

# Dynamics of wave equations with moving boundary

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## Abstract

This paper is concerned with long-time dynamics of weakly damped semilinear wave equations defined on domains with moving boundary. Since the boundary is a function of the time variable the problem is intrinsically non-autonomous. Under the hypothesis that the lateral boundary is time-like, the solution operator of the problem generates an evolution process  $U(t, \tau) : X_\tau \rightarrow X_t$ , where  $X_t$  are time-dependent Sobolev spaces. Then, by assuming the domains are expanding, we establish the existence of minimal pullback attractors with respect to a universe of tempered sets defined by the forcing terms. Our assumptions allow nonlinear perturbations with critical growth and unbounded time-dependent external forces.

**Keywords:** wave equation, non-cylindrical domain, non-autonomous system, pullback attractor, critical exponent.

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## 1 Introduction

This paper is concerned with long-time dynamics of semilinear wave equations defined on moving boundary domains. The problem involves a space-time domain

$$Q_\tau \subset \mathbb{R}^3 \times (\tau, \infty), \quad \tau \in \mathbb{R},$$

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such that its intersections with hyperplanes  $\{(x, s) \in \mathbb{R}^4 \mid s = t\}$  are bounded domains  $\Omega_t \subset \mathbb{R}^3$  with boundary  $\Gamma_t = \partial\Omega_t$ . Then  $Q_\tau$  and its lateral boundary  $\Sigma_\tau$  can be defined as

$$Q_\tau = \bigcup_{t>\tau} \{\Omega_t \times \{t\}\} \quad \text{and} \quad \Sigma_\tau = \bigcup_{t>\tau} \{\Gamma_t \times \{t\}\},$$

respectively. Since  $\Omega_t$  varies with respect to  $t$  we see that  $Q_\tau$  is, in general, non-cylindrical along the  $t$ -axis. We consider the mixed problem

$$\partial_t^2 u - \Delta u + \partial_t u + f(u) = g \quad \text{in } Q_\tau, \tag{1.1}$$

$$u = 0 \quad \text{on } \Sigma_\tau, \tag{1.2}$$

$$u(x, \tau) = u_\tau^0(x), \quad \partial_t u(x, t)|_{t=\tau} = u_\tau^1(x), \quad x \in \Omega_\tau, \tag{1.3}$$

where  $f$  and  $g = g(x, t)$  are forcing terms and  $u_\tau^0$  and  $u_\tau^1$  are initial data. Sometimes we write simply  $Q$  instead of  $Q_\tau$ .

This kind of wave equation was studied by several authors with  $\tau = 0$ . Indeed, the existence of a global solution was proved by Cooper and Bardos [10] under the assumption that there exists a one-to-one mapping transforming  $Q$  onto an expanding or contracting domain  $Q^*$ . One says that a domain  $Q$  is expanding if  $\Omega_s \subset \Omega_t$  whenever  $s \leq t$  and contracting in the reverse case. However, uniqueness of solutions is only known under the assumption that the exterior normal to  $\Sigma$  does not belong to the corresponding light cone, as proved in [10]. Writing the exterior normal as  $\nu = (\nu_x, \nu_t)$  this implies that  $|\nu_t| < |\nu_x|$  on  $\Sigma$ , which defines  $\Sigma$  as time-like. Roughly speaking, under suitable assumptions on  $f$  and  $g$ , problem (1.1)-(1.3) has a unique global solution if  $Q$  is smooth and its lateral boundary  $\Sigma$  is time-like.

On the other hand, the study of long-time dynamics is concerned with the behavior of the solutions as  $t \rightarrow \infty$ . In this direction, it was proved by Bardos and Chen [2] that the linear energy of the system increases when the domain  $Q$  is contracting and decreases when the domain is expanding. Therefore if we consider dissipative systems, it is natural to assume that  $Q$  is non-contracting. It is not clear whether a damping term can overcome the growth of energy produced by strictly contracting domains. The assumption that  $Q$  is expanding is used in the proof of an energy inequality (see Lemma 2.3 below).

Now, since the boundary of  $\Omega_t$  is a function of time, it follows that evolution equations on moving boundary domains are intrinsically non-autonomous, even if the external force  $g(x, t) = g(x)$  does not depend on  $t$ . In addition, given initial data  $(u_\tau^0, u_\tau^1)$  in  $H_0^1(\Omega_\tau) \times L^2(\Omega_\tau)$ , the (finite energy) solutions  $u$  of (1.1)-(1.3) satisfy

$$u(t) \in H_0^1(\Omega_t) \quad \text{and} \quad \partial_t u(t) \in L^2(\Omega_t), \quad \forall t \geq \tau,$$

where  $u(t)$  denotes  $u(\cdot, t)$ . Therefore, the solution operator of (1.1)-(1.3) generates an evolution process

$$U(t, \tau) : X_\tau \rightarrow X_t, \quad t \geq \tau,$$

where

$$X_t = H_0^1(\Omega_t) \times L^2(\Omega_t).$$

This contrasts with the standard theory for time-dependent attractors defined on a fixed space  $X$ . Our purpose is to establish the existence of time-dependent attractors to the problem (1.1)-(1.3).

Under this scenario, in order to study a class of parabolic problems on moving boundary domains, Kloeden, Marín-Rubio and Real [17] established an abstract theory to prove the existence of pullback attractors for processes defined on time-dependent spaces. With a different approach, Conti, Pata and Temam [9] and Di Plinio, Duane and Temam [12] also developed a theory of pullback attractors for processes defined on time-dependent spaces. Their work was concerned with a class of wave equations with time-dependent wave speed. The same approach was used in [8] to establish the existence of pullback attractors for a class of wave equations with time-dependent memory kernels.

Motivated by the above studies, our objective is to establish the existence of a pullback attractor to the problem (1.1)-(1.3) under the basic assumption that  $Q$  is time-like and expanding and  $f$  grows up to the critical level.

The main features of the paper can be summarized as follows:

(i) To the best of our knowledge, this is the first study concerned with attractors of wave equations on moving boundary domains. The problem is weakly damped and the nonlinear forcing is allowed to have critical growth. Then, as in [1, 4, 24], we set our analysis in a 3D framework and assume that  $|f(u)| \leq C(1 + |u|^3)$ ,  $u \in \mathbb{R}$ .

(ii) We show sufficient conditions on the external force  $g(x, t)$ , which may be unbounded, in order to obtain a minimal pullback  $\mathcal{D}$ -attractor which is unique within a universe of tempered sets defined by the growth of  $f(u)$ . Our main result is Theorem 4.1.

(iii) Since the present problem is hyperbolic, the proof of required compactness is very different from the ones in [17, 18], which enjoy higher regularity of parabolic systems. It is also different from [8, 9, 12] since our time-dependent spaces are typically  $X_t = H_0^1(\Omega_t) \times L^2(\Omega_t)$ , which makes difficult the use of fractional powers of the Laplacian in order to show asymptotic compactness of the processes. Therefore we present a slight variation of a compactness criterion (Theorem 3.2) based on the concept of weak quasi-stability [5, Section 2.2.3], which is appropriate for processes defined on time-dependent spaces in a pullback  $\mathcal{D}$ -attraction framework.

(iv) For the reader's convenience, we have provided a section with some definitions and abstract results for the existence of pullback  $\mathcal{D}$ -attractors, in the context of evolution processes defined on time-dependent spaces. This is presented in Section 3.1.

## 2 Preliminaries

In this section we present our assumptions on the parameters of the problem (1.1)-(1.3) and prove its well-posedness. Some remarks about function spaces related to moving boundary domains and a key energy estimate are also discussed.

## 2.1 Assumptions

**(H1) Assumptions on the domain:** Let  $\Omega$  be a bounded domain of  $\mathbb{R}^3$  with smooth boundary  $\Gamma$  and containing the origin. We define

$$\Omega_t = \{x \in \mathbb{R}^3 \mid x_i = h_i(t)y_i, \quad 1 \leq i \leq 3, \quad \text{with } y \in \Omega, \quad t \in \mathbb{R}\}, \quad (2.1)$$

where  $h_i \in C^3(\mathbb{R})$  are such that, there exist constants  $h_m, h_M > 0$  satisfying

$$h_m \leq h_i(t) \leq h_M, \quad \forall t \in \mathbb{R}, \quad 1 \leq i \leq 3. \quad (2.2)$$

Also, there exists  $\gamma \geq 0$  such that

$$0 \leq h'_i(t) \leq \gamma < D^{-1}, \quad \forall t \in \mathbb{R}, \quad 1 \leq i \leq 3, \quad (2.3)$$

with  $D = \max\{|y| \mid y \in \Omega\}$ .

**Remark 2.1.** (a) Since  $0 \in \Omega$ , condition (2.2) and (2.3) imply that there exist two bounded domains  $\Omega_*, \Omega^* \subset \mathbb{R}^3$  such that

$$\Omega_* \subset \Omega_\tau \subset \Omega_t \subset \Omega^*, \quad \forall \tau < t. \quad (2.4)$$

This means that  $\Omega_t$  are expanding with respect to  $t$ .

(b) The moving boundary domain is time-like. To see this, we note that the movement of a point  $x \in \Sigma$  is  $c(t) = (h_1(t)y_1, h_2(t)y_2, h_3(t)y_3, t)$  where  $y \in \Gamma$ . Consequently  $c'(t) = (h'_1(t)y_1, h'_2(t)y_2, h'_3(t)y_3, 1)$  is tangent to  $\Sigma$  and orthogonal to  $\nu$  and therefore

$$\nu_t = -(h'_1(t)y_1, h'_2(t)y_2, h'_3(t)y_3) \cdot \nu_x.$$

Using condition (2.3) we have  $|\nu_t| \leq \gamma D |\nu_x| < |\nu_x|$ .

(c) Let  $\lambda_1(\Omega)$  denote the first eigenvalue of  $-\Delta$  in  $H_0^1(\Omega)$ . Then, since  $\lambda_1(\Omega) \leq \lambda_1(\Omega')$  whenever  $\Omega' \subset \Omega$ , it follows from (2.4) that we have a uniform Poincaré inequality

$$\|u\|_{L^2(\Omega_t)}^2 \leq \frac{1}{\lambda_1^*} \|\nabla u\|_{L^2(\Omega_t)}^2, \quad \forall t \in \mathbb{R}, \quad (2.5)$$

where  $\lambda_1^* = \lambda_1(\Omega^*)$ .

(d) There exists a mapping  $r : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}^3$  defined by

$$r(y, t) = (h_1(t)y_1, h_2(t)y_2, h_3(t)y_3).$$

Its inverse  $\phi(\cdot, t) = r^{-1}(\cdot, t)$  is defined by

$$\phi(x, t) = (y, t) \quad \text{where } y = \left( \frac{x_1}{h_1(t)}, \frac{x_2}{h_2(t)}, \frac{x_3}{h_3(t)} \right). \quad (2.6)$$

Then  $\phi(\cdot, t)$  is a  $C^2$ -diffeomorphism with Jacobian

$$J_\phi(x, t) = \prod_{i=1}^3 \frac{1}{h_i(t)} \neq 0, \quad \forall t \in \mathbb{R}.$$

This shows that our moving boundary domain falls into the class considered in [19, 18].  $\square$

**(H2) Assumptions on the nonlinear forcing.** For the nonlinear term  $f(u)$ , we assume that  $f \in C^1(\Omega)$  and there exists  $C > 0$  such that,

$$|f(u) - f(v)| \leq C(1 + |u|^2 + |v|^2)|u - v|, \quad \forall u, v \in \mathbb{R}. \quad (2.7)$$

We also assume that there exist  $\beta \in (0, \lambda_1^*)$  and  $\rho > 0$  such that

$$F(u) \geq -\frac{\beta}{2}u^2 - \rho \quad \text{and} \quad f(u)u \geq F(u) - \frac{\beta}{2}u^2 - \rho, \quad \forall u \in \mathbb{R}, \quad (2.8)$$

where  $F(u) = \int_0^u f(s) ds$ .

**Remark 2.2.** The assumption (2.8) is often used in semilinear wave equations and it is satisfied if  $\lim_{|s| \rightarrow \infty} f'(s) > -\lambda_1^*$ .  $\square$

**(H3) Assumptions on the external force.** We assume that

$$g \in L_{\text{loc}}^2(Q) \quad (2.9)$$

and there exists  $\sigma_0 > 0$  such that

$$\int_{-\infty}^0 e^{\sigma_0 s} \|g(s)\|_{L^2(\Omega_s)}^2 ds < \infty, \quad (2.10)$$

with  $\sigma_0 \leq \sigma_1$ , where

$$\sigma_1 := \frac{2}{3} \min \left\{ \frac{\lambda_1^* - \beta}{2\lambda_1^*}, \frac{\lambda_1^* - \beta}{2 + 3(\lambda_1^* - \beta)} \right\}. \quad (2.11)$$

**Remark 2.3.** The assumption (2.11) is used to show certain dissipativeness of the system. An simpler condition could be

$$\sigma_0 \leq \frac{2}{3} \left( \frac{\lambda_1^* - \beta}{2 + 3\lambda_1^*} \right).$$

This is justified in Lemma 4.2.  $\square$

## 2.2 Function spaces with moving boundary

In the following we collect some definitions and properties of function spaces related to non-cylindrical domains of the form

$$Q_{\tau, T} = \bigcup_{t \in (\tau, T)} \{\Omega_t \times \{t\}\} \quad \text{and} \quad \Sigma_{\tau, T} = \bigcup_{t \in (\tau, T)} \{\Gamma_t \times \{t\}\},$$

with  $\Omega_t$  satisfying (2.1)-(2.2) and  $\tau < T$ . If  $B_t$  is a Banach space contained in  $L_{\text{loc}}^1(\Omega_t)$ , one defines, for  $q \geq 1$ ,

$$L^q(\tau, T; B_t) = \left\{ u \in L_{\text{loc}}^1(Q_{\tau, T}) \mid u(t) \in B_t \text{ for a.e. } t \in (\tau, T) \text{ and } \int_{\tau}^T \|u(t)\|_{B_t}^q dt < \infty \right\}$$

with

$$\|u\|_{L^q(\tau, T; B_t)} = \left( \int_{\tau}^T \|u(t)\|_{B_t}^q dt \right)^{\frac{1}{q}}.$$

Many of properties of  $L^q(\tau, T; L^p(\Omega_t))$ ,  $p, q \geq 1$ , can be proved by extending the functions (by zero) outside of  $\Omega_t$ . Indeed, as discussed in [18, Section 3], for  $u \in L^1_{\text{loc}}(Q_{\tau, T})$ , let  $\widehat{u}$  be its null extension

$$\widehat{u}(x, t) = \begin{cases} u(x, t), & x \in \Omega_t, \\ 0, & x \in \mathbb{R}^3 \setminus \Omega_t. \end{cases}$$

Clearly,

$$u \in L^q(\tau, T, L^p(\Omega_t)) \text{ implies } \widehat{u} \in L^q(\tau, T, L^p(\mathbb{R}^3)),$$

and

$$u \in L^q(\tau, T, H_0^1(\Omega_t)) \text{ implies } \widehat{u} \in L^q(\tau, T, H^1(\mathbb{R}^3)).$$

Therefore with respect to the spatial-derivative one has

$$\frac{\partial \widehat{u}}{\partial x_i} = \widehat{\frac{\partial u}{\partial x_i}}, \quad i = 1, 2, 3.$$

The weak time-derivative  $u' = \partial_t u$  is defined as

$$\langle u', \phi \rangle = - \int_{\tau}^T \int_{\Omega_t} u(x, t) \phi'(x, t) dx dt, \quad \phi \in C_c^\infty(Q_{\tau, T}),$$

and satisfies

$$\widehat{u}' = \widehat{u}'.$$

Now, with respect to the continuity in  $t$  we have the definition

$$C([\tau, T]; L^2(\Omega_t)) = \{u \in L^1(Q_{\tau, T}) \mid \widehat{u} \in C([\tau, T]; L^2(\mathbb{R}^3))\},$$

and

$$u_m \rightarrow u \text{ in } C([\tau, T]; L^2(\Omega_t)) \text{ if and only if } \widehat{u}_m \rightarrow \widehat{u} \text{ in } C([\tau, T]; L^2(\mathbb{R}^3)).$$

Similar definition applies with  $L^2$  replaced by  $H_0^1$ .

If assumption (2.3) is also satisfied, then we have more specialized properties. Using (2.6), given  $u \in L^1_{\text{loc}}(\mathbb{R}; L^q(\Omega_t))$  there exists a unique  $v \in L^1_{\text{loc}}(\mathbb{R}; L^q(\Omega))$  such that

$$v(y, t) = u(r(y, t), t) = u(x, t), \quad \forall t \in \mathbb{R}.$$

Then, there exist positive constants  $c_1, \dots, c_4$ , not dependent of  $t$ , but  $c_1, c_2$  dependent of  $q$ , such that

$$c_1 \|u(t)\|_{L^q(\Omega_t)} \leq \|v(t)\|_{L^q(\Omega)} \leq c_2 \|u(t)\|_{L^q(\Omega_t)}$$

and

$$c_3 \|\nabla u(t)\|_{L^2(\Omega_t)} \leq \|\nabla v(t)\|_{L^2(\Omega)} \leq c_4 \|\nabla u(t)\|_{L^2(\Omega_t)}.$$

These estimates imply that there exists an embedding constant  $\mu_q > 0$ , independent of  $t$ , such that

$$\mu_q \|u(t)\|_{L^q(\Omega_t)}^q \leq \|\nabla u(t)\|_{L^2(\Omega_t)}^q, \quad 1 \leq q \leq 6. \quad (2.12)$$

In particular we get (2.5). Moreover, it is proved in [18, Lemma 3.5] that for  $1 \leq p, q \leq \infty$ , we have

$$u \in L^q(\tau, T; L^p(\Omega_t)) \text{ if and only if } v \in L^q(\tau, T; L^p(\Omega)),$$

and there exist  $c_i = c_i(p, q, \tau, T)$ ,  $i = 5, 6$ , such that

$$c_5 \|u\|_{L^q(\tau, T; L^p(\Omega_t))} \leq \|v\|_{L^q(\tau, T; L^p(\Omega))} \leq c_6 \|u\|_{L^q(\tau, T; L^p(\Omega_t))}. \quad (2.13)$$

Analogously (cf. [18, Lemma 3.6]),

$$u \in L^2(\tau, T; H_0^1(\Omega_t)) \text{ if and only if } v \in L^2(\tau, T; H_0^1(\Omega)),$$

and there exist  $c_i = c_i(\tau, T)$ ,  $i = 7, 8$ , such that

$$c_7 \|u\|_{L^2(\tau, T; H_0^1(\Omega_t))} \leq \|v\|_{L^2(\tau, T; H_0^1(\Omega))} \leq c_8 \|u\|_{L^2(\tau, T; H_0^1(\Omega_t))}. \quad (2.14)$$

With (2.13) and (2.14) we can prove the following compactness lemma of Aubin-Lions type for non-cylindrical domains.

**Lemma 2.1.** *Suppose the assumption (H1) holds and let  $\{u_n\}$  be a bounded sequence of  $L^2(\tau, T; H_0^1(\Omega_t))$  such that  $\{\partial_t u_n\}$  is bounded in  $L^2(\tau, T; L^2(\Omega_t))$ . Then for any  $p \in [2, 6)$  there exists a subsequence  $\{u_{n_k}\}$  that converges strongly in  $L^2(\tau, T; L^p(\Omega_t))$ .*

*Proof.* Let  $v_n(\cdot, t) = u_n(r(\cdot), t)$ . From (2.13) and (2.14) we know that

$$\{v_n\} \text{ is bounded in } L^2(\tau, T; H_0^1(\Omega)),$$

$$\{\partial_t v_n\} \text{ is bounded in } L^2(\tau, T; L^2(\Omega)).$$

Then, from the classical Aubin-Lions Lemma [20, Theorem 1.5.1] there exists a subsequence  $\{v_{n_k}\}$  which converges strongly in  $L^2(\tau, T; L^p(\Omega))$ , say  $v_{n_k} \rightarrow v$ , for  $p \in [2, 6)$ . Using (2.13) again, with  $u(\cdot, t) = v(r^{-1}(\cdot), t)$ , we see that

$$\|u_{n_k} - u\|_{L^2(\tau, T; L^p(\Omega_t))} \leq \frac{1}{c_5} \|v_{n_k} - v\|_{L^2(\tau, T; L^p(\Omega))}.$$

Then  $\{u_{n_k}\}$  converges strongly in  $L^2(\tau, T; L^p(\Omega_t))$ . □

## 2.3 Energy estimates

**Definition 2.1.** *A function  $u$  is a weak solution of the problem (1.1)-(1.3) if for any pair  $\tau \leq T$ ,*

$$u \in C([\tau, T]; H_0^1(\Omega_t)) \cap C^1([\tau, T]; L^2(\Omega_t)),$$

$u(\tau) = u_\tau^0$ ,  $\partial_t u(\tau) = u_\tau^1$ , and

$$\int_\tau^T \int_{\Omega_t} \left\{ -\frac{\partial u}{\partial t} \frac{\partial \phi}{\partial t} + \nabla u \nabla \phi + \frac{\partial u}{\partial t} \phi + f(u) \phi - g \phi \right\} dx dt = 0,$$

for all  $\phi \in L^2(\tau, T; H_0^1(\Omega_t))$  satisfying  $\partial_t \phi \in L^2(\tau, T; L^2(\Omega_t))$  and  $\phi(\tau) = \phi(T) = 0$ . The above solution is called strong if in addition,

$$u \in L^\infty(\tau, T; H^2(\Omega_t) \cap H_0^1(\Omega_t)), \quad \partial_t u \in L^\infty(\tau, T; H_0^1(\Omega_t)), \quad \partial_t^2 u \in L^\infty(\tau, T; L^2(\Omega_t)).$$

**Lemma 2.2.** *Suppose that  $Q$  has regular lateral boundary  $\Sigma$  and that  $w \in C^1(\mathbb{R}; L^2(\Omega_t))$ . Then*

$$\frac{d}{dt} \int_{\Omega_t} w(x, t) dx = \int_{\Omega_t} \partial_t w dx - \int_{\Gamma_t} w(x, t) \nu_t d\sigma. \quad (2.15)$$

*Proof.* We know (e.g. [13]) that

$$\frac{d}{dt} \int_{\Omega_t} w(x, t) dx = \int_{\Omega_t} \partial_t w(x, t) dx + \int_{\Gamma_t} w(x, t) \dot{x} \cdot \nu_x d\sigma,$$

where  $\dot{x}$  is the velocity of a point  $x \in \Gamma_t$ . From assumption (H1), the velocity of the movement of  $x$  along  $\Sigma$  is  $(\dot{x}, 1)$ , which is tangent to  $\Sigma$ . It follows that

$$\dot{x} \cdot \nu_x = -\nu_t,$$

and therefore (2.15) holds.  $\square$

Now we prove an energy inequality that plays a key role in our study. The energy of the problem (1.1)-(1.3) is defined by

$$E(t) = \frac{1}{2} \int_{\Omega_t} (|\partial_t u|^2 + |\nabla u|^2) dx + \int_{\Omega_t} F(u) dx. \quad (2.16)$$

**Lemma 2.3.** *Under assumptions (H1)-(H2), the energy along any strong solution  $u$  satisfies*

$$\frac{d}{dt} E(t) \leq - \int_{\Omega_t} |\partial_t u|^2 dx + \int_{\Omega_t} g(x, t) \partial_t u dx. \quad (2.17)$$

*Proof.* We multiply equation (1.1) by  $\partial_t u$ . Then

$$\frac{1}{2} \partial_t |\partial_t u|^2 + \frac{1}{2} \partial_t |\nabla u|^2 + \partial_t F(u) = -|\partial_t u|^2 + g(x, t) \partial_t u + \operatorname{div}(\nabla u \partial_t u).$$

Now integrating the identity over  $\Omega_t$  and taking into account (2.15) we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega_t} |\partial_t u|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega_t} |\nabla u|^2 dx + \frac{d}{dt} \int_{\Omega_t} F(u) dx \\ = - \int_{\Omega_t} |\partial_t u|^2 dx + \int_{\Omega_t} g(x, t) \partial_t u dx + R, \end{aligned}$$

where

$$R = - \int_{\Gamma_t} F(u) \nu_t d\sigma - \frac{1}{2} \int_{\Gamma_t} (|\partial_t u|^2 + |\nabla u|^2) \nu_t d\sigma + \int_{\Omega_t} \operatorname{div}(\nabla u \partial_t u) dx.$$



On the other hand, it was observed in Bardos and Chen [2] that for  $u \in H^1(Q_{\tau,T})$  with  $u = 0$  on  $\Sigma_{\tau,T}$ , all tangential derivative of  $u$  also vanishes on  $\Sigma$ . Consequently the full gradient of  $u$  satisfies  $\nabla_{x,t}u = (\partial_\nu u)\nu$  which implies that

$$\partial_t u = (\partial_\nu u)\nu_t \quad \text{and} \quad \nabla u = (\partial_\nu u)\nu_x. \quad (2.18)$$

Using (2.18) we have

$$\frac{1}{2} \int_{\Gamma_t} (|\partial_t u|^2 + |\nabla u|^2)\nu_t d\sigma = \frac{1}{2} \int_{\Gamma_t} |\partial_\nu u|^2(|\nu_x|^2 + |\nu_t|^2)\nu_t d\sigma,$$

and

$$\int_{\Omega_t} \operatorname{div}(\nabla u \partial_t u) dx = \int_{\Gamma_t} |\partial_\nu u|^2 |\nu_x|^2 \nu_t d\sigma.$$

Then, since  $F(u) = 0$  on  $\Gamma_t$ , we conclude that

$$R = \frac{1}{2} \int_{\Gamma_t} |\partial_\nu u|^2 (|\nu_x|^2 - |\nu_t|^2) \nu_t d\sigma.$$

Finally, from Remark 2.1 we know that  $\nu_t \leq 0$  and  $|\nu_t| \leq |\nu_x|$  (expanding and time-like). Then,  $R \leq 0$  and therefore energy inequality (2.17) holds.  $\square$

**Lemma 2.4.** *Under assumptions (H1)-(H2), there exist positive constants  $\beta_0, C_f, C_F$ , independent of initial data and initial time, such that*

$$\beta_0 \|(u(t), \partial_t u(t))\|_{X_t}^2 - C_f \leq E(t) \leq C_F (1 + \|(u(t), \partial_t u(t))\|_{X_t}^4), \quad \forall t \in \mathbb{R}. \quad (2.19)$$

*Proof.* From assumption (2.8) and Poincaré's inequality (2.12) we have

$$\int_{\Omega_t} F(u) dx \geq -\frac{\beta}{2\lambda_1^*} \int_{\Omega_t} |\nabla u|^2 dx - \rho|\Omega^*|.$$

Then, the first inequality of (2.19) holds with

$$\beta_0 = \frac{1}{2} \left(1 - \frac{\beta}{\lambda_1^*}\right) \quad \text{and} \quad C_f = \rho|\Omega^*|. \quad (2.20)$$

To prove the second inequality we first note that from the growth assumption (2.7) there exists a constant  $C' > 0$  such that

$$F(u) \leq C'(1 + |u|^4), \quad \forall u \in \mathbb{R}.$$

Then we have

$$\int_{\Omega_t} F(u) dx \leq C'(|\Omega^*| + \|u\|_{L^4(\Omega_t)}^4),$$

and using (2.12) we obtain the second part of (2.19) with  $C_F = C' \max\{|\Omega^*|, \mu_4^{-1}\}$ .  $\square$

## 2.4 Well-posedness

The existence and uniqueness of solutions for semilinear wave equations with moving boundary is, roughly speaking, a standard result. Some minor comments are done in the following theorem, where we also prove the continuous dependence with respect to initial data.

**Theorem 2.5.** *Under assumptions (H1)-(H3), given an initial data  $(u_\tau^0, u_\tau^1) \in H_0^1(\Omega_\tau) \times L^2(\Omega_\tau)$ , problem (1.1)-(1.3) has a unique weak solution. If  $(u_\tau^0, u_\tau^1) \in [H^2(\Omega_\tau) \cap H_0^1(\Omega_\tau)] \times H_0^1(\Omega_\tau)$  and  $g \in H_{\text{loc}}^1(\mathbb{R}, L^2(\Omega_t))$ , then the above solution is strong. In addition, the weak solutions depend continuously on initial data.*

*Proof.* The existence of global solutions is essentially discussed in [10, 11, 15]. Accordingly, for each pair  $\tau < T$ , we must exhibit a “hyperbolic type”  $C^2$  mapping

$$\Phi : \mathbb{R}^3 \times (\tau, T) \rightarrow \mathbb{R}^3 \times (\tau, T),$$

such that  $\Phi(\overline{Q_{\tau, T}}) = \overline{\Omega} \times (\tau, T)$ . Writing  $\Phi(x, t) = (\phi_1(x, t), \dots, \phi_4(x, t))$ , this means that the sub-matrix

$$a_{ij} = \langle \nabla \phi_i, \nabla \phi_j \rangle - \frac{\partial \phi_i}{\partial t} \frac{\partial \phi_j}{\partial t}, \quad 1 \leq i, j \leq 3, \quad (2.21)$$

must be positive definite at each point of  $Q_{\tau, T}$ , where  $\nabla = \nabla_x$ , and also

$$\left( \frac{\partial \phi_4}{\partial t} \right)^2 - |\nabla \phi_4|^2 > 0. \quad (2.22)$$

In our case, taking into account (H1), we define

$$\phi_i(x, t) = \frac{x_i}{h_i(t)}, \quad 1 \leq i \leq 3 \quad \text{and} \quad \phi_4(x, t) = t.$$

Then, the condition (2.22) is clearly verified. To verify condition (2.21), we note that

$$\nabla \phi_i(x, t) = \frac{1}{h_i(t)} e_i \quad \text{and} \quad \frac{\partial \phi_i}{\partial t}(x, t) = -\frac{h_i'(t)}{h_i(t)} y_i, \quad 1 \leq i \leq 3,$$

where  $\{e_i\}$  is the standard basis of  $\mathbb{R}^3$ . Therefore,

$$a_{ij} = \frac{\delta_{ij} - h_i'(t)h_j'(t)y_i y_j}{h_i(t)h_j(t)}, \quad 1 \leq i, j \leq 3.$$

Let us denote  $\hat{y}_i = h_i' y_i$  and for any  $\xi \in \mathbb{R}^3$ , let  $\tilde{\xi}_i = \xi_i / h_i$ . Then,

$$\begin{aligned} \sum_{i,j=1}^3 a_{ij} \xi_i \xi_j &= |\tilde{\xi}|^2 - \langle \hat{y}, \tilde{\xi} \rangle^2 \\ &\geq (1 - \gamma^2 D^2) |\tilde{\xi}|^2 \\ &\geq \left( \frac{1 - \gamma^2 D^2}{h_m^2} \right) |\xi|^2, \end{aligned}$$

which, in view of (2.3), shows that  $(a_{ij})$  is positive definite. Therefore the existence and uniqueness of weak solutions of (1.1)-(1.3) follows from [10, Theorem 3.2]. Existence of strong solutions is considered in [11, 15].

In order to prove the continuous dependence of weak solutions with respect to initial data, we shall use the fact that  $Q$  is expanding. Let  $(u_{\tau_1}^0, u_{\tau_1}^1)$  and  $(u_{\tau_2}^0, u_{\tau_2}^1)$  be two pairs of initial data. By density arguments we can assume that they are in  $[H^2(\Omega_t) \times H_0^1(\Omega_t)] \times H_0^1(\Omega_t)$  so that we can perform all the calculus below. Then  $w = u_1 - u_2$ , where  $u_1, u_2$  are the corresponding solutions, satisfies

$$\partial_t^2 w - \Delta w + \partial_t w = f(u_2) - f(u_1), \quad x \in \Omega_t, \quad t \geq \tau, \quad (2.23)$$

with initial condition

$$w(\tau) = u_{\tau_1}^0 - u_{\tau_2}^0, \quad \partial_t w(\tau) = u_{\tau_1}^1 - u_{\tau_2}^1,$$

and Dirichlet boundary condition. We multiply the equation (2.23) by  $\partial_t w$  and arguing as in the proof of energy inequality (2.17) we infer that

$$\frac{d}{dt} \int_{\Omega_t} (|\partial_t w|^2 + |\nabla w|^2) dx + 2 \int_{\Omega_t} |\partial_t w|^2 dx \leq 2 \int_{\Omega_t} (f(u_2) - f(u_1)) \partial_t w dx.$$

On the other hand, from assumption (H2), and fixing a time interval  $[\tau, T]$ , there exists  $C_0 > 0$  such that

$$\begin{aligned} 2 \int_{\Omega_t} (f(u_2) - f(u_1)) \partial_t w dx &\leq C \left( 1 + \|u_1\|_{L^6(\Omega_t)}^2 + \|u_2\|_{L^6(\Omega_t)}^2 \right) \|w\|_{L^6(\Omega_t)} \|\partial_t w\|_{L^2(\Omega_t)} \\ &\leq C_0 \left( \|\nabla w(t)\|_{L^2(\Omega_t)}^2 + \|\partial_t w(t)\|_{L^2(\Omega_t)}^2 \right), \quad \tau \leq t \leq T. \end{aligned}$$

Then the Gronwall inequality implies that

$$\begin{aligned} &\|\nabla u_1(t) - \nabla u_2(t)\|_{L^2(\Omega_t)}^2 + \|\partial_t u_1(t) - \partial_t u_2(t)\|_{L^2(\Omega_t)}^2 \\ &\leq e^{C_0(t-\tau)} \left( \|\nabla u_{1\tau}^0 - \nabla u_{2\tau}^0\|_{L^2(\Omega_t)}^2 + \|u_{1\tau}^1 - u_{2\tau}^1\|_{L^2(\Omega_t)}^2 \right), \quad \tau \leq t \leq T. \end{aligned}$$

This completes the proof of the Theorem.  $\square$

### 3 Pullback dynamics

In order to establish our main result (Theorem 4.1) we first give some definitions and results about pullback dynamics. Our presentation is based on [17] that is concerned with dynamics of evolution processes defined in time-dependent spaces  $X_t$ . An alternative approach can be found in [9]. In the case of fixed metric space  $(X_t, d_t) = (X, d)$  (for all  $t \in \mathbb{R}$ ) a comprehensive theory of pullback attractors can be found in [3].

### 3.1 Attractors on time-dependent spaces

**Definition 3.1.** Let  $\{X_t\}_{t \in \mathbb{R}}$  be a family of non-empty metric spaces. A two-parameter operator  $U(t, \tau) : X_\tau \rightarrow X_t$ ,  $\tau \leq t$ , is called an evolution process if

$$(i) \quad U(\tau, \tau) = I_\tau, \quad \forall \tau \in \mathbb{R} \text{ (identity operator of } X_\tau),$$

$$(ii) \quad U(t, \tau) = U(t, s)U(s, \tau), \quad -\infty < \tau \leq s \leq t < \infty.$$

In addition, it is called closed if whenever a sequence  $x_n \rightarrow x$  in  $X_\tau$  and  $U(t, \tau)x_n \rightarrow y$  in  $X_t$ , then  $U(t, \tau)x = y$ .

**Remark 3.1.** (a) In general, if an evolution PDE is well-posed with respect to the phase space, then it generates a continuous evolution process. Of course, continuous processes are closed. Here, the notion of closed process is a natural extension of that for closed semigroups, introduced in [23]. Other notions of continuity such as strong-weak (norm-to-weak) can also be defined to the non-cylindrical framework.

(b) In particular, under the assumptions of Theorem 2.5, problem (1.1)-(1.3) generates a continuous evolution process  $U(t, \tau) : X_\tau \rightarrow X_t$  with  $X_t = H_0^1(\Omega_t) \times L^2(\Omega_t)$ . Here,  $X_t$  will be equipped with its natural inner-product,

$$((u_1, v_1), (u_2, v_2))_{X_t} = \int_{\Omega_t} \nabla u_1 \cdot \nabla u_2 \, dx + \int_{\Omega_t} v_1 v_2 \, dx. \quad \square$$

In the following we define the universe of the objects that are to be attracted by the attractors.

**Notation.** By a capital letter with circumflex, say  $\widehat{D}$ , we mean a family

$$\widehat{D} = \{D(t)\}_{t \in \mathbb{R}} \text{ with } D(t) \subset X_t, \quad t \in \mathbb{R}.$$

This will be used several times. □

**Definition 3.2.** A universe with respect to a family  $\{X_t\}_{t \in \mathbb{R}}$  of metric spaces is a class  $\mathcal{D}$  of elements  $\widehat{D} = \{D(t)\}_{t \in \mathbb{R}}$  such that each section  $D(t)$  is a non-empty subset of  $X_t$ ,  $t \in \mathbb{R}$ . We say that a universe  $\mathcal{D}$  is inclusion closed if whenever  $\widehat{D} \in \mathcal{D}$  and  $\widehat{C}$  is such that,

$$C(t) \subset X_t, \quad C(t) \subset D(t), \quad \forall t \in \mathbb{R},$$

then  $\widehat{C} \in \mathcal{D}$ .

**Definition 3.3.** A family  $\widehat{A}$  is called a pullback  $\mathcal{D}$ -attractor for a process  $U(t, \tau) : X_\tau \rightarrow X_t$  if,

$$(i) \quad \text{for any } t \in \mathbb{R}, \quad A(t) \text{ is a non-empty compact subset of } X_t,$$

$$(ii) \quad \text{the family } \widehat{A} \text{ is pullback } \mathcal{D}\text{-attracting, that is, for any } \widehat{D} \in \mathcal{D},$$

$$\lim_{\tau \rightarrow -\infty} \text{dist}_{X_t}(U(t, \tau)D(\tau), A(t)) = 0, \quad \forall t \in \mathbb{R},$$

(iii) the family  $\widehat{A}$  is invariant, that is,

$$U(t, \tau)A(\tau) = A(t), \quad -\infty < \tau \leq t < \infty.$$

Moreover, a pullback attractor is said to be minimal if whenever  $\widehat{C}$  is a  $\mathcal{D}$ -attracting family of non-empty closed sets, then  $A(t) \subset C(t)$  for all  $t \in \mathbb{R}$ .

**Remark 3.2.** (a) We observe that the definition of pullback  $\mathcal{D}$ -attractors does not imply uniqueness (see [21] or [17, Remark 22]). In order to ensure uniqueness one needs either to include the minimality property or to impose additional conditions, as for instance, that the attractor belongs to the same family  $\mathcal{D}$ . As will be shown in Theorem 3.1, it is possible under very general hypotheses to ensure the existence of a pullback  $\mathcal{D}$ -attractor which is minimal.

(b) If  $X_t = X$  is a fixed space for all  $t \in \mathbb{R}$  and  $\mathcal{D}$  is the class of all bounded subsets of  $X$ , then pullback  $\mathcal{D}$ -attraction becomes the usual pullback attraction, that is,  $A \subset X$  is attracting at time  $t$  if,

$$\lim_{\tau \rightarrow -\infty} \text{dist}_X(U(t, \tau)D, A) = 0,$$

for any bounded set  $D \subset X$ . □

The next two definitions are about dissipativeness and compactness in the  $\mathcal{D}$ -pullback sense.

**Definition 3.4.** A family  $\widehat{B}$  of non-empty sets is called pullback  $\mathcal{D}$ -absorbing for a process  $U(t, \tau) : X_\tau \rightarrow X_t$  if for any  $t \in \mathbb{R}$  and  $\widehat{D} \in \mathcal{D}$ , there exists  $\tau_0(t, \widehat{D}) \leq t$  such that

$$U(t, \tau)D(\tau) \subset B(t) \text{ if } \tau \leq \tau_0(t, \widehat{D}).$$

**Definition 3.5.** Given a family  $\widehat{D}$ , a process  $U(t, \tau) : X_\tau \rightarrow X_t$  is pullback  $\widehat{D}$ -asymptotically compact if, whenever  $t \in \mathbb{R}$ ,  $\{\tau_n\} \subset (-\infty, t]$  with  $\tau_n \rightarrow -\infty$  and  $y_n \in D(\tau_n)$ , the sequence  $\{U(t, \tau_n)y_n\}$  has a convergent subsequence in  $X_t$ . If a process is pullback  $\widehat{D}$ -asymptotically compact for any  $\widehat{D} \in \mathcal{D}$ , then we say it is pullback  $\mathcal{D}$ -asymptotically compact.

The following existence theorem is based on a result for  $X_t = X$  presented in [14, Theorem 3.11]. It is a slight generalization of the one proved in [17, Theorem 23].

**Theorem 3.1.** Let  $U(t, \tau) : X_\tau \rightarrow X_t$  be a closed evolution process defined on a family of metric spaces  $\{X_t\}_{t \in \mathbb{R}}$ . Consider a universe  $\mathcal{D}$  with respect to the family  $\{X_t\}_{t \in \mathbb{R}}$  and suppose that  $U$  admits a pullback  $\mathcal{D}$ -absorbing family  $\widehat{B}_0$  and that  $U$  is pullback  $\widehat{B}_0$ -asymptotically compact. Then, the family  $\mathcal{A}_{\mathcal{D}} = \{A_0(t)\}_{t \in \mathbb{R}}$  defined by

$$A_0(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}^{X_t}, \quad (3.1)$$

where  $\Lambda$  denotes the pullback omega-limit

$$\Lambda(\widehat{D}, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau) D(\tau)}^{X_t},$$

is the minimal pullback  $\mathcal{D}$ -attractor for  $U$ . If  $\widehat{B}_0 \in \mathcal{D}$ , then

$$A_0(t) = \Lambda(\widehat{B}_0, t) \subset \overline{B_0(t)}^{X_t}. \quad (3.2)$$

In addition, if  $B_0(t)$  is closed for all  $t \in \mathbb{R}$  and the universe  $\mathcal{D}$  is inclusion closed, then the pullback attractor  $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ .

*Proof.* The proof of the existence of the minimal attractor follows from the same arguments of [14, Theorem 3.11], making suitable changes as in [17, Theorem 23]. It is worth mentioning that existence of a minimal pullback  $\mathcal{D}$ -attracting family is proved without assuming that the process is closed (continuous). The assumption that the process is closed (continuous) is only required in order to prove the invariance of the attractors. The details are left to the reader. Here we present a self-contained proof of (3.2) and the final claim of the fact that the attractor belongs to the universe.

Given  $\widehat{D} \in \mathcal{D}$ , let us take  $y \in \Lambda(\widehat{D}, t)$ . Then there exist a sequence  $\tau_n \leq t$ ,  $\tau_n \rightarrow -\infty$ , and a sequence  $x_n \in D(\tau_n)$ , such that

$$y = \lim_{n \rightarrow \infty} U(t, \tau_n) x_n.$$

Now, for each integer  $k \geq 1$ , because  $\widehat{B}_0$  is pullback  $\mathcal{D}$ -absorbing, there exists  $n_k > k$  such that  $z_k = U(\tau_k, \tau_{n_k}) x_{n_k} \in B_0(\tau_k)$ . Then

$$y = \lim_{k \rightarrow \infty} U(t, \tau_{n_k}) x_{n_k} = \lim_{k \rightarrow \infty} U(t, \tau_k) z_k \in \Lambda(\widehat{B}_0, t), \quad (3.3)$$

which shows that  $\Lambda(\widehat{D}, t) \subset \Lambda(\widehat{B}_0, t)$  for all  $\widehat{D} \in \mathcal{D}$ . In particular, from (3.1) we see that  $A_0(t) \subset \Lambda(\widehat{B}_0, t)$ , since  $\Lambda(\widehat{B}_0, t)$  is closed from its definition. If we assume that  $\widehat{B}_0 \in \mathcal{D}$ , it follows from (3.1) that  $\Lambda(\widehat{B}_0, t) \subset A_0(t)$ . Therefore we proved that  $A_0(t) = \Lambda(\widehat{B}_0, t)$ . To conclude, under assumption that  $\widehat{B}_0 \in \mathcal{D}$ , we have for  $k$  sufficiently large,

$$y_k = U(t, \tau_k) z_k \in B_0(t).$$

Then from second limit in (3.3) we have  $y \in \overline{B_0(t)}^{X_t}$ , and therefore (3.2) holds. Moreover, if each  $B_0(t)$  is closed and  $\mathcal{D}$  is inclusion closed, then this implies that  $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ .  $\square$

### 3.2 A criterion for pullback $\mathcal{D}$ -asymptotic compactness

In this section we recall a well-known compactness criterion established in Chueshov and Lasiecka [6, Proposition 3.2] and [7, Proposition 2.10], for autonomous systems. Non-autonomous versions of that result were presented in [24, 26], with  $X_t = X$ , and in [22] with time-dependent spaces. To our purpose, we consider this compactness criterion in a  $\mathcal{D}$ -universe framework.

**Definition 3.6.** Let  $X$  be a metric space. Then we say that a function  $\Psi : X \times X \rightarrow \mathbb{R}$  is contractive on a bounded subset  $B$  of  $X$  if for any sequence  $\{x_n\}$  of  $B$  there exists a subsequence  $\{x_{n_k}\}$  such that

$$\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \Psi(x_{n_k}, x_{n_l}) = 0.$$

**Theorem 3.2.** Let  $\{X_t\}_{t \in \mathbb{R}}$  be a family of Banach spaces and let  $U(t, \tau) : X_\tau \rightarrow X_t$  be an evolution process that possesses a pullback  $\mathcal{D}$ -absorbing family  $\widehat{B}_0 = \{B_0(\tau)\}_{\tau \in \mathbb{R}}$ . Suppose that for any  $t \in \mathbb{R}$  and  $\epsilon > 0$  there exists a time  $\tau_\epsilon \leq t$  and a contractive function  $\Psi_\epsilon : B_0(\tau_\epsilon) \times B_0(\tau_\epsilon) \rightarrow \mathbb{R}$ , such that

$$\|U(t, \tau_\epsilon)x - U(t, \tau_\epsilon)y\|_{X_t} \leq \epsilon + \Psi_\epsilon(x, y), \quad \forall x, y \in B_0(\tau_\epsilon).$$

Then the process is pullback  $\mathcal{D}$ -asymptotically compact.

*Proof.* Let  $t \in \mathbb{R}$  and  $\widehat{D} \in \mathcal{D}$  be fixed, and consider  $\{\tau_n\}$  with  $\tau_n < t$  for all  $n$ , and with  $\tau_n \rightarrow -\infty$  and  $x_n \in D(\tau_n)$ . We are going to show that  $\{U(t, \tau_n)x_n\}_{n \in \mathbb{N}}$  has a convergent subsequence in  $X_t$ .

Consider a sequence  $\{\epsilon_n\}$  with  $\epsilon_n \downarrow 0$ .

(a) Given  $\epsilon_1$ , by hypothesis there exists a time  $\tau_{\epsilon_1} \leq t$  and a contractive function  $\Psi_{\epsilon_1} : B_0(\tau_{\epsilon_1}) \times B_0(\tau_{\epsilon_1}) \rightarrow \mathbb{R}$  such that

$$\|U(t, \tau_{\epsilon_1})x - U(t, \tau_{\epsilon_1})y\|_{X_t} \leq \frac{\epsilon_1}{2} + \Psi_{\epsilon_1}(x, y), \quad \forall x, y \in B_0(\tau_{\epsilon_1}).$$

(b) Because  $\{B_0(\tau)\}_{\tau \in \mathbb{R}}$  is pullback  $\mathcal{D}$ -absorbing, there exists  $n_1 \in \mathbb{N}$  such that  $\tau_{n_1} \leq \tau_{\epsilon_1}$  and

$$U(\tau_{\epsilon_1}, \tau_n)D(\tau_n) \subset B_0(\tau_{\epsilon_1}), \quad n \geq n_1,$$

and then

$$y_n = U(\tau_{\epsilon_1}, \tau_n)x_n \in B_0(\tau_{\epsilon_1}), \quad n \geq n_1.$$

Now, since  $\Psi_{\epsilon_1}$  is contractive, we can choose subsequences  $\{x_n^1\}$  of  $\{x_n\}$  and  $\{\tau_n^1\}$  of  $\{\tau_n\}$ , such that

$$y_n^1 = U(\tau_{\epsilon_1}, \tau_n^1)x_n^1 \in B_0(\tau_{\epsilon_1}) \quad \text{and satisfies} \quad \Psi_{\epsilon_1}(y_k^1, y_l^1) \leq \frac{\epsilon_1}{2}, \quad \forall k, l \geq 1.$$

(c) Therefore,

$$\begin{aligned} \|U(t, \tau_k^1)x_k^1 - U(t, \tau_l^1)x_l^1\|_{X_t} &= \|U(t, \tau_{\epsilon_1})U(\tau_{\epsilon_1}, \tau_k^1)x_k^1 - U(t, \tau_{\epsilon_1})U(\tau_{\epsilon_1}, \tau_l^1)x_l^1\|_{X_t} \\ &= \|U(t, \tau_{\epsilon_1})y_k^1 - U(t, \tau_{\epsilon_1})y_l^1\|_{X_t} \\ &\leq \frac{\epsilon_1}{2} + \Psi_{\epsilon_1}(y_k^1, y_l^1) \leq \epsilon_1, \quad \forall k, l \geq 1. \end{aligned}$$

By induction, there exist subsequences  $\{x_n^m\}$  of  $\{x_n^{m-1}\}$  and  $\{\tau_n^m\}$  of  $\{\tau_n^{m-1}\}$  such that

$$\|U(t, \tau_k^m)x_k^m - U(t, \tau_l^m)x_l^m\|_{X_t} \leq \epsilon_m, \quad \forall k, l \geq m.$$

As a consequence, since  $\epsilon_m \rightarrow 0$ , the diagonal subsequence

$$\{U(t, \tau_k^k)x_k^k\}_{k \in \mathbb{N}}$$

is a Cauchy sequence in  $X_t$ . This concludes the proof.  $\square$

**Remark 3.3.** The above theorem was presented, for simplicity, in a Banach space framework because the definition of  $X_t$  in our moving boundary problem. However the result is valid for a family of complete metric spaces.  $\square$

## 4 Wave equation with moving boundary

### 4.1 Existence of pullback $\mathcal{D}$ -attractors

We begin with the description of our universe  $\mathcal{D}$ . Given a function  $\rho : \mathbb{R} \rightarrow \mathbb{R}^+$ , we can define a family of closed balls

$$\overline{B}_{X_t}(0, \rho(t)) = \{z \in X_t \mid \|z\|_{X_t} \leq \rho(t)\}, \quad t \in \mathbb{R}.$$

In particular we consider the class of such balls satisfying

$$\lim_{\tau \rightarrow -\infty} |\rho(\tau)|^4 e^{\sigma_1 \tau} = 0, \quad (4.1)$$

where  $\sigma_1 > 0$  is a decay coefficient, depending only on  $\beta$  and  $\lambda_1^*$ , given in (2.11). Then we define our universe as

$$\mathcal{D} = \{\widehat{D} \mid D(t) \neq \emptyset \text{ and } D(t) \subset \overline{B}_{X_t}(0, \rho_{\widehat{D}}(t)) \text{ with } \rho_{\widehat{D}} \text{ satisfying (4.1)}\}. \quad (4.2)$$

Our main result is the following.

**Theorem 4.1.** *Suppose that assumptions (H1)-(H3) hold. Then the process associated to the problem (1.1)-(1.3) admits a minimal pullback  $\mathcal{D}$ -attractor, where  $\mathcal{D}$  is defined in (4.2). If in addition,  $\sigma_0 < \sigma_1/2$ , then above pullback  $\mathcal{D}$ -attractor belongs to  $\mathcal{D}$ .*

The proof of Theorem 4.1 relies on the existence of an absorbing family and asymptotic compactness for the corresponding process.

### 4.2 Pullback $\mathcal{D}$ -absorbing

**Lemma 4.2.** *There exist constants  $C_1, C_2, C_3 > 0$ , not depending on  $\tau \leq t$ ,  $z \in X_\tau$ , such that*

$$\|U(t, \tau)z\|_{X_t}^2 \leq C_1(1 + \|z\|_{X_\tau}^4) e^{-\sigma_1(t-\tau)} + C_2 \int_\tau^t e^{-\sigma_1(t-s)} \|g(s)\|_{L^2(\Omega_s)}^2 ds + C_3 C_f, \quad (4.3)$$

where  $\sigma_1 > 0$  is defined in (2.11).

*Proof.* The proof is divided into several steps.

*Step 1: Perturbed energy.* Let us define

$$\psi(t) = \int_{\Omega_t} u \partial_t u \, dx \quad \text{and} \quad E_\varepsilon(t) = E(t) + \varepsilon \psi(t),$$



where  $\varepsilon > 0$  is a constant. Then there exists  $\varepsilon_0 > 0$  such that

$$\frac{1}{2}E(t) - \frac{1}{2}C_f \leq E_\varepsilon(t) \leq \frac{3}{2}E(t) + \frac{1}{2}C_f, \quad \varepsilon \leq \varepsilon_0, \quad t \in \mathbb{R}. \quad (4.4)$$

To prove this we note that from (2.5) and (2.19)

$$\begin{aligned} |\psi(t)| &\leq \frac{1}{2} \max \left\{ 1, \frac{1}{\lambda_1^*} \right\} \|(u, \partial_t u)\|_{X_t}^2 \\ &\leq \frac{1}{2\beta_0} \max \left\{ 1, \frac{1}{\lambda_1^*} \right\} (E(t) + C_f). \end{aligned}$$

Then, taking

$$\varepsilon_0 = \beta_0 \min\{1, \lambda_1^*\}, \quad (4.5)$$

we see that (4.4) holds.

*Step 2.* The following inequality holds.

$$\frac{d}{dt}\psi(t) \leq -E(t) + \frac{1}{2} \left( 3 + \frac{1}{\beta_0 \lambda_1^*} \right) \int_{\Omega_t} |\partial_t u|^2 dx + \frac{1}{2\beta_0 \lambda_1^*} \|g(t)\|_{L^2(\Omega_t)}^2 + C_f. \quad (4.6)$$

Indeed, since  $u\partial_t u = 0$  on  $\Gamma_t$ , from Lemma 2.2 we have

$$\frac{d}{dt}\psi(t) = \int_{\Omega_t} |\partial_t u|^2 dx + \int_{\Omega_t} \partial_t^2 u u dx.$$

Then using equation (1.1) and the definition of  $E(t)$  in (2.16), we have

$$\begin{aligned} \frac{d}{dt}\psi(t) &= -E(t) + \frac{3}{2} \int_{\Omega_t} |\partial_t u|^2 dx - \frac{1}{2} \int_{\Omega_t} |\nabla u|^2 dx \\ &\quad - \int_{\Omega_t} \partial_t u u dx + \int_{\Omega_t} [F(u) - f(u)u] dx + \int_{\Omega_t} g u dx. \end{aligned} \quad (4.7)$$

From (2.8) and (2.20) we have

$$\int_{\Omega_t} [F(u) - f(u)u] dx \leq \frac{\beta}{2\lambda_1^*} \int_{\Omega_t} |\nabla u|^2 dx + C_f.$$

From (2.5), Hölder and Young inequalities,

$$- \int_{\Omega_t} \partial_t u u dx \leq \frac{\beta_0}{2} \int_{\Omega_t} |\nabla u|^2 dx + \frac{1}{2\beta_0 \lambda_1^*} \int_{\Omega_t} |\partial_t u|^2 dx,$$

and

$$\int_{\Omega_t} g u dx \leq \frac{\beta_0}{2} \int_{\Omega_t} |\nabla u|^2 dx + \frac{1}{2\beta_0 \lambda_1^*} \|g(t)\|_{L^2(\Omega_t)}^2.$$

So, plugging the above inequalities into (4.7) and in view of definition of  $\beta_0$  in (2.20), we obtain (4.6).

*Step 3: Conclusion.* Note that from (2.17) we have

$$\frac{d}{dt}E(t) \leq -\frac{1}{2} \int_{\Omega_t} |\partial_t u|^2 dx + \frac{1}{2} \|g(t)\|_{L^2(\Omega_t)}^2,$$

and we take

$$\varepsilon = \min \left\{ \varepsilon_0, \left( 3 + \frac{1}{\beta_0 \lambda_1^*} \right)^{-1} \right\}. \quad (4.8)$$

Then, from (4.6),

$$\frac{d}{dt}E_\varepsilon(t) \leq -\varepsilon E(t) + \|g(t)\|_{L^2(\Omega_t)}^2 + \varepsilon C_f.$$

Taking into account the second inequality in (4.4),

$$\frac{d}{dt}E_\varepsilon(t) \leq -\frac{2\varepsilon}{3} E_\varepsilon(t) + \|g(t)\|_{L^2(\Omega_t)}^2 + \frac{4\varepsilon}{3} C_f.$$

Therefore, integrating over  $[\tau, t]$  and since  $\int_\tau^t e^{-k(t-s)} ds \leq 1/k$ , we get

$$E_\varepsilon(t) \leq E_\varepsilon(\tau) e^{-\frac{2\varepsilon}{3}(t-\tau)} + \int_\tau^t e^{-\frac{2\varepsilon}{3}(t-s)} \|g(s)\|_{L^2(\Omega_s)}^2 ds + 2C_f.$$

Using again (4.4) we obtain

$$E(t) \leq 3E(\tau) e^{-\frac{2\varepsilon}{3}(t-\tau)} + 2 \int_\tau^t e^{-\frac{2\varepsilon}{3}(t-s)} \|g(s)\|_{L^2(\Omega_s)}^2 ds + 6C_f. \quad (4.9)$$

Now, inspecting the definition of  $\varepsilon$  in (4.8) we infer that  $2\varepsilon/3 = \sigma_1$ . Then, estimate (4.3) follows from (4.9) and both sides of (2.19) with  $C_1 = 3C_F \beta_0^{-1}$ ,  $C_2 = 2\beta_0^{-1}$  and  $C_3 = 7\beta_0^{-1}$ .  $\square$

**Lemma 4.3.** *Let us define  $\rho_0(t)$  such that*

$$|\rho_0(t)|^2 = C_2 \int_{-\infty}^t e^{-\sigma_0(t-s)} \|g(s)\|_{L^2(\Omega_s)}^2 ds + C_3 C_f + 1. \quad (4.10)$$

*Then the family  $\widehat{B}_0 = \{B_0(t)\}_{t \in \mathbb{R}}$  defined by the closed balls*

$$B_0(t) = \overline{B}_{X_t}(0, \rho_0(t))$$

*is pullback  $\mathcal{D}$ -absorbing.*

*Proof.* Firstly we observe that assumptions (2.9)-(2.10) imply that

$$\int_{-\infty}^t e^{-\sigma_0(t-s)} \|g(s)\|_{L^2(\Omega_s)}^2 ds < \infty, \quad \forall t \in \mathbb{R},$$

and therefore (4.10) is well-defined. Now, let  $\widehat{D} \in \mathcal{D}$  and  $t \in \mathbb{R}$ . Since  $\sigma_0 \leq \sigma_1$ , we can majorate  $e^{-\sigma_1(t-s)}$  by  $e^{-\sigma_0(t-s)}$  in the integral appearing in (4.3). Then we have,

$$\begin{aligned} \|U(t, \tau) z_\tau\|_{X_t}^2 &\leq C_1 (1 + |\rho_{\widehat{D}}(\tau)|^4) e^{-\sigma_1(t-\tau)} + C_2 \int_\tau^t e^{-\sigma_0(t-s)} \|g(s)\|_{L^2(\Omega_s)}^2 ds + C_3 C_f \\ &\leq C_1 e^{-\sigma_1 t} (1 + |\rho_{\widehat{D}}(\tau)|^4) e^{\sigma_1 \tau} + |\rho_0(t)|^2 - 1, \end{aligned} \quad (4.11)$$

for all  $z_\tau \in D(\tau)$ . Then, because  $(1 + |\rho_{\widehat{D}}(\tau)|^4)e^{\sigma_1\tau} \rightarrow 0$  as  $\tau \rightarrow -\infty$ , there exists  $\tau_0(t, \widehat{D}) < t$  such that

$$\|U(t, \tau)z_\tau\|_{X_t}^2 \leq |\rho_0(t)|^2, \quad \text{if } \tau < \tau_0(t, \widehat{D}), \quad z_\tau \in D(\tau),$$

that is,

$$U(t, \tau)D(\tau) \subset B_0(t) \quad \text{if } \tau < \tau_0(t, \widehat{D}).$$

This shows that  $\widehat{B}_0$  is a pullback  $\mathcal{D}$ -absorbing family.  $\square$

### 4.3 Pullback $\mathcal{D}$ -asymptotic compactness

To simplify the presentation we first prove a stabilization inequality.

**Lemma 4.4.** *Let  $\widehat{B}_0$  the pullback  $\mathcal{D}$ -absorbing family given by Lemma 4.3. Then, there exists a constant  $\sigma_2 > \sigma_1$ , and a constant  $C_{\tau,t} > 0$ , depending on  $\tau \leq t$ , such that*

$$\begin{aligned} \|U(t, \tau)z_1 - U(t, \tau)z_2\|_{X_t}^2 &\leq 3|\rho_0(\tau)|^2 e^{-\sigma_2(t-\tau)} + C_{\tau,t} \int_\tau^t \|u_1(s) - u_2(s)\|_{L^4(\Omega_s)}^2 ds \\ &\quad + 4 \int_\tau^t \int_{\Omega_s} (f(u_2) - f(u_1)) \partial_t w \, dx \, ds, \end{aligned} \quad (4.12)$$

where  $z_i \in B_0(\tau)$  and  $U(t, \tau)z_i = (u_i(t), \partial_t u_i(t))$ ,  $i = 1, 2$ .

*Proof.* Part of the arguments of the proof are similar to the ones of Lemma 4.2. First we observe that  $w = u_1 - u_2$  is a weak solution of

$$\partial_t^2 w - \Delta w + \partial_t w = f(u_2) - f(u_1), \quad x \in \Omega_t, \quad t \geq \tau, \quad (4.13)$$

with Dirichlet boundary condition and initial conditions

$$w(0) = u_1(0) - u_2(0) \quad \text{and} \quad \partial_t w(0) = \partial_t u_1(0) - \partial_t u_2(0).$$

Next we define the energy

$$G(t) = \frac{1}{2} \|(w(t), \partial_t w(t))\|_{X_t}^2.$$

Then by multiplying equation (4.13) by  $\partial_t w$  and integrating over  $\Omega_t$ , there exists  $C_2 > 0$  such that

$$\frac{d}{dt} G(t) \leq - \int_{\Omega_t} |\partial_t w|^2 \, dx + \int_{\Omega_t} (f(u_2) - f(u_1)) \partial_t w \, dx. \quad (4.14)$$

Let us set

$$\chi(t) = \int_{\Omega_t} \partial_t w w \, dx \quad \text{and} \quad G_\alpha(t) = G(t) + \alpha \chi(t), \quad \alpha > 0.$$

It is easy to see that

$$|\chi(t)| \leq \max \left\{ 1, \frac{1}{\lambda_1^*} \right\} G(t).$$

Then taking

$$\alpha_0 = \frac{1}{2} \min\{1, \lambda_1^*\}, \quad (4.15)$$

we have

$$\frac{1}{2}G(t) \leq G_\alpha(t) \leq \frac{3}{2}G(t), \quad \alpha \leq \alpha_0, \quad t \in \mathbb{R}. \quad (4.16)$$

Arguing as in the proof of Lemma 4.2, we obtain

$$\chi'(t) = -G(t) + \left(\frac{3}{2} + \frac{1}{\lambda_1^*}\right) \int_{\Omega_t} |\partial_t w|^2 dx + \int_{\Omega_t} (f(u_2) - f(u_1))w dx.$$

Since

$$\begin{aligned} \int_{\Omega_t} (f(u_2) - f(u_1))w dx &\leq C \left(1 + \|u_1(t)\|_{L^4(\Omega_t)}^2 + \|u_2(t)\|_{L^4(\Omega_t)}^2\right) \|w(t)\|_{L^4(\Omega_t)}^2 \\ &\leq C \left(1 + \|U(t, \tau)z_1\|_{X_t}^2 + \|U(t, \tau)z_2\|_{X_t}^2\right) \|w(t)\|_{L^4(\Omega_t)}^2, \end{aligned}$$

and from (4.11),

$$\|U(t, \tau)z_i\|_{X_t}^2 \leq C_1(1 + |\rho_0(\tau)|^4)e^{-\sigma_1(t-\tau)} + |\rho_0(t)|^2 - 1,$$

then there exists a constant  $k(\tau, t) > 0$  such that

$$\chi'(t) \leq -G(t) + \left(\frac{3}{2} + \frac{1}{\lambda_1^*}\right) \int_{\Omega_t} |\partial_t w|^2 dx + k(\tau, t)\|w(t)\|_{L^4(\Omega_t)}^2.$$

In view of (4.14), taking

$$\alpha = \min\left\{\alpha_0, \left(\frac{3}{2} + \frac{1}{\lambda_1^*}\right)^{-1}\right\}, \quad (4.17)$$

we see that, since  $\alpha \leq 1$ ,

$$\frac{d}{dt}G_\alpha(t) \leq -\alpha G(t) + k(\tau, t)\|w(t)\|_{L^4(\Omega_t)}^2 + \int_{\Omega_t} (f(u_2) - f(u_1))\partial_t w dx.$$

Then, using (4.16) and integrating over  $[\tau, t]$ , we obtain as before,

$$\begin{aligned} G_\alpha(t) &\leq G_\alpha(\tau)e^{-\frac{2\alpha}{3}(t-\tau)} + \sup_{s \in [\tau, t]} k(\tau, s) \int_\tau^t e^{-\frac{2\alpha}{3}(t-\tau)} \|w(s)\|_{L^4(\Omega_s)}^2 ds \\ &\quad + \int_\tau^t e^{-\frac{2\alpha}{3}(t-\tau)} \int_{\Omega_s} (f(u_2) - f(u_1))\partial_t w dx ds. \end{aligned}$$

Then, defining  $\sigma_2 = 2\alpha/3$ , and using (4.16) again, we obtain

$$\begin{aligned} G(t) &\leq 3G(\tau)e^{-\sigma_2(t-\tau)} + 2 \sup_{s \in [\tau, t]} k(\tau, s) \int_\tau^t \|w(s)\|_{L^4(\Omega_s)}^2 ds \\ &\quad + 2 \int_\tau^t \int_{\Omega_s} (f(u_2) - f(u_1))\partial_t w dx ds. \end{aligned}$$

In addition, from definition of  $G(t)$  we have  $G(\tau) \leq \frac{1}{2}|\rho_0(\tau)|^2$ , and therefore (4.12) follows with  $C_{\tau, t} = 4 \sup_{s \in [\tau, t]} k(\tau, s)$ . To complete the proof of the Lemma, from (4.5) and (4.15) we see that  $\varepsilon_0 < \alpha_0$ , since  $\beta_0 < 1/2$ . Then from (4.8) and (4.17) we conclude that  $\varepsilon < \alpha$ . This shows that  $\sigma_2 > \sigma_1$ .  $\square$

**Lemma 4.5.** *Under the assumptions of Theorem 4.1 the corresponding process is pullback  $\mathcal{D}$ -asymptotically compact.*

*Proof.* At the light of Lemma 4.4, we will construct a contractive function in order to apply Theorem 3.2. Given  $t \in \mathbb{R}$  and  $\epsilon > 0$ , from (4.10) we can write

$$|\rho_0(\tau)|^2 \leq \left( C_2 \int_{-\infty}^{\tau} e^{\sigma_0 s} \|g(s)\|_{L^2(\Omega_s)}^2 ds \right) e^{-\sigma_0 \tau} + C_3 C_f + 1, \quad \tau \leq t.$$

Since the integral in the above relation does not increase as  $\tau$  decreases, and  $\sigma_0 < \sigma_2$ , we have

$$\lim_{\tau \rightarrow -\infty} |\rho_0(\tau)|^2 e^{-\sigma_2(t-\tau)} = \lim_{\tau \rightarrow -\infty} e^{(\sigma_2 - \sigma_0)\tau} \left( C_2 \int_{-\infty}^{\tau} e^{\sigma_0 s} \|g(s)\|_{L^2(\Omega_s)}^2 ds \right) e^{-\sigma_2 t} = 0.$$

Therefore, there exists  $\tau_\epsilon = \tau_\epsilon(t, \epsilon) \leq t$  such that

$$3|\rho_0(\tau_0)|^2 e^{-\sigma_2(t-\tau_\epsilon)} < \epsilon^2.$$

Then, we define  $\Psi_\epsilon : B_0(\tau_\epsilon) \times B_0(\tau_\epsilon) \rightarrow \mathbb{R}$  by

$$\begin{aligned} \Psi_\epsilon(z_1, z_2)^2 &= C_{\tau_\epsilon, t} \int_{\tau_\epsilon}^t \|u_1(s) - u_2(s)\|_{L^4(\Omega_s)}^2 ds \\ &\quad + 4 \left| \int_{\tau_\epsilon}^t \int_{\Omega_s} (f(u_2) - f(u_1)) (\partial_t u_1 - \partial_t u_2) dx ds \right|, \end{aligned}$$

where  $C_{\tau_\epsilon, t} > 0$  is defined in (4.12). Then, from Lemma 4.4 we see that

$$\|U(t, \tau_\epsilon)y - U(t, \tau_\epsilon)z\|_{X_\tau} \leq \epsilon + \Psi_\epsilon(y, z), \quad \forall y, z \in B_0(\tau_\epsilon).$$

It remains to show that  $\Psi_\epsilon$  is contractive on  $B_0(\tau_\epsilon)$ . To this end, given any sequence  $\{z_n\}$  of  $B_0(\tau_\epsilon)$  we have, from (4.3) for instance, that

$$\|U(s, \tau_\epsilon)z_n\|_{X_s} \leq M_{\tau_\epsilon, t}, \quad \forall s \in [\tau_\epsilon, t].$$

Using the notation  $(u_n(s), \partial_t u_n(s)) = U(s, \tau_\epsilon)z_n$ , it follows that for some  $C_5 > 0$ ,

$$\|u_n\|_{L^2(\tau_\epsilon, t; H_0^1(\Omega_t))} \leq C_5 \quad \text{and} \quad \|\partial_t u_n\|_{L^2(\tau_\epsilon, t; L^2(\Omega_s))} \leq C_5.$$

Then, since  $H_0^1(\Omega_s)$  is compactly embedded in  $L^4(\Omega_s)$ , from Lemma 2.1 (Aubin-Lions) we conclude that there is  $u$  and a subsequence  $\{u_{n_k}\}$  such that

$$u_{n_k} \rightarrow u \quad \text{strongly in } L^2(\tau_\epsilon, t; L^4(\Omega_s)). \quad (4.18)$$

That is,

$$\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \int_{\tau_\epsilon}^t \|u_{n_k}(s) - u_{n_l}(s)\|_{L^4(\Omega_s)}^2 ds = 0. \quad (4.19)$$

Finally, we estimate the  $f$  term in  $\Psi_\epsilon$ . By integration,

$$\int_{\tau_\epsilon}^t \int_{\Omega_s} (f(u_{n_k}) - f(u_{n_l})) (\partial_t u_{n_k} - \partial_t u_{n_l}) dx ds = A(k, l) + B(k, l),$$

where

$$A(k, l) = \int_{\Omega_t} [F(u_{n_k}(t)) + F(u_{n_l}(t))] dx - \int_{\Omega_{\tau_\epsilon}} [F(u_{n_k}(\tau_\epsilon)) + F(u_{n_l}(\tau_\epsilon))] dx,$$

and

$$B(k, l) = - \int_{\tau_\epsilon}^t \int_{\Omega_s} [f(u_{n_k}) \partial_t u_{n_l} + f(u_{n_l}) \partial_t u_{n_k}] dx ds.$$

Since  $|F(u)| \leq C'(1 + |u|^4)$ , its Nemytskii map  $N_F : L^4(\Omega_s) \rightarrow L^1(\Omega_s)$  is continuous, for any fixed  $s$ . Then, the convergence (4.18) implies that

$$\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} A(k, l) = 2 \int_{\Omega_t} F(u(t)) dx - 2 \int_{\Omega_{\tau_\epsilon}} F(u(\tau_\epsilon)) dx.$$

Also, since the Nemytskii map  $N_f : H_0^1(\Omega_s) \rightarrow L^2(\Omega_s)$  maps bounded sets in bounded sets, for any fixed  $s$ , we see that  $f(u_n) \rightharpoonup f(u)$  weakly in  $L^2(\Omega_s)$ . Then, we infer that

$$\begin{aligned} \lim_{k \rightarrow \infty} \left( \lim_{l \rightarrow \infty} B(k, l) \right) &= -2 \int_{\tau_\epsilon}^t \int_{\Omega_s} f(u(s)) \partial_t u(s) dx ds \\ &= -2 \int_{\Omega_t} F(u(t)) dx + 2 \int_{\Omega_{\tau_\epsilon}} F(u(\tau_\epsilon)) dx. \end{aligned}$$

Therefore

$$\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \int_{\tau_\epsilon}^t \int_{\Omega_s} (f(u_{n_k}) - f(u_{n_l})) (\partial_t u_{n_k} - \partial_t u_{n_l}) dx ds = 0.$$

Combining this with (4.19) we see that  $\{z_n\}$  has a subsequence that

$$\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \Psi_\epsilon(z_{n_k}, z_{n_l}) = 0.$$

Then, the result follows from Theorem 3.2.  $\square$

## 4.4 Proof of main result

**Proof of Theorem 4.1** From Lemmas 4.3 and 4.5, the evolution process generated by the problem (1.1)-(1.3), with  $\sigma_0 \leq \sigma_1$  admits a pullback  $\mathcal{D}$ -absorbing family and it is pullback  $\mathcal{D}$ -asymptotically compact. Then, Theorem 3.1 guarantees the existence of the minimal pullback  $\mathcal{D}$ -attractor.

It remains to prove that the pullback attractor belongs to  $\mathcal{D}$  under further assumption  $\sigma_0 < \sigma_1/2$ . Clearly  $\mathcal{D}$  is inclusion closed. Then we must show that  $\widehat{B}_0 \in \mathcal{D}$ , that is

$$\lim_{\tau \rightarrow -\infty} |\rho_0(\tau)|^4 e^{\sigma_1 \tau} = 0. \quad (4.20)$$

From (4.10),

$$|\rho_0(\tau)|^2 e^{\frac{\sigma_1}{2}\tau} = \left( C_2 \int_{-\infty}^{\tau} e^{\sigma_0 s} \|g(s)\|_{L^2(\Omega_s)}^2 ds \right) e^{(\frac{\sigma_1}{2} - \sigma_0)\tau} + (C_3 C_f + 1) e^{\frac{\sigma_1}{2}\tau}.$$

Now, since the integral in the above relation is non-increasing as  $\tau$  decreases, and  $\frac{\sigma_1}{2} - \sigma_0 > 0$ , we see that

$$\lim_{\tau \rightarrow -\infty} |\rho_0(\tau)|^2 e^{\frac{\sigma_1}{2}\tau} = 0.$$

Therefore (4.20) holds and  $\widehat{B}_0 \in \mathcal{D}$ . Hence the second part of Theorem 3.1 guarantees that the attractor belongs to  $\mathcal{D}$ .  $\square$

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