Capítulo 14

The aftermath of Abraham de Moivre's Doctrine of Chances and Annuities on Lives in 18th-century Europe

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Some biographical data of De Moivre[i]

Abraham Moivre was born in May 26, 1667 as the son of a Protestant surgeon in Vitry-le-François in the Champagne. He spent the first 20 years of his life in France where he was educated in different Huguenot institutions until they were closed in the early 1680s. Quite early he had turned to mathematics. At 16 he had studied amongst other things Huygens' tract "De ratiociniis in ludo aleae". In Paris in the 1680s he was taught by the private teacher of mathematics Jacques Ozanam, who might have offered to Moivre a model for how to make his living when he had to support himself shortly afterwards.

After the revocation of the Edict of Nantes in 1685 hundreds of thousands of Huguenots left France. Amongst them was Moivre who went to England together with his brother Daniel in December 1687. In England he began his occupation as a tutor in mathematics. Here he added a "de" to his name. De Moivre mastered Newton's Principia from 1687 very early and became a true and loyal Newtonian. In 1692 he met Halley and Newton and in 1697 he was chosen a Fellow of the Royal Society. He was naturalized in 1705, and, when Newton had nominated him a member of the commission to decide the priority dispute between himself and Leibniz in favour of Newton in 1712, he gave up every hope of finding a position as a university professor for mathematics on the continent. But even in England he had to learn that for him the only way to make a living was to work as a private teacher and as a consultant for problems concerning different forms of annuities. In order to attract well-to-do clients he had to make a reputation as a mathematician. He soon found out that it was much easier to do that in the calculus of games of chance than in the new field of the infinitesimal calculus. Francis Robartes, one of de Moivre's clients, draw de Moivre's attention to the first edition of

Montmort's Essay d'analyse sur les jeux de hazard from 1708, which raised de Moivre's interest in the theory of games of chance and probability. In the Philosophical Transactions for 1711 de Moivre published a longer article De mensura sortis, seu, De Probabilitate Eventuum in Ludis a Casu Fortuito Pendentibus on the subject, which was followed by his Doctrine of Chances, the first edition of which was published in 1718. A second edition from 1738 contained de Moivre's normal approximation to the binomial distribution, which he had found in 1733. The third edition from 1756 contained as a second part the Annuities on Lives, which had been published as a monograph for the first time in 1725.

The Doctrine of Chances is in part the result of a competition between de Moivre on the one hand and Montmort together with Niklaus Bernoulli on the other. De Moivre claimed - very much to the annoyance of Montmort, but justified by his later work - that his representation of the solutions of the then current problems tended to be more general than those of Montmort. This holds at least for the second and third edition of the Doctrine of Chances, which offered so many new results that Montmort's contributions to the subject fell justly into oblivion.

Precursors of de Moivre

Christiaan Huygens with his tract De ratiociniis in ludo aleae[ii] of 1657 was the first author who acquainted de Moivre with the calculus of games of chance. In England de Moivre read a small booklet Of the Laws of Chance, or, a Method of Calculation of the Hazards of Game, published anonymously by John Arbuthnot in 1692, which was largely inspired by Huygens' tract. When in 1708 the Essai d'Analyse sur le Jeux de Hasard of the French nobleman Pierre Rémond de Montmort appeared anonymously, de Moivre hurried to read this work written in his native tongue.

Edmond Halley's "An Estimate of the Degrees of the Mortality of Mankind", published in the Philosophical Transactions for 1693[iii], might have become known to de Moivre quite early, since he had met Halley in 1692 in person. When de Moivre wrote De Mensura Sortis[iv], his first tract concerning games of chance and probability, he had no knowledge of Jakob Bernoulli's Ars Conjectandi, which was published in 1713, eight years after Bernoulli's death. In the preface of the Ars Conjectandi Nikolaus Bernoulli, Jakob Bernoulli's nephew, had asked de Montmort and de Moivre to continue his uncle's work, the application of the calculus of probabilities "to economical and political uses". Shortly after the Ars Conjectandi the second edition of Montmort's Essai of 1714 appeared, which contained, apart from some improvements, an extension of the first edition and the correspondence between Montmort and Nikolaus Bernoulli up to the end of 1713.

The long preface for the first edition of the Doctrine of Chances of 1718 shows that de Moivre understood the second edition of the Essai as a challenge to replace the solutions offered by Montmort and Nikolaus Bernoulli by his own. This is one reason why he showed no interest in continuing Jakob Bernoulli's project of applying the theory of probability to economics and politics as outlined in the Ars conjectandi.

Surprisingly, de Moivre seems to have been unfamiliar with the correspondence between Pascal and Fermat from 1654 concerning games of chance, part of which was published still in the 17th century[v], and with Blaise Pascal's Traité du triangle arithmétique published in 1665. In constrast to Huygens, Pascal and Fermat used combinatorial methods for the solution of the problems discussed by them. But even Jakob Bernoulli, who had devoted the second part of the Ars Conjectandi to combinatorics, was obviously totally ignorant of Pascal's contributions to combinatorics.

De Moivre's main contributions to stochastics

De Moivre's main contributions to stochastics are contained first of all in his The Doctrine of Chances: Or, A Method for Calculating the Probability of Events in Play, with editions in 1718, 1738, and 1756 and in his Annuities on Lives, which appeared first in 1725; later editions came out in 1731, 1743, 1750, 1752 and together with the Doctrine of Chances in 1756.

The Miscellanea analytica of 1730 dealt inter alia with open issues in the controversy with Montmort who, however, had died already in 1719. In a supplement to the Miscellanea analytica de Moivre had started, in competition with James Stirling, to find an approximation for n-factorial when n is large. This led in 1733 to the first formulation of the central limit theorem in the Approximatio ad Summam Terminorum Binomii $(a+b)^n$ in Seriem expansi which de Moivre declared as a private communication for some friends as distinct from a publication. The publication of this result occurred only in the second and third edition of the Doctrine of chances[vi].

The main achievements contained in the Doctrine are a first theory of probability in the introduction[vii], which contains the principal concepts like probability, conditional probability, expectation, dependent and independent events, the multiplication rule, and the binomial distribution.

De Moivre had also developed algebraical and analytical tools for the theory of probability for instance a "new algebra" for the solution of the problem of coincidences, which forshadowed Boolean algebra[viii], the theory of recurrent series for the solution of difference equations, and the method of generating functions. He used the method of generating functions, for example, for solving the general dicing problem of finding the number of chances to get s points with a throw of n dice each having f faces[ix]. De Moivre determines the number by taking the coefficient of x^s in the development of

$$\left(\sum_{i=1}^{f} x^i\right)^n .$$

He had developed for the solution of the problem of the duration of play the theory of what he called recurrent series, which are in modern terminology homogeneous linear difference equations with constant coefficients.

De Moivre's greatest mathematical achievement is considered a form of the central limit theorem, which he found in 1733 at the age of 66. He understood his central limit theorem as a generalization and a sharpening of Bernoulli's main theorem in the fourth part of the Ars Conjectandi, which was later named the law of large numbers by Poisson[x].

Starting from Bernoulli's law of large numbers in the form

$$1 > P\left(\left|r_n - p\right| \le \varepsilon\right) > \frac{c}{c+1}$$

where r_n is the relative frequency of the occurrence of independent events E_i, $i = 1,\ldots,n$ with $P(E_i) = p$, ε arbitrarily small and c arbitralily large quantities in a sufficiently large number n of trials, de Moivre found on the basis of an approximation of log n! for large n and of Bernoulli's formula for the sum of powers of integers the equivalent of

$$\lim_{n\to\infty} P\left(\left|r_n - p\right| \le c\sqrt{\frac{2p(1-p)}{n}}\right) = \frac{2}{\sqrt{\pi}}\int_0^c e^{-t^2}\,dt.$$

De Moivre calculated the numerical values 0.682.., 0.954.. and 0.998..of this integral for c = 1, 2, and 3. He concluded from this result that "altho' Chance produces Irregularities" those irregularities shall cancel out with time in order to reveal the "Order which naturally results from Original Design" and so the existence of God, "the great Maker and Governour of all"[xi]. Thus he interpreted his central limit theorem as a proof for the existence of God, an interpretation, which constituted a challenge for theologians, especially if they were sceptical of natural religion.

Next to games of chance de Moivre considered problems dealing with human mortality as the proper subject of probability theory. He collected his results in this field in his Annuities on lives. His starting point for the Annuities on lives was Halley's table of the population of Breslau published in the Philosophical Transactions for 1693, which consisted of the numbers of those living at age 1 to over 80 and was interpreted as a mortality table. Halley's table induced de Moivre to assume a constant decrement of the living in every year after age 12, which means that the probability $p_x(m)$ to survive with age x the next m years is

$$\frac{l_{x+m}}{l_x} = \frac{a_{\max} - (x+m)}{a_{\max} - x} \quad \text{where} \quad 12 \le x < a_{\max},\ 0 \le m \le a_{\max} - x \quad \text{and} \quad a_{\max} = 86.$$

If i is the rate of interest and the amount paid out every year is 1, the present value of an annuity on lives for a person aged x is

$$a_x = \sum_{j=1}^{a_{\max}-x} (1+i)^{-j} \cdot p_x(j),$$

for which de Moivre found the following recurrent series:

$$a_x = (1+i)^{-j} \cdot p_x(1) \cdot (1 + a_{x+1}).$$

For the determination of the annuities of joint lives he used, for the sake of ease of calculation, the hypothesis of what he called fictitious lives. According to this hypothesis the sequence of the probabilities to stay alive for the next m years at age x is a geometrical series, that is to say:

$$\frac{l_{x+m}}{l_x} = \left(\frac{l_{x+1}}{l_x}\right)^m.$$

Because of the sometimes great differences between the annuities calculated with "fictitious" lives and annuities calculated on the basis of "real" lives this hypothesis was eventually given up.

The impact of de Moivre's main publications on relevant English publications

Since de Moivre's most important works in stochastics the Doctrine of Chances and the Annuities on lives were written in English his impact on British authors was by far the most intense and lasting. The English mathematician who perhaps owed most to de Moivre's work and took advantage of de Moivre's achievement already during de Moivre's lifetime was Thomas Simpson. Simpson worked like de Moivre as a teacher of mathematics; he wrote many mathematical textbooks amongst them two concerning probability theory and annuities. He plagiarized and further developed de Moivre's work in The Nature and Laws of Chance[xii] and in The Doctrine of Annuities and Reversions[xiii].

The nature and laws of chance of 1740 contains the results of the Doctrine of Chances - without the descriptions of the different games of chance -, the general solution of the problem of the duration of play and the determination of annuities on lives; although de Moivre is mentioned respectfully in the preface as the man who had developed the subject, his name does not appear in the text in order to mark the results taken from de Moivre's Doctrine. Simpson's book was much cheaper than de Moivre's, in order to attract buyers to the detriment of de Moivre.

Simpson's Doctrine of annuities and reversions follows to a great deal de Moivre's Annuities on lives but can claim for its second edition a better solution for annuities of joint lives compared with de Moivre.

Whereas Simpson's indebtedness towards de Moivre's work is obvious, the influence of de Moivre on Thomas Bayes' Essay, which was published posthumously in 1764[xiv], remains hypothetical, despite many possible direct or indirect relationships between de Moivre and Bayes, via Colin MacLaurin and the Earl of Stanhope[xv], or in meetings of the Royal Society of which both de Moivre and Bayes were fellows - relationships determined by mathematical and theological issues[xvi]. A possible motive for Bayes to write his Essay was Moivre's second remark with respect to his central limit theorem[xvii]:
"As upon the Supposition of a certain determinate Law according to which any Event is to happen, we demonstrate that the Ratio of Happenings will continually approach to that Law, as the Experiments or Observations are multiplied: so, conversely, if from numberless Observations we find the Ratio of the Events to converge to a determinate quantity, as to the Ratio of P to Q; then we conclude that this Ratio expresses the determinate Law according to which the Event is to happen."

Obviously de Moivre here addressed the inverse problem in the form to conclude from the outcome of many trials the unknown probability of an event. Since normally the possibility to undertake many trials is restricted or even non-existent Bayes became interested in the inverse problem when the number of trials may even be small:
"Given the number of times in which an unknown event has happened and failed: Required the chance that the probability of its happening in a single trial lies somewhere between any two degrees of probability that can be named."

De Moivre had also addressed all the topics dealt with in the published papers of Bayes. These are Bayes' first publication on God's benevolence[xviii], to which de Moivre's two remarks following his central limit theorem in 1738 and 1756 could be considered as a reaction. In addition there is a paper of Bayes published posthumously in the Philosophical Transactions for 1763[xix]. It deals with the (asymptotic) series for $\log n!$ in the form developed by de Moivre in 1730 in his Miscellaneis Analyticis Supplementum. De Moivre had touched upon the semiconvergent character of this series, but claimed in the end that it is convergent[xx]. The series was used by de Moivre in order to prove the central limit theorem in 1733.

Bayes' posthumously published papers were edited by Richard Price (1723-1791), a very prominent Presbyterian Minister, who became known at first by his publications on ethical and political questions. He was made a Fellow of the Royal Society in 1765 after his edition of the two papers of the late Thomas Bayes, which were published in the Philosophical Transactions in 1764 together with an introduction and some remarks and proofs by Price. In 1771 Price's Observations on reversionary payments with a long subtitle appeared. This book had several editions in a few years and remained a standard text for actuarial science for the next hundred years. In it references to de Moivre's contributions to annuities on lives appear throughout.

James Dodson (ca. 1705-1757), a former student of de Moivre, called himself in his major work the Mathematical Repository "Accomptant, and Teacher of the Mathematics". Dodson's Repository was published in three volumes in 1748, 1753, and 1755. The first volume was dedicated to De Moivre; the second contained as the first of three parts "Indetermined Questions, solved generally, by an elegant method communicated by Mr De Moivre". The two other parts of the second volume deal with questions "relating to Chances and Lotteries" as well as those "concerning Annuities for Lives". In the Preface to the second volume from 1753 Dodson defended de Moivre's "truly admirable hypothesis, that the decrements of life may be esteemed nearly equal, after a certain age"; which is to say "that the number of those who die within a year is constant or that the number of living after x, x+1, x+2, ... years constitutes an arithmetical series". When De Moivre's critics had mantained that his hypothesis does not hold for the recently available data for London for the years from 1744 to 1753, for which the Reverend William Brakenridge had calculated a new life table, Dodson found that de Moivre's hypothesis holds even in this case.

The Repository not only earned Dodson election to the Royal Society in January 1755, but also allowed him to make a reputation as an actuary who formulated the principles on which the Equitable Society, one of the first life insurance companies in England was founded in 1762 five years after his death. Richard Price, who survived Dodson, became a consultant for the Equitable Life Assurance Society. So through his two mediators, James Dodson and Richard Price, de Moivre's work on life insurance influenced the development of the first insurance society in England, which was based on mortality statistics and on mathematics.

Apart from authors like Simpson, Dodson, Price and Bayes, who can claim at least a modest place in the history of mathematics, there are many British authors in the 18th century, who, forgotten for a long time, refer to de Moivre and his results in probability theory in textbooks as well as in popular books on games of chance and annuities on lives. To these belongs at his time the very popular Edmund Hoyle - "according to Hoyle" - in his Essay towards making the Doctrine of Chances Easy to those who understand Vulgar Arithmetic only (1754) or in his Polite Gamester (1761). An example how the Doctrine was exploited is given by Edmund Hoyle (1671-1769). In Problem III of his Essay he asks for the number of trials necessary to make the probability that an event, the chances of its happening to its failing being as a : b, is at least 1/2.

$$\frac{b^x}{(a+b)^x} = \frac{1}{2} \Rightarrow x\left[\ln(a+b) - \ln b\right] = \ln 2,$$

or for $q = \frac{b}{a} \quad \Rightarrow \quad x \ln\left(1 + \frac{1}{q}\right) = \ln 2$ and if $q \gg 1 \Rightarrow x \approx q \ln 2 \approx q \cdot 0.7$.

In his third query Hoyle asks "In how many Throws may you undertake upon an Equality of Chance, to throw two Sixes upon two Dice?" and gives the answer: "The whole Number of Chances upon two Dice are thirty-six, out of which there is but one Chance for throwing two Sixes; therefore, (according to Mr. Demoivre's Doctrine) you are to multiply 35 by 0.7, which solves the question."

The result 24.5 leads to the somehow irritating final statement of Hoyle: "You may undertake to do it in twenty-four Throws and a half."

In the same way Hoyle solved in chapter 8 the problem "In a Lottery to find out the Number of Tickets, which is requisite to entitle you to a Prize, upon an Equality of Chance" for different numbers of blanks and one prize.

Still in de Moivre's lifetime in 1741 Richard Hayes, an accomptant and writing master, had published his The gentleman's complete book-keeper. Its chapter 27 "Shews the Ways of casting up the Value of Annuities for successive Lives, and joint Lives, by Common Arithmetick, according to Mr. De Moivre's and Mr. Lea's Methods".

After his death most references to de Moivre in the English literature concern his work on life annuities. So James Ferguson (1710-1776), a lecturer on natural philosophy and inventor of mathematical instruments and FRS, published in 1767 his Tables and Tracts, Relative to Several Arts and Sciences. He refers to de Moivre's Annuities on Lives and the Doctrine for the solution of the problem[xxi]: "To find at what Rate of Interest I ought to lay out a sum S, so as it may increase1/3 for Instance, or become 4/3 S in 7 years." According to Ferguson "In questions concerning the Values of Lives any how combined, recourse must be had to Mr. De Moivre's last Edition of his Treatise on Annuities."

Another example for de Moivre's lasting authority concerning actuarial problems is offered by Robert Hamilton (1743-1829), a political economist and mathematician, who received in 1779 the chair of natural philosophy of Aberdeen University and in 1817 the chair of mathematics after he had taught mathematics at Aberdeen for many years. He published An Introduction to Merchandize, which came out in 1777 as the first of a whole series of practical treatises; in its first edition de Moivre is mentioned several times in chapter VIII of Part II which deals with "Annuities for lives" but not in the preceding chapter entitled "Doctrine of Chances". However, credit is given to De Moivre's general solution by a generating function of Hamiliton's problem V "Required the chance of throwing any proposed number with a given number of dice" in the chapter concerning the "Doctrine of Chances" in the second edition from 1788[xxii]. In the third edition from 1797 the same problem is solved by de Moivre's formula, however without mentioning his name. In the chapter on annuities there are again several references to de Moivre.

De Moivre is also mentioned by Hamilton in his Mathematical Tables from 1790 in "A Table of the value of an annuity of Life for a single life"[xxiii].

Francis Maseres (1731-1824), who came from a Huguenot family and whose plans to receive the Lucasian chair for mathematics in Cambridge failed, later worked as a lawyer and as a judge in London. He wrote a book The principles of the doctrine of life-annuities; explained in a familiar manner, so as to be intelligible to persons not acquainted with the doctrine of chances, which appeared in 1783. Maseres refered to de Moivre in this book very often, especially to de Moivre's hypothesis of the constant "decrements of human life". He was concerned with the differences between the annuities calculated according to de Moivre's hypothesis and those based on other assumptions.

As late as 1793 an anonymously published tract in Englisch on Faro and Rouge et Noir came out which contained an explanation of the two games together with a table showing the chances against the "punter" or Ponte in Faro or Pharao taken from de Moivre's Doctrine.

Most English 18th century publications use the expression "doctrine of chances" when refering to the theory of probability and so confirm even without mentioning de Moivre the influence on the subject of a man who had coined this expression. So Francis Maseres published in 1795 with The doctrine of permutations and combinations being an essential part of the doctrine of chances "as it is delivered by Mr. JAMES BERNOULLI, in his excellent Treatise on the Doctrine of Chances, intitled, Ars Conjectandi, and by the celebrated Dr. JOHN WALLIS, of Oxford, in a Tract intitled from the Subject, and published at the end of his Treatise on Algebra."

During de Moivre's lifetime nobody would have failed to name de Moivre when talking about the doctrine of chances, as one can see for instance from letters of the Abbé Jean Bernard Le Blanc which were originally published in French and appeared in English as Letters on the English and French nations, containing curious and useful observations on their constitutions natural and political in two volumes in 1747. Le Blanc reports[xxiv]: "I have had several conversations upon this subject with the famous M. DE MOIVRE, the greatest calculator of chances now in England: but I did not perceive that he had ever calculated the effects of gaming, with regard to morality, though that is a much more essential thing than the theory of chances."

He continues a bit later[xxv]: "I must add that the great gamesters of this country, who are not usually great geometricians, have a custom of consulting those who are reputable calculators upon the games of hazard. M. DE MOIVRE gives opinions of this sort every day at Slaughter's coffee-house, as some physicians give their advice upon diseases at several other coffee-houses about London."

One should add that le Blanc is an involuntary witness of de Moivre's prominence in the calculus of probabilities in England, since he was much more concerned with the bad effects of gaming, which makes according to him every gamester a loser.

De Moivre's influence on the continent

De Moivre influenced the development of mathematics outside Great Britain much less than in his second homeland for different reasons. For the Bernoullis, who had taken sides with Leibniz in the priority dispute with Newton, de Moivre stood on the wrong side. And even if de Moivre was considered by some as the leading mathematician after Newton in England, his mathematical output seemed modest compared with that of the leading mathematicians in continental Europe in the generations after Leibniz. Most of the continental mathematicians, who published mainly in Latin or French, were not prepared to read English, the language in which de Moivre's main works were published. None the less there are direct references to de Moivre's work in probability calculus and traces of its impact in continental Europe.

The most prolific mathematician in the 18th century, Leonhard Euler, certainly knew at least part of de Moivre's work, but we find nearly no direct hints to de Moivre in Euler's comparatively small oeuvre concerning probability theory[xxvi]. Euler was certainly an honest man but he was not very good in referring to the work of others. Euler's contributions to stochastics were to a great extent stimulated by requests from outside; as soon as he was confronted with a problem he would try to solve it no matter if others had solved it already. An example for this is the problem of finding the probabilities of two players engaged in the game of Rencontre which Maupertuis had posed to Euler. Euler hurried to answer Maupertuis the next day beginning with the remark[xxvii]: "I found the solution so difficult that I doubt very much that a more difficult problem was ever solved". Obviously Euler did not know, or did not care to know, that solutions were found years before by Montmort, Niklaus Bernoulli, and de Moivre. There are no direct hints to the Doctrine or the Annuities in Euler's works. He was probably familiar with the Mensura Sortis and certainly with the Miscellanea Analytica; this one can assume from Euler's terminology and references to the Miscellanea Analytica in his work. Similarity (and difference) in the treatment of several problems concerning lotteries, the coincidence (rencontre) in the game of Treize, the chances in Pharao, the duration of play, annuities on lives and human mortality and many others in his notebooks, like the determination of the probability of getting s points by throwing m dice each with n sides can, but are not necessarily to be interpreted as some familiarity with the relevant works of de Moivre.

De Moivre's impact on the further development of the analytical tools used in probability theory by Lagrange and Laplace is more easily proved, because both refer to de Moivre. In addition, Lagrange and Laplace tried, presumably independently of one another, to translate the Doctrine into French[xxviii], a project that apparently was never finished. Laplace refers in the introduction to the Théorie analytique des probabilités to de Moivre's determination of the general term of a recurring series and its summation as the first step in a development leading to Lagrange's recurro-recurrent series and his own theory of generating functions[xxix]. In his "note historique" Laplace honours even more de Moivre's extension of Bernoulli's law of large numbers and his contributions to annuities on lives.[xxx]

Whereas it is easy to disappear in the shadow of mathematical celebrities like Euler, Lagrange, or Laplace, because they are creative enough not to depend on the ideas of others, the less gifted are perhaps more prone to use and to refer to – if they are honest – the work of predecessors. For this group of minor figures I shall give two examples: The first concerns the two Cistercian Fathers Don Roberto Gaeta and Don Gregorio Fontana who translated de Moivre's Annuities on Lives into Italian and published it in 1776 in Milano as La Dottrina degli Azzardi Applicata ai Problemi della Probabilita della Vita, delle Pensioni Vitalizie, Reversioni, Tontine, ec. Di Abramo Moivre. This very well-informed book was used for a course in mathematics at the university of Pavia.

The book contains the most recent tables concerning human mortality and the numerous notes refer to the literature published after de Moivre; it contains the best bibliography available at the time, which includes the contributions of Euler, Süßmilch and Lambert to the subject. In addition one finds allusions to the Doctrine of Chances concerning the vignette on its first page and de Moivre's attempt to prove the existence of God.

The second example pertains to German probabilists stemming from the philosophical school of Christian Wolff. Representative for this group is Ludwig Martin Kahle, who in his Elementa Logicae probabilium methodo mathematica in usum scientiarum et vitae adornata[xxxi] refers in the preface to the reader to de Moivre's Mensura sortis.

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[i] For more extensive biographical information see Ivo SCHNEIDER, Abraham de Moivre (1667-1754). In: Oxford Dictionary of National Biography (hrsg. v. H.C.G. MATTHEW und Brian HARRISON) vol. 38, Oxford 2004, S. 522 f. and Ivo SCHNEIDER, Der Mathematiker Abraham de Moivre (1667 bis 1754). In: Archive for History of Exact Sciences 5, 1968, S. 177-317.

[ii] Christiaan HUYGENS, De ratiociniis in ludo aleae, in: Frans VAN SCHOOTEN, Exercitationum mathematicarum libri quinque, Leiden 1657, p. 517-534.

[iii] Edmund HALLEY, An estimate of the Degrees of the Mortality of Mankind, drawn from curious tables of the births and funerals at the city of Breslaw; with an attempt to ascertain the price of annuities upon lives, in: Philosophical Transactions Nr. 196 (1693), p. 596-610.
[iv] Abraham DE MOIVRE, De mensura sortis, seu, de probabilitate eventuum in ludis a casu fortuito pendentibus, in: Philosophical Transactions Nr. 329 (1711), p. 213-264.
[v] Pierre DE FERMAT, Varia opera mathematica, Toulouse 1679.
[vi] Doctrine of chances, London 1738, p. 235-243, and London 1756, p.243-254.
[vii] Doctrine 1738, p. 1-30, 1756, p. 1-33.
[viii] Doctrine 1718, p. 61-63, 1738, p. 96-98, 1756, p. 110-112.
[ix] Doctrine 1738, p. 37-39, 1756, p. 41-43.
[x] S.-D. POISSON, Recherches sur la probabilité des jugements en matière civile, Paris 1837, p. 7.
[xi] Doctrine 1738, p.243, 1756, p. 251 and 252.
[xii] London 1740.
[xiii] London 1742, second edition London 1775.
[xiv] Thomas BAYES, An essay towards solving a problem in the doctrine of chances, in: Philosophical Transactions vol. 53 (1763), p. 370-418.
[xv] See David BELLHOUSE, The Reverend Thomas Bayes, FRS: A Biography to Celebrate the Tercentenary of His Birth, in: Statistical Science 19 (2004), no. 1, p. 3–43.
[xvi] See Ivo SCHNEIDER, De Moivre's limit theorem and its possible connection with Bayes' Essay. In: Physica et historia – Festschrift für Andreas Kleinert zum 65. Geburtstag (hrsg. von Susan SPLINTER, Sybille GERSTENGARBE, Horst REMANE und Benno PARTHIER) (= Acta Historica Leopoldina, Nr. 45), Halle (Saale) 2005, p. 155-161.
[xvii] Doctrine 1756, p. 251.
[xviii] Thomas BAYES, Divine Benevolence, or an attempt to prove that the principal end of the divine providence and government is the happiness of his creatures. London 1731.
[xix] A letter from the late Reverend Mr. Thomas Bayes, F.R.S. to John Canton, M.A. and F.R.S. in: Philosophical Transactions vol. 53 (1763), p. 269-271.
[xx] Abraham DE MOIVRE, Miscellaneis analyticis supplementum, London 1730, p. 9.
[xxi] p. 286.
[xxii] p. 183.
[xxiii] p. 126.
[xxiv] vol. II, p. 307.
[xxv] vol. II, p. 309.
[xxvi] Ivo SCHNEIDER, I contributi di Euler alla stocastica nel contesto della letteratura contemporanea. In: Quaderni della Accademia delle Scienze di Torino, 2008, p. 103-121.
[xxvii] Letter from Euler to Maupertuis of April 16, 1752. See: Leonhard Euler Opera omnia IV, 6, Basel 1986, p. 203 f.
[xxviii] See letter from Lagrange to Laplace of December 30, 1776. In: Oeuvres de Lagrange vol. 14, Paris 1892, p. 66.
[xxix] See Pierre Simon LAPLACE, Théorie analytique des probabilités, third edition, Paris 1820, p. XXIX.
[xxx] Ibidem p. CXLVII.
[xxxi] Published in Magdeburg 1735.