

Time-dependent attractors for non-autonomous nonlocal reaction-diffusion equations

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Dedicated to Karin Wahl, in Memoriam

In this paper, the existence and uniqueness of weak and strong solutions for a non-autonomous nonlocal reaction-diffusion equation is proved. Next, the existence of minimal pullback attractors in the L^2 -norm in the frameworks of universes of fixed bounded sets and those given by a tempered growth condition, and some relationships between them are established. Finally, we prove the existence of minimal pullback attractors in the H^1 -norm and study relationships among these new families and those given previously in the L^2 - context. The results are also new in the autonomous framework in order to ensure the existence of global compact attractors, as a particular case.

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1. Introduction and statement of the problem

Over the last few decades the study of nonlocal problems has taken a keen interest (e.g., cf. [25, 6, 27, 41, 5] among many others), especially those of diffusion type (see e.g. [18, 32, 15, 13, 3]). In particular, several authors have been interested in the problem

$$\begin{cases} \frac{du}{dt} - a(l(u))\Delta u = f & \text{in } \Omega \times (0, T), \\ u(\cdot, t) \in \tilde{V} & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with boundary $\partial\Omega$. Here \tilde{V} denotes the set $\{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}$, where $\Gamma_0 \subset \partial\Omega$ has positive superficial measure.

This problem is not a trivial variant of the heat equation, because due to the presence of the nonlocal term, it is not possible to guarantee the existence of a Lyapunov structure (cf. [21]). In [16], Chipot and Lovat establish the existence and uniqueness of a weak solution to this problem using the Faedo-Galerkin approximations and compactness arguments. To prove the existence, they assume that

$a \in C(\mathbb{R}; (0, \infty))$ and there exist positive constants m, M such that

$$0 < m \leq a(s) \leq M \quad \forall s \in \mathbb{R}. \quad (1.2)$$

Furthermore, l is a continuous linear form from $L^2(\Omega)$ into \mathbb{R} . Namely,

$$l(u) = l_g(u) = \int_{\Omega} g(x)u(x)dx \quad \text{for some } g \in L^2(\Omega).$$

The assumptions made on the function a are sufficient in order to avoid that the solutions exist only in finite-time intervals (see [32] for more details). They do not make any additional assumptions on the function a because it depends on the type of problem intended to model. For example, in population dynamics, the monotonicity of the function must be adapted to the behaviour of the species we want to model (see [16]). To prove the uniqueness, they suppose that the function a is globally Lipschitz due to the presence of the nonlocal term. They also study the asymptotic behaviour of the solutions when $f \in V'$ and under additional assumptions. Later, in [17], Chipot and Molinet generalize the results obtained in [16], dealing with a continuum of steady states using dynamical systems.

Many authors have been interested in analyzing some variants of problem (1.1). In [1, 2], Andami Ovono and Rougirel study a problem in which the nonlocal operator does not act in the whole domain. In these two papers, the existence of radial solutions, bifurcation analysis, and their stability are analyzed. In [19], Chipot and Siegwart study the asymptotic behaviour of the solutions to problems with nonlocal diffusion and mixed boundary conditions. In [11], Chipot and Chang are also interested in the asymptotic behaviour of the solutions to nonlocal problems with two nonlocal terms and mixed boundary conditions. In particular, they prove results which establish relationships between the solution of the evolution problem and stationary solutions. These results are similar to those given in a simpler framework (see [16, Theorem 4.1] for more details). In [20], Chipot et al. consider a problem in which the nonlocal term depends on a *Dirichlet integral*. In this particular case, they are able to find a Lyapunov structure, which is used to study the asymptotic behaviour of the solutions.

Despite all the cited advances in the case of f independent of the solution, the generalization to the case of nonlinearities $f(u)$ involves many more difficulties. Actually, only a few references deal with such situation, and the results are only partial (e.g., cf. [36, 14]). Indeed, due to the extra troubles, it makes full sense to do attempts of dynamical studies in more general frameworks. An appropriate one to study the long-time behaviour of the solutions of different versions of problem (1.1) is through the theory of attractors. However, the study will be different depending on whether there are time-dependent terms or not. Whereas in the autonomous case, the compact global attractor is the natural object to seek and study, when a non-autonomous problem is dealt with, there exist several approaches, as uniform attractors and their kernel sections (cf. [12]), skew-product flows (cf. [38]), and pullback attractors, a very recent theory which has been vastly developed in the last decade in a wide range of problems (cf. [10, 7, 29, 4]). This approach allows us to establish not only the asymptotic behaviour of the dynamical system but also what the current attractions sections are when the initial data come from long time

ago in the past. Within this framework, some authors are interested in studying the pullback attractor in the classical sense, i.e. the pullback attractor of solutions starting in “fixed” bounded sets. Others, though, and particularly motivated by random dynamics, employ the concept of attraction related to a class of families, called universe \mathcal{D} , made up by sets which are allowed to move in time and usually defined in terms of a tempered condition (cf. [9, 10, 26]). Some relationships between these two kind of attractors have also been established (cf. [34]).

Some first and partial results concerning existence of global attractors for autonomous nonlocal problems with linear force (similar to (1.1)) were addressed by Lovat [32] and Andami Ovono [1].

In this paper we will focus on proving the existence of solutions and pullback \mathcal{D} -attractors in the L^2 and H^1 norms for the dynamical system associated to the non-autonomous nonlocal reaction-diffusion problem

$$\begin{cases} \frac{du}{dt} - a(l(u))\Delta u = f(u) + h(t) & \text{in } \Omega \times (\tau, \infty), \\ u = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x) & \text{in } \Omega, \end{cases} \quad (1.3)$$

where Ω is a bounded open subset of \mathbb{R}^N .

Observe that the assumptions on f have been weakened along several papers. In [36], Menezes proves the existence and uniqueness of weak solution to (1.3) by fixed point arguments, considering that the function f is globally Lipschitz. Later, the analogous existence and uniqueness result is proved in [8] without any assumptions of smoothness on the boundary of Ω and the nonlinearity $f(u)$ is just continuous, sublinear, and fulfills a monotonicity condition.

In this paper, we assume that $f \in C^1(\mathbb{R})$ and there exist positive constants α_1 , α_2 , κ , η , and $p > 2$ such that

$$f'(s) \leq \eta \quad \forall s \in \mathbb{R}, \quad (1.4)$$

$$-\kappa - \alpha_1|s|^p \leq f(s)s \leq \kappa - \alpha_2|s|^p \quad \forall s \in \mathbb{R}. \quad (1.5)$$

Observe that the assumption (1.4) can be weakened (see remark 5.9 (ii) below).

From (1.5), we deduce that there exists $\beta > 0$ such that

$$|f(s)| \leq \beta(|s|^{p-1} + 1) \quad \forall s \in \mathbb{R}. \quad (1.6)$$

Under these assumptions, in addition to proving the existence of weak solutions, we also show the existence of pullback attractors in the L^2 -norm for the dynamical system associated to (1.3). While in the sublinear framework we also needed to impose the usual assumption $\limsup_{|s| \rightarrow \infty} f(s)/s < m\lambda_1$ (see (17) in [8]), here condition (1.5) will be enough.

Finally, we also show the existence of pullback attractors in the H^1 -norm.

Note that the case $p \in [1, 2]$ is not considered here, since both goals, existence of solutions and pullback attractors follow by applying the results in [8]. As far as we know, there are no references in the previous literature devoted to the existence of neither global or pullback attractors in $L^2(\Omega)$ nor in $H_0^1(\Omega)$ for the associated dynamical systems. Our results are stated in a non-autonomous framework, but

they can be immediately applied to the autonomous case (h independent of time) in order to obtain global compact attractors, which is also new.

Regarding the assumptions made on the function a , it is locally Lipschitz and fulfil (1.2) (see Remark 5.9 (ii) for a more general setting).

The structure of the paper is as follows. The existence and uniqueness of a weak solution is analyzed in §2. There, we will also prove the existence and uniqueness of a strong solution and the regularizing effect of the equation. To prove the existence of a solution, we will use the Faedo-Galerkin approximations and compactness arguments (cf. [31, 37]). The core of the paper concerning the asymptotic behaviour of solutions is developed in the following paragraphs. Section 3 is devoted to briefly recalling some abstract results on pullback attractors that we will use in the following sections. Under an extra suitable assumption for h (cf. (4.1)), the existence of pullback attractors in the phase space $L^2(\Omega)$ is proved in §4 by using an energy method which relies on the continuity of the solutions (e.g. cf. [28, 33, 35, 26]). In §5, we prove the existence of pullback attractors in $H_0^1(\Omega)$, again via an energy method analogous to that of the previous section. Finally, we establish relationships between these families of pullback attractors and those given in §4 using the regularizing effect of the equation.

Before continuing, let us introduce some notation. We will denote by (\cdot, \cdot) the inner product in $L^2(\Omega)$, and by $|\cdot|$ its associated norm (we also represent in this way the Lebesgue measure of a subset of \mathbb{R}^N since no confusion arises). We will denote by $((\cdot, \cdot))$ the inner product in $H_0^1(\Omega)$ given by the product in $(L^2(\Omega))^N$ of the gradients and by $\|\cdot\|$ its associated norm. The duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ will be represented by $\langle \cdot, \cdot \rangle$, and by $\|\cdot\|_*$ the norm in $H^{-1}(\Omega)$. Identifying $L^2(\Omega)$ with its dual, we have the chain of dense and compact embeddings $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. As a consequence of the previous identification, the operator l acting on u must be understood as (l, u) , but we keep the notation $l(u)$, the usual one in the existing previous literature. We will also denote by (\cdot, \cdot) the duality product between $L^p(\Omega)$ and $L^q(\Omega)$ (where q is the conjugate exponent of p) and by $\|\cdot\|_{L^p(\Omega)}$ the norm in the space $L^p(\Omega)$. Analogously, the norm in $L^r(\tau, T; X)$, where $r \geq 1$ and X is a separable Banach space, will be denoted by $\|\cdot\|_{L^r(\tau, T; X)}$.

In what follows we assume that $h \in L_{loc}^2(\mathbb{R}; H^{-1}(\Omega))$ and $u_\tau \in L^2(\Omega)$.

Definition 1.1. *A weak solution to (1.3) is a function $u \in L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ for all $T > \tau$, with $u(\tau) = u_\tau$, and such that for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$*

$$\frac{d}{dt}(u(t), v) + a(l(u(t)))((u(t), v)) = (f(u(t)), v) + \langle h(t), v \rangle, \quad (1.7)$$

where the previous equation must be understood in the sense of $\mathcal{D}'(\tau, \infty)$.

Remark 1.2. *If u is a weak solution to (1.3), then from (1.2), (1.6), and (1.7) it fulfills for any $T > \tau$ that $u' \in L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$, and therefore $u \in C([\tau, \infty); L^2(\Omega))$ (cf. [31, Théorème 2, p. 575]). Hence the initial datum in*

(1.3) makes sense. Moreover, the following energy equality holds

$$|u(t)|^2 + 2 \int_s^t a(l(u(r))) \|u(r)\|^2 dr = |u(s)|^2 + 2 \int_s^t (f(u(r)), u(r)) dr + 2 \int_s^t \langle h(r), u(r) \rangle dr \quad (1.8)$$

for all $\tau \leq s \leq t$ (cf. [23, Théorème 2, p. 575] or [42, Lemma 3.2, p. 71]).

A notion of more regular solution is also suitable for problem (1.3).

Definition 1.3. A strong solution to (1.3) is a weak solution u such that $u \in L^2(\tau, T; D(-\Delta)) \cap L^\infty(\tau, T; H_0^1(\Omega) \cap L^p(\Omega))$ for all $T > \tau$.

Remark 1.4. If $h \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$ and u is a strong solution to (1.3), then $u' \in L^2(\tau, T; L^2(\Omega))$ for all $T > \tau$, and, consequently, $u \in C([\tau, +\infty); H_0^1(\Omega))$. In addition, the following energy equality holds:

$$\|u(t)\|^2 + 2 \int_s^t a(l(u(r))) |-\Delta u(r)|^2 dr = \|u(s)\|^2 + 2 \int_s^t (f(u(r)) + h(r), -\Delta u(r)) dr \quad (1.9)$$

for all $\tau \leq s \leq t$.

2. Existence and uniqueness of solution

In this section the existence and uniqueness of weak and strong solutions to (1.3) and the regularizing effect of the equation will be studied. To prove the existence of solutions, we will use the Faedo-Galerkin approximations and pass to the limit by using compactness arguments.

Theorem 2.1. Suppose that the function a is locally Lipschitz and satisfies (1.2), $f \in C^1(\mathbb{R})$ fulfills (1.4) and (1.5), $h \in L_{loc}^2(\mathbb{R}; H^{-1}(\Omega))$, and $l \in L^2(\Omega)$. Then, given $u_\tau \in L^2(\Omega)$, there exists a unique weak solution to the problem (1.3), which will be denoted by $u(\cdot; \tau, u_\tau)$, and satisfies the energy equality (1.8).

Proof. We split the proof into two steps.

Step 1. Uniqueness of weak solution. Suppose that there exist two weak solutions, $u_1(\cdot; \tau, u_\tau)$ and $u_2(\cdot; \tau, u_\tau)$, to (1.3). For short, we will denote $u_i(\cdot) = u_i(\cdot; \tau, u_\tau)$ for $i = 1, 2$. Once that a weak solution exists, the energy equality is an immediate consequence (cf. remark 1.2). Then, we deduce

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |u_1(t) - u_2(t)|^2 + a(l(u_1(t))) \|u_1(t) - u_2(t)\|^2 \\ &= [a(l(u_2(t))) - a(l(u_1(t)))] ((u_2(t), u_1(t) - u_2(t))) + (f(u_1(t)) - f(u_2(t)), u_1(t) - u_2(t)) \end{aligned}$$

a.e. $t \in [\tau, T]$.

Since $u_1, u_2 \in C([\tau, T]; L^2(\Omega))$, we have that $u_1(t), u_2(t) \in S$ for all $t \in [\tau, T]$, where S is a bounded subset of $L^2(\Omega)$. Besides, as $l \in L^2(\Omega)$, it holds $\{l(u_i(t))\}_{t \in [\tau, T]} \subset [-R, R]$ for $i = 1, 2$, and for some $R > 0$. Hence, using (1.2), (1.4), the locally Lipschitz continuity of the function a , and the Cauchy inequality (cf. [24, Appendix B, p. 622]), we obtain

$$\frac{d}{dt} |u_1(t) - u_2(t)|^2 \leq \frac{(L_a(R))^2 \|l\|^2 \|u_2(t)\|^2 + 4m\eta}{2m} |u_1(t) - u_2(t)|^2 \quad \text{a.e. } t \in (\tau, T),$$

where $L_a(R)$ is the Lipschitz constant of the function a in $[-R, R]$. Then, uniqueness follows.

Step 2. Existence of weak solution. Assume that $u_\tau \in L^2(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. Using the spectral theory, we deduce that there exists a sequence $\{w_i\}_{i \geq 1}$ of eigenfunctions of $-\Delta$ in $H^1_0(\Omega)$, which is a Hilbert basis of $L^2(\Omega)$. Then, for each integer $n \geq 1$, the function $u_n(t; \tau, u_\tau) = \sum_{j=1}^n \varphi_{nj}(t) w_j$ ($u_n(t)$ for short) will denote the local solution of

$$\begin{cases} \frac{d}{dt}(u_n(t), w_j) + a(l(u_n(t)))(u_n(t), w_j) = (f(u_n(t)), w_j) + \langle h(t), w_j \rangle, & t \in (\tau, \infty) \\ (u_n(\tau), w_j) = (u_\tau, w_j), & j = 1, \dots, n. \end{cases} \quad (2.1)$$

Now, multiplying by φ_{nj} in (2.1), summing from $j = 1$ to n , and using (1.2), (1.5), and the Cauchy inequality, we obtain

$$\frac{d}{dt}|u_n(t)|^2 + m\|u_n(t)\|^2 + 2\alpha_2\|u_n(t)\|_{L^p(\Omega)}^p \leq 2\kappa|\Omega| + \frac{1}{m}\|h(t)\|_*^2 \quad \text{a.e. } t \in (\tau, t_n).$$

Integrating between τ and t with $\tau < t < t_n$, we obtain

$$\begin{aligned} & |u_n(t)|^2 + m \int_\tau^t \|u_n(s)\|^2 ds + 2\alpha_2 \int_\tau^t \|u_n(s)\|_{L^p(\Omega)}^p ds \\ & \leq |u_\tau|^2 + 2\kappa|\Omega|(T - \tau) + \frac{1}{m} \int_\tau^T \|h(s)\|_*^2 ds. \end{aligned}$$

From this a priori estimate, we deduce that $\{u_n\}$ is well defined and bounded in $L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ for all $T > \tau$. Taking into account this estimate and (1.2), we deduce that the sequence $\{-a(l(u_n))\Delta u_n\}$ is bounded in $L^2(\tau, T; H^{-1}(\Omega))$. In addition, using the boundedness of $\{u_n\}$ in $L^p(\tau, T; L^p(\Omega))$ and (1.6), it yields that $\{f(u_n)\}$ is bounded in $L^q(\tau, T; L^q(\Omega))$.

Finally, to obtain the boundedness of $\{u'_n\}$, we need first to define two additional projection operators related to $P_n : L^2(\Omega) \rightarrow V_n := \text{span}\{w_1, \dots, w_n\}$ given by $P_n \phi = \sum_{j=1}^n (\phi, w_j) w_j$ for all $\phi \in L^2(\Omega)$. In order to do this, denote by $A = -\Delta$ with homogeneous Dirichlet boundary condition, i.e. the isomorphism from $H^1_0(\Omega)$ into $H^{-1}(\Omega)$ (also seen as an unbounded operator in $L^2(\Omega)$). There, consider $k \geq 1$ such that $H^k_0(\Omega) \hookrightarrow L^p(\Omega)$ and the domain of fractional power of A , $D(A^{k/2}) = \{u \in L^2(\Omega) : \sum_{j \geq 1} \lambda_j^k (u, w_j)^2 < +\infty\}$. Then, the first operator is given by $L^q(\Omega) \ni v \mapsto \tilde{P}_n(v) \in D(A^{-k/2})$, where $\langle \tilde{P}_n(v), \phi \rangle_{D(A^{-k/2}), D(A^{k/2})} = (v, P_n \phi)$. Analogously, the second one is defined as $H^{-1}(\Omega) \ni v \mapsto \hat{P}_n(v) \in H^{-1}(\Omega)$, where $\langle \hat{P}_n(v), \phi \rangle = (v, P_n \phi)$. Observe that \tilde{P}_n and \hat{P}_n are the continuous extensions in $L^q(\Omega)$ and $H^{-1}(\Omega)$ of P_n , respectively. Then, in what follows, we will make an abuse of notation and denote both projections by P_n .

Using the boundedness of $\{f(u_n)\}$ in $L^q(\tau, T; L^q(\Omega))$, it holds that $\{P_n f(u_n)\}$ is bounded in $L^q(\tau, T; H^{-k}(\Omega))$. Since $\{P_n h\}$ is bounded in $L^2(\tau, T; H^{-1}(\Omega))$, we have that the sequence $\{u'_n\}$ is bounded in $L^q(\tau, T; H^{-k}(\Omega))$.

Therefore, from compactness arguments and the Aubin-Lions Lemma, there exist a subsequence of $\{u_n\}$ (reabeled the same) and a function $u \in L^\infty(\tau, T; L^2(\Omega)) \cap$

$L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ with $u' \in L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$, such that

$$\left\{ \begin{array}{l} u_n \overset{*}{\rightharpoonup} u \text{ weakly-star in } L^\infty(\tau, T; L^2(\Omega)), \\ u_n \rightharpoonup u \text{ weakly in } L^2(\tau, T; H_0^1(\Omega)), \\ u_n \rightharpoonup u \text{ weakly in } L^p(\tau, T; L^p(\Omega)), \\ f(u_n) \rightharpoonup f(u) \text{ weakly in } L^q(\tau, T; L^q(\Omega)), \\ a(l(u_n))u_n \rightharpoonup a(l(u))u \text{ weakly in } L^2(\tau, T; H_0^1(\Omega)), \\ u'_n \rightharpoonup u' \text{ weakly in } L^q(\tau, T; H^{-k}(\Omega)), \\ u_n \rightarrow u \text{ strongly in } L^2(\tau, T; L^2(\Omega)), \end{array} \right. \quad (2.2)$$

for all $T > \tau$. Observe that the limits of the sequences $\{f(u_n)\}$ and $\{-a(l(u_n))\Delta u_n\}$ have been obtained applying [31, Lemme 1.3, p. 12].

Then, taking into account (2.2), we can pass to the limit in (2.1) and using that $\{w_i\}$ is dense in $H_0^1(\Omega) \cap L^p(\Omega)$, it holds (1.7) for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$. Therefore, to prove that u is a weak solution to (1.3), we only need to check that $u(\tau) = u_\tau$, which makes sense since $u \in C([\tau, T]; L^2(\Omega))$ (see remark 1.2). To that end, consider fixed $\varphi \in H^1(\tau, T)$ with $\varphi(T) = 0$ and $\varphi(\tau) \neq 0$. Now, we multiply by φ in (2.1), integrate between τ and T , and pass to the limit. Comparing this limiting equation with the expression obtained multiplying (1.7) by φ and integrating between τ and T , we conclude that $u(\tau) = u_\tau$. \square

Remark 2.2. *From the uniqueness of weak solutions to (1.3), it turns out that the whole sequence $\{u_n\}$ converges to u weakly in $L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ and weakly-star in $L^\infty(\tau, T; L^2(\Omega))$. Analogously, it also fulfills that the whole sequence $\{u'_n\}$ converges to u' weakly in $L^q(\tau, T; H^{-k}(\Omega))$.*

Now, the existence and uniqueness of a strong solution to (1.3) and the regularizing effect of the equation will be analyzed.

Theorem 2.3. *Suppose that the function a is locally Lipschitz and satisfies (1.2), $f \in C^1(\mathbb{R})$ fulfills (1.4) and (1.5), and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and $l \in L^2(\Omega)$ are given. Then, for each $u_\tau \in H_0^1(\Omega) \cap L^p(\Omega)$, there exists a unique strong solution u to (1.3), with $u' \in L^2(\tau, T; L^2(\Omega))$.*

Proof. By theorem 2.1, we know that there exists a unique weak solution u to (1.3). Now, let us prove that $u \in L^2(\tau, T; D(-\Delta)) \cap L^\infty(\tau, T; H_0^1(\Omega) \cap L^p(\Omega))$ for all $T > \tau$.

Multiplying by $\lambda_j \varphi_{n_j}$ in (2.1), summing from $j = 1$ to n , adding $\pm f(0)$, and using (1.2), we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_n(t)\|^2 + m |-\Delta u_n(t)|^2 \leq (f(u_n(t)) - f(0), -\Delta u_n(t)) + (f(0) + h(t), -\Delta u_n(t))$$

a.e. $t \in (\tau, T)$.

Integrating by parts, and using (1.4) and the Cauchy inequality, we deduce

$$\frac{d}{dt} \|u_n(t)\|^2 + m |-\Delta u_n(t)|^2 \leq 2\eta \|u_n(t)\|^2 + \frac{1}{m} |f(0) + h(t)|^2 \quad \text{a.e. } t \in (\tau, T).$$

Integrating between τ and $t \in [\tau, T]$, we obtain

$$\|u_n(t)\|^2 + m \int_{\tau}^t |-\Delta u_n(s)|^2 ds \leq \|u_{\tau}\|^2 + 2\eta \int_{\tau}^T \|u_n(s)\|^2 ds + \frac{1}{m} \int_{\tau}^T |f(0) + h(t)|^2 dt.$$

Taking into account that $\{u_n\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$ (cf. theorem 2.1), we deduce that $\{u_n\}$ is bounded in $L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; D(-\Delta))$. Then, thanks to the uniqueness of weak solutions, it holds that u_n converge to u weakly-star in $L^\infty(\tau, T; H_0^1(\Omega))$ and weakly in $L^2(\tau, T; D(-\Delta))$. Thus, u is a strong solution.

Now, to prove the regularity of u' , we consider the auxiliary problem

$$(P_{\delta}) \begin{cases} \frac{du}{dt} - a^{\delta}(l(u))\Delta u = f(u) + h(t) & \text{in } \Omega \times (\tau, \infty), \\ u = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_{\tau}(x) & \text{in } \Omega. \end{cases}$$

where the function $a^{\delta} \in C^1(\mathbb{R})$ is a mollification of a , i.e. $a^{\delta} = \rho_{\delta} * a$, and so it fulfils $a^{\delta} \rightarrow a$ uniformly on compact sets. It can be checked that u^{δ} is the strong solution to (P_{δ}) fulfilling

$$\frac{du^{\delta}}{dt} - a^{\delta}(l(u^{\delta}))\Delta u^{\delta} = f(u^{\delta}) + h(t) \in L^2(\tau, T; L^2(\Omega)). \quad (2.3)$$

Multiplying (2.3) by u^{δ} , we obtain that the sequence $\{u^{\delta}\}$ converges to an element u , which is the weak solution to (1.3). In fact, we can prove that u is the strong solution to (1.3), multiplying (2.3) by $-\Delta u^{\delta}$. Now, we are ready to show that $u' \in L^2(\tau, T; L^2(\Omega))$ making use of the sequence $\{u^{\delta}\}$. Observe that multiplying (2.3) by $(u^{\delta})' \in L^2(\tau, T; L^2(\Omega))$ (this can be done rigorously via the Galerkin approximations), we deduce

$$|(u^{\delta}(t))'|^2 + \langle -a(l(u^{\delta}(t)))\Delta u^{\delta}, (u^{\delta}(t))' \rangle = (f(u^{\delta}(t)), (u^{\delta}(t))') + (h(t), (u^{\delta}(t))'). \quad (2.4)$$

Observe that

$$(f(u^{\delta}(t)), (u^{\delta}(t))') = \frac{d}{dt} \int_{\Omega} \mathcal{F}(u^{\delta}(t)) dt,$$

with

$$\mathcal{F}(s) = \int_0^s f(r) dr,$$

fulfilling

$$-\tilde{\kappa} - \tilde{\alpha}_1 |s|^p \leq \mathcal{F}(s) \leq \tilde{\kappa} - \tilde{\alpha}_2 |s|^p \quad \forall s \in \mathbb{R}, \quad (2.5)$$

with $\tilde{\kappa}$, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ positive constants.

Regarding the nonlocal term, note that

$$\begin{aligned} & \langle -a(l(u^{\delta}(t)))\Delta u^{\delta}(t), (u^{\delta}(t))' \rangle \\ & \geq \frac{1}{2} \frac{d}{dt} (a(l(u^{\delta}(t))) \|u^{\delta}(t)\|^2) - \varepsilon |(u^{\delta}(t))'|^2 - \frac{|l|^2 L_a^2}{16\varepsilon} \|u^{\delta}(t)\|^4 \end{aligned}$$

a.e. $t \in [\tau, T]$, where $\varepsilon \in (0, 1)$ and $L_a(|l|C_\infty)$ (L_a for short) denotes the uniform bound of Lipschitz constants of a^δ , i.e. $\sup_{\delta>0} \max_{s \in [-|l|C_\infty, |l|C_\infty]} |(a^\delta)'(s)|$ being $|u^\delta(t)| \leq C_\infty$ for all $t \in [\tau, T]$.

Then, taking this into account, from (2.4) we have

$$\begin{aligned} & (1 - \varepsilon)|(u^\delta(t))'|^2 + \frac{d}{dt}(a^\delta(l(u^\delta(t))))\|u^\delta(t)\|^2 \\ & \leq 2 \frac{d}{dt} \int_{\Omega} \mathcal{F}(u^\delta(x, t)) dx + \frac{|l|^2 L_a^2}{8\varepsilon} \|u^\delta(t)\|^4 + \frac{1}{1 - \varepsilon} |h(t)|^2, \end{aligned}$$

a.e. $t \in (\tau, T)$. Integrating the previous expression between τ and T , making use of (1.2) and (2.5), we obtain

$$\begin{aligned} & (1 - \varepsilon) \int_{\tau}^T |(u^\delta(t))'|^2 dt + m \|u^\delta(T)\|^2 + 2\tilde{\alpha}_2 \|u^\delta(T)\|_{L^p(\Omega)}^p \\ & \leq M \|u_\tau\|^2 + 4\tilde{\kappa}|\Omega| + 2\tilde{\alpha}_1 \|u_\tau\|_{L^p(\Omega)}^p + \frac{|l|^2 L_a^2}{8\varepsilon} \int_{\tau}^T \|u^\delta(t)\|^4 dt + \frac{1}{1 - \varepsilon} \int_{\tau}^T |h(t)|^2 dt. \end{aligned}$$

Finally, as argued above, the sequence $\{u^\delta\}$ is bounded in $L^\infty(\tau, T; H_0^1(\Omega))$. So we deduce that the sequence $\{(u^\delta)'\}$ converges to u' weakly in $L^2(\tau, T; L^2(\Omega))$ and $\{u^\delta\}$ converges to u weakly-star in $L^\infty(\tau, T; L^p(\Omega))$. \square

The following result shows the regularizing effect of the equation. We omit the proof for the sake of brevity, since it is close to that in [8, Theorem 5] and theorem 2.3.

Theorem 2.4. *Assume that the function a is locally Lipschitz and fulfills (1.2), $f \in C^1(\mathbb{R})$ satisfies (1.4) and (1.5), $h \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$, and $l \in L^2(\Omega)$. Then, for any initial datum $u_\tau \in L^2(\Omega)$, the weak solution u ensured by theorem 2.1 belongs to $L^2(\tau + \varepsilon, T; D(-\Delta)) \cap L^\infty(\tau + \varepsilon, T; H_0^1(\Omega) \cap L^p(\Omega))$ and u' belongs to $L^2(\tau + \varepsilon, T; L^2(\Omega))$ for every $\varepsilon > 0$ and $T > \tau + \varepsilon$.*

3. Abstract results on the theory of Pullback Attractors

In this section, we will recall briefly some results from the theory of pullback attractors that we will use to prove some of the main results of this paper (e.g. cf. [9, 10, 34, 26]).

Consider given a metric space (X, d_X) and denote $\mathbb{R}_d^2 = \{(t, \tau) \in \mathbb{R}^2 : \tau \leq t\}$.

Definition 3.1. *A process on X (also called a two-parameter semigroup) is a mapping U such that $\mathbb{R}_d^2 \times X \ni (t, \tau, x) \mapsto U(t, \tau)x \in X$ with $U(\tau, \tau)x = x$ for any $(\tau, x) \in \mathbb{R} \times X$, and $U(t, r)(U(r, \tau)x) = U(t, \tau)x$ for any $\tau \leq r \leq t$ and all $x \in X$.*

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and consider \mathcal{D} a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. \mathcal{D} will be called a universe in $\mathcal{P}(X)$.

Definition 3.2. A process U on X is said to be

- (a) continuous if for any pair $(t, \tau) \in \mathbb{R}_d^2$, the mapping $U(t, \tau) : X \rightarrow X$ is continuous.
- (b) strong-weak (also known as norm-to-weak) continuous if for any pair $(t, \tau) \in \mathbb{R}_d^2$, the map $U(t, \tau)$ is continuous from X with the strong topology into X with the weak topology.
- (c) closed if for any pair $(t, \tau) \in \mathbb{R}_d^2$, and any sequence $\{x_n\} \subset X$ such that $x_n \rightarrow x \in X$ and $U(t, \tau)x_n \rightarrow y \in X$, then $U(t, \tau)x = y$.
- (d) pullback \mathcal{D} -asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \rightarrow -\infty$ and $x_n \in D(\tau_n)$ for all n , the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X .

Observe that every continuous process is strong-weak continuous, and every strong-weak continuous process is closed.

Now, we consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. We do not require any additional condition on these sets as compactness nor boundedness.

Definition 3.3. The family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback \mathcal{D} -absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists $\tau_0(\widehat{D}, t) < t$ such that

$$U(t, \tau)D(\tau) \subset D_0(t) \quad \forall \tau \leq \tau_0(\widehat{D}, t).$$

Observe that in the previous definition \widehat{D}_0 does not necessarily belong to the universe \mathcal{D} .

Definition 3.4. A family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is called the minimal pullback \mathcal{D} -attractor for the process U if the following properties are satisfied:

- (a) the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X for any $t \in \mathbb{R}$,
- (b) $\mathcal{A}_{\mathcal{D}}$ is pullback \mathcal{D} -attracting, i.e. $\lim_{\tau \rightarrow -\infty} \text{dist}_X(U(t, \tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0$ for all $\widehat{D} \in \mathcal{D}$, $t \in \mathbb{R}$, where $\text{dist}_X(\cdot, \cdot)$ denotes the Hausdorff semi-distance in X between two subsets of X ,
- (c) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e. $U(t, \tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $\tau \leq t$,
- (d) $\mathcal{A}_{\mathcal{D}}$ is minimal, i.e. if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets which is pullback \mathcal{D} -attracting, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

The uniqueness of the minimal pullback \mathcal{D} -attractor comes from its own definition (cf. (d)). See also remark 3.6 (i).

Now, we establish the main result of this section. The following theorem guarantees the existence of the minimal pullback attractor (see [26, Theorem 3.11]).

Theorem 3.5. Assume that $U : \mathbb{R}_d^2 \times X \rightarrow X$ is a closed process, \mathcal{D} is a universe in $\mathcal{P}(X)$, and $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a pullback \mathcal{D} -absorbing family for

U , and moreover suppose that U is pullback \widehat{D}_0 -asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined for $t \in \mathbb{R}$ by

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}^X,$$

where $\Lambda(\widehat{D}, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)D(\tau)}^X$ for all $t \in \mathbb{R}$ and $\{\dots\}^X$ denotes the closure in X , is the minimal pullback \mathcal{D} -attractor for the process U . In addition, if $\widehat{D}_0 \in \mathcal{D}$, then it holds that $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.

Remark 3.6. (i) If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in \mathcal{D} that satisfies (b) and (c). A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the universe \mathcal{D} is inclusion-closed, which means that if $\widehat{D} \in \mathcal{D}$ and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ satisfies that $D'(t) \subset D(t)$ for all $t \in \mathbb{R}$, then $\widehat{D}' \in \mathcal{D}$.

(ii) The universe of fixed nonempty bounded subsets of X is denoted by \mathcal{D}_F^X . Then, the corresponding minimal pullback \mathcal{D}_F^X -attractor for the process U is the attractor defined by Crauel, Debussche and Flandoli (cf. [22, Theorem 1.1]).

The following two results allow us to establish relationships between pullback attractors (see [34]).

Corollary 3.7. Under the assumptions of theorem 3.5, if $\mathcal{D}_F^X \subset \mathcal{D}$, then the minimal pullback attractors $\mathcal{A}_{\mathcal{D}_F^X}$ and $\mathcal{A}_{\mathcal{D}}$ exist and $\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$.

Besides, if for some $T \in \mathbb{R}$ the set $\bigcup_{t \leq T} D_0(t)$ is a bounded subset of X , then $\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \leq T$.

Thanks to the following result, we can compare two attractors for a process (see [26, Theorem 3.15]).

Theorem 3.8. Let $\{X_i, d_{X_i}\}_{i=1,2}$ be two metric spaces such that $X_1 \subset X_2$ with continuous injection, and \mathcal{D}_i is a universe in $\mathcal{P}(X_i)$ for $i = 1, 2$, with $\mathcal{D}_1 \subset \mathcal{D}_2$. Assume that a map U acts as a process in both cases, i.e. $U : \mathbb{R}_d^2 \times X_i \rightarrow X_i$ for $i = 1, 2$ is a process. For each $t \in \mathbb{R}$, denote

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i \in \mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i}, \quad i = 1, 2,$$

where the subscript i in the symbol of the omega-limit set Λ_i is used to denote the dependence on the respective topology. Then, $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

If moreover $\mathcal{A}_1(t)$ is a compact subset of X_1 for all $t \in \mathbb{R}$, and for any $\widehat{D}_2 \in \mathcal{D}_2$ and any $t \in \mathbb{R}$, there exist a family $\widehat{D}_1 \in \mathcal{D}_1$ and a $t_{\widehat{D}_1}^*$ such that U is pullback \widehat{D}_1 -asymptotically compact, and for any $s \leq t_{\widehat{D}_1}^*$ there exists a $\tau_s < s$ such that

$$U(s, \tau)D_2(\tau) \subset D_1(s) \quad \forall \tau \leq \tau_s,$$

then $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

4. Pullback attractors in $L^2(\Omega)$

Now, we are ready to study the existence of pullback attractors in the phase space $L^2(\Omega)$ using the abstract results given in the previous section.

The first goal is straightforward defining $U : \mathbb{R}_d^2 \times L^2(\Omega) \rightarrow L^2(\Omega)$ as

$$U(t, \tau)u_\tau = u(t; \tau, u_\tau) \quad \forall u_\tau \in L^2(\Omega) \quad \forall \tau \leq t,$$

where $u(\cdot; \tau, u_\tau)$ denotes the weak solution to (1.3).

It is not difficult to check that U is a process on $L^2(\Omega)$. Moreover, as a consequence of theorem 2.1, we have the following result.

Proposition 4.1. *Assume that the function a is locally Lipschitz and satisfies (1.2), $f \in C^1(\mathbb{R})$ fulfills (1.4) and (1.5), $h \in L_{loc}^2(\mathbb{R}; H^{-1}(\Omega))$, and $l \in L^2(\Omega)$. Then, for any pair $(t, \tau) \in \mathbb{R}_d^2$, the map $U(t, \tau)$ is continuous from $L^2(\Omega)$ into itself.*

Lemma 4.2. *Suppose that the assumptions in proposition 4.1 hold and consider $u_\tau \in L^2(\Omega)$. Then, for any $\mu \in (0, 2\lambda_1 m)$, the solution u to (1.3) satisfies*

$$|u(t)|^2 \leq e^{-\mu(t-\tau)}|u_\tau|^2 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu t}}{2m - \mu\lambda_1^{-1}} \int_\tau^t e^{\mu s} \|h(s)\|_*^2 ds \quad \forall t \geq \tau.$$

Proof. Applying the Cauchy-Schwartz inequality, (1.2) and (1.5) to the energy equality, and adding $\pm\mu|u(t)|^2$, we obtain

$$\frac{d}{dt}|u(t)|^2 + \mu|u(t)|^2 + 2m\|u(t)\|^2 \leq 2\kappa|\Omega| + \mu|u(t)|^2 + 2\|h(t)\|_*\|u(t)\|.$$

Using the Poincaré and Cauchy inequalities, and multiplying by $e^{\mu t}$ in the above expression, it holds

$$\frac{d}{dt}(e^{\mu t}|u(t)|^2) \leq 2\kappa|\Omega|e^{\mu t} + \frac{1}{2m - \mu\lambda_1^{-1}}e^{\mu t}\|h(t)\|_*^2.$$

The result then follows integrating on $[\tau, t]$. □

Thanks to the previous estimate, now we can define a suitable tempered universe in $\mathcal{P}(L^2(\Omega))$.

Definition 4.3. *For each $\mu > 0$, we denote by $\mathcal{D}_\mu^{L^2}$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ such that*

$$\lim_{\tau \rightarrow -\infty} \left(e^{\mu\tau} \sup_{v \in D(\tau)} |v|^2 \right) = 0.$$

Observe that $\mathcal{D}_F^{L^2} \subset \mathcal{D}_\mu^{L^2}$ and $\mathcal{D}_\mu^{L^2}$ is inclusion-closed.

Now, if we assume that h satisfies a suitable growth condition, using the above estimates, we can prove the existence of a $\mathcal{D}_\mu^{L^2}$ -absorbing family for the process U .

Proposition 4.4. *Under the assumptions of proposition 4.1, if h also satisfies that there exists some $\mu \in (0, 2\lambda_1 m)$ such that*

$$\int_{-\infty}^0 e^{\mu s} \|h(s)\|_*^2 ds < \infty, \quad (4.1)$$

the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_{L^2}(0, R_{L^2}^{1/2}(t))$, the closed ball in $L^2(\Omega)$ of center zero and radius $R_{L^2}^{1/2}(t)$, where

$$R_{L^2}(t) = 1 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu t}}{2m - \mu\lambda_1^{-1}} \int_{-\infty}^t e^{\mu s} \|h(s)\|_*^2 ds,$$

is pullback $\mathcal{D}_\mu^{L^2}$ -absorbing for the process $U : \mathbb{R}_d^2 \times L^2(\Omega) \rightarrow L^2(\Omega)$. Besides, $\widehat{D}_0 \in \mathcal{D}_\mu^{L^2}$.

Finally, to prove the existence of the minimal pullback attractor for the process $U : \mathbb{R}_d^2 \times L^2(\Omega) \rightarrow L^2(\Omega)$, we only need to check the pullback asymptotic compactness in $L^2(\Omega)$ for the universe $\mathcal{D}_\mu^{L^2}$. To that end, we firstly establish the following result.

Lemma 4.5. *Under the assumptions of proposition 4.4, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, there exists $\tau_1(\widehat{D}, t) < t - 2$ such that, for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$,*

$$\left\{ \begin{array}{l} |u(r; \tau, u_\tau)|^2 \leq \rho_1(t) \quad \forall r \in [t - 2, t], \\ \int_{r-1}^r \|u(s; \tau, u_\tau)\|^2 ds \leq \rho_2(t) \quad \forall r \in [t - 1, t], \\ \int_{r-1}^r \|u(s; \tau, u_\tau)\|_{L^p(\Omega)}^p ds \leq \frac{m}{2\alpha_2} \rho_2(t) \quad \forall r \in [t - 1, t], \end{array} \right. \quad (4.2)$$

where

$$\rho_1(t) = 1 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu(t-2)}}{2m - \mu\lambda_1^{-1}} \int_{-\infty}^t e^{\mu s} \|h(s)\|_*^2 ds,$$

$$\rho_2(t) = \frac{1}{m} \left(\rho_1(t) + 2\kappa|\Omega| + \frac{1}{m} \max_{r \in [t-1, t]} \int_{r-1}^r \|h(s)\|_*^2 ds \right).$$

Proof. The first inequality in (4.2) with the expression ρ_1 in the right hand side is similar to the proof of lemma 4.2, if $\tau \leq \tau_1(\widehat{D}, t) < t - 2$ (far enough pullback in time) because of our choice of tempered universe and taking into account (4.1). We therefore omit the details. Observe that in fact this estimate also holds for the Galerkin approximations, which have already been used in §2.

To obtain the other two inequalities in (4.2), we will proceed with the Galerkin approximations and then passing to the limit by compactness arguments. Multiplying by φ_{nj} in (2.1), summing from $j = 1$ to n , and using (1.2) and the Cauchy inequality, we obtain

$$\frac{d}{ds} |u_n(s)|^2 + m \|u_n(s)\|^2 + 2\alpha_2 \|u_n(s)\|_{L^p(\Omega)}^p \leq 2\kappa|\Omega| + \frac{1}{m} \|h(s)\|_*^2 \quad \text{a.e. } s > \tau.$$

Now, from the above inequality, it holds

$$\begin{aligned} & |u_n(r)|^2 + m \int_{r-1}^r \|u_n(s)\|^2 ds + 2\alpha_2 \int_{r-1}^r \|u_n(s)\|_{L^p(\Omega)}^p ds \\ & \leq |u_n(r-1)|^2 + 2\kappa|\Omega| + \frac{1}{m} \int_{r-1}^r \|h(s)\|_*^2 ds \end{aligned} \quad (4.3)$$

for all $\tau \leq r-1$.

In particular, from (4.3) we obtain for any $n \geq 1$

$$\int_{r-1}^r \|u_n(s)\|^2 ds \leq \rho_2(t) \quad \forall r \in [t-1, t], \quad \tau \leq \tau_1(\widehat{D}, t), \quad u_\tau \in D(\tau), \quad (4.4)$$

where $\rho_2(t)$ is given in the statement.

Also, from (4.3) we can deduce that for any $n \geq 1$,

$$\int_{r-1}^r \|u_n(s)\|_{L^p(\Omega)}^p ds \leq \frac{m}{2\alpha_2} \rho_2(t) \quad \forall r \in [t-1, t], \quad \tau \leq \tau_1(\widehat{D}, t), \quad u_\tau \in D(\tau). \quad (4.5)$$

Taking now into account the fact (cf. theorem 2.1 and remark 2.2) that u_n converge to $u(\cdot; \tau, u_\tau)$ weakly in $L^2(r-1, r; H_0^1(\Omega)) \cap L^p(r-1, r; L^p(\Omega))$ for all $r \in [t-1, t]$, and the estimates (4.4) and (4.5), then (4.2) follows. \square

Now we will prove that the process U is pullback $\mathcal{D}_\mu^{L^2}$ -asymptotically compact using an energy method with continuous functions (e.g. cf. [28, 33, 35, 26]).

Proposition 4.6. *Under the assumptions of proposition 4.4, the process $U : \mathbb{R}_d^2 \times L^2(\Omega) \rightarrow L^2(\Omega)$ is pullback $\mathcal{D}_\mu^{L^2}$ -asymptotically compact.*

Proof. Let be given $t \in \mathbb{R}$, a family $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, a sequence $\{\tau_n\} \subset (-\infty, t-2]$ with $\tau_n \rightarrow -\infty$, and $u_{\tau_n} \in D(\tau_n)$ for all n . Let us prove that the sequence $\{u(t; \tau_n, u_{\tau_n})\}$ is relatively compact in $L^2(\Omega)$. For short, we denote $u^n(\cdot) = u(\cdot; \tau_n, u_{\tau_n})$.

From lemma 4.5 we know that there exists $\tau_1(\widehat{D}, t) < t-2$ satisfying that, if $n_1 \geq 1$ is such that $\tau_n \leq \tau_1(\widehat{D}, t)$ for all $n \geq n_1$, $\{u^n\}_{n \geq n_1}$ is bounded in $L^\infty(t-2, t; L^2(\Omega)) \cap L^2(t-2, t; H_0^1(\Omega)) \cap L^p(t-2, t; L^p(\Omega))$. Besides, taking into account (1.2), it holds that $\{-a(l(u^n))\Delta u^n\}_{n \geq n_1}$ is bounded in $L^2(t-2, t; H^{-1}(\Omega))$. From (1.6) we deduce that $\{f(u^n)\}_{n \geq n_1}$ is bounded in $L^q(t-2, t; L^q(\Omega))$. As a consequence of the above uniform estimates and the equality satisfied by u^n , it yields that $\{(u^n)'\}_{n \geq n_1}$ is bounded in $L^2(t-2, t; H^{-1}(\Omega)) + L^q(t-2, t; L^q(\Omega))$. Then, using the Aubin-Lions compactness Lemma, analogously as in the proof of theorem 2.1, it holds that there exists $u \in L^\infty(t-2, t; L^2(\Omega)) \cap L^2(t-2, t; H_0^1(\Omega)) \cap L^p(t-2, t; L^p(\Omega))$, with $u' \in L^2(t-2, t; H^{-1}(\Omega)) + L^q(t-2, t; L^q(\Omega))$, such that

for a subsequence (reabeled the same) it satisfies

$$\left\{ \begin{array}{l} u^n \overset{*}{\rightharpoonup} u \text{ weakly-star in } L^\infty(t-2, t; L^2(\Omega)), \\ u^n \rightharpoonup u \text{ weakly in } L^2(t-2, t; H_0^1(\Omega)), \\ u^n \rightharpoonup u \text{ weakly in } L^p(t-2, t; L^p(\Omega)), \\ (u^n)' \rightharpoonup u' \text{ weakly in } L^2(t-2, t; H^{-1}(\Omega)) + L^q(t-2, t; L^q(\Omega)), \\ u^n \rightarrow u \text{ strongly in } L^2(t-2, t; L^2(\Omega)), \\ u^n(s) \rightarrow u(s) \text{ strongly in } L^2(\Omega) \text{ a.e. } s \in (t-2, t), \\ a(l(u^n))u^n \rightharpoonup a(l(u))u \text{ weakly in } L^2(t-2, t; H_0^1(\Omega)), \\ f(u^n) \rightharpoonup f(u) \text{ weakly in } L^q(t-2, t; L^q(\Omega)). \end{array} \right. \quad (4.6)$$

Observe that $u \in C([t-2, t]; L^2(\Omega))$, and due to (4.6), u fulfills (1.7) in the interval $(t-2, t)$.

From (4.6) we can also deduce that $\{u^n\}$ is equicontinuous in $H^{-1}(\Omega) + L^q(\Omega)$ on $[t-2, t]$. Moreover, it holds that $\{u^n\}_{n \geq n_1}$ is bounded in $C([t-2, t]; L^2(\Omega))$ and the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega) + L^q(\Omega)$ is compact. Therefore, applying the Ascoli-Arzelà Theorem, we have (for another sequence, relabeled again the same) that

$$u^n \rightarrow u \text{ strongly in } C([t-2, t]; H^{-1}(\Omega) + L^q(\Omega)). \quad (4.7)$$

Thanks to the boundedness of $\{u^n\}_{n \geq n_1}$ in $C([t-2, t]; L^2(\Omega))$, for any sequence $\{s_n\} \subset [t-2, t]$ with $s_n \rightarrow s_*$ it holds

$$u^n(s_n) \rightharpoonup u(s_*) \text{ weakly in } L^2(\Omega), \quad (4.8)$$

where we have used (4.7) to identify the weak limit.

Now we will prove that

$$u^n \rightarrow u \text{ strongly in } C([t-1, t]; L^2(\Omega)), \quad (4.9)$$

which implies that the process U is pullback $\mathcal{D}_\mu^{L^2}$ -asymptotically compact.

We establish (4.9) by contradiction. Let us suppose that there exist $\varepsilon > 0$, a sequence $\{t_n\} \subset [t-1, t]$, without loss of generality converging to some t_* , with

$$|u^n(t_n) - u(t_*)| \geq \varepsilon \quad \forall n \geq 1. \quad (4.10)$$

On the other hand, applying the Cauchy inequality, (1.2) and (1.5) to the energy equality (1.8), we deduce

$$|z(s)|^2 \leq |z(r)|^2 + 2\kappa|\Omega|(s-r) + \frac{1}{2m} \int_r^s \|h(\xi)\|_*^2 d\xi \quad \forall t-2 \leq r \leq s \leq t,$$

where z may be replaced by u or any u^n .

Now we define the following functions

$$\begin{aligned} J_n(s) &= |u^n(s)|^2 - 2\kappa|\Omega|s - \frac{1}{2m} \int_{t-2}^s \|h(r)\|_*^2 dr, \\ J(s) &= |u(s)|^2 - 2\kappa|\Omega|s - \frac{1}{2m} \int_{t-2}^s \|h(r)\|_*^2 dr. \end{aligned}$$

It is clear from the regularity of u and all u^n that these functions are continuous on $[t-2, t]$. In addition, using the above inequality, it is not difficult to check that J and all J_n are non-increasing on $[t-2, t]$. From this and (4.6), it turns out that

$$J_n(s) \rightarrow J(s) \quad \text{a.e. } s \in (t-2, t).$$

In particular, we can consider a sequence $\{\tilde{t}_k\} \subset (t-2, t_*)$ such that $\tilde{t}_k \rightarrow t_*$ when $k \rightarrow \infty$ and such that the above convergence holds for any \tilde{t}_k . Now, fix an arbitrary value $\epsilon > 0$. Since the function J is continuous on $[t-2, t]$, there exists $k(\epsilon) \geq 1$ such that

$$|J(\tilde{t}_k) - J(t_*)| < \frac{\epsilon}{2} \quad \forall k \geq k(\epsilon).$$

Thereupon, we consider $n(\epsilon) \geq 1$ such that

$$t_n \geq \tilde{t}_{k(\epsilon)} \quad \text{and} \quad |J_n(\tilde{t}_{k(\epsilon)}) - J(\tilde{t}_{k(\epsilon)})| < \frac{\epsilon}{2} \quad \forall n \geq n(\epsilon).$$

Then, since the functions J_n are non-increasing, it holds for all $n \geq n(\epsilon)$

$$\begin{aligned} J_n(t_n) - J(t_*) &\leq J_n(\tilde{t}_{k(\epsilon)}) - J(t_*) \\ &\leq |J_n(\tilde{t}_{k(\epsilon)}) - J(\tilde{t}_{k(\epsilon)})| + |J(\tilde{t}_{k(\epsilon)}) - J(t_*)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

As $\epsilon > 0$ is arbitrary, it yields $\limsup_{n \rightarrow \infty} J_n(t_n) \leq J(t_*)$. Then we deduce that $\limsup_{n \rightarrow \infty} |u^n(t_n)| \leq |u(t_*)|$, which, together with (4.8), allows us to prove that that $u^n(t_n)$ converge to $u(t_*)$ strongly in $L^2(\Omega)$, in contradiction with (4.10). Therefore, (4.9) holds. \square

As a consequence, we have the following result.

Theorem 4.7. *Assume that the function a is locally Lipschitz and (1.2) holds, $f \in C^1(\mathbb{R})$ satisfies (1.4) and (1.5), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ fulfills condition (4.1) for some $\mu \in (0, 2\lambda_1 m)$, and $l \in L^2(\Omega)$. Then, there exist the minimal pullback $\mathcal{D}_F^{L^2}$ -attractor $\mathcal{A}_{\mathcal{D}_F^{L^2}} = \{\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) : t \in \mathbb{R}\}$, and the minimal pullback $\mathcal{D}_\mu^{L^2}$ -attractor $\mathcal{A}_{\mathcal{D}_\mu^{L^2}} = \{\mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) : t \in \mathbb{R}\}$ for the process $U : \mathbb{R}_d^2 \times L^2(\Omega) \rightarrow L^2(\Omega)$ associated to (1.3). In addition, the family $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ belongs to $\mathcal{D}_\mu^{L^2}$ and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) \subset \overline{B}_{L^2}(0, R_{L^2}^{1/2}(t)) \quad \forall t \in \mathbb{R}.$$

Moreover, if h fulfills

$$\sup_{s \leq 0} \left(e^{-\mu s} \int_{-\infty}^s e^{\mu \xi} \|h(\xi)\|_*^2 d\xi \right) < \infty, \quad (4.11)$$

then $\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)$ for all $t \in \mathbb{R}$.

Proof. The existence of $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$, $\mathcal{A}_{\mathcal{D}_F^{L^2}}$, and the relationship between both attractors are due to corollary 3.7. Namely, the continuity of the process (cf. proposition 4.1),

the relationship between the universes, the existence of an absorbing family in $\mathcal{D}_\mu^{L^2}$ (cf. proposition 4.4) and the asymptotic compactness of this universe in the L^2 -norm (cf. proposition 4.6) hold.

From theorem 3.5, we also obtain that $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) \subset \overline{B}_{L^2}(0, R_{L^2}^{1/2}(t))$ for all $t \in \mathbb{R}$. Moreover, the family $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ belongs to $\mathcal{D}_\mu^{L^2}$ (cf. remark 3.6).

Finally, under assumption (4.11), we deduce that $\bigcup_{t \leq T} R_{L^2}(t)$ is bounded for each $T \in \mathbb{R}$, where R_{L^2} is given in proposition 4.4. Thus, corollary 3.7 implies that both families of attractors coincide. \square

Analogously as in [8, Remark 23 (ii)], we can extend the above result to new larger universes and obtaining new attractors.

Corollary 4.8. *Under the assumptions of theorem 4.7, for any $\sigma \in (\mu, 2\lambda_1 m)$, there exists the corresponding pullback $\mathcal{D}_\sigma^{L^2}$ -attractor $\mathcal{A}_{\mathcal{D}_\sigma^{L^2}}$, which satisfies $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) \subset \mathcal{A}_{\mathcal{D}_\sigma^{L^2}}(t)$ for all $t \in \mathbb{R}$. In addition, if h fulfills (4.11), then $\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\sigma^{L^2}}(t)$ for all $t \in \mathbb{R}$ and any $\sigma \in (\mu, 2\lambda_1 m)$.*

5. Pullback attractors in $H_0^1(\Omega)$

In this section, we will prove the existence of pullback attractors for the dynamical systems associated to (1.3) in the phase space $H_0^1(\Omega)$.

Observe that when $h \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$, thanks to theorem 2.3, the restriction of the process U to $\mathbb{R}_d^2 \times H_0^1(\Omega)$ is a process in $H_0^1(\Omega)$. Since no confusion arises, we will not modify the notation and continue denoting this process by U .

Now, we will prove that this process defined on $H_0^1(\Omega)$ still fulfills properties to apply the results given in §3. Firstly, we check that the process U is strong-weak continuous in $H_0^1(\Omega)$.

Proposition 5.1. *Suppose that the function a is locally Lipschitz and (1.2) holds, $f \in C^1(\mathbb{R})$ satisfies (1.4) and (1.5), and $h \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$ and $l \in L^2(\Omega)$ are given. Then, the process U is strong-weak continuous in $H_0^1(\Omega)$.*

Proof. Consider fixed $(t, \tau) \in \mathbb{R}_d^2$ and let $\{u_{\tau_k}\}$ be a sequence of initial data such that $u_{\tau_k} \rightarrow u_\tau$ strongly in $H_0^1(\Omega)$. We will prove that $U(t, \tau)u_{\tau_k} \rightharpoonup U(t, \tau)u_\tau$ weakly in $H_0^1(\Omega)$. To do this, we use the Galerkin approximations and pass to the limit by compactness arguments. Multiplying (2.1) by $\lambda_j \varphi_{nj}$, summing from $j = 1$ to n , adding $\pm(f(0), -\Delta u_n(t))$, and making use of (1.2), (1.4), and the Cauchy inequality, we deduce

$$\frac{d}{dt} \|u_n(t)\|^2 \leq 2\eta \|u_n(t)\|^2 + \frac{1}{2m} |f(0) + h(t)|^2 \quad \text{a.e. } t \geq \tau.$$

Integrating between τ and t , and applying the Gronwall lemma, we have

$$\|u_n(t)\|^2 \leq \left(\|u_{\tau_k}\|^2 + \frac{1}{2m} \int_\tau^t |f(0) + h(s)|^2 ds \right) e^{2\eta(t-\tau)}.$$

Now, since $\{u_n\}$ is bounded in $L^\infty(\tau, t; H_0^1(\Omega))$, $u_n(\cdot; \tau, u_{\tau_k}) \rightharpoonup u(\cdot; \tau, u_{\tau_k})$ weakly in $L^2(\tau, t; H_0^1(\Omega))$, and $u(\cdot; \tau, u_{\tau_k}) \in C([\tau, t]; L^2(\Omega))$ (cf. theorem 2.1), taking into

account [37, Lemma 11.2, p. 288], we deduce

$$\|u(t; \tau, u_{\tau_k})\|^2 \leq \left(\|u_{\tau_k}\|^2 + \frac{1}{2m} \int_{\tau}^t |f(0) + h(s)|^2 ds \right) e^{2\eta(t-\tau)}.$$

From this, taking into account that the map $U(t, \tau)$ is continuous from $L^2(\Omega)$ into itself (cf. proposition 4.1), it fulfills

$$U(t, \tau)u_{\tau_k} \rightharpoonup U(t, \tau)u_{\tau} \quad \text{weakly in } H_0^1(\Omega).$$

□

In order to prove that the process $U : \mathbb{R}_2^d \times H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is pullback asymptotically compact, we previously establish some uniform estimates of the solutions in a finite-time interval up to t when the initial datum is shifted pullback far enough.

To clarify the statement of the following result, we introduce

$$\rho_1^{ext}(t) = 1 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu(t-3)}}{2m - \mu\lambda_1^{-1}} \int_{-\infty}^t e^{\mu\xi} \|h(\xi)\|_*^2 d\xi, \quad (5.1)$$

$$\rho_2^{ext}(t) = \frac{1}{m} \left(\rho_1^{ext}(t) + 2\kappa|\Omega| + \frac{1}{m} \max_{r \in [t-2, t]} \int_{r-1}^r \|h(\xi)\|_*^2 d\xi \right).$$

Lemma 5.2. *Under the assumptions of proposition 5.1, if h satisfies (4.1) for some $\mu \in (0, 2\lambda_1 m)$, then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, there exists $\tau_2(\widehat{D}, t) < t - 3$ such that for any $\tau \leq \tau_2(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, the following estimates hold*

$$\left\{ \begin{array}{l} \|u(r; \tau, u_\tau)\|^2 \leq \tilde{\rho}_1(t) \quad \forall r \in [t-2, t], \\ \int_{r-1}^r |-\Delta u(\xi; \tau, u_\tau)|^2 d\xi \leq \tilde{\rho}_2(t) \quad \forall r \in [t-1, t], \\ \int_{r-1}^r |u'(\xi; \tau, u_\tau)|^2 d\xi \leq \tilde{\rho}_3(t) \quad \forall r \in [t-1, t], \end{array} \right. \quad (5.2)$$

with

$$\tilde{\rho}_1(t) = (1 + 2\eta)\rho_2^{ext}(t) + \frac{1}{m} \max_{r \in [t-2, t]} \int_{r-1}^r |f(0) + h(\xi)|^2 d\xi,$$

$$\tilde{\rho}_2(t) = \frac{1}{m} \left(\tilde{\rho}_1(t) + 2\eta\rho_2^{ext}(t) + \frac{1}{m} \max_{r \in [t-1, t]} \int_{r-1}^r |f(0) + h(\xi)|^2 d\xi \right),$$

$$\begin{aligned} \tilde{\rho}_3(t) &= \frac{1}{1-\varepsilon} \left[4\tilde{\kappa}|\Omega| + M\tilde{\rho}_1(t) + \frac{4\tilde{\alpha}_1\tilde{\kappa}|\Omega|}{\tilde{\alpha}_2} + \left(\frac{\tilde{\alpha}_1^2 m}{\tilde{\alpha}_2 \alpha_2} + \frac{M\tilde{\alpha}_1}{\tilde{\alpha}_2} \right) \rho_2^{ext}(t) \right] \\ &\quad + \frac{1}{1-\varepsilon} \left(\frac{\tilde{\alpha}_1 |l|^2 L_a^2}{8\tilde{\alpha}_2 \varepsilon} + \frac{|l|^2 L_a^2}{8\varepsilon} \right) (\tilde{\rho}_1(t))^2 \\ &\quad + \frac{\tilde{\alpha}_1}{\tilde{\alpha}_2 (1-\varepsilon)^2} \max_{r \in [t-2, t]} \int_{r-1}^r |h(\xi)|^2 d\xi + \frac{1}{(1-\varepsilon)^2} \max_{r \in [t-1, t]} \int_{r-1}^r |h(\xi)|^2 d\xi, \end{aligned}$$

where L_a (for short) is denoting $L_a(|l|\rho_1^{ext}(t))$, the Lipschitz constant of a in the interval $[-|l|\rho_1^{ext}(t), |l|\rho_1^{ext}(t)]$.

Proof. Analogously as in the proof of lemma 4.5, we can obtain uniform estimates for the solutions in a longer time-interval. Actually, there exists $\tau_2(\widehat{D}, t) < t - 3$ such that for any $\tau \leq \tau_2(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, it holds

$$\begin{aligned} |u(r; \tau, u_\tau)| &\leq \rho_1^{ext}(t) \quad \forall r \in [t - 3, t], \\ \int_{r-1}^r \|u(\xi; \tau, u_\tau)\|^2 d\xi &\leq \rho_2^{ext}(t) \quad \forall r \in [t - 2, t], \\ \int_{r-1}^r \|u(\xi; \tau, u_\tau)\|_{L^p(\Omega)}^p d\xi &\leq \frac{m}{2\alpha_2} \rho_2^{ext}(t) \quad \forall r \in [t - 2, t], \end{aligned}$$

where $\{\rho_i^{ext}\}_{i=1,2}$ are given in (5.1). Observe that these estimates also hold for the Galerkin approximations $u_n(\cdot; \tau, u_\tau)$, which have already been used in §2 and §4.

Multiplying by $\lambda_j \varphi_{nj}$ in (2.1), summing from $j = 1$ to n and making use of (1.2), (1.4), and the Cauchy inequality, we deduce

$$\frac{d}{d\xi} \|u_n(\xi)\|^2 + m |-\Delta u_n(\xi)|^2 \leq 2\eta \|u_n(\xi)\|^2 + \frac{1}{m} |f(0) + h(\xi)|^2 \quad \text{a.e. } \xi > \tau. \quad (5.3)$$

Integrating between r and s with $\tau \leq r - 1 \leq s \leq r$, it holds in particular

$$\|u_n(r)\|^2 \leq \|u_n(s)\|^2 + 2\eta \int_{r-1}^r \|u_n(\xi)\|^2 d\xi + \frac{1}{m} \int_{r-1}^r |f(0) + h(\xi)|^2 d\xi.$$

Integrating the last inequality in s between $r - 1$ and r , we have

$$\|u_n(r)\|^2 \leq (1 + 2\eta) \int_{r-1}^r \|u_n(s)\|^2 ds + \frac{1}{m} \int_{r-1}^r |f(0) + h(\xi)|^2 d\xi,$$

for all $\tau \leq r - 1$. Thus, from the estimate on the solutions by ρ_2^{ext} given above, we deduce

$$\|u_n(r; \tau, u_\tau)\|^2 \leq \tilde{\rho}_1(t) \quad \forall r \in [t - 2, t], \quad \tau \leq \tau_2(\widehat{D}, t), \quad u_\tau \in D(\tau), \quad (5.4)$$

where $\tilde{\rho}_1(t)$ is given in the statement.

Now, integrating between $r - 1$ and r in (5.3), we obtain in particular

$$\begin{aligned} &\int_{r-1}^r |-\Delta u_n(\xi)|^2 d\xi \\ &\leq \frac{1}{m} \left(\|u_n(r - 1)\|^2 + 2\eta \int_{r-1}^r \|u_n(\xi)\|^2 d\xi + \frac{1}{m} \int_{r-1}^r |f(0) + h(\xi)|^2 d\xi \right), \end{aligned}$$

for all $\tau \leq r - 1$. Then, for any $n \geq 1$,

$$\int_{r-1}^r |-\Delta u_n(\xi)|^2 d\xi \leq \tilde{\rho}_2(t) \quad \forall r \in [t - 1, t], \quad \tau \leq \tau_2(\widehat{D}, t), \quad u_\tau \in D(\tau), \quad (5.5)$$

where $\tilde{\rho}_2(t)$ is given in the statement. Now, observe that $u \in C([t-2, t]; H_0^1(\Omega))$, u_n converge to $u(\cdot; \tau, u_\tau)$ weakly-star in $L^\infty(t-2, t; H_0^1(\Omega))$ and weakly in $L^2(r-1, r; D(-\Delta))$ for all $r \in [t-1, t]$ (cf. theorem 2.3). From all this, together with (5.4) and (5.5), we obtain the first two inequalities of (5.2).

Finally, we show the last estimate of (5.2). Arguing as done at the end of the proof of theorem 2.3, we have

$$(1-\varepsilon)|(u^\delta(\xi))'|^2 + \frac{d}{d\xi}(a^m(l(u^\delta(\xi))))\|u^\delta(\xi)\|^2 \\ \leq 2\frac{d}{d\xi} \int_{\Omega} \mathcal{F}(u^\delta(x, \xi))dx + \frac{|l|^2 L_a^2}{8\varepsilon} \|u^\delta(\xi)\|^4 + \frac{1}{1-\varepsilon} |h(\xi)|^2 \quad \text{a.e. } \xi > t-3,$$

where u^δ is the solution to (P_δ) (see for more details the proof of theorem 2.3).

Integrating between s and r with $t-2 \leq r-1 \leq s \leq r$, we obtain

$$(1-\varepsilon) \int_s^r |(u^\delta(\xi))'|^2 d\xi + m\|u^\delta(r)\|^2 + 2\tilde{\alpha}_2 \|u^\delta(r)\|_{L^p(\Omega)}^p \\ \leq 4\tilde{\kappa}|\Omega| + 2\tilde{\alpha}_1 \|u^\delta(s)\|_{L^p(\Omega)}^p + M\|u^\delta(s)\|^2 + \frac{|l|^2 L_a^2}{8\varepsilon} \int_{r-1}^r \|u^\delta(\xi)\|^4 d\xi \\ + \frac{1}{1-\varepsilon} \int_{r-1}^r |h(\xi)|^2 d\xi. \quad (5.6)$$

Now, integrating the previous expression in s between $r-1$ and r we deduce in particular

$$\|u^\delta(r)\|_{L^p(\Omega)}^p \\ \leq \frac{2\tilde{\kappa}|\Omega|}{\tilde{\alpha}_2} + \frac{\tilde{\alpha}_1}{\tilde{\alpha}_2} \int_{r-1}^r \|u^\delta(s)\|_{L^p(\Omega)}^p ds + \frac{M}{2\tilde{\alpha}_2} \int_{r-1}^r \|u^\delta(s)\|^2 ds + \frac{|l|^2 L_a^2}{16\tilde{\alpha}_2 \varepsilon} \int_{r-1}^r \|u^\delta(\xi)\|^4 d\xi \\ + \frac{1}{2\tilde{\alpha}_2(1-\varepsilon)} \int_{r-1}^r |h(\xi)|^2 d\xi.$$

Then, making use of the previous estimates,

$$\|u^\delta(r)\|_{L^p(\Omega)}^p \leq \rho_{\infty, p}(t) \quad \forall r \in [t-2, t], \quad (5.7)$$

with

$$\rho_{\infty, p}(t) := \frac{2\tilde{\kappa}|\Omega|}{\tilde{\alpha}_2} + \left[\frac{\tilde{\alpha}_1 m}{2\tilde{\alpha}_2 \alpha_2} + \frac{M}{2\tilde{\alpha}_2} \right] \rho_2^{ext}(t) + \frac{|l|^2 L_a^2}{16\tilde{\alpha}_2 \varepsilon} (\tilde{\rho}_1(t))^2 \\ + \frac{1}{2\tilde{\alpha}_2(1-\varepsilon)} \max_{r \in [t-2, t]} \int_{r-1}^r |h(\xi)|^2 d\xi,$$

where ρ_2^{ext} and $\tilde{\rho}_1$ are independent of δ .

Now, taking $s = r-1$ in (5.6) and making use of the first inequality of (5.2) and (5.7), passing to the limit in δ , we obtain in particular

$$\int_{r-1}^r |u'(s)|^2 ds \leq \tilde{\rho}_3(t) \quad \forall r \in [t-1, t], \quad \forall \tau \leq \tau_2(\hat{D}, t), \quad \forall u_\tau \in D(\tau),$$

where $\tilde{\rho}_3$ is given in the statement. \square

Now, we introduce the following universe in $\mathcal{P}(H_0^1(\Omega))$.

Definition 5.3. For each $\mu > 0$, we denote by $\mathcal{D}_\mu^{L^2, H_0^1}$ the class of all families of nonempty subsets $\widehat{D}_{H_0^1} = \{D(t) \cap H_0^1(\Omega) : t \in \mathbb{R}\}$, where $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^{L^2}$.

Again, according to the notation in §3, we denote by $\mathcal{D}_F^{H_0^1}$ the universe of families (parameterized in time but constant for all $t \in \mathbb{R}$) of nonempty fixed bounded subsets of $H_0^1(\Omega)$. Observe that $\mathcal{D}_F^{H_0^1} \subset \mathcal{D}_\mu^{L^2, H_0^1}$ and $\mathcal{D}_\mu^{L^2, H_0^1}$ is inclusion-closed.

Now, from the existence of a pullback $\mathcal{D}_\mu^{L^2}$ -absorbing family (cf. proposition 4.4) and the regularizing effect of the equation (cf. theorem 2.4), the following result is straightforward.

Corollary 5.4. Under the assumptions of lemma 5.2, the family

$$\widehat{D}_{0, H_0^1} = \{\overline{B}_{L^2}(0, R_{L^2}^{1/2}(t)) \cap H_0^1(\Omega) : t \in \mathbb{R}\}$$

belongs to $\mathcal{D}_\mu^{L^2, H_0^1}$ and satisfies that, for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, there exists $\tau_3(\widehat{D}, t) < t$ such that

$$U(t, \tau)D(\tau) \subset D_{0, H_0^1}(t) \quad \forall \tau \leq \tau_3(\widehat{D}, t).$$

In particular, the family \widehat{D}_{0, H_0^1} is pullback $\mathcal{D}_\mu^{L^2, H_0^1}$ -absorbing for the process $U : \mathbb{R}_d^2 \times H_0^1(\Omega) \rightarrow H_0^1(\Omega)$.

The following result establishes that the process U defined on $H_0^1(\Omega)$ as phase-space is pullback $\mathcal{D}_\mu^{L^2, H_0^1}$ -asymptotically compact. To that end, we apply again an energy method with continuous functions (cf. proposition 4.6).

Proposition 5.5. Under the assumptions of lemma 5.2, the process $U : \mathbb{R}_d^2 \times H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is pullback $\mathcal{D}_\mu^{L^2, H_0^1}$ -asymptotically compact.

Proof. We only point out the main ideas because of its similarity to the proof of proposition 4.6. In this more regular setting, we take into account the energy equality (1.9) and these continuous functions

$$\begin{aligned} J_n(s) &= \|u^n(s)\|^2 - 2\eta \int_{t-2}^s \|u^n(r)\|^2 dr - \frac{1}{2m} \int_{t-2}^s |f(0) + h(r)|^2 dr, \\ J(s) &= \|u(s)\|^2 - 2\eta \int_{t-2}^s \|u(r)\|^2 dr - \frac{1}{2m} \int_{t-2}^s |f(0) + h(r)|^2 dr. \end{aligned}$$

□

Now, thanks to above results, we establish attraction in $H_0^1(\Omega)$ and some relationships among the new pullback attractors and those given in theorem 4.7.

Theorem 5.6. Assume that the function a is locally Lipschitz and satisfies (1.2), the function $f \in C^1(\mathbb{R})$ fulfils (1.4) and (1.5), $h \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$ satisfies condition

(4.1) for some $\mu \in (0, 2\lambda_1 m)$, and $l \in L^2(\Omega)$. Then, there exist the minimal pullback $\mathcal{D}_F^{H_0^1}$ -attractor $\mathcal{A}_{\mathcal{D}_F^{H_0^1}} = \{\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) : t \in \mathbb{R}\}$ and the minimal pullback $\mathcal{D}_\mu^{L^2, H_0^1}$ -attractor $\mathcal{A}_{\mathcal{D}_\mu^{L^2, H_0^1}} = \{\mathcal{A}_{\mathcal{D}_\mu^{L^2, H_0^1}}(t) : t \in \mathbb{R}\}$ for the process $U : \mathbb{R}_d^2 \times H_0^1(\Omega) \rightarrow H_0^1(\Omega)$, and the following relationship holds

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) \subset \mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2, H_0^1}}(t) \quad \forall t \in \mathbb{R}, \quad (5.8)$$

where $\mathcal{A}_{\mathcal{D}_F^{L^2}}$ and $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ are respectively the minimal pullback $\mathcal{D}_F^{L^2}$ -attractor and the minimal pullback $\mathcal{D}_\mu^{L^2}$ -attractor for the process $U : \mathbb{R}_d^2 \times L^2(\Omega) \rightarrow L^2(\Omega)$, whose existences are guaranteed by theorem 4.7. In particular, we have the following pullback attraction result in $H_0^1(\Omega)$,

$$\lim_{\tau \rightarrow -\infty} \text{dist}_{H_0^1}(U(t, \tau)D(\tau), \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)) = 0 \quad \forall t \in \mathbb{R} \quad \forall \widehat{D} \in \mathcal{D}_\mu^{L^2}. \quad (5.9)$$

Finally, if moreover h satisfies

$$\sup_{s \leq 0} \left(e^{-\mu s} \int_{-\infty}^s e^{\mu r} |h(r)|^2 dr \right) < \infty, \quad (5.10)$$

then the following chain of equalities holds

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) = \mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2, H_0^1}}(t) \quad \forall t \in \mathbb{R},$$

and for any bounded subset B of $L^2(\Omega)$,

$$\lim_{\tau \rightarrow -\infty} \text{dist}_{H_0^1}(U(t, \tau)B, \mathcal{A}_{\mathcal{D}_F^{L^2}}(t)) = 0 \quad \forall t \in \mathbb{R}. \quad (5.11)$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}$ and $\mathcal{A}_{\mathcal{D}_\mu^{L^2, H_0^1}}$ is a consequence of corollary 3.7. Indeed, the process U is strong-weak continuous (cf. proposition 5.1), the relationship between the universes, the existence of an absorbing family (cf. corollary 5.4) and the asymptotic compactness (cf. proposition 5.5) hold.

The chain of inclusions (5.8) follows from corollary 3.7 and theorem 3.8. In fact, the equality for all $t \in \mathbb{R}$ between $\mathcal{A}_{\mathcal{D}_F^{L^2}}(t)$ and $\mathcal{A}_{\mathcal{D}_\mu^{L^2, H_0^1}}(t)$ is also due to theorem 3.8, using corollary 5.4. Then, (5.9) obviously holds.

When h satisfies (4.11), it holds equality $\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)$ for all $t \in \mathbb{R}$ (cf. theorem 4.7). The equality $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) = \mathcal{A}_{\mathcal{D}_F^{L^2}}(t)$ is again due to theorem 3.8. To that end we assume (5.10), which is stronger than (4.11). Therefore, (5.11) is straightforward. \square

Analogously to corollary 4.8, we have the following result.

Corollary 5.7. *Under the assumptions of theorem 5.6, for any $\sigma \in (\mu, 2\lambda_1 m)$ there exists the corresponding $\mathcal{D}_\sigma^{L^2, H_0^1}$ -pullback attractors $\mathcal{A}_{\mathcal{D}_\sigma^{L^2, H_0^1}}$ and the equality $\mathcal{A}_{\mathcal{D}_\sigma^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\sigma^{L^2, H_0^1}}(t)$ holds for all $t \in \mathbb{R}$. Moreover, if h fulfills (5.10), then the families $\mathcal{A}_{\mathcal{D}_\sigma^{L^2, H_0^1}}$ and $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}$ coincide.*

Another immediate consequence of theorem 5.6 is an improvement in the regularity of the attractor in an autonomous framework. Namely, we have the following

Corollary 5.8. *Suppose that $h \equiv 0$ in (1.3). Under the assumptions of theorem 5.6, there exist the global attractors \mathcal{A}_{L^2} and $\mathcal{A}_{H_0^1}$ for the associated autonomous dynamical system in $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively, and they coincide. Furthermore,*

$$\|u\|_{L^\infty(\Omega)} \leq \left(\frac{\kappa}{\alpha_2}\right)^{1/p} \quad \forall u \in \mathcal{A}_{L^2}. \quad (5.12)$$

Proof. The existence of global attractors is guaranteed by theorem 5.6. In addition, thanks to the regularizing effect of the equation (cf. theorem 2.4), we deduce that $\mathcal{A}_{L^2} = \mathcal{A}_{H_0^1}$. Finally, the estimate (5.12) follows arguing as in [37, Theorem 11.6, p. 292]. \square

Remark 5.9. (i) *In the context of [8], for f sublinear, the global attractor also satisfies the estimate (5.12) in $L^\infty(\Omega)$, provided that $f(s)s \leq \kappa - \alpha_2|s|^p$ for all $s \in \mathbb{R}$ for some $p \geq 1$.*

(ii) *Observe that the assumptions (1.2) and (1.4) can be weakened along this paper. Concerning (1.2), it is not difficult to deduce that the upper bound M can be removed thanks to the uniform bound of the solutions in $L^\infty(\tau, T; L^2(\Omega))$. Regarding the assumption (1.4), it can be replaced by the monotonicity condition*

$$(f(s) - f(r))(s - r) \leq \eta(s - r)^2 \quad \forall s, r \in \mathbb{R},$$

where f is just simply a continuous function. To that end, using mollifiers, we construct $\{f_\epsilon\}$ which does fulfil (1.4), and by compactness arguments we recover the desired results for f .

Despite the cited improvements on the assumptions made on f and a , the results are proved in the less general setting for the sake of clarity and simplicity.

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