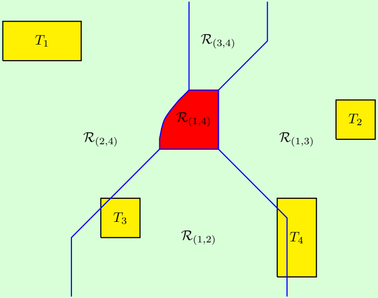


- Taking $I = \{1, 4\}$, $\delta = (\frac{1}{2}, 0, 0, \frac{1}{2})$, $p_1 = (1, -1)$, $p_4 = (-1, 1)$; (I, δ, p) is a suitable triplet.

Example:

$$\Phi_4(\overline{D}(x)) = \sum_{i=3}^{4} \bar{d}_{(i)}(x)$$

$d_{(i)}(x) = d_{\sigma_i}(x)$ with σ a permutation of $\{1, \ldots, 4\}$, such that, $d_{\sigma_1}(x) \leq \ldots \leq d_{\sigma_4}(x)$.



- Taking $I = \{1, 4\}$, $\delta = (1, 0, 0, 1)$, $p_1 = (1, -1)$, $p_4 = (-1, 1)$; $\overline{C}_I(p)$ is the rectangle defined by the closest vertices of T_1 and T_4.

Discretization

Result:

1. $\forall \varepsilon > 0$ there exist countable sets $A \subseteq T$, $\{w_a \geq 0\}_{a \in A}$, such that, the solutions of $F_W^*(A)$ are ε-solution set of $P_\phi(T)$, with

$$F_W^*(A) = \min_{x \in X} F_{W,A}(x) := \sum_{a \in A} w_a \gamma_a(x - a)$$

2. $\forall \varepsilon > 0$ there exist a countable set $A \subseteq T$, $W_i = \{w_{i,a}\}_{a \in A}$ $\forall i$, such that, the solutions of

$$\min_{x \in X} \Phi(F_{W_1,A}(x), \ldots, F_{W_M,A}(x)), \qquad (P_\Phi(A))$$

are ε-solution set of Problem $P_\Phi(\Upsilon)$.

Discretization

Result:

1. $\forall \varepsilon > 0$ there exist countable sets $A \subseteq T$, $\{w_a \geq 0\}_{a \in A}$, such that, the solutions of $F^*_W(A)$ are ε-solution set of $P_\phi(T)$, with

$$F^*_W(A) = \min_{x \in X} F_{W,A}(x) := \sum_{a \in A} w_a \gamma_a(x - a)$$

2. $\forall \varepsilon > 0$ there exist a countable set $A \subseteq T$, $W_i = \{w_{i,a}\}_{a \in A}$ $\forall i$, such that, the solutions of

$$\min_{x \in X} \Phi(F_{W_1,A}(x), \ldots, F_{W_M,A}(x)), \qquad (P_\Phi(A))$$

are ε-solution set of Problem $P_\Phi(\Upsilon)$.

Proof: (X, γ) separable Banach space has a de Possel net.[Ioffe72]

- $E_a \cap E_{a'} = \emptyset$, $a \neq a'$, and $\bigcup_{a \in A} E_a = X$;
- $int(E_a) \neq \emptyset$, $E_a \subset cl(int(E_a))$, $a \in A$;
- $\sup_{a \in A} diam\,(E_a) < \varepsilon/(2\mu(T))$.

Discretization

Result:

1. $\forall \varepsilon > 0$ there exist countable sets $A \subseteq T$, $\{w_a \geq 0\}_{a \in A}$, such that, the solutions of $F_W^*(A)$ are ε-solution set of $P_\phi(T)$, with
$$F_W^*(A) = \min_{x \in X} F_{W,A}(x) := \sum_{a \in A} w_a \gamma_a(x - a)$$
2. $\forall \varepsilon > 0$ there exist a countable set $A \subseteq T$, $W_i = \{w_{i,a}\}_{a \in A}\ \forall i$, such that, the solutions of
$$\min_{x \in X} \Phi(F_{W_1,A}(x), \ldots, F_{W_M,A}(x)), \qquad (P_\Phi(A))$$
are ε-solution set of Problem $P_\Phi(\Upsilon)$.

Remark: If T were a compact set, $|A| < \infty$.