

ROBUSTNESS OF TIME-DEPENDENT ATTRACTORS IN H^1 -NORM FOR NONLOCAL PROBLEMS

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To Professor Igor Chueshov, in Memoriam

ABSTRACT. In this paper, the existence of regular pullback attractors as well as their upper semicontinuous behaviour in H^1 -norm are analysed for a parameterized family of non-autonomous nonlocal reaction-diffusion equations without uniqueness, improving previous results [*Nonlinear Dyn.* **84** (2016), 35–50].

1. Introduction and existence results. Many nonlocal problems have been analysed in the last few decades due to their usefulness in real applications (e.g. cf. [23, 4, 25, 40, 3]). Namely, many authors have been interested in studying the nonlocal parabolic equation

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f,$$

where a is a continuous function and $l \in (L^2(\Omega))'$, i.e.

$$l(u) = l_g(u) = \int_{\Omega} g(x)u(x)dx.$$

From a biological point of view, the function u might represent the density of a population. Additional assumptions could be imposed on the function a to better reflect the behaviour of the community. For instance, to model species with a tendency to leave crowded zones, a natural assumption would be to assume that a is an increasing function of its argument. On the other hand, if we are dealing with species attracted by growing population, one would assume a to decrease. This equation has been used in epidemic theory and from a physical point of view, to study the heat propagation (for more details cf. [13, 14]).

It is worth highlighting that the above equation is not a trivial perturbation of the heat equation and serious difficulties arise in different contexts. For instance, the existence of a Lyapunov function is not guaranteed in a general framework. Additional requirements (see [14] for more details) or more specific nonlocal operators,

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which are strongly related to the diffusion terms (see [17, 15]), are needed to build this structure.

Many authors have been interested in studying the asymptotic behaviour of the solutions to problems of the same kind as the one presented above. For instance, in [13] Chipot and Lovat establish a comparison result between two stationary solutions and the solution of the evolution problem. Then, using this result, they prove the convergence of the solution of the evolution problem towards a stationary solution. Later, Chipot and Molinet [14] generalise the results obtained in [13] dealing with a continuum of steady states, making use of dynamical systems. In [16] Chipot and Siegwart consider a more general elliptic operator to study the long-time behaviour of the solutions. Chang and Chipot [11] analyse the same kind of results as in [13], however, in this case the authors deal with two nonlocal operators. Another interesting study is the one by Chipot and Zheng in [18], where the authors analyse the convergence of the solution of the evolution problem to one of the equilibria without assuming uniqueness of stationary solutions. Among several other results, Andami Ovono [1] analyses the existence of the compact global attractor in $L^2(\Omega)$.

If f also depends on the unknown u , the study of the stationary solutions is not trivial at all, and the natural generalization for the analysis of the long-time behaviour of the solutions is to consider the theory of attractors. Although an autonomous approach is almost new in this setting, it might be meaningful (and more general) to consider time-dependent terms in the model and then there are several approaches from the point of view of non-autonomous dynamical systems, like skew-product flows (see Sell [38]), uniform attractors and their kernel sections (cf. Chepyzhov and Vishik [12]) and pullback attractors (see Kloeden and Schmalfuß [30, 31] and Kloeden [28], also related to random dynamical systems [21]). This last approach allows us to minimize the assumptions on the forcing terms and the resultant objects are strictly invariant in a suitable *non-autonomous-dynamical-system sense*, unlike what happens with uniform attractors. Furthermore, pullback attractors let us study the behaviour of the current system considering initial times that come from the past (e.g. cf. [29] for more details). Many new results have appeared over the last years related to pullback attractors. Some authors have been interested in studying the pullback attractor in the classical sense, i.e. the pullback attractor of solutions starting in fixed bounded sets. Others, though, have employed the concept of attraction related to a class of families, called universe \mathcal{D} , which is made up of sets which are allowed to move in time and are usually defined in terms of a tempered condition (e.g. cf. [19, 9, 10]). This last approach has been used recently to study nonlocal problems (cf. [5, 7]). For instance, in [5] the existence of pullback attractors in $L^2(\Omega)$ and $H_0^1(\Omega)$ is established for a non-autonomous parabolic equation with nonlocal diffusion and sublinear terms. Later in [7], continuing in a single-valued framework (the nonlocal viscosity is locally Lipschitz), the existence of these families is analysed for a non-autonomous nonlocal reaction-diffusion equation.

In addition, in [6] the existence of pullback attractors in $L^2(\Omega)$ is analysed in a multi-valued framework for the non-autonomous nonlocal reaction-diffusion problem

$$\begin{cases} \frac{\partial u}{\partial t} - (1 - \varepsilon)a(l(u))\Delta u = f(u) + \varepsilon h(t) & \text{in } \Omega \times (\tau, \infty), \\ u = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x) & \text{in } \Omega, \end{cases}$$

where $\varepsilon \in [0, 1]$ and no locally Lipschitz assumption on the function a or monotonicity on the nonlinearity f are imposed (so lack of uniqueness). Moreover, the upper semicontinuous behaviour of attractors in L^2 -norm is studied. Namely, it is proved that the family of pullback attractors converges to the compact global attractor associated to the autonomous problem when ε goes to 0. [Do not confuse this notion, the upper-semicontinuous behaviour of attractors, with upper-semicontinuous process (see Definition 4). While being an upper-semicontinuous multi-valued process is a property only related with a two-parameter semigroup, the upper-semicontinuous behaviour of attractors takes into account all the processes indexed by the parameter in which it is taken the limit and it analyses an asymptotic property of a whole family of problems.]

In this paper we improve the results given in [6], analysing existence of pullback attractors in $H_0^1(\Omega)$ as well as their upper semicontinuous behaviour in H^1 -norm. According to [6, Section 6] we consider the parameterized family ($\varepsilon \in [0, 1]$) of non-autonomous nonlocal reaction-diffusion problems

$$(P_\varepsilon) \begin{cases} \frac{\partial u}{\partial t} - g_1(\varepsilon)a(l(u))\Delta u = \tilde{g}_1(\varepsilon)f(u) + g_0(\varepsilon)h(t) & \text{in } \Omega \times (\tau, \infty), \\ u = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded open subset of \mathbb{R}^N of class $C^{1,1}$, $\tau \in \mathbb{R}$, the nonlocal diffusion term fulfils that there exists $m > 0$ such that

$$a \in C(\mathbb{R}; [m, \infty)), \quad (1)$$

$$l \in (L^2(\Omega))'. \quad (2)$$

Concerning the family of perturbed coefficients, varying with the parameter ε , suppose that

$$g_1 \in C([0, 1]; (0, \infty)), \quad \tilde{g}_1 \in C([0, 1]; [0, \infty)), \quad g_0 \in C([0, 1]). \quad (3)$$

Actually, at some stages of the paper –not from the very beginning– it will be imposed that $g_1(0) = \tilde{g}_1(0) = 1$ and $g_0(0) = 0$ (the values of g_1 and \tilde{g}_1 at 0 are just for the sake of simplicity when dealing with the limit problem –see Theorems 7 and 8–; indeed any other values can be, after rearrangement, easily translated to this situation).

Regarding the nonlinearity, we assume that the function $f \in C^1(\mathbb{R})$ (cf. Remark 4 for a more general setting) and there exist positive constants α_1, α_2 and $\kappa, \eta \geq 0$ and $p \geq 2$ such that

$$-\kappa - \alpha_1|s|^p \leq f(s)s \leq \kappa - \alpha_2|s|^p \quad \forall s \in \mathbb{R}. \quad (4)$$

$$f'(s) \leq \eta \quad \forall s \in \mathbb{R}, \quad (5)$$

From (4) we deduce that there exists $\beta > 0$ such that

$$|f(s)| \leq \beta(|s|^{p-1} + 1) \quad \forall s \in \mathbb{R}. \quad (6)$$

The structure of this paper is as follows. In the rest of this Section 1, the setting of the problem is established as well as the existence of strong solutions and the regularising effect of the equation. Section 2 is devoted to providing abstract results on multi-valued non-autonomous dynamical systems which are essential to prove the existence of pullback attractors in $L^2(\Omega)$ and $H_0^1(\Omega)$ in the last two sections. Namely, in Section 3 we generalise some results of [6], which guarantee the existence of pullback attractors in $L^2(\Omega)$ and will be crucial in what follows. In Section 4,

we show the existence of pullback attractors in H^1 -norm and analyse their upper semicontinuity with respect to the parameter. Indeed, we prove that both families of attractors, those given in $L^2(\Omega)$ and $H_0^1(\Omega)$, converge to the compact global attractor associated to the autonomous problem (P_0) in H^1 -norm when ε goes to 0.

Regarding the notation, the inner product in $L^2(\Omega)$ is represented by (\cdot, \cdot) and its associated norm by $|\cdot|$ (since no confusion arises, this also denotes the Lebesgue measure of a subset of \mathbb{R}^N). The inner product in $H_0^1(\Omega)$, given by the product of the gradients in $(L^2(\Omega))^N$, is represented by $((\cdot, \cdot))$, and by $\|\cdot\|$ the associated norm. The duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$ and by $\|\cdot\|_*$, the norm in $H^{-1}(\Omega)$. Identifying $L^2(\Omega)$ with its dual, the usual chain of dense and compact embeddings $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ holds. Observe that, by the Riesz theorem, we can obtain $\tilde{l} \in L^2(\Omega)$ with $\langle l, u \rangle_{(L^2(\Omega))', L^2(\Omega)} = (\tilde{l}, u)$; here on, thanks to the identification $(L^2(\Omega))' \equiv L^2(\Omega)$, we just use l instead of \tilde{l} , but at the same time we keep the usual notation in the existing previous literature $l(u)$. The duality product between $L^p(\Omega)$ and $L^q(\Omega)$, where p and q are conjugate exponents, is denoted by (\cdot, \cdot) and the norm in $L^p(\Omega)$ is represented by $|\cdot|_p$. Finally, $\|\cdot\|_{L^s(\tau, T; X)}$ denotes the norm in $L^s(\tau, T; X)$ where $s \geq 1$ and X is a separable Banach space.

To start with the weak-solution framework, we assume that $u_\tau \in L^2(\Omega)$ and $h \in L_{loc}^2(\mathbb{R}; H^{-1}(\Omega))$.

Definition 1. A weak solution to problem (P_ε) is a function u that belongs to $L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ for all $T > \tau$, with $u(\tau) = u_\tau$, and such that for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$

$$\frac{d}{dt}(u(t), v) + g_1(\varepsilon)a(l(u(t)))(u(t), v) = \tilde{g}_1(\varepsilon)(f(u(t)), v) + g_0(\varepsilon)\langle h(t), v \rangle, \quad (7)$$

where the previous equation must be understood in the sense of $\mathcal{D}'(\tau, \infty)$.

When u is a weak solution to (P_ε) , making use of the continuity of the function a , (2), (6) and (7), it holds that $u' \in L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$ for any $T > \tau$. Therefore, $u \in C([\tau, \infty); L^2(\Omega))$ and the initial datum in (P_ε) makes sense. Furthermore, the following energy equality holds

$$\begin{aligned} & |u(t)|^2 + 2g_1(\varepsilon) \int_s^t a(l(u(r)))\|u(r)\|^2 dr \\ &= |u(s)|^2 + 2\tilde{g}_1(\varepsilon) \int_s^t (f(u(r)), u(r)) dr + 2g_0(\varepsilon) \int_s^t \langle h(r), u(r) \rangle dr \end{aligned} \quad (8)$$

for all $\tau \leq s \leq t$ (cf. [22, Théorème 2, p. 575] or [41, Lemma 3.2, p. 71]).

The existence of weak solutions to (P_ε) has been proved in [6, Theorem 1] (as commented in the introduction, the lack of Lipschitz character on the function a does not allow to ensure uniqueness).

Theorem 1. *Assume that (1)–(4) hold and $h \in L_{loc}^2(\mathbb{R}; H^{-1}(\Omega))$. Then, for any $u_\tau \in L^2(\Omega)$, there exists at least one weak solution to (P_ε) .*

Now, in a more regular framework, we will show the regularising effect of the equation and the existence of strong solutions. In order to do that we assume that the function f also fulfils (5) (anyway this assumption can be weakened, see Remark 4 for more details), and $h \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$.

Definition 2. A strong solution to (P_ε) is a weak solution which also belongs to $L^2(\tau, T; D(-\Delta)) \cap L^\infty(\tau, T; H_0^1(\Omega))$ for all $T > \tau$.

Now we show the regularising effect of the equation for any ε and the existence of strong solutions.

Theorem 2. Assume that (1)–(5) hold and $h \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$. Then, for any $u_\tau \in L^2(\Omega)$, each weak solution u belongs to $L^\infty(\tau + \varepsilon, T; H_0^1(\Omega)) \cap L^2(\tau + \varepsilon, T; D(-\Delta))$ for every $\varepsilon > 0$ and $T > \tau + \varepsilon$. In addition, if the initial datum $u_\tau \in H_0^1(\Omega)$, then the weak solutions to (P_ε) are in fact strong solutions.

Proof. We split the proof into two steps.

Step 1. Regularising effect.

Fix a weak solution $u(\cdot; \tau, u_\tau)$ to (P_ε) , for short denoted by $u(\cdot)$. Then, we consider the problem

$$(P_{\varepsilon, u}) \begin{cases} \frac{dy}{dt} - g_1(\varepsilon)a(l(u))\Delta y = \tilde{g}_1(\varepsilon)f(y) + g_0(\varepsilon)h(t) & \text{in } \Omega \times (\tau, \infty), \\ y = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\ y(x, \tau) = u_\tau(x) & \text{in } \Omega. \end{cases}$$

Observe that there exists a unique solution to $(P_{\varepsilon, u})$ thanks to the monotonicity of the Laplacian and the assumption (5) made on f (cf. [33, Chapter II]). Thus, more regular (a posteriori) estimates as well as using the Galerkin approximations make complete sense. Moreover, it holds that $y = u$ since u solves (P_ε) , and $(P_{\varepsilon, u})$ possesses a unique solution.

Now, making use of spectral theory and regularity results (cf. [39, 32]), consider a sequence $\{w_i\}_{i \geq 1}$ of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, which is a Hilbert basis of $L^2(\Omega)$. For each integer $n \geq 1$, we define the function $u_n(t; \tau, u_\tau) = \sum_{j=1}^n \varphi_{nj}(t)w_j$ ($u_n(t)$ for short), which is the local solution to

$$\begin{cases} \frac{d}{dt}(u_n(t), w_j) + g_1(\varepsilon)a(l(u(t)))(u_n(t), w_j) = (\tilde{g}_1(\varepsilon)f(u_n(t)) + g_0(\varepsilon)h(t), w_j) \\ t \in (\tau, \infty), \\ (u_n(\tau), w_j) = (u_\tau, w_j), \\ j = 1, \dots, n. \end{cases} \quad (9)$$

Multiplying (9) by $\varphi_{nj}(t)$ and summing from $j = 1$ until n , we deduce

$$\frac{1}{2} \frac{d}{dt} |u_n(t)|^2 + g_1(\varepsilon)a(l(u(t))) \|u_n(t)\|^2 = \tilde{g}_1(\varepsilon)(f(u_n(t)), u_n(t)) + g_0(\varepsilon)(h(t), u_n(t))$$

a.e. $t \in (\tau, T)$.

Integrating the previous expression between τ and T , and making use of (1) and (4), we obtain

$$|u_n(T)|^2 + 2g_1(\varepsilon)m \int_\tau^T \|u_n(t)\|^2 dt \leq |u_\tau|^2 + 2\tilde{g}_1(\varepsilon)(T-\tau)\kappa|\Omega| + 2g_0(\varepsilon) \int_\tau^T (h(t), u_n(t)) dt.$$

Using the Cauchy inequality,

$$\int_\tau^T \|u_n(t)\|^2 dt \leq \frac{1}{g_1(\varepsilon)m} |u_\tau|^2 + \frac{2\tilde{g}_1(\varepsilon)(T-\tau)\kappa|\Omega|}{g_1(\varepsilon)m} + \frac{(g_0(\varepsilon))^2}{\lambda_1(g_1(\varepsilon))^2 m^2} \int_\tau^T |h(t)|^2 dt. \quad (10)$$

On the other hand, multiplying (9) by $\lambda_j \varphi_{nj}(t)$, summing from $j = 1$ until n and using (1), we have

$$\frac{1}{2} \frac{d}{dt} \|u_n(t)\|^2 + g_1(\varepsilon)m |-\Delta u_n(t)|^2 \leq \tilde{g}_1(\varepsilon)(f(u_n(t)), -\Delta u_n(t)) + g_0(\varepsilon)(h(t), -\Delta u_n(t))$$

a.e. $t \in (\tau, T)$.

Applying (5) and the Cauchy inequality, it holds

$$\frac{d}{dt} \|u_n(t)\|^2 + g_1(\varepsilon)m |-\Delta u_n(t)|^2 \leq 2\tilde{g}_1(\varepsilon)\eta \|u_n(t)\|^2 + \frac{|\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(t)|^2}{g_1(\varepsilon)m} \quad (11)$$

a.e. $t \in (\tau, T)$.

Integrating between s and t , with $\tau < s \leq t \leq T$, we obtain

$$\begin{aligned} & \|u_n(t)\|^2 + g_1(\varepsilon)m \int_s^t |-\Delta u_n(r)|^2 dr \\ & \leq \|u_n(s)\|^2 + 2\tilde{g}_1(\varepsilon)\eta \int_\tau^T \|u_n(r)\|^2 dr + \frac{1}{g_1(\varepsilon)m} \int_\tau^T |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(r)|^2 dr. \end{aligned}$$

Now, integrating w.r.t. s between τ and t , we have in particular

$$\begin{aligned} & (t - \tau) \|u_n(t)\|^2 \\ & \leq (1 + 2\tilde{g}_1(\varepsilon)(T - \tau)\eta) \int_\tau^T \|u_n(r)\|^2 dr + \frac{T - \tau}{g_1(\varepsilon)m} \int_\tau^T |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(r)|^2 dr \end{aligned}$$

for all $t \in [\tau + \varepsilon, T]$ with $\varepsilon \in (0, T - \tau)$.

Then, from this and making use of (10), it holds that the sequence $\{u_n\}$ is bounded in $L^\infty(\tau + \varepsilon, T; H_0^1(\Omega)) \cap L^2(\tau + \varepsilon, T; D(-\Delta))$. Therefore, by the uniqueness of weak solution to $(P_{\varepsilon, u})$, it fulfils

$$\begin{cases} u_n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau + \varepsilon, T; H_0^1(\Omega)), \\ u_n \rightharpoonup u & \text{weakly in } L^2(\tau + \varepsilon, T; D(-\Delta)). \end{cases}$$

Step 2. Strong solution. Assume that $u_\tau \in H_0^1(\Omega)$. We will see that $u \in L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; D(-\Delta))$ and actually also $u' \in L^q(\tau, T; L^q(\Omega))$ for all $T > \tau$.

Integrating (11) between τ and $t \in [\tau, T]$, we obtain

$$\begin{aligned} & \|u_n(t)\|^2 + g_1(\varepsilon)m \int_\tau^t |-\Delta u_n(r)|^2 dr \\ & \leq \|u_\tau\|^2 + 2\tilde{g}_1(\varepsilon)\eta \int_\tau^T \|u_n(r)\|^2 dr + \frac{1}{g_1(\varepsilon)m} \int_\tau^T |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(r)|^2 dr. \end{aligned}$$

Then, taking into account that $\{u_n\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$, we deduce that $\{u_n\}$ is bounded in $L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; D(-\Delta))$. Thanks to the uniqueness of weak solution to $(P_{\varepsilon, u})$, we have

$$\begin{cases} u_n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau, T; H_0^1(\Omega)), \\ u_n \rightharpoonup u & \text{weakly in } L^2(\tau, T; D(-\Delta)). \end{cases}$$

On the other hand, since

$$\frac{\partial u}{\partial t} - g_1(\varepsilon)a(l(u))\Delta u = \tilde{g}_1(\varepsilon)f(u) + g_0(\varepsilon)h(t) \quad \text{in } L^q(\tau, T; L^q(\Omega)),$$

it holds that $u' \in L^q(\tau, T; L^q(\Omega))$. \square

2. Set-valued non-autonomous dynamical systems and pullback attractors. In this section, we provide abstract results on multi-valued non-autonomous dynamical systems (cf. [37, 8, 36, 2]) which are crucial to prove the existence of minimal pullback attractors. Furthermore, results which establish relationships between the families of pullback attractors are also stated (cf. [36]).

To set our abstract framework, we consider a metric space (X, d_X) and the set $\mathbb{R}_d^2 = \{(t, s) \in \mathbb{R}^2 : t \geq s\}$. In addition, let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and consider a universe \mathcal{D} , which is a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 3. A multi-valued map $U : \mathbb{R}_d^2 \times X \mapsto \mathcal{P}(X)$ is a multi-valued process on X (also called multi-valued non-autonomous dynamical system) if it fulfils

- (i) $U(\tau, \tau)x = \{x\} \quad \forall \tau \in \mathbb{R} \quad \forall x \in X$,
- (ii) $U(t, \tau)x \subset U(t, s)(U(s, \tau)x) \quad \forall \tau \leq s \leq t \quad \forall x \in X$, where

$$U(t, \tau)W := \bigcup_{y \in W} U(t, \tau)y \quad \forall W \subset X.$$

When the relationship established in (ii) is an equality instead of an inclusion, the multi-valued process U is called strict.

Definition 4. A multi-valued process U on X is upper-semicontinuous if for all $(t, \tau) \in \mathbb{R}_d^2$, the mapping $U(t, \tau)$ is upper-semicontinuous from X into $\mathcal{P}(X)$, that is, for each $x \in X$ and any neighbourhood $\mathcal{N}(U(t, \tau)x)$ of $U(t, \tau)x$, there exists a neighbourhood \mathcal{M} of x such that $U(t, \tau)y \subset \mathcal{N}(U(t, \tau)x)$ for any $y \in \mathcal{M}$.

Definition 5. A universe \mathcal{D} is called inclusion-closed if given two families \widehat{D} and \widehat{D}' such that $\widehat{D} \in \mathcal{D}$ and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all $t \in \mathbb{R}$, it fulfils that $\widehat{D}' \in \mathcal{D}$.

Now, we consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. We do not require any additional condition on these sets such as compactness or boundedness.

Definition 6. The family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ is pullback \mathcal{D} -absorbing for a multi-valued process U if for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$, there exists $\tau(\widehat{D}, t) \leq t$ such that

$$U(t, \tau)D(\tau) \subset D_0(t) \quad \forall \tau \leq \tau(\widehat{D}, t).$$

Definition 7. Given a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$, a multi-valued process U on X is said to be pullback \widehat{D}_0 -asymptotically compact if for any $t \in \mathbb{R}$, every sequence $\{\tau_n\} \subset (-\infty, t]$ such that $\tau_n \rightarrow -\infty$ and any sequence $\{x_n\} \subset X$ with $x_n \in D_0(\tau_n)$ for all $n \in \mathbb{N}$, it fulfils that any sequence $\{y_n\}$, with each $y_n \in U(t, \tau_n)x_n$, is relatively compact in X .

Given a universe \mathcal{D} , a multi-valued process U on X is said to be pullback \mathcal{D} -asymptotically compact if it is pullback \widehat{D} -asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

Definition 8. A family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is called the minimal pullback \mathcal{D} -attractor for a multi-valued process U if the following properties are fulfilled:

1. $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X for any $t \in \mathbb{R}$,
2. $\mathcal{A}_{\mathcal{D}}$ is pullback \mathcal{D} -attracting, i.e. $\lim_{\tau \rightarrow -\infty} \text{dist}_X(U(t, \tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0$ for all $\widehat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$, where $\text{dist}_X(\cdot, \cdot)$ denotes the Hausdorff semi-distance in X between two subsets of X ,

3. $\mathcal{A}_{\mathcal{D}}$ is negatively invariant, i.e. $\mathcal{A}_{\mathcal{D}}(t) \subset U(t, \tau)\mathcal{A}_{\mathcal{D}}(\tau)$ for all $(t, \tau) \in \mathbb{R}_d^2$,
4. For any family of closed sets $\widehat{C} = \{C(t) : t \in \mathbb{R}\}$ which is pullback \mathcal{D} -attracting, the relationship $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$ holds for all $t \in \mathbb{R}$.

To continue our analysis, we define the omega limit of the family \widehat{D} in time t by

$$\Lambda(\widehat{D}, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)D(\tau)}^X \quad \forall t \in \mathbb{R},$$

where $\overline{\{\dots\}}^X$ denotes the closure in X .

Now, we have the main result of this section, which ensures the existence of the minimal pullback \mathcal{D} -attractor for a multi-valued process U (cf. [6, Theorem 2]).

Theorem 3. *Consider an upper-semicontinuous multi-valued process U which has closed values, a pullback \mathcal{D} -absorbing family called $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ and also assume that U is pullback \widehat{D}_0 -asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by*

$$\mathcal{A}_{\mathcal{D}}(t) = \bigcup_{\widehat{D} \in \mathcal{D}} \overline{\Lambda(\widehat{D}, t)}^X \quad \forall t \in \mathbb{R}$$

is the minimal pullback \mathcal{D} -attractor and $\mathcal{A}_{\mathcal{D}}(t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$. In addition, if $\widehat{D}_0 \in \mathcal{D}$, each $D_0(t)$ is closed and the universe \mathcal{D} is inclusion-closed, then the family $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$. Moreover, when U is strict, the family $\mathcal{A}_{\mathcal{D}}$ is invariant under the multi-valued process U , i.e. $\mathcal{A}_{\mathcal{D}}(t) = U(t, \tau)\mathcal{A}_{\mathcal{D}}(\tau)$ for all $(t, \tau) \in \mathbb{R}_d^2$.

Now, we are going to establish relationships between pullback attractors (cf. [36]), but first we need to introduce some notation. We denote by \mathcal{D}_F^X the universe of fixed nonempty bounded subsets of X , i.e. the class of all families \widehat{D} of the form $\widehat{D} = \{D(t) = B : t \in \mathbb{R}\}$, where B is a fixed nonempty bounded subset of X .

Corollary 1. *Under the assumptions of Theorem 3, if $\mathcal{D}_F^X \subset \mathcal{D}$, then the following relationship holds*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_{\mathcal{D}}(t) \quad \forall t \in \mathbb{R},$$

where

$$\mathcal{A}_{\mathcal{D}_F^X}(t) = \overline{\bigcup_{B \text{ bounded}} \Lambda(B, t)}^X \quad \forall t \in \mathbb{R}$$

is the minimal pullback \mathcal{D}_F^X -attractor for the multi-valued process U .

In addition, if there exists $T \in \mathbb{R}$ such that the set $\bigcup_{t \leq T} D_0(t)$ is bounded in X , then $\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \leq T$.

Thanks to the following result, we can compare two attractors for a process (see [24, Theorem 3.15] for a proof in the single-valued framework).

Theorem 4. *Suppose that $\{(X_i, d_{X_i})\}_{i=1,2}$ are two metric spaces such that $X_1 \subset X_2$ with continuous injection, \mathcal{D}_i is a universe in $\mathcal{P}(X_i)$ for $i = 1, 2$, and $\mathcal{D}_1 \subset \mathcal{D}_2$. Assume that U is a multi-valued map that acts as a multi-valued process in both cases, i.e. $U : \mathbb{R}_d^2 \times X_i \rightarrow \mathcal{P}(X_i)$ for $i = 1, 2$ is a multi-valued process. For each $t \in \mathbb{R}$,*

$$\mathcal{A}_i(t) = \bigcup_{\widehat{D}_i \in \mathcal{D}_i} \overline{\Lambda_i(\widehat{D}_i, t)}^{X_i} \quad i = 1, 2,$$

where the subscript i in the symbol of the omega-limit set Λ_i is used to denote the dependence on the respective topology. Then, $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

If moreover

- (i) $\mathcal{A}_1(t)$ is a compact subset of X_1 for all $t \in \mathbb{R}$,
- (ii) for any $\widehat{D}_2 \in \mathcal{D}_2$ and $t \in \mathbb{R}$, there exist a family $\widehat{D}_1 \in \mathcal{D}_1$ and a $t_{\widehat{D}_1}^*$ such that U is pullback \widehat{D}_1 -asymptotically compact, and for any $s \leq t_{\widehat{D}_1}^*$ there exists a $\tau_s < s$ such that $U(s, \tau)D_2(\tau) \subset D_1(s)$ for all $\tau \leq \tau_s$,

then $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

3. Previous results on the asymptotic behaviour in L^2 -norm. In [6] the existence of pullback attractors to (P_ε) in $L^2(\Omega)$ is analysed. Namely, in what follows we will recall the main results that guarantee the existence of these families. This is the first step in order to state our regularity results in $H_0^1(\Omega)$ in Section 4. Observe that the results are provided without proofs since (P_ε) is a slight generalization of the one analysed in [6]. However, we include the (adapted) statements here for the sake of clarity when reading the next section.

Thanks to the existence of weak solutions to (P_ε) (cf. [6, Theorem 1]), we can define a multi-valued map $U^\varepsilon : \mathbb{R}_d^2 \times L^2(\Omega) \rightarrow \mathcal{P}(L^2(\Omega))$ as

$$U^\varepsilon(t, \tau)u_\tau = \{u(t) : u \in \Phi^\varepsilon(\tau, u_\tau)\}, \quad \tau \leq t, \quad u_\tau \in L^2(\Omega),$$

where $\Phi^\varepsilon(\tau, u_\tau)$ denotes the set of weak solutions to (P_ε) in $[\tau, \infty)$ with initial datum $u_\tau \in L^2(\Omega)$.

The following result is the natural generalization of [6, Lemma 1, Proposition 2] to this setting.

Proposition 1. *Assume that (1)–(4) hold and $h \in L_{loc}^2(\mathbb{R}; H^{-1}(\Omega))$. Then, $U^\varepsilon : \mathbb{R}_d^2 \times L^2(\Omega) \rightarrow \mathcal{P}(L^2(\Omega))$ is a strict upper-semicontinuous multi-valued process with closed values for all $\varepsilon \in [0, 1]$.*

From now on, for any $\mu > 0$, the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ such that

$$\lim_{\tau \rightarrow -\infty} \left(e^{\mu\tau} \sup_{v \in D(\tau)} |v|^2 \right) = 0$$

is denoted by $\mathcal{D}_\mu^{L^2}$. It is worth noting that $\mathcal{D}_F^{L^2} \subset \mathcal{D}_\mu^{L^2}$ and $\mathcal{D}_\mu^{L^2}$ is inclusion-closed.

From now on, we assume a condition to simplify the exposition and the form of the limit problem (P_0) , namely

$$g_1(0) = 1. \tag{12}$$

To prove the existence of a pullback absorbing family, we assume that there exists $\bar{\mu} \in (0, 2\lambda_1 m)$ such that

$$\int_{-\infty}^0 e^{\bar{\mu}s} \|h(s)\|_*^2 ds < \infty. \tag{13}$$

Remark 1. From the continuity of g_1 and (12) it is immediate to deduce that there exists $\bar{\varepsilon} \in (0, 1]$ such that $\bar{\mu} < 2g_1(\varepsilon)\lambda_1 m$ for all $\varepsilon \in [0, \bar{\varepsilon}]$. Indeed, this last condition will be used in the sequel to construct the absorbing families in suitable universes $\mathcal{D}_{\mu_\varepsilon}^{L^2}$. Furthermore, it is consistent with the final goal of studying the limit of problems (P_ε) when ε goes to 0. In what follows the parameter μ_ε is taken in $[\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$ since (13) also holds with the weight $e^{\bar{\mu}s}$ replaced by $e^{\mu_\varepsilon s}$ and $\mathcal{D}_{\bar{\mu}}^{L^2} \subset \mathcal{D}_{\mu_\varepsilon}^{L^2}$, so a larger class of objects will be attracted.

Then, it holds the following result (cf. [6, Proposition 4] for a similar proof).

Proposition 2. *Assume that (1)–(4) and (12) hold, and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ fulfils (13) for some $\bar{\mu} \in (0, 2\lambda_1 m)$. Then, there exists $\bar{\varepsilon} \in (0, 1]$ such that for any $\varepsilon \in [0, \bar{\varepsilon}]$ and $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$ the family $\widehat{D}_0^\varepsilon = \{D_0^\varepsilon(t) : t \in \mathbb{R}\}$ defined by $D_0^\varepsilon(t) = \overline{B}_{L^2}(0, (R_{L^2}^\varepsilon(t))^{1/2})$, the closed ball in $L^2(\Omega)$ of center zero and radius $(R_{L^2}^\varepsilon(t))^{1/2}$, where*

$$R_{L^2}^\varepsilon(t) = 1 + \frac{2\tilde{g}_1(\varepsilon)\kappa|\Omega|}{\mu_\varepsilon} + \frac{(g_0(\varepsilon))^2 e^{-\mu_\varepsilon t}}{2g_1(\varepsilon)m - \lambda_1^{-1}\mu_\varepsilon} \int_{-\infty}^t e^{\mu_\varepsilon s} \|h(s)\|_*^2 ds, \quad (14)$$

is pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2}$ -absorbing for the multi-valued process $U^\varepsilon : \mathbb{R}_d^2 \times L^2(\Omega) \rightarrow \mathcal{P}(L^2(\Omega))$. Besides, $\widehat{D}_0^\varepsilon \in \mathcal{D}_{\mu_\varepsilon}^{L^2}$.

Now to prove the pullback asymptotic compactness we first establish the following estimates (cf. [6, Lemma 2] for an analogous proof).

Lemma 1. *Under the assumptions of Proposition 2, there exists $\bar{\varepsilon} \in (0, 1]$ such that for any $\varepsilon \in [0, \bar{\varepsilon}]$, $t \in \mathbb{R}$, $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$ and $\widehat{D} \in \mathcal{D}_{\mu_\varepsilon}^{L^2}$, there exists $\tau_1(\widehat{D}, t) < t - 2$, such that for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, the solutions to (P_ε) satisfy*

$$\begin{cases} |u(r; \tau, u_\tau)|^2 \leq \rho_1^\varepsilon(t) & \forall r \in [t - 2, t], \\ \int_{r-1}^r \|u(s; \tau, u_\tau)\|^2 ds \leq \rho_2^\varepsilon(t) & \forall r \in [t - 1, t], \\ \int_{r-1}^r |u(s; \tau, u_\tau)|_p^p ds \leq \frac{g_1(\varepsilon)m}{2\alpha_2\tilde{g}_1(\varepsilon)} \rho_2^\varepsilon(t) & \forall r \in [t - 1, t], \end{cases}$$

where

$$\rho_1^\varepsilon(t) = 1 + \frac{2\tilde{g}_1(\varepsilon)\kappa|\Omega|}{\mu_\varepsilon} + \frac{(g_0(\varepsilon))^2 e^{-\mu_\varepsilon(t-2)}}{2g_1(\varepsilon)m - \lambda_1^{-1}\mu_\varepsilon} \int_{-\infty}^t e^{\mu_\varepsilon s} \|h(s)\|_*^2 ds,$$

$$\rho_2^\varepsilon(t) = \frac{1}{g_1(\varepsilon)m} \left(\rho_1^\varepsilon(t) + 2\tilde{g}_1(\varepsilon)\kappa|\Omega| + \frac{(g_0(\varepsilon))^2}{g_1(\varepsilon)m} \max_{r \in [t-1, t]} \int_{r-1}^r \|h(s)\|_*^2 ds \right).$$

The proof of the following result is very close to that of [6, Proposition 5] under minor modifications.

Corollary 2. *Under the assumptions and notation of Lemma 1, the multi-valued process U^ε is pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2}$ -asymptotically compact for any $\varepsilon \in [0, \bar{\varepsilon}]$ and $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$.*

Now the existence of pullback attractors in $L^2(\Omega)$ is guaranteed.

Theorem 5. *Assume that (1)–(4) and (12) hold, and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ fulfils (13) for some $\bar{\mu} \in (0, 2\lambda_1 m)$. Then, there exists $\bar{\varepsilon} \in (0, 1]$ such that for any $\varepsilon \in [0, \bar{\varepsilon}]$ and $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$, the process U^ε possesses the minimal pullback $\mathcal{D}_F^{L^2}$ -attractor $\mathcal{A}_{\mathcal{D}_F^{L^2}}^\varepsilon = \{\mathcal{A}_{\mathcal{D}_F^{L^2}}^\varepsilon(t) : t \in \mathbb{R}\}$ and the minimal pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2}$ -attractor $\mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}^\varepsilon = \{\mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}^\varepsilon(t) : t \in \mathbb{R}\}$, which is strictly U^ε -invariant.*

In addition, the family $\mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}^\varepsilon$ belongs to $\mathcal{D}_{\mu_\varepsilon}^{L^2}$ and the following relationships hold

$$\mathcal{A}_{\mathcal{D}_F^{L^2}}^\varepsilon(t) \subset \mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}^\varepsilon(t) \subset \overline{B}_{L^2}(0, (R_{L^2}^\varepsilon(t))^{1/2}) \quad \forall t \in \mathbb{R} \quad \forall \varepsilon \in [0, \bar{\varepsilon}].$$

Moreover, if there exists some $\tilde{\mu} \in (0, 2\lambda_1 m)$ such that h fulfils

$$\sup_{s \leq 0} \left(e^{-\tilde{\mu}s} \int_{-\infty}^s e^{\tilde{\mu}\theta} \|h(\theta)\|_*^2 d\theta \right) < \infty, \quad (15)$$

then there exists $\tilde{\varepsilon} \in (0, 1]$ such that for any $\varepsilon \in [0, \tilde{\varepsilon}]$, $\mathcal{A}_{\mathcal{D}_F^{\varepsilon} L^2}^{\varepsilon}(t) = \mathcal{A}_{\mathcal{D}_{\mu\varepsilon}^{\varepsilon} L^2}(t)$ for all $t \in \mathbb{R}$.

4. Existence of pullback attractors and their upper semicontinuous behaviour in H^1 -norm. Now, we are ready to analyse the existence of pullback attractors in $H_0^1(\Omega)$. Observe that $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, thanks to the assumptions made on the domain Ω (indeed $C^{1,1}$ is a sufficient condition, and it can be weakened, e.g. cf. [26, Theorem 9.15, p. 241]). Therefore, in what follows we will use either the norm of $D(-\Delta)$ or the norm of $H^2(\Omega) \cap H_0^1(\Omega)$ since both are equivalent.

Under the assumptions imposed in Theorem 2, it does not seem possible to guarantee that $u \in C([\tau, T]; H_0^1(\Omega))$ due to the fact that it is unknown whether or not u' belongs to $L^2(\tau, T; L^2(\Omega))$. To ensure a positive answer to this question, we will make the most of the known regularity for strong solutions, interpolation results (cf. [42, Lemma II.4.1, p. 72]) and a certain growth condition on the nonlinearity f . Namely, we assume that

$$|f(s)| \leq C(1 + |s|^{\gamma+1}) \quad \forall s \in \mathbb{R}, \quad (16)$$

where $\gamma = 4$ when $N = 3$, $\gamma = 2$ when $N = 4$ and $\gamma = 4/(N - 2)$ when $N \geq 5$. This way, given $u \in L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega))$, we obtain

$$\begin{aligned} & \int_{\tau}^T \int_{\Omega} |f(u(x, t))|^2 dx dt \\ & \leq 2 \int_{\tau}^T \int_{\Omega} C^2 (1 + |u(x, t)|^{2\gamma+2}) dx dt \\ & \leq 2C^2 |\Omega| (T - \tau) + 2C^2 \|u\|_{L^\infty(\tau, T; L^p(N, H_0^1))}^{2\hat{b}} \|u\|_{L^2(\tau, T; L^p(N, H^2))}^{2\tilde{b}} \\ & \leq 2C^2 \left[|\Omega| (T - \tau) + (C_{H_0^1}(N))^{2\hat{b}} (C_{H^2}(N))^{2\tilde{b}} \|u\|_{L^\infty(\tau, T; H_0^1(\Omega))}^{2\hat{b}} \|u\|_{L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega))}^{2\tilde{b}} \right], \end{aligned} \quad (17)$$

where $\hat{b} = (1 - \theta)(\gamma + 1)$, $\tilde{b} = \theta(\gamma + 1)$, $C_{H_0^1}(N)$ and $C_{H^2}(N)$ are the constants of the continuous embeddings of $H_0^1(\Omega)$ and $H^2(\Omega)$ into L^p -spaces respectively and $\theta \in [0, 1]$.

Remark 2. Observe that condition (16) only concerns dimensions $N \geq 3$. The cases $N = 1, 2$ do not require any additional restriction since $H_0^1(\Omega) \subset L^p(\Omega)$ for all $p < \infty$, and in view of (6) and the fact that $u \in L^\infty(\tau, T; H_0^1(\Omega))$, it is immediate that $f(u) \in L^2(\tau, T; L^2(\Omega))$.

Then, under the previous assumptions it fulfils that $u' \in L^2(\tau, T; L^2(\Omega))$. Therefore, it satisfies that $u \in C([\tau, T]; H_0^1(\Omega))$ and it holds

$$\begin{aligned} & \|u(t)\|^2 + 2g_1(\varepsilon) \int_s^t a(l(u(r))) |-\Delta u(r)|^2 dr \\ & = \|u(s)\|^2 + 2\tilde{g}_1(\varepsilon) \int_s^t (f(u(r)), -\Delta u(r)) dr + 2g_0(\varepsilon) \int_s^t (h(r), -\Delta u(r)) dr. \end{aligned} \quad (18)$$

Thanks to Theorem 2, the restriction of U^ε to $\mathbb{R}_d^2 \times H_0^1(\Omega)$ defines a strict multi-valued process into $H_0^1(\Omega)$. Since no confusion arises, we will not modify the notation and continue denoting this process by U^ε .

Now to prove that the multi-valued process U^ε is upper-semicontinuous with closed values in $H_0^1(\Omega)$ for any ε fixed, we first provided the following auxiliary result.

Proposition 3. *Assume that (1)–(5) and (16) hold, and $h \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$. Then, if $\{u_\tau^n\} \subset H_0^1(\Omega)$ is such that $u_\tau^n \rightarrow u_\tau$ strongly in $H_0^1(\Omega)$, for any sequence $\{u^n\}$ with $u^n \in \Phi^\varepsilon(\tau, u_\tau^n)$ for all $n \geq 1$, there exist a subsequence of $\{u^n\}$ (relabelled the same) and $u \in \Phi^\varepsilon(\tau, u_\tau)$ such that*

$$u^n(t) \rightarrow u(t) \quad \text{strongly in } H_0^1(\Omega) \quad \forall t \geq \tau. \quad (19)$$

Proof. Consider fixed $\tau < T$. In view of the energy equality (8) and (1), we deduce

$$\frac{1}{2} \frac{d}{dt} |u^n(t)|^2 + g_1(\varepsilon)m \|u^n(t)\|^2 \leq \tilde{g}_1(\varepsilon)(f(u^n(t)), u^n(t)) + g_0(\varepsilon)\langle h(t), u^n(t) \rangle$$

a.e. $t \in (\tau, T)$. Then, bearing in mind

$$\begin{aligned} (f(u^n(t)), u^n(t)) &\leq \kappa|\Omega| - \alpha_2|u^n(t)|_p^p, \\ g_0(\varepsilon)\langle h(t), u^n(t) \rangle &\leq \frac{(g_0(\varepsilon))^2 \|h(t)\|_*^2}{2g_1(\varepsilon)m} + \frac{g_1(\varepsilon)m}{2} \|u^n(t)\|^2, \end{aligned}$$

we have

$$\frac{d}{dt} |u^n(t)|^2 + g_1(\varepsilon)m \|u^n(t)\|^2 + 2\alpha_2\tilde{g}_1(\varepsilon)|u^n(t)|_p^p \leq 2\tilde{g}_1(\varepsilon)\kappa|\Omega| + \frac{(g_0(\varepsilon))^2}{g_1(\varepsilon)m} \|h(t)\|_*^2$$

a.e. $t \in (\tau, T)$. Integrating between τ and $t \in (\tau, T]$,

$$\begin{aligned} |u^n(t)|^2 + g_1(\varepsilon)m \int_\tau^t \|u^n(s)\|^2 ds + 2\alpha_2\tilde{g}_1(\varepsilon) \int_\tau^t |u^n(s)|_p^p ds \\ \leq |u_\tau^n|^2 + 2\tilde{g}_1(\varepsilon)\kappa|\Omega|(T - \tau) + \frac{(g_0(\varepsilon))^2}{g_1(\varepsilon)m} \int_\tau^T \|h(s)\|_*^2 ds. \end{aligned}$$

From the previous inequality, we obtain that the sequence $\{u^n\}$ is bounded in $L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$. Taking this into account together with the fact that each $u^n \in C([\tau, T]; L^2(\Omega))$, we deduce that there exists a constant $C_\infty > 0$ such that

$$|u^n(t)| \leq C_\infty \quad \forall t \in [\tau, T] \quad \forall n \geq 1.$$

Now, since the function $a \in C(\mathbb{R}; [0, \infty))$ and $l \in L^2(\Omega)$, there exists a constant $M_{C_\infty} > 0$ such that

$$a(l(u^n(t))) \leq M_{C_\infty} \quad \forall t \in [\tau, T] \quad \forall n \geq 1. \quad (20)$$

Now, making use of (5) and the Cauchy inequality in the energy equality (18) for the Galerkin approximations associated to the problems (P_{ε, u^n}) , integrating between τ and $t \in [\tau, T]$ and passing to the limit, we have

$$\begin{aligned} \|u^n(t)\|^2 + g_1(\varepsilon)m \int_\tau^t |-\Delta u^n(s)|^2 ds \\ \leq \|u_\tau^n\|^2 + 2\tilde{g}_1(\varepsilon)\eta \int_\tau^T \|u^n(s)\|^2 ds + \int_\tau^T \frac{|\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(s)|^2}{g_1(\varepsilon)m} ds. \end{aligned} \quad (21)$$

Therefore, taking into account that $\{u^n\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$, it is also bounded in $L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega))$. Bearing this in mind together with (20), we deduce that $\{-a(l(u^n))\Delta u^n\}$ is bounded in $L^2(\tau, T; L^2(\Omega))$. In addition, $\{f(u^n)\}$ is bounded in $L^2(\tau, T; L^2(\Omega))$, thanks to (16). As a consequence, $\{(u^n)'\}$ is bounded in $L^2(\tau, T; L^2(\Omega))$. Then, applying the Aubin-Lions lemma, there exist a subsequence of $\{u^n\}$ (reabeled the same) and an element $u \in L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega))$ with $u' \in L^2(\tau, T; L^2(\Omega))$, such that

$$\left\{ \begin{array}{l} u^n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(\tau, T; H_0^1(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega)), \\ u^n \rightarrow u \quad \text{strongly in } L^2(\tau, T; H_0^1(\Omega)), \\ u^n(s) \rightarrow u(s) \quad \text{strongly in } H_0^1(\Omega) \quad \text{a.e. } (\tau, T), \\ (u^n)' \rightharpoonup u' \quad \text{weakly in } L^2(\tau, T; L^2(\Omega)), \\ f(u^n) \rightharpoonup f(u) \quad \text{weakly in } L^2(\tau, T; L^2(\Omega)), \\ -a(l(u^n))\Delta u^n \rightharpoonup -a(l(u))\Delta u \quad \text{weakly in } L^2(\tau, T; L^2(\Omega)), \end{array} \right. \quad (22)$$

where the limits of the last two convergences have been identified using [33, Lemma 1.3, p. 12].

Making use of the previous convergences, it holds that u fulfils (7) in the interval (τ, T) and $u(\tau) = u_\tau$. Therefore, $u \in \Phi^\varepsilon(\tau, u_\tau)$.

Now, we are ready to prove the convergence (19).

On the one hand, observe that the sequence $\{u^n\}$ is equicontinuous in $L^2(\Omega)$ on $[\tau, T]$, thanks to the boundedness of $\{(u^n)'\}$ in $L^2(\tau, T; L^2(\Omega))$. In addition, since the sequence $\{u^n\}$ is bounded in $C([\tau, T]; H_0^1(\Omega))$ and the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, making use of the Ascoli-Arzelà Theorem, it holds for another subsequence (reabeled again the same) the following convergence

$$u^n \rightarrow u \quad \text{strongly in } C([\tau, T]; L^2(\Omega)). \quad (23)$$

Furthermore, using the boundedness of $\{u^n\}$ in $C([\tau, T]; H_0^1(\Omega))$, we have

$$u^n(t) \rightharpoonup u(t) \quad \text{weakly in } H_0^1(\Omega) \quad \forall t \in [\tau, T], \quad (24)$$

where (23) has been used to identify the weak limit.

Now, we define the following continuous functions on $[\tau, T]$

$$\begin{aligned} J_n(t) &= \|u^n(t)\|^2 - 2\tilde{g}_1(\varepsilon)\eta \int_\tau^t \|u^n(r)\|^2 dr - \int_\tau^t \frac{|\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(r)|^2}{2g_1(\varepsilon)m} dr, \\ J(t) &= \|u(t)\|^2 - 2\tilde{g}_1(\varepsilon)\eta \int_\tau^t \|u(r)\|^2 dr - \int_\tau^t \frac{|\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(r)|^2}{2g_1(\varepsilon)m} dr. \end{aligned}$$

Observe that all the functions J_n are non-increasing on $[\tau, T]$ thanks to the energy equality (18) for u^n . In addition, from (22), we deduce

$$J_n(t) \rightarrow J(t) \quad \text{a.e. } t \in (\tau, T).$$

In fact, taking this into account together the continuity of J on $[\tau, T]$ and the non-increasing character of all J_n , it holds

$$J_n(t) \rightarrow J(t) \quad \forall t \in [\tau, T].$$

Then, bearing in mind the definitions of J and J_n , we deduce

$$\limsup_{n \rightarrow \infty} |u^n(t)|^2 \leq |u(t)|^2 \quad \forall t \in [\tau, T].$$

From this and (24), (19) holds in $[\tau, T]$. Successive iterations of this procedure in $[\tau, T+1]$, $[\tau, T+2]$, and so on, and a diagonal argument, yield (19) for all $t \geq \tau$ for a suitable subsequence. \square

As a consequence of the previous result, we obtain the following result (cf. [6, Proposition 2]).

Proposition 4. *Under the assumptions of Proposition 3, the multi-valued process U^ε is upper-semicontinuous with closed values in $H_0^1(\Omega)$ for all $\varepsilon \in [0, 1]$.*

Now we introduce new universes that involve more regularity.

Definition 9. For each $\mu > 0$, $\mathcal{D}_\mu^{L^2, H_0^1}$ denotes the class of all families of nonempty subsets $\widehat{D}_{H_0^1} = \{D(t) \cap H_0^1(\Omega) : t \in \mathbb{R}\}$, where $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^{L^2}$.

Observe that $\mathcal{D}_F^{H_0^1} \subset \mathcal{D}_\mu^{L^2, H_0^1}$ and $\mathcal{D}_\mu^{L^2, H_0^1}$ is inclusion-closed.

Thanks to the regularising effect of the equation (cf. Theorem 2) and the existence of a pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2}$ -absorbing family (cf. Proposition 2), it holds the following result.

Proposition 5. *Assume that (1)–(5), (12) and (16) hold, and $h \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$ satisfies (13) for some $\bar{\mu} \in (0, 2\lambda_1 m)$. Then, there exists $\bar{\varepsilon} \in (0, 1]$ such that for any $\varepsilon \in [0, \bar{\varepsilon}]$ and any $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$, the family*

$$\widehat{D}_{0, H_0^1}^\varepsilon = \{\overline{B}_{L^2}(0, (R_{L^2}^\varepsilon(t))^{1/2}) \cap H_0^1(\Omega) : t \in \mathbb{R}\},$$

where $R_{L^2}^\varepsilon$ is given in (14), belongs to $\mathcal{D}_{\mu_\varepsilon}^{L^2, H_0^1}$, and for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}_{\mu_\varepsilon}^{L^2}$, there exists $\tau_2(\widehat{D}, t) < t$ such that

$$U^\varepsilon(t, \tau)D(\tau) \subset D_{0, H_0^1}^\varepsilon(t) \quad \forall \tau \leq \tau_2(\widehat{D}, t).$$

Proof. Take $\bar{\varepsilon} \in (0, 1]$ as in Remark 1 and therefore $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$. Let us fix $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_{\mu_\varepsilon}^{L^2}$. By Proposition 2, there exists $\tau_2(\widehat{D}, t) < t$ such that

$$|u(t; \tau, u_\tau)|^2 \leq R_{L^2}^\varepsilon(t) \quad \forall u \in \Phi^\varepsilon(\tau, u_\tau) \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_2(\widehat{D}, t).$$

Moreover, thanks to the regularising effect of the equation (cf. Theorem 2), when $u_\tau \in L^2(\Omega)$, it fulfils that $u(\cdot; \tau, u_\tau) \in C((\tau, \infty); H_0^1(\Omega))$. As a result, $u(t; \tau, u_\tau) \in H_0^1(\Omega)$ if $t > \tau$. Therefore,

$$U^\varepsilon(t, \tau)D(\tau) \subset H_0^1(\Omega) \cap \overline{B}_{L^2}(0, (R_{L^2}^\varepsilon(t))^{1/2}) \quad \forall \tau \leq \tau_2(\widehat{D}, t).$$

\square

To prove that the process $U^\varepsilon : \mathbb{R}_2^d \times H_0^1(\Omega) \rightarrow \mathcal{P}(H_0^1(\Omega))$ is pullback asymptotically compact, we previously establish some uniform estimates of the solutions in a finite-time interval up to t when the initial datum is shifted pullback far enough.

To clarify the statement of the following result, we introduce the next two amounts

$$[(\rho_1^\varepsilon)^{ext}](t) = 1 + \frac{2\tilde{g}_1(\varepsilon)\kappa|\Omega|}{\mu_\varepsilon} + \frac{(g_0(\varepsilon))^2 e^{-\mu_\varepsilon(t-3)}}{2g_1(\varepsilon)m - \lambda_1^{-1}\mu_\varepsilon} \int_{-\infty}^t e^{\mu_\varepsilon s} \|h(s)\|_*^2 ds, \quad (25)$$

$$[(\rho_2^\varepsilon)^{ext}](t) = \frac{1}{g_1(\varepsilon)m} \left([(\rho_1^\varepsilon)^{ext}](t) + 2\tilde{g}_1(\varepsilon)\kappa|\Omega| + \frac{(g_0(\varepsilon))^2}{g_1(\varepsilon)m} \max_{r \in [t-2, t]} \int_{r-1}^r \|h(s)\|_*^2 ds \right).$$

Lemma 2. *Under the assumptions of Proposition 5, there exists $\bar{\varepsilon} \in (0, 1]$ such that for any $\varepsilon \in [0, \bar{\varepsilon}]$, $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$, $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_{\mu_\varepsilon}^{L^2}$, there exists $\tau_3(\widehat{D}, t) < t - 3$ such that for any $\tau \leq \tau_3(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, the following estimates hold*

$$\begin{cases} \|u(r; \tau, u_\tau)\|^2 \leq \tilde{\rho}_1^\varepsilon(t) & \forall r \in [t - 2, t], \\ \int_{r-1}^r |-\Delta u(s; \tau, u_\tau)|^2 ds \leq \tilde{\rho}_2^\varepsilon(t) & \forall r \in [t - 1, t], \\ \int_{r-1}^r |u'(s; \tau, u_\tau)|^2 ds \leq \tilde{\rho}_3^\varepsilon(t) & \forall r \in [t - 1, t], \end{cases} \quad (26)$$

with

$$\tilde{\rho}_1^\varepsilon(t) = (1 + 2\tilde{g}_1(\varepsilon)\eta)[(\rho_2^\varepsilon)^{ext}](t) + \frac{1}{g_1(\varepsilon)m} \max_{r \in [t-2, t]} \int_{r-1}^r |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(s)|^2 ds,$$

$$\begin{aligned} \tilde{\rho}_2^\varepsilon(t) = \frac{1}{g_1(\varepsilon)m} & \left(\tilde{\rho}_1^\varepsilon(t) + 2\tilde{g}_1(\varepsilon)\eta[(\rho_2^\varepsilon)^{ext}](t) \right. \\ & \left. + \frac{1}{g_1(\varepsilon)m} \max_{r \in [t-1, t]} \int_{r-1}^r |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(s)|^2 ds \right), \end{aligned}$$

$$\begin{aligned} \tilde{\rho}_3^\varepsilon(t) = 3(M_{[(\rho_1^\varepsilon)^{ext}]}(t))^2 & (g_1(\varepsilon))^2 \tilde{\rho}_2^\varepsilon(t) + 3(g_0(\varepsilon))^2 \max_{r \in [t-1, t]} \int_{r-1}^r |h(s)|^2 ds \\ & + 6(\tilde{g}_1(\varepsilon))^2 C^2[|\Omega| + (C_{H_0^1}(N))^{2\hat{b}}(C_{H^2}(N))^{2\tilde{b}}(\tilde{\rho}_1^\varepsilon(t))^{\hat{b}}(\tilde{\rho}_2^\varepsilon(t))^{\tilde{b}}], \end{aligned}$$

where \hat{b} , \tilde{b} , C_f and $M_{[(\rho_1^\varepsilon)^{ext}]}(t)$ are positive constants.

Proof. Take $\bar{\varepsilon} \in (0, 1]$ as in Remark 1, and μ_ε as given in the statement. Let us firstly observe that we may obtain uniform estimates for solutions in a time-interval longer than the one established in Lemma 1. Namely, there exists $\tau_3(\widehat{D}, t) < t - 3$, such that for any $\tau \leq \tau_3(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, we have that for any u solution to (P_ε) it holds

$$\begin{aligned} |u(r; \tau, u_\tau)| & \leq [(\rho_1^\varepsilon)^{ext}](t) \quad \forall r \in [t - 3, t], \quad (27) \\ \int_{r-1}^r \|u(\xi; \tau, u_\tau)\|^2 d\xi & \leq [(\rho_2^\varepsilon)^{ext}](t) \quad \forall r \in [t - 2, t], \end{aligned}$$

where $\{[(\rho_i^\varepsilon)^{ext}]\}_{i=1,2}$ are given in (25). Observe that these estimates also hold for the Galerkin approximations $u_n(\cdot; \tau, u_\tau)$.

In addition, from the continuity of the function a , the fact that $l \in L^2(\Omega)$ and the first inequality in (27), we deduce that there exists a constant $M_{[(\rho_1^\varepsilon)^{ext}]}(t) > 0$ such that

$$a(l(u_n(r))) \leq M_{[(\rho_1^\varepsilon)^{ext}]}(t, l) \quad \forall r \in [t - 3, t]. \quad (28)$$

Fix a solution u to (P_ε) and consider the problem $(P_{\varepsilon, u})$ stated in Theorem 2. Multiplying by $\lambda_j \varphi_{nj}$ in (9), summing from $j = 1$ to n and making use of (1), (5) and the Cauchy inequality, we deduce

$$\frac{d}{d\xi} \|u_n(\xi)\|^2 + g_1(\varepsilon)m |-\Delta u_n(\xi)|^2 \leq 2\tilde{g}_1(\varepsilon)\eta \|u_n(\xi)\|^2 + \frac{1}{g_1(\varepsilon)m} |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(\xi)|^2 \quad (29)$$

a.e. $\xi > \tau$. Integrating between r and s with $\tau \leq r-1 \leq s \leq r$, we obtain in particular

$$\begin{aligned} \|u_n(r)\|^2 &\leq \|u_n(s)\|^2 + 2\tilde{g}_1(\varepsilon)\eta \int_{r-1}^r \|u_n(\xi)\|^2 d\xi \\ &\quad + \frac{1}{g_1(\varepsilon)m} \int_{r-1}^r |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(\xi)|^2 d\xi. \end{aligned}$$

Integrating the last inequality w.r.t. s between $r-1$ and r ,

$$\|u_n(r)\|^2 \leq (1 + 2\tilde{g}_1(\varepsilon)\eta) \int_{r-1}^r \|u_n(s)\|^2 ds + \frac{1}{g_1(\varepsilon)m} \int_{r-1}^r |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(\xi)|^2 d\xi$$

for all $\tau \leq r-1$.

Therefore, making use of the estimate on the solutions given by $(\rho_2^\varepsilon)^{ext}$, it fulfils for any $n \geq 1$

$$\|u_n(r; \tau, u_\tau)\|^2 \leq \tilde{\rho}_1^\varepsilon(t) \quad \forall r \in [t-2, t] \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_3(\widehat{D}, t), \quad (30)$$

where $\tilde{\rho}_1^\varepsilon(t)$ is given in the statement. Now, taking limit inferior in (30) and using the well-known fact that u_n converge to $u(\cdot; \tau, u_\tau) \in C([t-2, t]; H_0^1(\Omega))$ weakly-star in $L^\infty(t-2, t; H_0^1(\Omega))$ (cf. Theorem 2), the first inequality in (26) holds.

Then integrating between $r-1$ and r in (29), we obtain in particular for any $n \geq 1$

$$\begin{aligned} \int_{r-1}^r |-\Delta u_n(\xi)|^2 d\xi &\leq \frac{1}{g_1(\varepsilon)m} \left(\|u_n(r-1)\|^2 + 2\tilde{g}_1(\varepsilon)\eta \int_{r-1}^r \|u_n(\xi)\|^2 d\xi \right. \\ &\quad \left. + \frac{1}{g_1(\varepsilon)m} \int_{r-1}^r |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(\xi)|^2 d\xi \right) \end{aligned}$$

for all $\tau \leq r-1$. Then,

$$\int_{r-1}^r |-\Delta u_n(\xi)|^2 d\xi \leq \tilde{\rho}_2^\varepsilon(t) \quad \forall r \in [t-1, t] \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_3(\widehat{D}, t), \quad (31)$$

where $\tilde{\rho}_2^\varepsilon(t)$ is given in the statement. Then, taking limit inferior in (31) and bearing in mind that u_n converge to u weakly in $L^2(r-1, r; H^2(\Omega) \cap H_0^1(\Omega))$ for all $r \in [t-1, t]$ (cf. Theorem 2), the second inequality in (26) holds.

Now, taking into account that f satisfies (16) (see also (17)) together with the previous estimates, it holds

$$\int_{r-1}^r |f(u_n(\xi))|^2 d\xi \leq 2C^2|\Omega| + 2C^2(C_{H_0^1}(N))^{2\hat{b}}(C_{H^2}(N))^{2\tilde{b}}(\tilde{\rho}_1^\varepsilon(t))^{\hat{b}}(\tilde{\rho}_2^\varepsilon(t))^{\tilde{b}} \quad (32)$$

for all $r \in [t-1, t]$, $u_\tau \in D(\tau)$ and $\tau \leq \tau_3(\widehat{D}, t)$, where $\hat{b} = (1-\theta)(\gamma+1)$, $\tilde{b} = \theta(\gamma+1)$, $C_{H_0^1}(N)$ and $C_{H^2}(N)$ are the constants of the continuous embeddings of $H_0^1(\Omega)$ and $H^2(\Omega)$ into L^p -spaces respectively and $\theta \in [0, 1]$.

Finally, observe that for all $\tau \leq r-1$

$$\begin{aligned} \int_{r-1}^r |u'_n(\xi)|^2 d\xi &\leq 3(g_1(\varepsilon))^2 \int_{r-1}^r | -a(l(u_n(\xi)))\Delta u_n(\xi) |^2 d\xi \\ &\quad + 3(\tilde{g}_1(\varepsilon))^2 \int_{r-1}^r |f(u_n(\xi))|^2 d\xi + 3(g_0(\varepsilon))^2 \int_{r-1}^r |h(\xi)|^2 d\xi. \end{aligned}$$

Then, using (28), (31) and (32), we obtain for any $n \geq 1$

$$\int_{r-1}^r |u'_n(\xi)|^2 d\xi \leq \tilde{\rho}_3^\varepsilon(t) \quad \forall r \in [t-1, t] \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_3(\widehat{D}, t),$$

where $\tilde{\rho}_3^\varepsilon$ is given in the statement. Finally, taking limit inferior in the above expression and using that u'_n converge to $u'(\cdot; \tau, u_\tau)$ weakly in $L^2(r-1, r; L^2(\Omega))$ for all $r \in [t-1, t]$, we deduce the last inequality in (26). \square

Now, to prove the pullback asymptotic compactness of U^ε in $H_0^1(\Omega)$ for the universe $\mathcal{D}_{\mu_\varepsilon}^{L^2, H_0^1}$, we apply an energy method which relies on the continuity of solutions (see [27, 35, 36, 24] for more details).

Proposition 6. *Under the assumptions of Proposition 5, there exists $\bar{\varepsilon} \in (0, 1]$ such that for any $\varepsilon \in [0, \bar{\varepsilon}]$ and $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$, the process $U^\varepsilon : \mathbb{R}_d^2 \times H_0^1(\Omega) \rightarrow \mathcal{P}(H_0^1(\Omega))$ is pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2, H_0^1}$ -asymptotically compact.*

Proof. According to Proposition 5 let us fix $\varepsilon \in [0, \bar{\varepsilon}]$, $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$, $t \in \mathbb{R}$, a family $\widehat{D}_{H_0^1} \in \mathcal{D}_{\mu_\varepsilon}^{L^2, H_0^1}$, sequences $\{\tau_n\} \subset (-\infty, t-3]$ with $\tau_n \rightarrow -\infty$ and $\{u_\tau^n\}$ with $u_\tau^n \in D(\tau_n)$ for all n . We aim to prove that any sequence $\{y_n\}$, where $y_n \in U^\varepsilon(t, \tau_n)u_\tau^n$ for all n , is relatively compact in $H_0^1(\Omega)$. In fact, since $y^n \in U^\varepsilon(t, \tau_n)u_\tau^n$, there exists $u^n \in \Phi^\varepsilon(\tau_n, u_\tau^n)$ such that $y^n = u^n(t)$. Therefore, we will show that the sequence $\{u^n(t)\}$ is relatively compact in $H_0^1(\Omega)$.

As a consequence of Lemma 2, there exists $\tau_3(\widehat{D}, t) < t-3$, such that $\tau_n \leq \tau_3(\widehat{D}, t)$ for all $n \geq n_1$, the sequence $\{u^n\}_{n \geq n_1}$ is bounded in $L^\infty(t-2, t; H_0^1(\Omega)) \cap L^2(t-2, t; H^2(\Omega))$, and $\{-a(l(u^n))\Delta u^n\}_{n \geq n_1}$, $\{f(u^n)\}_{n \geq n_1}$ and $\{(u^n)'\}_{n \geq n_1}$ are bounded in $L^2(t-2, t; L^2(\Omega))$. Then, using the Aubin-Lions lemma, there exists $u \in L^\infty(t-2, t; H_0^1(\Omega)) \cap L^2(t-2, t; H^2(\Omega) \cap H_0^1(\Omega))$ with $u' \in L^2(t-2, t; L^2(\Omega))$, such that for a subsequence (relabelled the same) it holds

$$\left\{ \begin{array}{l} u^n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(t-2, t; H_0^1(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^2(t-2, t; H^2(\Omega) \cap H_0^1(\Omega)), \\ (u^n)' \rightharpoonup u' \quad \text{weakly in } L^2(t-2, t; L^2(\Omega)), \\ u^n \rightarrow u \quad \text{strongly in } L^2(t-2, t; H_0^1(\Omega)), \\ u^n(s) \rightarrow u(s) \quad \text{strongly in } H_0^1(\Omega) \quad \text{a.e. } s \in (t-2, t), \\ f(u^n) \rightharpoonup f(u) \quad \text{weakly in } L^2(t-2, t; L^2(\Omega)), \\ -a(l(u^n))\Delta u^n \rightharpoonup -a(l(u))\Delta u \quad \text{weakly in } L^2(t-2, t; L^2(\Omega)), \end{array} \right. \quad (33)$$

where the last two convergences have been identified using [33, Lemma 1.3, p. 12]. Then we deduce that $u \in C([t-2, t]; H_0^1(\Omega))$ and fulfils (7) in the interval $(t-2, t)$.

Moreover, since $\{(u^n)'\}_{n \geq n_1}$ is bounded in $L^2(t-2, t; L^2(\Omega))$, it satisfies that $\{u^n\}_{n \geq n_1}$ is equicontinuous in $L^2(\Omega)$ on $[t-2, t]$. From this and taking into account that $\{u^n\}_{n \geq n_1}$ is bounded in $L^\infty(t-2, t; H_0^1(\Omega))$ and the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, applying the Ascoli-Arzelá Theorem we obtain

$$u^n \rightarrow u \quad \text{strongly in } C([t-2, t]; L^2(\Omega)). \quad (34)$$

On the other hand, using that $\{u^n\}_{n \geq n_2}$ is bounded in $C([t-2, t]; H_0^1(\Omega))$, we have that for any sequence $\{s_n\} \subset [t-2, t]$ with $s_n \rightarrow s_*$,

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } H_0^1(\Omega), \quad (35)$$

where (34) has been used to identify the weak limit.

If we prove

$$u^n \rightarrow u \quad \text{strongly in } C([t-1, t]; H_0^1(\Omega)), \quad (36)$$

in particular, we will deduce that the sequence $\{u^n(t)\}$ is relatively compact in $H_0^1(\Omega)$. To that end, we argue by contradiction. We suppose that there exist $\varepsilon > 0$, a sequence $\{t_n\} \subset [t-1, t]$, without loss of generality converging to some t_* , with

$$\|u^n(t_n) - u(t_*)\| \geq \varepsilon \quad \forall n \geq 1. \quad (37)$$

From (35), it holds

$$\|u(t_*)\| \leq \liminf_{n \rightarrow \infty} \|u^n(t_n)\|.$$

It is not difficult to prove, making use of the Galerkin approximations, that

$$\begin{aligned} \|u^n(s)\|^2 &\leq \|u^n(r)\|^2 - 2\tilde{g}_1(\varepsilon)\eta \int_r^s \|u^n(\xi)\|^2 d\xi \\ &\quad - \frac{1}{2g_1(\varepsilon)m} \int_r^s |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(\xi)|^2 d\xi \end{aligned} \quad (38)$$

for all $t-2 \leq r \leq s \leq t$.

Then, we define the following continuous functions on $[t-2, t]$

$$\begin{aligned} J_n(s) &= \|u^n(s)\|^2 - 2\tilde{g}_1(\varepsilon)\eta \int_{t-2}^s \|u^n(r)\|^2 dr - \frac{1}{2g_1(\varepsilon)m} \int_{t-2}^s |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(r)|^2 dr, \\ J(s) &= \|u(s)\|^2 - 2\tilde{g}_1(\varepsilon)\eta \int_{t-2}^s \|u(r)\|^2 dr - \frac{1}{2g_1(\varepsilon)m} \int_{t-2}^s |\tilde{g}_1(\varepsilon)f(0) + g_0(\varepsilon)h(r)|^2 dr. \end{aligned}$$

Observe that thanks to (38), all the functions J_n are non-increasing on the interval $[t-2, t]$. In addition, taking into account the definition of J_n and (33), it holds

$$J_n(s) \rightarrow J(s) \quad \text{a.e. } s \in (t-2, t).$$

Hence, there exists a sequence $\{\tilde{t}_k\} \subset (t-2, t_*)$ such that $\tilde{t}_k \rightarrow t_*$ when $k \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} J_n(\tilde{t}_k) = J(\tilde{t}_k) \quad \forall k \geq 1.$$

Consider fixed $\epsilon > 0$. Since the function J is continuous on $[t-2, t]$, there exists $k(\epsilon) \geq 1$ such that

$$|J(\tilde{t}_k) - J(t_*)| < \frac{\epsilon}{2} \quad \forall k \geq k(\epsilon).$$

Now, we consider $n(\epsilon) \geq 1$ such that

$$t_n \geq \tilde{t}_{k(\epsilon)} \quad \text{and} \quad |J_n(\tilde{t}_{k(\epsilon)}) - J(\tilde{t}_{k(\epsilon)})| < \frac{\epsilon}{2} \quad \forall n \geq n(\epsilon).$$

Since all the functions J_n are non-increasing, for all $n \geq n(\epsilon)$

$$\begin{aligned} J_n(t_n) - J(t_*) &\leq J_n(\tilde{t}_{k(\epsilon)}) - J(t_*) \\ &\leq |J_n(\tilde{t}_{k(\epsilon)}) - J(\tilde{t}_{k(\epsilon)})| \\ &\leq |J_n(\tilde{t}_{k(\epsilon)}) - J(\tilde{t}_{k(\epsilon)})| + |J(\tilde{t}_{k(\epsilon)}) - J(t_*)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Then, $\limsup_{n \rightarrow \infty} J_n(t_n) \leq J(t_*)$. Thus, it satisfies that $\limsup_{n \rightarrow \infty} \|u^n(t_n)\| \leq \|u(t_*)\|$ which, together with (35), allow us to prove that $\{u^n(t_n)\}$ converges to $u(t_*)$ strongly in $H_0^1(\Omega)$, in contradiction with (37). Therefore, (36) holds. \square

The following result shows the existence of pullback attractors in $H_0^1(\Omega)$ as well as some relationships between them.

Theorem 6. Assume that (1)–(5), (12) and (16) hold, and that $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ satisfies (13) for some $\bar{\mu} \in (0, 2\lambda_1 m)$. Then, there exists $\bar{\varepsilon} \in (0, 1]$ such that for any $\varepsilon \in [0, \bar{\varepsilon}]$ and $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$, there exist the minimal pullback $\mathcal{D}_F^{H_0^1}$ -attractor $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}^\varepsilon$ and the minimal pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2, H_0^1}$ -attractor $\mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2, H_0^1}}^\varepsilon$ for the multi-valued process $U^\varepsilon : \mathbb{R}_d^2 \times H_0^1(\Omega) \rightarrow \mathcal{P}(H_0^1(\Omega))$. Furthermore, it fulfils

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}}^\varepsilon(t) \subset \mathcal{A}_{\mathcal{D}_F^{L^2}}^\varepsilon(t) \subset \mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}^\varepsilon(t) = \mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2, H_0^1}}^\varepsilon(t) \quad \forall t \in \mathbb{R} \quad \forall \varepsilon \in [0, \bar{\varepsilon}]. \quad (39)$$

In particular, for any $\widehat{D} \in \mathcal{D}_{\mu_\varepsilon}^{L^2}$, the following pullback attraction in $H_0^1(\Omega)$ holds

$$\lim_{\tau \rightarrow -\infty} \text{dist}_{H_0^1}(U^\varepsilon(t, \tau)D(\tau), \mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}^\varepsilon(t)) = 0 \quad \forall t \in \mathbb{R}. \quad (40)$$

Finally, if there exists some $\tilde{\mu} \in (0, 2\lambda_1 m)$ such that

$$\sup_{s \leq 0} \left(e^{-\tilde{\mu}s} \int_{-\infty}^s e^{\tilde{\mu}r} |h(r)|^2 dr \right) < \infty, \quad (41)$$

then there exists $\bar{\varepsilon} \in (0, 1]$ such that

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}}^\varepsilon(t) = \mathcal{A}_{\mathcal{D}_F^{L^2}}^\varepsilon(t) = \mathcal{A}_{\mathcal{D}_{\tilde{\mu}}^{L^2}}^\varepsilon(t) = \mathcal{A}_{\mathcal{D}_{\tilde{\mu}}^{L^2, H_0^1}}^\varepsilon(t) \quad \forall t \in \mathbb{R} \quad \forall \varepsilon \in [0, \bar{\varepsilon}].$$

In addition,

$$\lim_{\tau \rightarrow -\infty} \text{dist}_{H_0^1}(U^\varepsilon(t, \tau)B, \mathcal{A}_{\mathcal{D}_F^{L^2}}^\varepsilon(t)) = 0 \quad \forall t \in \mathbb{R} \quad \forall B \in \mathcal{D}_F^{L^2} \quad \forall \varepsilon \in [0, \bar{\varepsilon}]. \quad (42)$$

Proof. As in the previous results, for $\varepsilon \in [0, \bar{\varepsilon}]$ and $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$, the existence of $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}^\varepsilon$ and $\mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2, H_0^1}}^\varepsilon$ is a consequence of Corollary 1. Indeed, the process U^ε is

upper-semicontinuous with closed values (cf. Proposition 4), the relation $\mathcal{D}_F^{H_0^1} \subset \mathcal{D}_{\mu_\varepsilon}^{L^2, H_0^1}$ is fulfilled, and the existence of an absorbing family (cf. Proposition 5) and the asymptotic compactness (cf. Proposition 6) hold.

The chain of inclusions (39) follows from Corollary 1 and Theorem 4. In fact, the equality for all $t \in \mathbb{R}$ between $\mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}^\varepsilon(t)$ and $\mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2, H_0^1}}^\varepsilon(t)$ is also due to Theorem 4, using Proposition 5. Then, (40) obviously holds.

After (41), the equality $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}^\varepsilon(t) = \mathcal{A}_{\mathcal{D}_F^{L^2}}^\varepsilon(t)$ is again due to Theorem 4, making use of the first estimate appearing in Lemma 2. Therefore, (42) is straightforward. \square

Till now, in the previous results the problem (P_0) could be non-autonomous. However we aim to consider (as done in [6]) the case of (P_0) being autonomous, approached by perturbed problems coming from *some noise* that affects the coefficients and in particular it includes time-dependent forces (i.e. h). So in the sequel we assume that

$$g_0(0) = 0. \quad (43)$$

Furthermore, for the sake of simplicity in the formulation of the limit problem, we also assume that

$$\tilde{g}_1(0) = 1. \quad (44)$$

Remark 3. (i) Under the new assumption (43), problem (P_0) becomes autonomous. Since the above results also hold for (P_0) , its associated pullback attractors are in fact the global compact attractor $\mathcal{A}_{\mathcal{D}_F^{L^2}}^0$ in $L^2(\Omega)$ for the multi-valued semiflow

\mathcal{S} , where $\mathcal{S}(t - \tau) = U^0(t, \tau)$. Namely, it can be seen as a pullback attractor for the universes $\mathcal{D}_F^{L^2}$ and $\mathcal{D}_{\mu_0}^{L^2}$ with $\mu_0 = 2\lambda_1 m$ (cf. Proposition 2). Indeed, $\mathcal{A}_{\mathcal{D}_{\mu_0}^{L^2}}^0(t) = \mathcal{A}_{\mathcal{D}_F^{L^2}}^0$ holds for all $t \in \mathbb{R}$.

(ii) Analogously there exists the compact global attractor $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}^0$ in $H_0^1(\Omega)$. This set can be seen as pullback attractor for the universes $\mathcal{D}_F^{H_0^1}$ and $\mathcal{D}_{\mu_0}^{L^2, H_0^1}$. Namely, $\mathcal{A}_{\mathcal{D}_{\mu_0}^{L^2, H_0^1}}^0(t) = \mathcal{A}_{\mathcal{D}_F^{H_0^1}}^0$ for all $t \in \mathbb{R}$.

Finally, the upper semicontinuous behaviour of the pullback attractors $\mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}^\varepsilon(t) = \mathcal{A}_{\mathcal{D}_{\mu_\varepsilon}^{L^2, H_0^1}}^\varepsilon(t)$ in H^1 -norm as ε goes to 0 for all $t \in \mathbb{R}$ is analysed. As done in [6], to prove these properties we first establish the following continuity (in ε) result of solutions to (P_ε) toward solutions of the limit problem

$$(P_0) \begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) & \text{in } \Omega \times (\tau, \infty), \\ u = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x) & \text{in } \Omega. \end{cases}$$

Theorem 7. *Assume that (1)–(5), (12), (16), (43) and (44) hold, $h \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$ and consider sequences $\{\varepsilon_n\}$ with $\lim_n \varepsilon_n = 0$ and $\{u_\tau^n\} \subset L^2(\Omega)$ such that $u_\tau^n \rightharpoonup u_\tau$ weakly in $L^2(\Omega)$. Then, there exist a subsequence of $\{u_\tau^n\}$ (relabelled the same), a sequence $\{u^{\varepsilon_n}\}$, with $u^{\varepsilon_n} \in \Phi^{\varepsilon_n}(\tau, u_\tau^n)$, and $u^0 \in \Phi^0(\tau, u_\tau)$ such that*

$$u^{\varepsilon_n}(t) \rightarrow u^0(t) \quad \text{strongly in } H_0^1(\Omega) \text{ for all } t > \tau. \quad (45)$$

Proof. Let $\{u_\tau^n\} \subset L^2(\Omega)$ be a sequence such that $u_\tau^n \rightharpoonup u_\tau$ weakly in $L^2(\Omega)$. Consider arbitrary values $T > \tau$ and $\delta \in (0, T - \tau)$.

Consider u^{ε_n} a weak solution to (P_{ε_n}) in $[\tau, T]$. Making use of the energy equality and (1), we obtain

$$\frac{1}{2} \frac{d}{ds} |u^{\varepsilon_n}(s)|^2 + g_1(\varepsilon_n)m \|u^{\varepsilon_n}(s)\|^2 \leq \tilde{g}_1(\varepsilon_n)(f(u^{\varepsilon_n}(s)), u^{\varepsilon_n}(s)) + g_0(\varepsilon_n)\langle h(s), u^{\varepsilon_n}(s) \rangle$$

a.e. $s \in [\tau, T]$.

Now, define $\gamma := \min_n \{g_1(\varepsilon_n)\} \in (0, 1]$. Then, taking this into account together with (4) and the Cauchy inequality, we have

$$\frac{d}{ds} |u^{\varepsilon_n}(s)|^2 + \gamma m \|u^{\varepsilon_n}(s)\|^2 + 2\alpha_2 \tilde{g}_1(\varepsilon_n) |u^{\varepsilon_n}(s)|_p^p \leq 2\tilde{g}_1(\varepsilon_n) \kappa |\Omega| + \frac{(g_0(\varepsilon_n))^2 \|h(s)\|_*^2}{\gamma m}$$

a.e. $s \in [\tau, T]$. Then, integrating between τ and t with $t \in [\tau, T]$, we deduce

$$\begin{aligned} & |u^{\varepsilon_n}(t)|^2 + \gamma m \int_\tau^t \|u^{\varepsilon_n}(s)\|^2 ds + 2\alpha_2 \tilde{g}_1(\varepsilon_n) \int_\tau^t |u^{\varepsilon_n}(s)|_p^p ds \\ & \leq |u_\tau^n|^2 + 2\tilde{g}_1(\varepsilon_n) \kappa |\Omega| (T - \tau) + \frac{(g_0(\varepsilon_n))^2}{\gamma m} \int_\tau^T \|h(s)\|_*^2 ds. \end{aligned}$$

Therefore, $\{u^{\varepsilon_n}\}$ is bounded in $L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$. In addition, since each $u^{\varepsilon_n} \in C([\tau, T]; L^2(\Omega))$ for all n , it holds

$$|u^{\varepsilon_n}(t)| \leq C_\infty \quad \forall t \in [\tau, T] \quad \forall n \geq 1,$$

where C_∞ is a positive constant independent of ε_n . Now, as $l \in L^2(\Omega)$ and $a \in C(\mathbb{R}; [0, \infty))$, there exists $M_{C_\infty} > 0$ such that

$$a(l(u^{\varepsilon_n}(t))) \leq M_{C_\infty} \quad \forall t \in [\tau, T] \quad \forall n \geq 1.$$

Then, it is standard to deduce that there exist a subsequence of $\{u^{\varepsilon_n}\}$ (reabeled the same) and $u^0 \in C([\tau, T]; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$, solution to (P_0) with $u^0(\tau) = u_\tau$.

On the other hand, making use of the strong energy equality (18) for the Galerkin approximations to problems $(P_{\varepsilon_n, u^{\varepsilon_n}})$, applying (1) and (5), and passing to the limit, we obtain

$$\begin{aligned} & \|u^{\varepsilon_n}(t)\|^2 + g_1(\varepsilon_n)m \int_{\tau+\delta}^t |-\Delta u^{\varepsilon_n}(r)|^2 dr \leq \|u^{\varepsilon_n}(\tau+\delta)\|^2 \\ & + 2\tilde{g}_1(\varepsilon_n)\eta \int_{\tau+\delta}^t \|u^{\varepsilon_n}(r)\|^2 dr + \frac{1}{g_1(\varepsilon_n)m} \int_{\tau+\delta}^t |\tilde{g}_1(\varepsilon_n)f(0) + g_0(\varepsilon_n)h(r)|^2 dr \end{aligned}$$

for all $\delta \in (0, t - \tau)$.

Taking this into account together with Theorem 2, the sequence $\{u^{\varepsilon_n}\}$ is bounded in $L^\infty(\tau+\delta, T; H_0^1(\Omega)) \cap L^2(\tau+\delta, T; H^2(\Omega) \cap H_0^1(\Omega))$. Then from (20), the sequence $\{-a(l(u^{\varepsilon_n}))\Delta u^{\varepsilon_n}\}$ is bounded in $L^2(\tau+\delta, T; L^2(\Omega))$. From (16) and the previous boundedness of $\{u^{\varepsilon_n}\}$, we deduce that $\{f(u^{\varepsilon_n})\}$ is bounded in $L^2(\tau+\delta, T; L^2(\Omega))$. Finally, bearing in mind the above estimates and the problems (P_{ε_n}) we obtain that $\{(u^{\varepsilon_n})'\}$ is bounded in $L^2(\tau+\delta, T; L^2(\Omega))$.

Then, we gain that the limit $u^0 \in L^\infty(\tau+\delta, T; H_0^1(\Omega)) \cap L^2(\tau+\delta, T; H^2(\Omega) \cap H_0^1(\Omega))$ with $(u^0)' \in L^2(\tau+\delta, T; L^2(\Omega))$. Now using the Aubin-Lions lemma, there exists a subsequence of $\{u^{\varepsilon_n}\}$ (reabeled the same) such that

$$\left\{ \begin{array}{l} u^{\varepsilon_n} \overset{*}{\rightharpoonup} u^0 \quad \text{weakly-star in } L^\infty(\tau+\delta, T; H_0^1(\Omega)), \\ u^{\varepsilon_n} \rightharpoonup u^0 \quad \text{weakly in } L^2(\tau+\delta, T; H^2(\Omega)), \\ u^{\varepsilon_n} \rightarrow u^0 \quad \text{strongly in } L^2(\tau+\delta, T; H_0^1(\Omega)), \\ u^{\varepsilon_n}(t) \rightarrow u^0(t) \quad \text{strongly in } H_0^1(\Omega) \quad \text{a.e. } t \in (\tau+\delta, T), \\ (u^{\varepsilon_n})' \rightharpoonup (u^0)' \quad \text{weakly in } L^2(\tau+\delta, T; L^2(\Omega)), \\ \tilde{g}_1(\varepsilon_n)f(u^{\varepsilon_n}) \rightharpoonup f(u^0) \quad \text{weakly in } L^2(\tau+\delta, T; L^2(\Omega)), \\ -g_1(\varepsilon_n)a(l(u^{\varepsilon_n}))\Delta u^{\varepsilon_n} \rightharpoonup -a(l(u^0))\Delta u^0 \quad \text{weakly in } L^2(\tau+\delta, T; L^2(\Omega)), \end{array} \right. \quad (46)$$

where the limits of the last two convergences have been identified using [33, Lemma 1.3, p. 12]. In fact, we may now repeat the arguments in the intervals $(\tau+\delta/2, T+1)$, $(\tau+\delta/3, T+2)$, etcetera, and making use of a diagonal argument, (46) holds in $(\tau+\tilde{\delta}, \tilde{T})$ for all $\tilde{T} > \tau$ and any $\tilde{\delta} \in (0, \tilde{T} - \tau)$.

Now, we complete the proof by showing (45). Given $t > \tau$, consider $T > t$ and $\delta \in (0, T - \tau)$.

On the one hand, taking into account that the sequences $\{u^{\varepsilon_n}\}$ and $\{(u^{\varepsilon_n})'\}$ are bounded in $C([\tau+\delta, T]; H_0^1(\Omega))$ and $L^2(\tau+\delta, T; L^2(\Omega))$ respectively, and the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the Ascoli-Arzelà Theorem implies that

$$u^{\varepsilon_n} \rightarrow u^0 \quad \text{strongly in } C([\tau+\delta, T]; L^2(\Omega)). \quad (47)$$

In fact, since $\{u^{\varepsilon_n}\}$ is bounded in $C([\tau+\delta, T]; H_0^1(\Omega))$, we obtain

$$u^{\varepsilon_n}(s) \rightharpoonup u^0(s) \quad \text{weakly in } H_0^1(\Omega) \quad \forall s \in [\tau+\delta, T], \quad (48)$$

where (47) has been used to identify the weak limit.

On the other hand, again from the energy equality (18) for the Galerkin approximations for $(P_{\varepsilon_n, u^{\varepsilon_n}})$, applying (1) and (5) and passing to the limit, we have for

all $\tau + \delta \leq r \leq s \leq T$

$$\begin{aligned} \|u^{\varepsilon_n}(s)\|^2 &\leq \|u^{\varepsilon_n}(r)\|^2 + 2\tilde{g}_1(\varepsilon_n)\eta \int_{\tau+\delta}^s \|u^{\varepsilon_n}(\xi)\|^2 d\xi \\ &\quad + \frac{1}{2g_1(\varepsilon_n)m} \int_{\tau+\delta}^s |\tilde{g}_1(\varepsilon_n)f(0) + g_0(\varepsilon_n)h(\xi)|^2 d\xi. \end{aligned} \quad (49)$$

Now, we define the following continuous functions on $[\tau + \delta, T]$

$$\begin{aligned} J_{\varepsilon_n}(s) &= \|u^{\varepsilon_n}(s)\|^2 - 2\tilde{g}_1(\varepsilon_n)\eta \int_{\tau+\delta}^s \|u^{\varepsilon_n}(\xi)\|^2 d\xi \\ &\quad - \frac{1}{2g_1(\varepsilon_n)m} \int_{\tau+\delta}^s |\tilde{g}_1(\varepsilon_n)f(0) + g_0(\varepsilon_n)h(\xi)|^2 d\xi, \\ J_0(s) &= \|u^0(s)\|^2 - 2\eta \int_{\tau+\delta}^s \|u^0(\xi)\|^2 d\xi - \frac{(f(0))^2|\Omega|[s - (\tau + \delta)]}{2m}. \end{aligned}$$

Observe that from (49) we deduce that all the functions J_n are non-increasing on $[\tau + \delta, T]$. Furthermore, since $u^{\varepsilon_n}(t) \rightarrow u^0(t)$ strongly in $H_0^1(\Omega)$ a.e. $t \in (\tau + \delta, T)$, J is continuous in $[\tau + \delta, T]$ and all J_n are non-increasing in $[\tau + \delta, T]$, it holds

$$J_{\varepsilon_n}(s) \rightarrow J_0(s) \quad \forall s \in [\tau + \delta, T].$$

From this, we deduce

$$\lim_{n \rightarrow \infty} \|u^{\varepsilon_n}(s)\|^2 = \|u^0(s)\|^2 \quad \forall s \in [\tau + \delta, T].$$

Taking this into account, together with (48), (45) holds. \square

In order to prove the upper semicontinuous behaviour of attractors in $H_0^1(\Omega)$ we introduce a last condition relating some terms involved in the formula for $R_{L^2}^\varepsilon$ when ε goes to 0, namely we assume that

$$\limsup_{\varepsilon \rightarrow 0} \frac{(g_0(\varepsilon))^2}{2g_1(\varepsilon)m - \lambda_1^{-1}\mu_\varepsilon} < \infty, \quad (50)$$

where μ_ε are chosen in $[\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$.

Observe that in [6] we did not specify how to choose μ_ε . Actually we just said that they could be taken equal to μ_{ε_0} . In this paper, condition (50) provides how close to zero the amount $2g_1(\varepsilon)m - \lambda_1^{-1}\mu_\varepsilon$ can be such that the whole fraction in (50) is $O(1)$. This fact will be essential in the proof of our main result.

Theorem 8. *Assume that (1)–(5), (12), (16), (43), (44) and (50) hold, and $h \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$ satisfies (13) for some $\bar{\mu} \in (0, 2\lambda_1 m)$. Then, there exists $\bar{\varepsilon} \in (0, 1]$ such that for any $\varepsilon \in [0, \bar{\varepsilon}]$ and any $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$, the family $\{\mathcal{A}_{\mathcal{D}^{\mu_\varepsilon}}^\varepsilon(t)\}_{\varepsilon \in (0, \bar{\varepsilon}]}$ converges upper semicontinuously to $\mathcal{A}_{\mathcal{D}_F^2}^0$ in $H_0^1(\Omega)$ as ε goes to 0, i.e.*

$$\lim_{\varepsilon \rightarrow 0} \text{dist}_{H_0^1}(\mathcal{A}_{\mathcal{D}^{\mu_\varepsilon}}^\varepsilon(t), \mathcal{A}_{\mathcal{D}_F^2}^0) = 0 \quad \forall t \in \mathbb{R}. \quad (51)$$

Proof. As in previous results, denote $\bar{\varepsilon} \in (0, 1]$ such that $\bar{\mu} < 2g_1(\varepsilon)\lambda_1 m$ for any $\varepsilon \in [0, \bar{\varepsilon}]$ and take $\mu_\varepsilon \in [\bar{\mu}, 2g_1(\varepsilon)\lambda_1 m)$. Without loss of generality, from (50) consider a constant $C > 0$ such that

$$\frac{(g_0(\varepsilon))^2}{2g_1(\varepsilon)m - \lambda_1^{-1}\mu_\varepsilon} < C \quad \forall \varepsilon \in [0, \bar{\varepsilon}].$$

We prove (51) arguing by contradiction. Suppose that there exist $\epsilon > 0$, $t \in \mathbb{R}$ and a sequence $\{\varepsilon_n\}_{n \geq 1} \subset (0, \bar{\varepsilon}]$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ such that

$$\text{dist}_{H_0^1}(\mathcal{A}_{\mathcal{D}_{\mu\varepsilon_n}^{L^2}}^{\varepsilon_n}(t), \mathcal{A}_{\mathcal{D}_F^{L^2}}^0) > \epsilon \quad \forall n \in \mathbb{N}.$$

Since the pullback attractors are negatively invariant (cf. Definition 8), there exists a sequence of solutions $\{u^{\varepsilon_n}\}_{n \geq 1}$ with $u^{\varepsilon_n}(t) \in \mathcal{A}_{\mathcal{D}_{\mu\varepsilon_n}^{L^2}}^{\varepsilon_n}(t)$ such that

$$d_{H_0^1}(u^{\varepsilon_n}(t), \mathcal{A}_{\mathcal{D}_F^{L^2}}^0) > \epsilon \quad \forall n \in \mathbb{N}, \quad (52)$$

where the distance in $H_0^1(\Omega)$ makes complete sense thanks to the regularising effect of the equation (cf. Theorem 2) and the fact that $\mathcal{A}_{\mathcal{D}_F^{L^2}}^0 = \mathcal{A}_{\mathcal{D}_F^{H_0^1}}^0$ thanks to the cited regularising effect.

Observe that $\mathcal{A}_{\mathcal{D}_{\mu\varepsilon_n}^{L^2}}^{\varepsilon_n}(t) \subset D_{0, H_0^1}^{\varepsilon_n}(t)$ (cf. Theorem 2 and Proposition 5) for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, and $D_{0, H_0^1}^{\varepsilon_n}(t) = \overline{B}_{L^2}(0, (R_{L^2}^{\varepsilon_n}(t))^{1/2}) \cap H_0^1(\Omega)$, where we remind that

$$R_{L^2}^{\varepsilon_n}(t) = 1 + \frac{2\tilde{g}_1(\varepsilon_n)\kappa|\Omega|}{\mu\varepsilon_n} + \frac{(g_0(\varepsilon_n))^2 e^{-\mu\varepsilon_n t}}{2g_1(\varepsilon_n) - \lambda_1^{-1}\mu\varepsilon_n} \int_{-\infty}^t e^{\mu\varepsilon_n s} \|h(s)\|_*^2 ds.$$

Taking into account (50), $\mu\varepsilon_n \in [\bar{\mu}, 2g_1(\varepsilon_n)\lambda_1 m]$, and denoting $\widetilde{M} := \max_{[0,1]} |\tilde{g}_1(\cdot)|$, we obtain that

$$R_{L^2}^{\varepsilon}(t) \leq R_{L^2}^0(t) := 1 + \frac{2\kappa\widetilde{M}|\Omega|}{\bar{\mu}} + C e^{\tilde{C}(-t)} \int_{-\infty}^t e^{\tilde{C}(s)} \|h(s)\|_*^2 ds \quad \forall t \in \mathbb{R} \quad \forall \varepsilon \in (0, \bar{\varepsilon}],$$

where

$$\tilde{C}(s) = \begin{cases} \bar{\mu}s & \text{if } s \leq 0, \\ 2\lambda_1 m s & \text{if } s > 0. \end{cases}$$

In particular, observe that for $t < 0$ we have that

$$R_{L^2}^0(t) = 1 + \frac{2\kappa\widetilde{M}|\Omega|}{\bar{\mu}} + C e^{-2\lambda_1 m t} \int_{-\infty}^t e^{\bar{\mu}s} \|h(s)\|_*^2 ds,$$

whence it is obvious that the family

$$\widetilde{D}_{H_0^1}^0 = \{D_{H_0^1}^0(t) := \overline{B}_{L^2}(0, (R_{L^2}^0(t))^{1/2}) \cap H_0^1(\Omega) : t \in \mathbb{R}\} \in \mathcal{D}_{2\lambda_1 m}^{L^2, H_0^1}.$$

From the above inclusions and definition we deduce that

$$\mathcal{A}_{\mathcal{D}_{\mu\varepsilon_n}^{L^2}}^{\varepsilon_n}(t) \subset D_{H_0^1}^0(t) \quad \forall t \in \mathbb{R} \quad \forall n \geq 1. \quad (53)$$

On the other hand, since $\widetilde{D}_{H_0^1}^0$ belongs to $\mathcal{D}_{2\lambda_1 m}^{L^2, H_0^1}$, there exists $\tau(t, \widetilde{D}_{H_0^1}^0, \epsilon) < t$ such that

$$\text{dist}_{H_0^1}(U^0(t, \tau)D_{H_0^1}^0(\tau), \mathcal{A}_{\mathcal{D}_F^{L^2}}^0) \leq \frac{\epsilon}{2} \quad \forall \tau \leq \tau(t, \widetilde{D}_{H_0^1}^0, \epsilon). \quad (54)$$

From (53), we deduce that the sequence $\{u^{\varepsilon_n}(\tau(t, \widetilde{D}_{H_0^1}^0, \epsilon))\}_{n \geq 1}$ is bounded and possesses a subsequence (relabelled the same) such that $u^{\varepsilon_n}(\tau(t, \widetilde{D}_{H_0^1}^0, \epsilon)) \rightharpoonup u_\tau$ weakly in $L^2(\Omega)$.

Now, applying Theorem 7, we deduce that there exists $u^0 \in \Phi^0(\tau, u_\tau)$ and a subsequence of $\{\varepsilon_n\}_{n \geq 1}$ (relabelled the same) such that (45) holds for all $t > \tau(t, \widetilde{D}_{H_0^1}^0, \epsilon)$. Thus, we deduce that there exists $n_0 \geq 1$ such that

$$\|u^{\varepsilon_n}(t) - u^0(t)\| \leq \frac{\epsilon}{2} \quad \forall n \geq n_0. \quad (55)$$

Now, using (54) and (55), we have

$$\begin{aligned} d_{H_0^1}(u^{\varepsilon_n}(t), \mathcal{A}_{\mathcal{D}L^2}^0) &\leq d_{H_0^1}(u^{\varepsilon_n}(t), u^0(t)) + d_{H_0^1}(u^0(t), \mathcal{A}_{\mathcal{D}L^2}^0) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n \geq n_0, \end{aligned}$$

which is contradictory with (52). \square

Remark 4. (i) The results concerning weak solutions and L^2 -attractors only make use of f continuous and fulfilling (4) (cf. [6]). However, in the strong framework, assumption (5) is used. Therefore, to avoid confusion in the exposition this regularity has been imposed from the beginning. Nevertheless, this last condition (5) can be replaced by the weaker one

$$(f(s) - f(r))(s - r) \leq \eta(s - r)^2 \quad \forall s, r \in \mathbb{R},$$

with f just continuous. To that end, just simply considering mollifiers ρ_δ , which implies that $f_\delta = \rho_\delta * f$ fulfils (5), and compactness arguments.

(ii) The values that γ takes in assumption (16) are larger if the interpolation result is applied not only to $L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega))$, but also to $L^\infty(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ and $L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$. Then, the improved values of γ are

$$\gamma = \begin{cases} \max \left\{ 4, \frac{p-2}{2} \right\} & \text{if } N = 3, \\ \max \left\{ 2, \frac{p-2}{2} \right\} & \text{if } N = 4, \\ \max \left\{ \frac{4}{N-2}, \frac{p-2}{2} \right\} & \text{if } N \geq 5. \end{cases}$$

Bearing in mind that the proofs are completely analogous with these new values of γ , we have decided to use just $L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega))$ along the paper for the sake of clarity and simplicity.

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