EQUILIBRIUM PROBLEMS ON RIEMANNIAN MANIFOLDS WITH APPLICATIONS

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Abstract. We study the equilibrium problem on general Riemannian manifolds. The results on existence of solutions and on the convex structure of the solution set are established. Our approach consists in relating the equilibrium problem to a suitable variational inequality problem on Riemannian manifolds, and is completely different from previous ones on this topic in the literature. As applications, the corresponding results for the mixed variational inequality and the Nash equilibrium are obtained. Moreover, we formulate and analyze the convergence of the proximal point algorithm for the equilibrium problem. In particular, correct proofs are provided for the results claimed in J. Math. Anal. Appl. 388, 61-77, 2012 (i.e., Theorems 3.5 and 4.9 there) regarding the existence of the mixed variational inequality and the domain of the resolvent for the equilibrium problem on Hadamard manifolds.

Key words. Riemannian manifold, equilibrium problem, variational inequality problem, proximal point algorithm

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1. Introduction. Let X be a metric space, $Q \subseteq X$ a nonempty subset and $F : X \times X \to (-\infty, +\infty]$ a bifunction. The equilibrium problem (introduced by Blum and Oettli in [6]), abbreviated as EP, associated to the pair (F, Q) is to find a point $\bar{x} \in Q$ such that

$$(1.1) \qquad\qquad F(\bar{x}, y) \geq 0 \quad \text{for any } y \in Q.$$

As shown in [6, 36], EP contains, as special cases, optimization problems, complementarity problems, fixed point problems, variational inequalities and problems of Nash equilibria; and it has been broadly applied in many areas, such as economics, image reconstruction, transportation, network, and elasticity. In recent years, EP has been studied extensively, including the issues regarding existence of solutions and iterative algorithms for finding solutions; see e.g., [5, 6, 12, 15, 21].

Since the classical existence results in EPs work for the case when Q is a convex set and the bifunction F is convex in the second variable, some authors focused on exploiting

the possible existence without the convexity assumption; see e.g., [3, 4, 9, 20]. One useful approach used in [23] and [24] is to embed the underlying nonconvex and/or nonsmooth Nash/Nash-type equilibrium problems into a suitable Riemannain manifold M to study the existence and the location problems of the Nash/Nash-type equilibrium points. In particular, Kristály established in [23], the existence results for Nash equilibrium points associated to strategy sets $\{Q_i \subseteq M_i\}_{i\in I}$ and loss-functions $\{f_i\}_{i\in I}$ ($I := \{1, 2, \ldots, n\}$) under the following assumption:

(A_K): each Q_i is a compact and geodesic convex set of M_i for all $i \in I$

(see item (e) in Definition 2.3 for the notion of the geodesic convexity). This class of approaches has also been used extensively in many optimization problems since some non-convex and/or nonsmooth problems of the constrained optimization in $\mathbb{R}^n$ can be reduced to convex and/or smooth unconstrained optimization problems on appropriate Riemannian manifolds; see, for examples, [14, 30, 31, 40, 42]. More about optimization techniques and notions in Riemannian manifolds can be found in [1, 2, 8, 18, 26, 27, 28, 43, 45] and the bibliographies therein.

For the equilibrium problem (1.1) on a Riemannian manifold M, Colao et. al, by generalizing the KKM lemma to a Hadamard manifold, established an existence result (i.e., [11, Theorem 3.2]) for solutions of EP under the following assumptions:

(A_C-1): M is a Hadamard manifold and Q is closed and convex;
(A_C-2): the set $\{y \in Q : F(x, y) \leq 0\}$ is convex for any $x \in Q$

(which was extensively studied in [47] for the generalized vector equilibrium problem). This existence result was applied there to solve the following problems:

(P1). the existence problem of solutions for mixed variational inequality problems;
(P2). the well-definedness of the resolvent and the proximal point algorithm for solving EP;
(P3). the existence problem of fixed points for set-valued mappings;
(P4). the existence problem of solutions for Nash equilibrium problems.

However, the applications to problems (P1)-(P3) above rely heavily on the following claim:

$$(1.2) \qquad \text{the function } y \mapsto \langle u_x, \exp_x^{-1} y \rangle \text{ is quasi-convex,}$$

where $x \in M$ and $u_x \in T_x M$; see the proofs for Theorems 3.5, 3.10 and 4.9 in [11]. Unfortunately, unlike in the linear space setting, claim (1.2) is not true in general as pointed out in [42, Theorem 2.1, p. 299] or [25]. Note that, for any $x \in M$, the function defined by (1.2) is convex at x (see Definition 2.4 (i)); this motivates us to introduce the new notion of the point-wise (weak) convexity for a bifunction on general manifolds (see Definition 3.1 (c)).

Our main purpose in the present paper is to develop a new approach (based on the new notion and the work in [28]) to study the issue on the existence and structure of solutions for equilibrium problems on general Riemannian manifolds, which, in particular, covers problems (1), (2) and (4) as special cases. In our approach, rather than assumptions (A_K) or (A_C-1)-(A_C-2), we make the following ones on the involved Q and F:

- Q is a closed and weakly convex subset of Riemannian manifold M;

- F is point-wise weakly convex on Q.

The technique used in the present paper for proving the main results is completely different from the ones used in [11, 47, 23]. Actually, our technique here is mainly focused on establishing the equivalence between the EP and a suitable variational inequality problem; and then apply the corresponding results in [28] for the variational inequality problem to study the existence of solutions and the convex structure of the solution set of the EP .

As applications to problems (P1), (P2) and (P4), we obtained some results on the existence of solutions and convexity of the solution sets for mixed variational inequality problems and Nash equilibrium problems (see Theorem 5.1 and 5.2), as well as the convergence of the proximal point algorithm for solving EP. In particular, the existence result for mixed variational inequality problems and the well-definedness results of the resolvent for solving EP on Hadamard manifolds provide correct proofs for the corresponding ones [11, Theorem 3.5 and 4.9] (see the explanations before Corollaries 4.5 and 5.2 in Section 4 and 5, respectively); while the existence result for the Nash equilibrium on general manifolds relaxes the geodesic convexity assumption made on $\{Q_i\}$ in [23, Theorem 1.1] to the weaker one that each Q_i is weakly convex. It is worthwhile to notice that the geodesic convexity assumption for $\{Q_i\}$ in [23] prevents its application to some special but important Riemannian manifolds, such as compact Stiefel manifolds $\text{St}(p, n)$ and Grassmann manifolds $\text{Grass}(p, n)$ $(p < n)$, in which there is no geodesic convex subset; see Remark 5.1 in Section 5. Moreover, to our best knowledge, the convex structure results on the solution set for mixed variational inequality problems and Nash equilibrium problems are new even in Hadamard manifold settings.

The paper is organized as follows. In the next section, we introduce some basic notions and notations on Riemannian manifolds, some properties about the (weakly) convex function and the results about the VIP in [28] which will be used in our approach. In section 3, we establish the existence and the uniqueness result of the solution and the convexity of the solution set of the EP on general Riemannian manifolds. Following these, the formulation of the proximal point algorithm for the equilibrium problem on general Riemnnannian manifolds is given and the convergence property about the algorithm is analyzed in section 4. The last section is devoted to the applications to the Nash equilibrium problem and the mixed variational inequality problem.

2. Notations and preliminary results.

2.1. Background of Riemannian manifolds. The notations used in the present paper are standard; and the readers are referred to some textbooks for more details, for example, [13, 39, 42].

Let M be a connected n-dimensional Riemannian manifold with the Levi-Civita connection ∇ on M. Let $x \in M$, and let $T_x M$ stand for the tangent space at x to M endowed with the scalar product $\langle,\rangle_x$ and the associated norm $\|.\|_x$, where the subscript x is sometimes

omitted. Thus the tangent bundle, denoted by TM, is defined by

$$TM := \{(x, v) : x \in M, v \in T_x M\}.$$

Fix $y \in M$, and let $\gamma : [0, 1] \to M$ be a piecewise smooth curve joining x to y. Then, the arc-length of γ is defined by $l(\gamma) := \int_0^1 \|\dot{\gamma}(t)\| dt$, while the Riemannian distance from x to y is defined by $\mathrm{d}(x, y) := \inf_\gamma l(\gamma)$, where the infimum is taken over all piecewise smooth curves $\gamma : [0, 1] \to M$ joining x to y. We use $\mathbf{B}(x, r)$ and $\overline{\mathbf{B}}(x, r)$ to denote, respectively, the open metric ball and the closed metric ball at x with radius r, that is,

$$\mathbf{B}(x, r) := \{y \in M : \mathrm{d}(x, y) < r\} \quad \text{and} \quad \overline{\mathbf{B}(x, r)} := \{y \in M : \mathrm{d}(x, y) \le r\}.$$

A vector field V is said to be parallel along γ if $\nabla_{\dot{\gamma}} V = 0$. In particular, for a smooth curve γ, if $\dot{\gamma}$ is parallel along itself, then γ is called a geodesic, that is, a smooth curve γ is a geodesic if an only if $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$. A geodesic $\gamma : [0, 1] \to M$ joining x to y is minimal if its arc-length equals its Riemannian distance between x and y. By the Hopf-Rinow theorem [13], if M is complete, then (M, d) is a complete metric space, and there is at least one minimal geodesic joining x to y. One of the important structures on M is the exponential map $\exp_x : T_x M \to M$, which is defined at $x \in M$ by $\exp_x v = \gamma_v(1, x)$ for each $v \in T_x M$, where $\gamma_v(\cdot, x)$ is the geodesic starting at x with velocity v. Then, $\exp_x tv = \gamma_v(t, x)$ for each real number t. Another useful tool is the parallel transport $P_{\gamma, \cdot, \cdot}$ on the tangent bundle TM along a geodesic γ, which is defined by

$$P_{\gamma, \gamma(b), \gamma(a)}(v) = V(\gamma(b)) \quad \text{for any } a, b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)} M,$$

where V is the unique vector field satisfying $V(\gamma(a)) = v$ and $\nabla_{\dot{\gamma}(t)} V = 0$ for all t. Then, for any $a, b \in \mathbb{R}$, $P_{\gamma, \gamma(b), \gamma(a)}$ is an isometry from $T_{\gamma(a)} M$ to $T_{\gamma(b)} M$. We will write $P_{y,x}$ instead of $P_{\gamma, y, x}$ in the case when γ is a minimal geodesic joining x to y and no confusion arises. The following lemma can be checked easily.

LEMMA 2.1. *Let $x_0 \in M$ and $\{x_k\} \subset M$ be such that $\lim_{k \to \infty} x_k = x_0$. Let $u_0, v_0 \in T_{x_0} M$ and let $\{u_k\}, \{v_k\}$ be sequences with each $u_k, v_k \in T_{x_k} M$ such that $u_k \to u_0$ and $v_k \to v_0$. Then*

$$\exp_{x_k} u_k \to \exp_{x_0} u_0 \quad \text{and} \quad \langle u_k, v_k \rangle \to \langle u_0, v_0 \rangle.$$

The following result is known in any textbook about Riemannain geometry, see e.g., [13, Corollary, p. 73] or [39, Exercise 5, p. 39].

LEMMA 2.2. *Let $\gamma : [a, b] \to M$ be a piecewise differentiable curve. If $l(\gamma) = \mathrm{d}(\gamma(a), \gamma(b))$, then γ is a geodesic joining $\gamma(a)$ and $\gamma(b)$.*

Consider a set $Q \subseteq M$ and $x, y \in Q$. The set of all geodesics $\gamma : [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$ satisfying $\gamma([0, 1]) \subseteq Q$ is denoted by Γ_{xy}^Q, that is

$$\Gamma_{xy}^Q := \{\gamma : [0, 1] \to Q : \ \gamma(0) = x, \ \gamma(1) = y \text{ and } \nabla_{\dot{\gamma}} \dot{\gamma} = 0\}.$$

In particular, we write Γ_{xy} for Γ_{xy}^M, and Γ_{xy} is nonempty for all $x, y \in M$ provided that M is complete. Furthermore, for a subset $\Gamma_0 \subseteq \Gamma_{xy}$, we use $\min - \Gamma_0$ to denote the subset of Γ_0

consisting of all minimal geodesics in Γ_0. Thus $\gamma_{xy} \in \min-\Gamma_0$ means that $\gamma_{xy} \in \Gamma_0$ and γ_{xy} is minimal.

Recall that a Hadamard manifold is a complete simply connected m-dimensional Riemannian manifold with nonpositive sectional curvatures. In a Hadamard manifold, the geodesic between any two points is unique and the exponential map at each point of M is a global diffeomorphism; see, e.g., [39, Theorem 4.1, p. 221]. Thus $\min-\Gamma_{xy}$ coincides with Γ_{xy} in a Hadamard manifold for any $x, y \in M$.

2.2. Convex analysis on Riemmanian manifolds. Definition 2.3 below presents the notions of the convexity for subsets in M, where item (e) is known in [23], and see e.g., [29, 43] for the others. As usual, we use $\overline{C}$ to stand for the closure of a subset $C \subseteq M$.

DEFINITION 2.3. *Let $Q \subseteq M$ be a nonempty set. The set Q is said to be*

(a) *weakly convex if, for any $x, y \in Q$, there is a minimal geodesic of M joining x to y and it is in Q;*

(b) *strongly convex if, for any $x, y \in Q$, the minimal geodesic in M joining x to y is unique and lies in Q;*

(c) *locally convex if, for any $x \in \overline{Q}$, there is a positive $\varepsilon > 0$ such that $Q \cap \mathbf{B}(x, \varepsilon)$ is strongly convex;*

(d) *r-convex if, for any $x, y \in Q$ with $\mathrm{d}(x, y) \leq r$, the minimal geodesic in M joining x to y is unique and lies in Q;*

(e) *geodesic convex if, for any $x, y \in Q$, the geodesic in M joining x to y is unique and lies in Q.*

REMARK 2.1. (a) *The following implications are obvious:*

geodesic convexity $\Rightarrow$ strong convexity $\Rightarrow$ r-convexity/weak convexity $\Rightarrow$ local convexity.

(b) *The intersection of a weakly convex set and a strongly convex set is strongly convex.*

(c) *All convexities (except the local convexity) in a Hadamard manifold coincide and are simply called the convexity.*

Recall that the convexity radius at x is defined by

$$(2.1) \qquad r_x := \sup \left\{ r > 0 : \begin{array}{l} \text{each ball in } \mathbf{B}(x, r) \text{ is strongly convex} \\ \text{and each geodesic in } \mathbf{B}(x, r) \text{ is minimal} \end{array} \right\}.$$

Then r_x is well defined and positive, and $r_x = +\infty$ for each $x \in M$ in the case when M is a Hadamard manifold. Moreover, for any compact subset $Q \subseteq M$, we have that

$$(2.2) \qquad r_Q := \inf\{r_x : x \in Q\} > 0;$$

see [39, Theorem 5.3, p. 169] or [29, Lemma 3.1].

Consider now an extended real-valued function $f : M \to \overline{\mathbb{R}} := (-\infty, \infty]$ and let $\mathcal{D}(f)$ denote its domain, that is, $\mathcal{D}(f) := \{x \in M : f(x) \neq \infty\}$. Write $\Gamma_{xy}^{f} := \Gamma_{xy}^{\mathcal{D}(f)}$ for simplicity, that is Γ_{xy}^{f} stands for the subset consisting of all $\gamma_{xy} \in \Gamma_{xy}$ such that $\gamma([0,1]]) \subseteq \mathcal{D}(f)$. In the following definition, we introduce the notions of the convexity for functions, where item (c) is known in [27, 28].

DEFINITION 2.4. *Let $f : M \to \overline{\mathbb{R}}$ be a proper function with a weakly convex domain $\mathcal{D}(f)$, and let $x \in \mathcal{D}(f)$. Then, f is said to be*

(a) *convex (resp. strictly convex) at x if, for any $y \in \mathcal{D}(f) \setminus \{x\}$ and any geodesic $\gamma_{xy} \in \Gamma_{xy}^{f}$ the composition $f \circ \gamma_{xy} : [0,1] \to \mathbb{R}$ is convex (resp. strictly convex) on $(0,1)$:*

$$(2.3) \qquad f \circ \gamma_{xy}(t) \leq (resp. \ <)(1-t)f(x) + tf(y) \quad \text{for all } t \in (0,1);$$

(b) *weakly convex (resp. weakly strictly convex) at x if, for any $y \in \mathcal{D}(f)$ there exists $\gamma_{xy} \in \min{-}\Gamma_{xy}^{f}$ such that (2.3) holds;*

(c) *weakly convex (resp. convex, strictly convex, weakly strictly convex) if so is it at each $x \in \mathcal{D}(f)$.*

Clearly, for a proper function f on M, the convexity implies the weak convexity, and the strict convexity implies the convexity.

Let $f : M \to \overline{\mathbb{R}}$ be proper and weakly convex at $x \in \mathcal{D}(f)$. The directional derivative in direction $u \in T_x M$ and the subdifferential of f at x are, respectively, defined by

$$f'(x; u) := \lim_{t \to 0^+} \frac{f(\exp_x tu) - f(x)}{t}$$

and

$$\partial f(x) := \{v \in T_x M : \langle v, u \rangle \leq f'(x; u) \text{ for any } u \in T_x M\}.$$

Then, by [27, Proposition 3.8(iii)], the following relationship holds between $\partial f(x)$ and $\mathrm{cl} f'(x; \cdot)$, the lower semi-continuous hull of $f'(x; \cdot)$:

$$(2.4) \qquad \mathrm{cl} f'(x; u) = \sup\{\langle u, v \rangle : v \in \partial f(x)\} \quad \text{for any } u \in T_x M.$$

LEMMA 2.5. *Let $f : M \to \overline{\mathbb{R}}$ be proper with a weakly convex domain $\mathcal{D}(f)$. Let $x \in \mathcal{D}(f)$ and $v \in T_x M$.*

(i) *If f is weakly convex (resp. weakly strictly convex) at x, then $v \in \partial f(x)$ if and only if, for some or any constant $r > 0$, and for any $y \in \mathcal{D}(f) \cap \mathbf{B}(x, r)$, there exists a geodesic $\gamma_{xy} \in \min{-}\Gamma_{xy}^{f}$ such that*

$$(2.5) \qquad f(y) \geq (resp. \ >)f(x) + \langle v, \dot{\gamma}_{xy}(0) \rangle.$$

(ii) *If f is convex (resp. strictly convex) at x, then $v \in \partial f(x)$ if and only if, for some or any constant $r > 0$, the inequality (2.5) holds for any $y \in \mathcal{D}(f) \cap \mathbf{B}(x, r)$ and any $\gamma_{xy} \in \min{-}\Gamma_{xy}^{f}$.*

Proof. We only prove assertion (i) (as the proof for assertion (ii) is similar). To do this, suppose that f is weakly convex (resp. weakly strictly convex) at x. It suffices to verify that the following statements are equivalent:

(a) $v \in \partial f(x)$.
(b) For any $r > 0$ and any $y \in \mathcal{D}(f) \cap \mathbf{B}(x,r)$, there exists a geodesic $\gamma_{xy} \in \min{-}\Gamma_{xy}^f$ such that (2.5) holds.
(c) There is some $r > 0$ such that for any $y \in \mathcal{D}(f) \cap \mathbf{B}(x,r)$, there exists a geodesic $\gamma_{xy} \in \min{-}\Gamma_{xy}^f$ such that (2.5) holds.

We shall complete the proof by showing the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). To do this, assume (a). Then $v \in \partial f(x)$, and by definition, we have that

$$(2.6) \qquad \langle v, u \rangle \le f'(x; u) \quad \text{for any } u \in T_x M.$$

Let $r > 0$ and $y \in \mathcal{D}(f) \cap \mathbf{B}(x,r)$ be arbitrary. Noting that f is weak convex (resp. weakly strictly convex) at x, there exists $\gamma_{xy} \in \min{-}\Gamma_{xy}^f$ such that the composite $f \circ \gamma_{xy} : [0,1] \to \mathbb{R}$ is convex (resp. strictly convex) on $[0,1]$. Therefore,

$$f'(x; \dot\gamma_{xy}(0)) = \inf_{t>0} \frac{f(\exp_x t\dot\gamma_{xy}(0)) - f(x)}{t} \le (resp. <) f(y) - f(x).$$

This, together with (2.6), yields that

$$\langle v, \dot\gamma_{xy}(0) \rangle \le f'(x; \dot\gamma_{xy}(0)) \le (resp. <) f(y) - f(x).$$

Hence (b) holds, and the implication (a) $\Rightarrow$ (b) is checked. Noting that the implication (b) $\Rightarrow$ (c) is evident, it remains to show the implication (c) $\Rightarrow$ (a). To this end, assume (c). Then one can choose $r > 0$ and $\gamma_{xy} \in \min{-}\Gamma_{xy}^f$ for any $y \in \mathcal{D}(f) \cap \mathbf{B}(x,r)$ such that (2.5) holds. Without loss of generality, one could assume that $r \le r_x$. Let $u \in T_x M \setminus \{0\}$ be arbitrary, and set $s_0 := \frac{r}{\|u\|}$. Then, for any $s \in (0, s_0)$, $y(s) := \exp_x(su) \in \mathbf{B}(x,r) \subseteq \mathbf{B}(x, r_x)$. It follows that the geodesic $\gamma_{xy(s)}$ joining x and $y(s)$ is unique. Therefore, $\gamma_{xy(s)}$ is of the form:

$$\gamma_{xy(s)}(t) = \exp_x(t(su)) \quad \text{for each } t \in [0,1],$$

and if $y(s) \in \mathcal{D}(f)$, then (2.5) holds with $y(s)$ in place of y:

$$\langle v, su \rangle = \langle v, \dot\gamma_{xy(s)}(0) \rangle \le f(y(s)) - f(x) = f(\exp_x(su)) - f(x) \quad \text{for any } s \in (0, s_0).$$

Note that the above inequality holds trivially if $y(s) \notin \mathcal{D}(f)$. Then, by definition, we get that $\langle v, u \rangle \le f'(x; u)$, and so (a) holds as $u \in T_x M$ is arbitrary. Thus, the implication (c) $\Rightarrow$ (a) is shown and the proof is complete. $\square$

Fix $\bar{x} \in \mathcal{D}(f)$ and recall that f is center Lipschitz continuous at $\bar{x}$ if there exits a neighborhood U of $\bar{x}$ and a constant L such that

$$|f(x) - f(\bar{x})| \le L \mathrm{d}(x, \bar{x}) \quad \text{for any } x \in U.$$

The center Lipschitz constant $L_{\bar{x}}^f$ at $\bar{x}$ is defined to be the minimum of all L such that above inequality holds for some neighborhood U of $\bar{x}$. Then it is clear that

$$L_{\bar{x}}^f = \lim_{\delta \to 0+} \sup \left\{ \frac{|f(x) - f(\bar{x})|}{\mathrm{d}(x, \bar{x})} : 0 < \mathrm{d}(x, \bar{x}) \le \delta \right\}.$$

The following properties about the subdifferential of a (weakly) convex function can be found in [28, Proposition 6.2] except (2.7) by definition.

LEMMA 2.6. *Let $f : M \to \overline{\mathbb{R}}$ be a proper function. Then the following assertions hold:*

(i) *If is weakly convex , then f is continuous on $\mathrm{int}\mathcal{D}(f)$.*

(ii) *If $\bar{x} \in \mathrm{int}\mathcal{D}(f)$ and f is weakly convex at $\bar{x}$, then $\partial f(\bar{x})$ is a nonempty, compact and convex set satisfying*

$$(2.7) \qquad\qquad \|v\| \le L_{\bar{x}}^f \quad \text{for any } v \in \partial f(\bar{x}).$$

The following lemma, which provides some sufficient conditions ensuring the sum rule of subdifferential, was proved in [27, Proposition 4.3].

LEMMA 2.7. *Let $f, g : M \to \overline{\mathbb{R}}$ be proper functions such that f, g and $f + g$ are weakly convex at $x \in \mathrm{int}\mathcal{D}(f) \cap \mathcal{D}(g)$. Then the following sum rule for the subdifferential holds:*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x).$$

2.3. VIP: existence and convexity properties of solution sets. Let $Q \subseteq M$ be a nonempty subset and let $A : Q \rightrightarrows TM$ be a set-valued vector field defined on Q, that is, $A(x) \subseteq T_x M$ is nonempty for each $x \in Q$. Consider the following variational inequality problem (VIP for short) associated to the pair (A, Q): To find a point $\bar{x} \in Q$ such that

$$(2.8) \qquad \exists \bar{v} \in A(\bar{x}) \text{ s.t. } \langle \bar{v}, \dot{\gamma}_{\bar{x}y}(0) \rangle \ge 0 \text{ for any } y \in Q \text{ and } \gamma_{\bar{x}y} \in \Gamma_{\bar{x}y}^Q.$$

Any point $\bar{x} \in Q$ satisfying (2.8) is called a solution of VIP, and the set of all solutions is denoted by $\mathrm{VIP}(A, Q)$.

Variational inequality problem (2.8) was first introduced in [49], for single-valued vector fields on Hadamard manifolds, and extended respectively in [29] and [28] for single-valued vector fields and multivalued vector fields on general Riemannian manifolds. As we have mentioned previously, our approach to solve the EP is founded strongly on some existence results about the VIP, which are taken from [28]. For this purpose, we recall some notions in the following definition; see, e.g, [26, 28].

DEFINITION 2.8. *Let $Q \subseteq M$ be a subset and $A : Q \rightrightarrows TM$ be a set-valued vector field on Q. A is said to be*

(a) *upper semi-continuous (usc for short) at x_0, if, for any open set U satisfying $A(x_0) \subseteq U \subseteq T_{x_0}M$, there exists an open neighborhood $U(x_0)$ of x_0 such that $P_{x_0,x}A(x) \subseteq U$ for any $x \in U(x_0) \cap Q$;*

(b) *upper Kuratowski semi-continuous (uKsc for short) at x_0 if, for any sequences $\{x_k\} \subset Q$ and $\{u_k\} \subset TM$ with each $u_k \in A(x_k)$, relations $\lim_{k\to\infty} x_k = x_0 \in Q$ and $\lim_{k\to\infty} u_k = u_0$ imply $u_0 \in A(x_0)$;*

(c) *usc (resp., uKsc) on Q if it is usc (resp., uKsc) at each $x \in Q$.*

By definition, it is evident that upper semi-continuity implies upper Kuratowski semi-continuity. In the following example, we provide two set-valued vector fields which are useful for our study in next sections of the present paper.

EXAMPLE 2.1. *Fix a point* $y \in M$, *and define vector fields* $\mathrm{Exp}_{(\cdot)}^{-1} y :\rightrightarrows TM$ *and* $\exp_{(\cdot)}^{-1} y :\rightrightarrows TM$ *respectively by*

$$\mathrm{Exp}_x^{-1} y := \{u \in T_x M : \exp_x u = y\} \quad \text{for each } x \in M,$$

and

(2.9) $$\exp_x^{-1} y := \{u \in \mathrm{Exp}_x^{-1} y : \|u\| = \mathrm{d}(x,y)\} \quad \text{for each } x \in M.$$

Then one can check easily by definition and Lemma 2.1 that $\mathrm{Exp}_{(\cdot)}^{-1} y$ *is uKsc on* M *and* $\exp_{(\cdot)}^{-1} y$ *is usc on* M.

Recall from [28] that a point $o \in Q$ is called a weak pole of Q if for each $x \in Q$, $\min{-}\Gamma_{ox}$ is a singleton and $\min{-}\Gamma_{ox} \subseteq Q$. Clearly, any subset with a weak pole is connected. The notions of the monotonicity in the following definition are well known; see for example [11, 28].

DEFINITION 2.9. *Let* $Q \subseteq M$ *be a subset and* $A : Q \rightrightarrows TM$ *be a set-valued vector field. The vector field* A *is said to be*

(a) *monotone on* Q *if, for any* $x, y \in Q$ *and* $\gamma_{xy} \in \Gamma_{xy}^Q$ *the following inequality holds:*

$$\langle v_x, \dot{\gamma}_{xy}(0) \rangle - \langle v_y, \dot{\gamma}_{xy}(1) \rangle \leq 0 \quad \text{for any } v_x \in A(x), \ v_y \in A(y);$$

(b) *strictly monotone on* Q *if it is monotone and, for any* $x, y \in Q$ *with* $x \neq y$ *and* $\gamma_{xy} \in \Gamma_{xy}^Q$ *the following inequality holds:*

$$\langle v_x, \dot{\gamma}_{xy}(0) \rangle - \langle v_y, \dot{\gamma}_{xy}(1) \rangle < 0 \quad \text{for any } v_x \in A(x), \ v_y \in A(y).$$

Let $Q \subseteq M$ be a closed connected and locally convex set. By [39, p. 170], there exists a connected (embedded) k-dimensional totally geodesic sub-manifold N of M such that $Q = \overline{N}$. Following [28], the set $\mathrm{int}_R Q := N$ is called the relative interior of Q. Moreover, as in [28], we say that a closed locally convex set Q has the BCC (bounded convex cover) property if there exists $o \in Q$ such that, for any $R \geq 0$, there exists a weakly convex compact subset of M containing $Q \cap \mathbf{B}(o, R)$.

REMARK 2.2. *We remark that the notion of the BCC property defined above is a litter stronger than that defined in [28, Definition 3.9], where it is required that the compact subset containing* $Q \cap \mathbf{B}(o, R)$ *is "locally convex" rather than "weakly convex". From its proof, one sees that the BCC property assumption defined in [28, Definition 3.9] seems insufficient for [28, Theorem 3.10], while the stronger version of the BCC property defined here is sufficient.*

For the remainder, we use $\mathcal{V}(Q)$ to denote the set of all uKsc set-valued vector fields A such that $A(x)$ is compact and convex for each $x \in Q$.

Proposition 2.10 below extends the corresponding existence result in [28, Theorem 3.10] (see the explanation made in Remark 2.3). The proof of Proposition 2.10 is similar to that for [28, Theorem 3.10], and is kept here for completeness.

PROPOSITION 2.10. *Let $Q \subseteq M$ be a closed locally convex subset with a weak pole $o \in \mathrm{int}_R Q$ and $A \in \mathcal{V}(Q)$. Then $\mathrm{VIP}(A, Q) \neq \emptyset$ provided one of the following assumptions holds:*

(a) *Q is compact;*

(b) *Q has the BCC property and there exists a compact subset $L \subseteq M$ such that*

$$(2.10) \qquad x \in Q \setminus L \Rightarrow [\forall v \in A(x), \exists y \in Q \cap L,\, \gamma_{xy} \in \min{-}\Gamma_{xy}^Q\ s.t.\ \langle v, \dot{\gamma}_{xy}(0) \rangle < 0].$$

Proof. It was known in [28, Theorem 3.6] in the case when Q is compact. Below we assume that assumption (b) holds. Then, there exists a compact subset L such that (2.10) holds. Then there exist $R > 0$ and a weakly convex and compact subset K_R of M such that $L \subset \mathbf{B}(o, R)$ and $Q \cap \mathbf{B}(o, R) \subseteq K_R$. Write $Q_R := Q \cap \overline{\mathbf{B}(o, R)}$, and $\hat{Q}_R := Q \cap K_R$ for saving the print space. Then

$$Q \cap L \subseteq Q \cap \mathbf{B}(o, R) \subseteq Q_R \subseteq \hat{Q}_R.$$

Thus, by (2.10), one checks that

$$(2.11) \qquad\qquad \mathrm{VIP}(A, \hat{Q}_R) \subseteq \mathrm{VIP}(A, Q_R) \subseteq Q \cap L \subseteq \mathbf{B}(o, R).$$

Moreover, since $o \in \mathrm{int}_R Q$ is a weak pole of Q (and so the minimal geodesic γ_{ox} joining o to x is unique) and K_R is weakly convex, one can check by definition that o is a weak pole of $\hat{Q}_R$ and $o \in \mathrm{int}_R \hat{Q}_R$ (noting that $Q \cap \mathbf{B}(o, R) \subseteq K_R$). Thus [28, Theorem 3.6] is applied (with $\hat{Q}_R$ in place of A) to get that $\mathrm{VIP}(A, \hat{Q}_R) \neq \emptyset$. In view of (2.11), $\emptyset \neq \mathrm{VIP}(A, Q_R) \subseteq \mathbf{B}(o, R)$, and it follows from [28, Proposition 3.2] that

$$\mathrm{VIP}(A, Q_R) = \mathrm{VIP}(A, Q_R) \cap \mathbf{B}(o, R) \subseteq \mathrm{VIP}(A, Q),$$

and so $\mathrm{VIP}(A, Q) \neq \emptyset$, completing the proof. $\square$

REMARK 2.3. *Let $Q \subseteq M$ be a locally convex subset with a weak pole $o \in \mathrm{int}_R Q$. Recall from [28] that the vector field A satisfies the coerciveness condition on Q if*

$$\sup_{v_o \in A(o), v_x \in A(x)} \frac{\langle v_x, \dot{\gamma}_{xo}(0) \rangle - \langle v_o, \dot{\gamma}_{xo}(1) \rangle}{\mathrm{d}(o, x)} \to -\infty \quad as\ \mathrm{d}(o, x) \to +\infty\ for\ x \in Q.$$

Then one checks directly by definition that the coerciveness condition for A implies that there exists a compact subset $L \subseteq M$ such that (2.10) in (b) of Proposition 2.10 holds (noting that $A(o)$ is compact). However, the converse is not true, in general, even in the Euclidean space setting. To see this, one may consider the simple mapping A on $Q := \mathbb{R}$ defined by $A(x) := [-1, 1]$ if $x = 0$ and $A(x) := \mathrm{sign}(x)$ otherwise. Thus Proposition 2.10 is an extension of the corresponding existence result in [28, Theorem 3.10].

As usual, we set $D_\kappa := \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$ and $D_\kappa := +\infty$ if $\kappa \leq 0$ (see e.g., [28, 39]). The following proposition lists some results on the structure of the solution set $\mathrm{VIP}(A, Q)$, which are known in [28, Theorems 3.13, 4.6 and 4.8], respectively.

PROPOSITION 2.11. *Suppose that $A \in \mathcal{V}(Q)$ is monotone on $Q \subseteq M$ and $\mathrm{VIP}(A, Q) \neq \emptyset$. Then the following assertion holds:*

(i) *If Q is a locally convex subset, then the solution set $\mathrm{VIP}(A, Q)$ is locally convex.*

(ii) *If A is strictly monotone on Q, then $\mathrm{VIP}(A, Q)$ is a singleton.*

(iii) *If M is of the sectional curvatures bounded from above by some $\kappa \in [0, +\infty)$ and Q is a D_κ-convex subset, then the solution set $\mathrm{VIP}(A, Q)$ is D_κ-convex. .*

3. Equilibrium problem. Throughout the whole section, we always assume that

• $Q \subseteq M$ is a nonempty closed and locally convex subset;

• $F : M \times M \to \overline{\mathbb{R}}$ is a proper bifunction with $0 \leq F(x, x) < +\infty$ for any $x \in Q$. The domain $\mathcal{D}(F)$ of F is defined by

$$\mathcal{D}(F) := \{(x, y) \in M \times M : -\infty < F(x, y) < +\infty\}.$$

Recall that the EP associated to the pair (F, Q) is to find a point $\bar{x} \in Q$ such that $F(\bar{x}, y) \geq 0$ for any $y \in Q$. Any point $\bar{x} \in Q$ satisfying (1.1) is called a solution of EP (1.1), and the set of all solutions is denoted by $\mathrm{EP}(F, Q)$.

3.1. Properties of bifunctions. In the following definition we introduce some monotonicity and convexity notions for bifuctions on Riemannian manifolds. In particular, the corresponding ones of items (a) and (b) in linear spaces are refereed to, for example, [10, 22]; while item (c) as far as we know are new and plays a key role in our study in the present paper.

DEFINITION 3.1. *The bifunction F is said to be*

(a) *monotone on $Q \times Q$ if $F(x, y) + F(y, x) \leq 0$ for any $(x, y) \in Q \times Q$;*

(b) *strictly monotone on $Q \times Q$ if $F(x, y) + F(y, x) < 0$ for any $(x, y) \in Q \times Q$ with $x \neq y$ and*

(3.1) $$F(x, x) = 0 \quad \text{for any } x \in Q;$$

(c) *point-wise weakly convex (resp. point-wise convex) on Q if, for any $x \in Q$, the function $F(x, \cdot) : M \to \overline{\mathbb{R}}$ is weakly convex (resp. convex) at x.*

Note that if F is monotone on $Q \times Q$, then (3.1) holds (as $F(x, x) \geq 0$ for any $x \in Q$ by assumption).

Let $V : Q \rightrightarrows TM$ be a vector field. Associated to V, we define the bifunction $G_V : M \times M \to \overline{\mathbb{R}}$ by

(3.2) $$G_V(x, y) := \sup_{u \in V(x), v \in \exp_x^{-1} y} \langle u, v \rangle \quad \text{for any } (x, y) \in M \times M,$$

where for any $(x,y) \in M \times M$, $\exp_x^{-1} y$ is defined by (2.9), and we adopt the the convention that $\sup \emptyset = +\infty$. Proposition 3.2 below provides some properties of the bifunctions G_V that will be used in the sequel. As usual, for a subset Z of $T_x M$, we use $\overline{\mathrm{co}}Z$ to denotes the closed and convex hull of the set Z in $T_x M$.

PROPOSITION 3.2. *Suppose $V(x) \subseteq T_x M$ is nonempty for each $x \in Q$, and let G_V be defined by (3.2). Then the following assertions hold:*

(i) *If $V(x)$ is compact-valued, then*

$$(3.3) \qquad \mathcal{D}(G_V) = Q \times M \quad and \quad G_V(x,x) = 0 \quad for\ any\ x \in Q.$$

(ii) *$G_V(x,\cdot) \circ \gamma_{xy}$ is convex on $[0,1]$ for any x, $y \in Q$ and any geodesic $\gamma_{xy} \in \min-\Gamma_{xy}$.*

(iii) *If $G : Q \times Q \to \overline{\mathbb{R}}$ is point-wise weakly convex on Q, then so is $G_V + G$.*

(iv) *$\partial G_V(x,\cdot)(x) = \overline{\mathrm{co}}V(x)$ for any $x \in Q$.*

(v) *If $V(x)$ is compact-valued and V usc on Q, then the function $x \mapsto G_V(x,y)$ is usc on Q for each $y \in Q$.*

Proof. Assertion (i) is clear by definition. To show assertion (ii), fix $x, y \in Q$ and let $\gamma_{xy} \in \min-\Gamma_{xy}$ and $y_t := \gamma_{xy}(t)$. Then we have that

$$(3.4) \qquad \exp_x^{-1} y_t \subseteq t\,\exp_x^{-1} y \quad for\ each\ t \in (0,1).$$

Indeed, let $v_t \in \exp_x^{-1} y_t$ with some $t \in (0,1)$. Then, $\|v_t\| = \mathrm{d}(x, y_t) = t\mathrm{d}(x,y)$ and $\exp_x v_t = y_t$. Define a curve $\beta : [0,1] \to M$ by

$$\beta(s) := \begin{cases} \exp_x \frac{s}{t} v_t, & s \in [0,t], \\ \gamma_{xy}(s), & s \in (t,1]. \end{cases}$$

Then $l(\beta) = \|v_t\| + \mathrm{d}(y_t, y) = \mathrm{d}(x,y)$. This means that $\beta \in \min-\Gamma_{xy}$ thanks to Lemma 2.2. Therefore, $\frac{1}{t}v_t \in \exp_x^{-1} y$ by definition because $\dot\beta(0) = \frac{1}{t}v_t$; hence (3.4) holds. Thus

$$G_V(x, \gamma_{xy}(t)) = \sup_{u \in V(x), v_t \in \exp_x^{-1} y_t} \langle u, v_t \rangle \leq t \sup_{u \in V(x), v \in \exp_x^{-1} y} \langle u, v \rangle = tG(x,y).$$

This shows that $G_V(x, \gamma_{xy}(\cdot))$ is convex on $[0,1]$ (noting that $G_V(x,x) = 0$), and assertion (ii) is shown as $\gamma_{xy} \in \min-\Gamma_{xy}$ is arbitrary.

Assertion (iii) follows immediately from assertion (ii). Now, we verify assertion (iv). To proceed, let $x \in Q$ and $\xi \in T_x M$. Then for any $t > 0$ small enough, one has that $\exp_x^{-1} \exp_x t\xi = \{t\xi\}$. Thus noting that $G_V(x,x) = 0$, we have by definition that

$$(3.5) \qquad G_V(x,\cdot)'(x;\xi) = \lim_{t \to 0^+} \frac{\sup_{u \in V(x)} \langle u, t\xi \rangle}{t} = \sup_{u \in V(x)} \langle u, \xi \rangle = \sup_{u \in \overline{\mathrm{co}}V(x)} \langle u, \xi \rangle.$$

This, together with (2.4), implies that

$$\sup_{u \in \overline{\mathrm{co}}V(x)} \langle u, \xi \rangle \geq \sup_{u \in \partial G_V(x,\cdot)(x)} \langle u, \xi \rangle \quad for\ any\ \xi \in T_x M,$$

and so $\partial G_V(x, \cdot)(x) \subseteq \overline{\mathrm{co}} V(x)$ by [37, Corollary 13.1.1, p.113]. Moreover, by (3.5), one have by definition that

$$\partial G_V(x, \cdot)(x) = \partial G_V(x, \cdot)'(x; 0) \supseteq \overline{\mathrm{co}} V(x).$$

Thus assertion (iv) is shown.

It remains to show assertion (v). To this end, fix $y \in Q$. Let $\varepsilon > 0$, $x \in Q$ and let $\{x_n\} \subseteq Q$ be such that $\lim_{n \to \infty} x_n = x$. Since V and $\exp_{(\cdot)}^{-1} y$ are usc at x (see Lemma 2.1), there is $K \in \mathbb{N}$ such that

(3.6) $P_{x,x_n} V(x_n) \subseteq \mathbf{B}(V(x), \varepsilon)$ and $P_{x,x_n} \exp_{x_n}^{-1} y \subseteq \mathbf{B}(\exp_x^{-1} y, \varepsilon)$ for each $n \geq K$.

Set $R := \max\{|V(x)|, \mathrm{d}(x,y)\}$ (where $|V(x)| := \max_{v \in V(x)}\{\|v\|\} < +\infty$ as $V(x)$ is compact), and, without loss of generality, assume that $\varepsilon < R$. Then, it follows from (3.6) that, for any $n > K$,

$$\sup_{v_n \in V(x_n), u_n \in \exp_{x_n}^{-1} y} \langle v_n, u_n \rangle \leq \sup_{v \in \mathbf{B}(V(x), \varepsilon), u \in \mathbf{B}(\exp_x^{-1} y, \varepsilon)} \langle v, u \rangle \leq \sup_{v \in V(x), u \in \exp_x^{-1} y} \langle v, u \rangle + 3\varepsilon R,$$

and so $\overline{\lim}_{n \to \infty} G_V(x_n, y) \leq G_V(x, y) + 3R\varepsilon$. Thus, assertion (v) holds as $\varepsilon > 0$ is arbitrary and the proof is complete. $\square$

3.2. Relationship between VIP and EP. For the remainder of the paper, we will make use of the following hypotheses for the bifunction F, where, as usual, we use $\delta_C(\cdot)$ to denote the indicator function of the nonempty subset C defined by $\delta_C(x) := 0$ if $x \in C$ and $+\infty$ otherwise:

(H1) F is point-wise weakly convex on Q and, $x \in \mathrm{int} \mathcal{D}(F(x, \cdot))$ for each $x \in Q$.

(H2) $F + \delta_{Q \times Q}$ is point-wise weakly convex on Q.

(H3) For any $y \in Q$, the function $x \mapsto F(x, y)$ is usc on Q.

(H4) The function $x \mapsto F(x, x)$ is lower semi-continuous (lsc for short) on Q.

REMARK 3.1. *We remark that the latter part of hypothesis* (H1) *is particularly satisfied if $Q \times Q \subseteq \mathrm{int} \mathcal{D}(F)$. The first part of hypothesis* (H1) *and hypothesis* (H2) *are satisfied in the case when Q is weakly convex and $F(x, \cdot)$ is (weakly) convex for any $x \in Q$, which, together with Hypothesis* (H3) *are standard assumption for the* EP *(see, e.g, [10, 11, 12, 20, 47]); while hypothesis* (H4) *is particularly satisfied if $F(x, x) = 0$ for any $x \in Q$ (which was used in [10, 12, 20]).*

Note that, by definition, the following implication holds:

(3.7)   (H2)$\Longrightarrow Q$ is weakly convex.

Associated to the pair (F, Q), we define the set-valued vector field $A_F : Q \rightrightarrows TM$ by

(3.8) $A_F(x) := \partial F(x, \cdot)(x)$ for any $x \in Q$.

Then the following proposition is clear from Lemma 2.6 (ii).

PROPOSITION 3.3. *Suppose that F satisfies (H1). Then, the set-valued vector field A_F is well-defined, compact convex-valued on Q and satisfies*

$$\max_{v \in A_F(x)} \|v\| \le L_x^F \quad \text{for each } x \in Q,$$

where L_x^F stands for the center Lipschitz constant of $F(x, \cdot)$ at x.

The following proposition establishes the relationship between the EP associated to the pair (F, Q) and the VIP associated to the pair (A_F, Q).

PROPOSITION 3.4. *Suppose that F satisfies* (H1) *and* (H2). *Then*

(3.9) $$\text{VIP}(A_F, Q) \subseteq \text{EP}(F, Q),$$

and the equality holds if (3.1) *is additionally assumed.*

Proof. Let $\bar{x} \in Q$ and note that F satisfies (H1) and (H2). Then, by implication (3.7), Q is weakly convex, and then the same argument for proving [28, Proposition 6.4] (with $F(\bar{x}, \cdot)$, Q in place of f, A there) works for the following equivalence:

(3.10) $$\bar{x} \in \text{VIP}(A_F, Q) \iff [F(\bar{x}, y) \ge F(\bar{x}, \bar{x}) \text{ for any } y \in Q].$$

Thus (3.9) follows from the assumption that $F(\bar{x}, \bar{x}) \ge 0$; while the converse inclusion of (3.9) holds trivially by (3.10) if (3.1) is additionally assumed. The proof is complete. $\square$

PROPOSITION 3.5. *Suppose that F satisfies* (H1). *Then the following assertions hold:*

(i) *If F is monotone (resp. strictly monotone) on $Q \times Q$, then so is A_F on Q.*

(ii) *If F satisfies* (H3) *and* (H4)*, then A_F is uKsc on Q; hence $A_F \in \mathcal{V}(Q)$.*

Proof. (i). Suppose that F is monotone on $Q \times Q$. Let $x, y \in Q$, $u_x \in A_F(x)$, $u_y \in A_F(y)$ and let $\gamma_{xy} \in \Gamma_{xy}^Q$. We have to show

(3.11) $$\langle u_x, \dot{\gamma}_{xy}(0) \rangle - \langle u_y, \dot{\gamma}_{xy}(1) \rangle \le 0.$$

To do this, subdivide γ_{xy} into n subsegments with the equal length determined by the consecutive points

$$x = x_0 < x_1 < \ldots < x_{n-1} < x_n = y$$

such that

$$d(x_{i-1}, x_i) = \frac{l(\gamma_{xy})}{n} \le \bar{r}. \quad i = 1, 2, \ldots, n,$$

where $\bar{r} := \min\{r_z : z \in \gamma_{xy}[0, 1]\} > 0$ by (2.2). Thus, for each $i = 1, 2, \ldots, n$, $\exp_{x_{i-1}}^{-1} x_i$ is a singleton, and

(3.12) $$\min-\Gamma_{x_{i-1}x_i} = \{\gamma_{x_{i-1}x_i}\} \quad \text{with} \quad \gamma_{x_{i-1}x_i}(\cdot) := \exp_{x_{i-1}}(\cdot)(\exp_{x_{i-1}}^{-1} x_i).$$

Moreover, we have that

(3.13) $$\exp_{x_0}^{-1} x_1 = \frac{1}{n}\dot{\gamma}_{xy}(0), \quad \exp_{x_n}^{-1} x_{n-1} = -\frac{1}{n}\dot{\gamma}_{xy}(1),$$

and

$$(3.14) \qquad \exp_{x_i}^{-1} x_{i+1} + \exp_{x_i}^{-1} x_{i-1} = 0 \quad \text{for each } i = 1, 2, \ldots, n-1.$$

To proceed, set $u_0 := u_x$, $u_n := u_y$ and take $u_i \in A_F(x_i)$ for each $i = 1, 2, \ldots, n-1$. Now fix $i = 1, 2, \ldots, n$. Then, by assumption (H1), Lemma 2.5 (i) is applicable, and thus, thanks to (3.12), we have that

$$F(x_{i-1}, x_i) \geq \langle u_{i-1}, \exp_{x_{i-1}}^{-1} x_i \rangle \quad \text{and} \quad F(x_i, x_{i-1}) \geq \langle u_i, \exp_{x_i}^{-1} x_{i-1} \rangle,$$

as $F(x_i, x_i) \geq 0$. This, together with the monotonicity of F, implies that $\langle u_{i-1}, \exp_{x_{i-1}}^{-1} x_i \rangle + \langle u_i, \exp_{x_i}^{-1} x_{i-1} \rangle \leq 0$; hence,

$$(3.15) \qquad \sum_{i=1}^{n} \left(\langle u_{i-1}, \exp_{x_{i-1}}^{-1} x_i \rangle + \langle u_i, \exp_{x_i}^{-1} x_{i-1} \rangle \right) \leq 0.$$

Since by (3.14), $\langle u_i, \exp_{x_i}^{-1} x_{i-1} \rangle + \langle u_i, \exp_{x_i}^{-1} x_{i+1} \rangle) = 0$ for each $i = 1, 2, \ldots, n$, and since

$$\langle u_0, \exp_{x_0}^{-1} x_1 \rangle + \sum_{i=1}^{n-1} \left(\langle u_i, \exp_{x_i}^{-1} x_{i-1} \rangle + \langle u_i, \exp_{x_i}^{-1} x_{i+1} \rangle \right) + \langle u_n, \exp_{x_n}^{-1} x_{n-1} \rangle$$
$$= \sum_{i=1}^{n} \left(\langle u_{i-1}, \exp_{x_{i-1}}^{-1} x_i \rangle + \langle u_i, \exp_{x_i}^{-1} x_{i-1} \rangle \right),$$

it follows from (3.15) that $\langle u_0, \exp_{x_0}^{-1} x_1 \rangle + \langle u_n, \exp_{x_n}^{-1} x_{n-1} \rangle \leq 0$. Thus (3.11) is seen to hold by (3.13), and the proof for assertion (i) is complete.

(ii). Let $x_0 \in Q$ and let $\{x_k\} \subset Q$, $\{u_k\} \subset TM$ with each $u_k \in A_F(x_k)$ such that

$$(3.16) \qquad \lim_{k \to \infty} x_k = x_0 \quad \text{and} \quad \lim_{k \to \infty} P_{x_0, x_k} u_k = u_0.$$

It suffices to show $u_0 \in A_F(x_0)$. To do this, write $r_{\mathbf{B}} := r_{\mathbf{B}(x_0, r_{x_0})} > 0$ (see (2.2)). Without loss of generality, we may assume that $x_k \in \mathbf{B}(x_0, \frac{r_{\mathbf{B}}}{2})$ for all k. Let $y \in \mathbf{B}(x_0, \frac{r_{\mathbf{B}}}{2})$. Then, for each k,

$$d(x_k, y) \leq d(x_0, y) + d(x_0, x_k) \leq r_{\mathbf{B}},$$

and so $\exp_{x_k}^{-1} y$ is a singleton. It immediately follows from (3.16) and Lemma 2.1 that

$$\lim_{k \to \infty} \langle u_k, \exp_{x_k}^{-1} y \rangle = \langle u_0, \exp_{x_0}^{-1} y \rangle.$$

Now, suppose hypotheses (H3) and (H4) are satisfied. Then,

$$(3.17) \qquad F(x_0, y) \geq \overline{\lim}_{k \to \infty} F(x_k, y) \quad \text{and} \quad F(x_0, x_0) \leq \underline{\lim}_{k \to \infty} F(x_k, x_k).$$

Recalling that each $u_k \in A_F(x_k) = \partial F(x_k, \cdot)(x_k)$, we get by (H1) that

$$F(x_k, y) \geq F(x_k, x_k) + \langle u_k, \exp_{x_k}^{-1} y \rangle \quad \text{for each } k;$$

hence

$$\overline{\lim}_{k \to \infty} F(x_k, y) \geq \underline{\lim}_{k \to \infty} (F(x_k, x_k) + \lim_{k \to \infty} \langle u_k, \exp_{x_k}^{-1} y \rangle)$$
$$= \underline{\lim}_{k \to \infty} F(x_k, x_k) + \langle u_0, \exp_{x_0}^{-1} y \rangle.$$

This, together with (3.17), yields that

$$F(x_0, y) \geq F(x_0, x_0) + \langle u_0, \exp_{x_0}^{-1} y \rangle.$$

Thus, Lemma 2.5 is applicable to concluding that $u_0 \in \partial F(x_0, \cdot)(x_0) = A_F(x_0)$ as $y \in \mathbf{B}(x_0, \frac{r_\mathbf{B}}{2})$ is arbitrary, and the upper Kuratowski semi-continuity of A_F is proved. Furthermore, by Proposition 3.3, $A_F(x)$ is nonempty, compact and convex for each $x \in Q$. Hence $A_F \in \mathcal{V}(Q)$. Thus the proof is complete. $\square$

3.3. Existence and convexity properties of the solution set. Let $F : M \times M \to \overline{\mathbb{R}}$ and $Q \subseteq M$ satisfy the conditions assumed at the beginning of the present section. We have the following existence result on the solution of EP associated to the pair (F, Q).

THEOREM 3.6. *Suppose that Q contains a weak pole $o \in \mathrm{int}_R Q$ and that F satisfies* (H1)-(H4). *Then* $\mathrm{EP}(F, Q) \neq \emptyset$ *provided that Q is compact, or assumptions* (b) *in Proposition 2.10 is satisfied with A_F in place of A.*

Proof. By hypotheses (H1) and (H2), we see from Proposition 3.4 that

(3.18) $$\mathrm{VIP}(A_F, Q) \subseteq \mathrm{EP}(F, Q).$$

Moreover, by hypotheses (H3) and (H4), we get by Proposition 3.5 (ii) that $A_F \in \mathcal{V}(Q)$. Thus, by assumption, Proposition 2.10 is applicable to getting that

$$\mathrm{VIP}(A_F, Q) \neq \emptyset.$$

The result follows immediately from (3.18) and the proof is complete. $\square$

REMARK 3.2. *Assumption* (b) *in Proposition 2.10 is satisfied with A_F in place of A if Q has the BBC and one of the following assumptions holds (in particular, assumption (b2) was used by Colao et al in [11]):*

(b1) *A_F satisfies the coerciveness condition on Q.*
(b2) *There exists a compact set $L \subseteq M$ such that*

(3.19) $$x \in Q \setminus L \Rightarrow [\exists y \in Q \cap L \text{ s.t. } F(x, y) < 0].$$

In fact, it is clear from Remark 2.3 in the case of (b1). To check this for the case of (b2), let $L \subseteq M$, $x \in Q \setminus L$ and let $y \in Q \cap L$ be given by (3.19) such that $F(x, y) < 0$. Then,

(3.20) $$F(x, y) - F(x, x) \leq F(x, y) < 0.$$

By assumption (H2) and the definition of A_F, we see that for any $v \in A_F(x)$, there exists a minimal geodesic $\gamma_{xy} \in \min{-}\Gamma_{xy}^Q$ such that $F(x, y) \geq F(x, x) + \langle v, \dot{\gamma}_{xy}(0) \rangle$. This, together with (3.20), implies that $\langle v, \dot{\gamma}_{xy}(0) \rangle < 0$. Hence condition (2.10) is satisfied as $x \in Q \setminus L$ is arbitrary, and the proof is complete.

The following theorem provides the convexity properties of the solution set $\mathrm{EP}(F, Q)$, which is a direct consequence of Propositions 3.4, 3.5 and 2.11 (noting by (3.7) that Q is weakly convex).

THEOREM 3.7. *Suppose that F satisfies* (H1)-(H3) *and* $\mathrm{EP}(F, Q) \neq \emptyset$. *Suppose further that A_F is monotone on Q with (3.1) (e.g., F is monotone on $Q \times Q$). Then the following assertions hold:*

(i) *The solution set* $\mathrm{EP}(F, Q)$ *is locally convex.*

(ii) *If* A_F *is strictly monotone on* Q *(e.g.,* F *is strictly monotone on* $Q \times Q$*), then* $\mathrm{EP}(F, Q)$ *is a singleton.*

(iii) *If* M *is of the sectional curvatures bounded above by* $\kappa > 0$*, then* $\mathrm{EP}(F, Q)$ *is* D_κ*-convex.*

In particular, in the case when M is a Hadamard manifold, hypothesis (H1) implies (H2), and every convex subset has both weak poles and the BCC property. Thus the following corollary is immediate from Theorems 3.6 and 3.7.

COROLLARY 3.8. *Let* M *be a Hadamard manifold. Suppose that* F *satisfies* (H1) *and* (H3). *Then the following assertions hold:*

(i) *If* (H4) *holds, then* $\mathrm{EP}(F, Q) \neq \emptyset$ *provided that* Q *is compact, or one of* (b1) *and* (b2) *in Remark 3.2 holds.*

(ii) *If* F *is monotone on* $Q \times Q$ *with* $\mathrm{EP}(F, Q) \neq \emptyset$*, then* $\mathrm{EP}(F, Q)$ *is convex.*

REMARK 3.3. *Assertion* (ii) *in Corollary 3.8 seems new even in the Hadamard manifold setting; while assertion* (i) *was established in* [11, *Theorem 3.2] under the following assumptions:*

(c1) *there exists a compact set* $L \subseteq M$ *and* $y_0 \in Q \cap L$ *such that* $F(x, y_0) < 0$ $\forall x \in Q \setminus L$;
(c2) *the set* $\{y \in Q : F(x, y) < 0\}$ *is convex for each* $x \in Q$.

Clearly assumption (c1) *implies our assumption* (b2) *in Remark 3.2. Moreover, as will be seen in the application to the proximal point algorithm in the next section and to the mixed variational inequalities in Subsection 5.2, assumption* (c2) *is not satisfied, in general (thus* [11, *Theorem 3.2] is not applicable); while Corollary 3.8 is applicable because our assumptions* (H1) *and* (H2) *presented here are satisfied there.*

4. Resolvent and proximal point algorithm for EP. As in the previous section, we always assume that $F : M \times M \to \overline{\mathbb{R}}$ and $Q \subseteq M$ satisfy the conditions assumed at the beginning of Section 3. Recall the equilibrium problem is defines by (1.1) and its solution set is denoted by $\mathrm{EP}(F, Q)$. The aim of this section is to introduce the resolvent and the proximal point algorithm for EP (1.1) on general manifolds and show convergence of this algorithm. The applications of the proximal point method to solve many different problems in the Riemannian context could be fond in e.g., [26, 28, 46, 48].

Fix $z \in M$ and define the bifunction $G_z : M \times M \to \overline{\mathbb{R}}$ by

$$G_z(x, y) := \sup_{u \in \exp_x^{-1} z, v \in \exp_x^{-1} y} \langle -u, v \rangle_x \quad \text{for any } (x, y) \in M \times M.$$

In the following definition, we extend the notion of the resolvent defined in [11, definition 4.6] for the bifunction F on Hadamard manifolds to the general manifold setting. Let $\lambda > 0$.

DEFINITION 4.1. *The resolvent* $J_\lambda^F : M \rightrightarrows Q$ *of* F *is defined by*

$$(4.1) \qquad\qquad J_\lambda^F(z) := \mathrm{EP}(F_{\lambda, z}, Q) \quad \text{for any } z \in M,$$

where the bifunction $F_{\lambda,z} : M \times M \to \overline{\mathbb{R}}$ is defined as

$$F_{\lambda,z}(x,y) := \lambda F(x,y) + G_z(x,y) \quad \text{for any } (x,y) \in M \times M.$$

For the remainder, we always assume that M is of the sectional curvature bounded above by $\kappa \geq 0$. Recall that $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$ and $D_\kappa = +\infty$ if $\kappa = 0$. Then, for any $z \in M$, $\mathbf{B}(z, \frac{D_\kappa}{4})$ is strongly convex (see, e.g., [39, p. 169]), and so $\exp^{-1}_{(\cdot)} z$ is a singleton on $\mathbf{B}(z, \frac{D_\kappa}{4})$.

Recall that A_F is the set valued vector field associated to the bifunction F (see (3.8)). Following [28], the resolvent $J_\lambda^{A_F} : M \rightrightarrows Q$ of A_F is defined by

$$(4.2) \qquad\qquad J_\lambda^{A_F}(z) := \text{VIP}(A_{\lambda,z}^F, Q) \quad \text{for any } z \in M,$$

where $A_{\lambda,z}^F : M \rightrightarrows TM$ is defined by

$$A_{\lambda,z}^F(x) := \lambda A_F(x) - E_z^Q(x) \quad \text{for any } x \in Q,$$

with the set-valued vector field $E_z^Q : M \rightrightarrows TM$ defined by

$$E_z^Q(x) := \{u \in \exp^{-1}_x z : \exp_x tu \in Q \ \forall t \in [0,1]\} \quad \text{for any } x \in Q$$

The following theorem provides an estimate for the domain of the resolvent J_λ^F. Recall that L_z^F denotes the center Lipschitz constant of $F(z, \cdot)$ at $z \in M$. Set

$$(4.3) \qquad\qquad \mathcal{D}_\lambda^F := \{z \in Q : \lambda L_z^F < \frac{D_\kappa}{4}\}.$$

THEOREM 4.2. *Suppose that F satisfies hypotheses* (H1)-(H3) *and is monotone on $Q \times Q$. Then,*

(i) $\lambda \mathrm{d}(0, A_F(z)) < \frac{D_\kappa}{4}$ *for each $z \in \mathcal{D}_\lambda^F$.*

(ii) $\mathcal{D}_\lambda^F \subseteq \mathcal{D}(J_\lambda^F)$.

(iii) $J_\lambda^F(z) \cap \mathbf{B}(z, \frac{D_\kappa}{4}) = J_\lambda^{A_F}(z)$ *is a singleton for each $z \in \mathcal{D}_\lambda^F$.*

Proof. Note that Q is weakly convex by implication (3.7). Moreover, by assumptions made for F, one sees by Propositions 3.3 and 3.5 that $A_F \in \mathcal{V}(Q)$ is monotone, and

$$\mathrm{d}(0, A_F(z)) \leq L_z^F \quad \text{for each } z \in Q.$$

Thus assertion (i) follows from the definition of $\mathcal{D}_\lambda^F$ in (4.3). Below we show assertions (ii) and (iii). To do this, let $z \in \mathcal{D}_\lambda^F$. Then, $\lambda \mathrm{d}(0, A_F(z)) < \frac{D_\kappa}{4}$ by (i), and it follows from [28, Lemma 4.3 and Corollary 5.4] (applied to A_F, Q in place of V, A there) that

$$(4.4) \qquad\qquad J_\lambda^{A_F}(z) \text{ is a singleton and } J_\lambda^{A_F}(z) \subseteq \mathbf{B}(z, \frac{D_\kappa}{4}).$$

Moreover, thanks to hypotheses (H1) and (H2), we have by Proposition 3.2 (i) and (iii) that the bifunction $F_{\lambda,z} = \lambda F + G_z$ satisfies hypotheses (H1) and (H2) (with $F_{\lambda,z}$ in place of F).

Furthermore, for any $x \in Q$, $F(x, x) = 0$ by the monotonicity of F and $G_z(x, x) = 0$ by of Proposition 3.2 (i); hence $F_{\lambda, z}(x, x) = 0$. Thus one can apply Proposition 3.4 to get that

$$(4.5) \qquad \text{EP}(F_{\lambda, z}, Q) = \text{VIP}(A_{F_{\lambda, z}}, Q).$$

Noting that $\mathcal{D}(G_z(x, \cdot)) = M$ by Proposition 3.2 (i), we see from Lemma 2.7 and Proposition 3.2 (iv) that, for any $x \in Q$,

$$(4.6) \qquad A_{F_{\lambda, z}}(x) := \partial(\lambda F(x, \cdot) + G_z(x, \cdot))(x) = \lambda A_F(x) - \overline{\text{co}} E_z^Q(x).$$

Hence $A_{\lambda, z}^F(x) \subseteq A_{F_{\lambda, z}}(x)$ for any $x \in Q$, and then $\text{VIP}(A_{\lambda, z}^F, Q) \subseteq \text{VIP}(A_{F_{\lambda, z}}, Q)$. By defintion (see (4.1) and (4.2)) and (4.5), it follows that

$$(4.7) \qquad J_\lambda^{A_F}(z) = \text{VIP}(A_{\lambda, z}^F, Q) \subseteq \text{VIP}(A_{F_{\lambda, z}}, Q) = \text{EP}(F_{\lambda, z}, Q) = J_\lambda^F(z).$$

In light of (4.4), we see that $J_\lambda^F(z) \neq \emptyset$, and so assertion (ii) holds as $z \in \mathcal{D}_\lambda^F$ is arbitrary. To show assertion (iii), note that $A_{\lambda, z}^F(x) = A_{F_{\lambda, z}}(x)$ if $\mathrm{d}(x, z) < D_\kappa$ by (4.6). It follows from (4.7) that

$$J_\lambda^{A_F}(z) \cap \mathbf{B}(z, \frac{D_\kappa}{4}) = J_\lambda^F(z) \cap \mathbf{B}(z, \frac{D_\kappa}{4}).$$

This, together with (4.4), implies that $J_\lambda^F(z) \cap \mathbf{B}(z, \frac{D_\kappa}{4}) = J_\lambda^{A_F}(z)$ is a singleton, and so assertion (iii) holds. The proof is complete. $\square$

The following theorem provides sufficient conditions for $\mathcal{D}(J_\lambda^F) = M$. In particular, in the Hadamard manifold setting, this result was claimed in [11, Theorem 4.9] under the additional assumption (c1) in Remark 3.3 but the proof presented there is not correct.

THEOREM 4.3. *Suppose that F satisfies hypotheses* (H1)-(H3) *and is monotone on* $Q \times Q$. *Then, $\mathcal{D}(J_\lambda^F) = M$ provided that one of the following assumptions holds:*

(a) *Q is compact and contains a weak pole $o \in \mathrm{int}_R Q$;*

(b) *M is a Hadamard manifold.*

Proof. Let $z \in M$. Then by the assumptions made for F and Proposition 3.2 (i), (iii) and (v), one can checks easily that the bifunction $F_{\lambda, z} = \lambda F + G_z$ satisfies (H1)-(H4). To complete the proof, it suffices to verify that $J_\lambda^F(z) \neq \emptyset$, which is true by Theorem 3.6 in case (a). Thus we only consider case (b). To do this, we assume that M is a Hadamard manifold. Then, for any $x, y \in M$, $\exp_x^{-1} y$ is a singleton and $F_{\lambda, z}(x, y)$ is reduced to

$$F_{\lambda, z}(x, y) := \lambda F(x, y) - \langle \exp_x^{-1} z, \exp_x^{-1} y \rangle.$$

Recalling from [11, (2.7)] that

$$\langle \exp_x^{-1} z, \exp_x^{-1} y \rangle + \langle \exp_y^{-1} w, \exp_y^{-1} x \rangle \geq \mathrm{d}^2(x, y) \quad \text{for any } x, y \in M$$

and that F is monotone on $Q \times Q$, we get that

$$(4.8) \qquad F_{\lambda, z}(x, y) + F_{\lambda, z}(y, x) \leq -\mathrm{d}^2(x, y) \quad \text{for any } (x, y) \in Q \times Q.$$

Below we show that $F_{\lambda,z}$ satisfies (b2) in Remark 3.2: there is a compact subset $L \subseteq M$ such that

(4.9) $x \in Q \setminus L \Rightarrow [\exists \, y \in Q \cap L \text{ s.t. } F_{\lambda,z}(x,y) < 0].$

Granting this, we get $J_\lambda^F(z) = \mathrm{EP}(F_{\lambda,z}, Q) \neq \emptyset$ by Corollary 3.8, and the proof is complete. To show (4.9), take $y \in Q$ and set $R := L_y^{F_{\lambda,z}}$. Then $R < +\infty$ as $F_{\lambda,z}$ satisfies (H1), and $L := \overline{\mathbf{B}(y, R)}$ is as desired. To show this, let $x \in Q \setminus L$ and $v \in A_{F_{\lambda,z}}(y)$. Then, $\mathrm{d}(x,y) > R$, and $\|v\| \leq R$ by Proposition 3.3. Therefore, we have that

$$F_{\lambda,z}(y,x) \geq F_{\lambda,z}(y,y) + \langle v, \exp_y^{-1} x \rangle \geq -R\mathrm{d}(x,y)$$

(noting that $F_{\lambda,z}(y,y) = 0$). This, together with (4.8), implies

$$F_{\lambda,z}(x,y) \leq -\mathrm{d}^2(x,y) - F_{\lambda,z}(y,x) \leq (R - \mathrm{d}(x,y))\,\mathrm{d}(x,y) < 0.$$

Thus, (4.9) is shown, and the proof is complete. ☐

To define the proximal point algorithm for solving EP (1.1), let $x_0 \in Q$ and $\{\lambda_k\} \subset (0, +\infty)$. Thus the proximal point algorithm can be formulated as follows.

Algorithm P Letting $k = 1, 2, \ldots$ and having x_k, choose x_{k+1} such that

$$x_{k+1} \in J_{\lambda_k}^F(x_k) \cap \mathbf{B}(x_k, \frac{D_\kappa}{4}).$$

Clearly, in the case when M is a Hadamard manifold, **Algorithm P** is reduced to the one defined in [11]:

$$x_{k+1} \in J_{\lambda_k}^F(x_k) \quad \text{for each } k \in \mathbb{N}.$$

The convergence result of **Algorithm P** is as follows.

THEOREM 4.4. *Suppose that F satisfies hypotheses (H1)-(H3) and is monotone on $Q \times Q$ with $\mathrm{EP}(F, Q) \neq \emptyset$. Let $x_0 \in Q$ and $\{\lambda_k\} \subset (0, \infty)$ be such that*

(4.10) $\mathrm{d}(x_0, \mathrm{EP}(F, Q)) < \dfrac{D_\kappa}{8},$

(4.11) $\Sigma_{k=0}^\infty \lambda_k^2 = \infty \quad and \quad \lambda_k L_{x_k}^F < \dfrac{D_\kappa}{4} \quad \text{for all } k \in \mathbb{N}.$

*Then, **Algorithm P** is well-defined, and converges to a point in $\mathrm{EP}(F, Q)$.*

Proof. Recall that $A_F : Q \rightrightarrows TM$ is defined by (3.8). By assumption, Propositions 3.4 and 3.5 are applicable; hence A_F is monotone, $A_F \in \mathcal{V}(Q)$, and

(4.12) $\mathrm{VIP}(A_F, Q) = \mathrm{EP}(F, Q)$

(noting that (3.1) hold by the monotonicity assumption). Then, thanks to (4.10), one sees that

(4.13) $\mathrm{d}(x_0, \mathrm{VIP}(A_F, Q)) < \dfrac{D_\kappa}{8}.$

Let $\{\tilde{x}_k\}$ be a sequence generated by the following proximal algorithm with initial point $\tilde{x}_0 := x_0$, which was introduced in [28] for finding a point in $\mathrm{VIP}(A_F, Q)$:

$$(4.14) \qquad \tilde{x}_{k+1} \in J_{\lambda_k}^A(\tilde{x}_k) \quad \text{for each } k \in \mathbb{N}.$$

In view of the second assumption in (4.11), and applying Theorem 4.2 (iii), we can check inductively that **Algorithm P** is well-defined and that the generated sequence $\{x_k\}$ coincides with $\{\tilde{x}_k\}$ and satisfies

$$\lambda_k \mathrm{d}(0, A_F(\tilde{x}_k)) < \frac{D_\kappa}{4} \quad \text{for each } k \in \mathbb{N}.$$

This, together with the first assumption in (4.11) and (4.13), implies that [28, Corollary 5.8] (with A_F, Q in place of V, A) is applicable, and the sequence $\{\tilde{x}_k\}$ and so $\{x_k\}$ converges to a point in $\mathrm{VIP}(A_F, Q)$. Thus the conclusion follows immediately from (4.12), and the proof is complete. $\square$

In the special case when M is a Hadamard manifold, assumption (4.10) and the second one in (4.11) are satisfied automatically. Therefore the following corollary is direct from Theorem 4.4, which was claimed in [11, Theorem 4.9, 4.10] (for constant parameters $\lambda_k \equiv \lambda > 0$) but with an incorrect proof there as we explained in Section 1).

COROLLARY 4.5. *Suppose that M is a Hadamard manifold, and that F satisfies hypotheses* (H1)-(H3) *and is monotone on $Q \times Q$ with $\mathrm{EP}(F, Q) \neq \emptyset$. Let $\{\lambda_k\} \subset (0, \infty)$ be such that $\Sigma_{k=0}^\infty \lambda_k^2 = \infty$. Then, **Algorithm P** is well-defined, and converges to a solution in $\mathrm{EP}(F, Q)$.*

5. Applications. This section is devoted to two applications of the results regarding the solution set of the EP in the previous sections: One is to the Nash equilibrium and the other to the mixed variational inequality.

5.1. Nash equilibrium. We consider the Nash equilibrium problem (NEP for short) on Riemannian manifolds in this subsection, which is formulated as follow. Let $I = \{1, 2, \ldots, m\}$ be a finite index set which denotes the set of players, and let (M_i, d_i), $i \in I$, be a Riemannian manifold. For each $i \in I$, let $Q_i \subseteq M_i$ be the strategy set of the i-th player, and $f_i : M \to \overline{\mathbb{R}}$ be his loss-function, where $M := M_1 \times M_2 \times \cdots \times M_m$ is the product manifold with the standard Riemannian product metric. The Nash equilibrium problem associated to $Q := Q_1 \times Q_2 \times \cdots \times Q_m \subseteq M$ and $\{f_i\}_{i \in I}$ consists of finding a point $\bar{x} = (\bar{x}_i) \in Q$ such that

$$(5.1) \qquad f_i(\bar{x}) = \min_{y_i \in Q_i} f_i(\bar{x}_1, \ldots, \bar{x}_{i-1}, y_i, \bar{x}_{i+1}, \ldots, \bar{x}_m) \quad \text{for each } i \in I.$$

Any point $\bar{x} \in Q$ satisfying (5.1) is called a Nash equilibrium point of the NEP, and we denote the set of all Nash equilibrium points by $\mathrm{NEP}(\{f_i\}_{i \in I}, Q)$.

The most well-known existence results for the classical NEP in the linear space setting is due to Nash [33, 34], where it is assumed that each Q_i is compact and convex and each f_i is (quasi)convex in the i-th variable. Further extensions and applications of Nash's original work could be founded in [16, 32, 35, 41] and references therein. Kristály seems the first

one to consider the existence and localization of NEP in the framework of Riemannian manifolds; see [23]. Recently, Kristály used in [24] a variational approach to analyzed the NEP with nonconvex strategy sets and nonconvex /nonsmooth payoff functions in Hadamard manifolds.

To proceed, we assume for the whole subsection that

(H_N-**a**) Q_i is closed and weakly convex in M_i and $Q \subseteq \mathrm{int} \bigcap_{i \in I} \mathcal{D}(f_i)$;

(H_N-**b**) for each $i \in I$, f_i is continuous on Q;

(H_N-**c**) for each $i \in I$, f_i and $f_i + \delta_Q$ are weakly convex in the i-th variable.

To apply our results in the previous sections, we, following [11] and [38], reformulate NEP (5.1) as an EP as follows. Let $r := (r_i) \in \mathbb{R}^m_{++} := \{(r_i) \in \mathbb{R}^m : \text{each } r_i > 0\}$, and define the bifunction $F_r : M \times M \to \mathbb{R}$ as the weighted positive sum of the functions $\{f_i\}$:

$$F_r(x, y) := \sum_{i \in I} r_i \left(f_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) - f_i(x) \right)$$

for any $x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in M$. Then, it is easy to check that

(5.2) \hspace{2cm} $\mathrm{EP}(F_r, Q) = \mathrm{NEP}(\{f_i\}_{i \in I}, Q)$.

In the spirit of the idea in [38] for the NEP in the Euclidean space setting, we introduce the pseudosubgradient mapping $g_r : M \to TM$ for functions $\{f_i\}$ in the Riemannian manifold setting, which is defined by

$$g_r(x) := (r_1 \partial_1 f_1(x), r_2 \partial_2 f_2(x), \ldots, r_m \partial_m f_m(x)) \quad \text{for each } x \in M,$$

where, for each $i \in I$ and $x \in M$, $\partial_i f_i(x)$ stands for the subdifferential of the function $f_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_m)$ at x_i, that is

$$\partial_i f_i(x) := \partial f_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_m)(x_i).$$

By definition, we check that

(5.3) \hspace{2cm} $A_{F_r}(x) := \partial F_r(x, \cdot)(x) = g_r(x) \quad$ for any $x \in Q$.

The main theorem in this subsection is as follows, which provides the results on the existence, the uniqueness and the convexity of the Nash equilibrium point.

THEOREM 5.1. *Let $r \in \mathbb{R}^m_{++}$. Then the following assertions hold:*

(i) *Suppose that Q contains a weak pole $o \in \mathrm{int}_R Q$. Then $\mathrm{NEP}(\{f_i\}_{i \in I}, Q) \neq \emptyset$ provided that Q is compact, or Q has the BCC property and that there exists a compact subset $L \subseteq M$ such that*

(5.4) \hspace{1cm} $x \in Q \setminus L \Rightarrow [\forall v \in g_r(x), \exists y \in Q \cap L, \gamma_{xy} \in \min{-}\Gamma^Q_{xy} \text{ s.t. } \langle v, \dot{\gamma}_{xy}(0) \rangle < 0]$.

(ii) *If g_r is strictly monotone on Q, then $\mathrm{NEP}(\{f_i\}_{i \in I}, Q)$ is at most a singleton.*

(iii) *If g_r is monotone on Q and $\mathrm{NEP}(\{f_i\}_{i \in I}, Q) \neq \emptyset$, then $\mathrm{NEP}(\{f_i\}_{i \in I}, Q)$ is locally convex, and $\mathrm{NEP}(\{f_i\}_{i \in I}, Q)$ is D_κ-convex if M is additionally assumed to be of the sectional curvatures bounded above by some $\kappa \geq 0$.*

Proof. In view of (5.2), (5.3), (5.4) and thanks to Theorems 3.6 and 3.7 (applied to F_r in place of F), it suffices to show that F_r satisfies hypotheses (H1)-(H4) made in Section 3. Note that (H3) follows trivially from assumption $(\mathrm{H_N}$-b); while (H4) is clear as $F_r(x, x) = 0$ for any $x \in Q$. Thus we only need to show that F_r satisfies hypotheses (H1) and (H2). To do this, let $x = (x_i)_{i \in I} \in Q$, and write

$$\mathcal{D}_i := \mathcal{D}(f_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_m)) \quad \text{for each } i \in I.$$

Then

(5.5) $$\mathcal{D}(F_r(x, \cdot)) = \mathcal{D}_1 \times \ldots \times \mathcal{D}_i \times \ldots \times \mathcal{D}_m.$$

By assumption $(\mathrm{H_N}$-a), each $Q_i \subseteq \mathrm{int}\mathcal{D}_i$ and so

$$x \in Q \subseteq \mathrm{int}\mathcal{D}(F_r(x, \cdot)).$$

Furthermore, in light of assumption $(\mathrm{H_N}$-c), one sees that each $\mathcal{D}_i$ is weakly convex in M_i. This, together with (5.5), implies that $\mathcal{D}(F_r(x, \cdot))$ is weakly convex in M. We claim that $F_r(x, \cdot)$ and $F_r(x, \cdot) + \delta_{Q \times Q}(x, \cdot)$ are weakly convex in M. Granting this, (H1) and (H2) are checked. In fact, let $y = (y_i), z = (z_i) \in \mathcal{D}(F_r(x, \cdot))$. Then, by assumption $(\mathrm{H_N}$-c), for each $i \in I$, there is a geodesic $\gamma_i \in \min{-}\Gamma_{z_i y_i}^{\mathcal{D}_i}$ such that

(5.6) $$f_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_m) \circ \gamma_i \text{ is convex on } [0, 1].$$

Define $\gamma_{zy}[0, 1] \to M$ by $\gamma_{zy}(t) := (\gamma_1(t), \gamma_2(t), \ldots, \gamma_m(t))$ for each $t \in [0, 1]$. Then, $\gamma_{zy} \in \min{-}\Gamma_{zy}^{F_r(x, \cdot)}$ (see, e.g., [7]), and

$$F_r(x, \cdot) \circ \gamma_{zy} = \sum_{i \in I} f_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_m) \circ \gamma_i.$$

This means that $F_r(x, \cdot) \circ \gamma_{zy}$ is clearly convex thanks to (5.6), and so $F_r(x, \cdot)$ is weakly convex in M. Similarly, one can checks that $F_r(x, \cdot) + \delta_{Q \times Q}(x, \cdot)$ is also weakly convex in M. Thus the claim stands, and the proof is complete. □

REMARK 5.1. *Assertion* (i) *extends the corresponding one in [23, Theorem 1.1], which was proved under the assumption that each Q_i is compact and geodesic convex. It is worthy remarking that the geodesic convexity assumption for Q_i prevents its application to some special but important Riemannian manifolds, such as compact Stiefel manifolds $\mathrm{St}(p, n)$ and Grassmann manifolds $\mathrm{Grass}(p, n)$ $(p < n)$, in which there is no geodesic convex subset (see [1, p. 104 (5.27)]).*

Example 5.1 below provides the case where our existence result of Theorem 5.1 is applicable but not [23, Theorem 1.1]. Note also that the NEP in Example 5.1 is originally defined on the Euclidean space, and the corresponding existing results in the Euclidean space setting (see, e.g., [16, 32, 33, 34]), to the best our knowledge, are nor applicable because the set Q_2 involved is not convex in the usual sense.

EXAMPLE 5.1. *Consider the Nash equilibrium problem* (5.1) *with the associated* $Q := Q_1 \times Q_2 \subseteq \mathbb{R} \times \mathbb{R}^3$ *and* $\{f_i\}_{i=1,2}$ *defined respectively as follows:*

$$Q_1 := [-1, 1], \quad Q_2 := \{(t_1, t_2, t_3) : t_1^2 + t_2^2 + t_3^2 = 1, t_1 > 0, |t_2| \leq \frac{1}{2}, t_3 > 0\},$$

$$f_1(x_1, x_2) = (x_1 - t_3)^2 \quad and \quad f_2(x_1, x_2) = \arccos t_1 \quad for\ any\ x_1 \in \mathbb{R},\ x_2 = (t_1, t_2, t_3) \in \mathbb{R}^3.$$

Clearly $Q_2 \subset \mathbb{R}^3$ is not convex, and so the existence results in the Euclidean space setting are not applicable.

Below, we shall consider the problem on the Riemannian manifold $M := \mathbb{R} \times \mathbb{S}^2$, where

$$\mathbb{S}^2 := \left\{(t_1, t_2, t_3) \in \mathbb{R}^3 \big|\ t_1^2 + t_2^2 + t_3^2 = 1\right\}$$

is the 2-dimensional unit sphere. Denote $\mathbf{x} := (0, 0, 1)$, $\mathbf{y} := (0, 0, -1)$, and consider system of coordinates $\Phi \colon (0, \pi) \times [0, 2\pi] \subset \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{\mathbf{x}, \mathbf{y}\}$ around $x \in \mathbb{S}^2 \setminus \{\mathbf{x}, \mathbf{y}\}$ defined by

$$\Phi(\theta, \varphi) := (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^T \quad for\ each\ (\theta\varphi) \in (0, \pi) \times [0, 2\pi].$$

Then the Riemannian metric on $\mathbb{S}^2 \setminus \{\mathbf{x}, \mathbf{y}\}$ is given by

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = \sin^2\theta \quad for\ each\ \theta \in (0, \pi)\ and\ \varphi \in [0, 2\pi],$$

and the geodesics of $\mathbb{S}^2 \setminus \{\mathbf{x}, \mathbf{y}\}$ are great circles or semicircles; see [42, p. 84] for more details.

Restricting f_1 and f_2 to $M = \mathbb{R} \times \mathbb{S}^2$, one can check by definition that assumptions $(\mathrm{H_N}\text{-}a)$-$(\mathrm{H_N}\text{-}c)$ are satisfied (noting that $\arccos t_1 = \mathrm{d}(x_2, z_0)$ for each $x_2 = (t_1, t_2, t_3) \in \mathbb{S}^2$, where $z_0 := (1, 0, 0)$), and that $Q \subset M$ is compact and has a weak pole in $\mathrm{int}Q$. Thus, Theorem 5.1 is applicable and guarantees $\mathrm{NEP}(\{f_1, f_2\}, Q_1 \times Q_2) \neq \emptyset$. Indeed, by a simple calculation, we see that $\mathrm{NEP}(\{f_1, f_2\}, Q_1 \times Q_2) = \{(1, (1, 0, 0))\}$. However, the existence result in [23, Theorem 1.1] is not applicable because there is no geodesic subset on $\mathbb{S}^2$.

As explained before (see the paragraph right before Corollary 3.8), the following corollary is a direct consequence of Theorem 5.1. In particular, assertion (i) was proved in [11, Theorem 3.12] with each Q_i being compact; while assertion (ii) is new even in the Hadamard manifold setting.

COROLLARY 5.2. *Suppose that each M_i is a Hadamard manifold. Then, the following assertions hold:*

(i) *The solution set $\mathrm{NEP}(\{f_i\}_{i \in I}, Q) \neq \emptyset$ provided Q is compact, or there exists a compact subset $L \subseteq M$ such that (5.4) holds for some $r \in \mathbb{R}_{++}^m$.*

(ii) *If there exists some $r \in \mathbb{R}_{++}^m$ such that g_r is monotone on Q, then $\mathrm{NEP}(\{f_i\}_{i \in I}, Q)$ is convex.*

REMARK 5.2. *In view of (5.3), one checks by Remark 3.2 (applied to g_r, F_r in place of A_F, F) that a compact subset L exists such that (5.4) holds provided one of the following assumptions holds:*

(a) *g_r satisfies the coerciveness condition on Q;*

(b) *there exists a compact set $L \subseteq M$ such that*

$$x \in Q \setminus L \Rightarrow [\exists y \in Q \cap L\ \text{s.t.}\ F_r(x, y) < 0].$$

5.2. Mixed variational inequalities. Let $Q \subset M$ be a nonempty closed subset. Given a vector field $V : Q \to TM$ and a real-valued function $f : M \to \overline{\mathbb{R}}$. The mixed variational inequality problem (MVIP for short) associated to V and f is to find $\bar{x} \in Q$, called a solution of the MVIP, such that

$$(5.7) \qquad \langle V(\bar{x}), \dot{\gamma}_{\bar{x}y}(0) \rangle + f(y) - f(\bar{x}) \geq 0 \quad \text{for any } y \in Q, \ \gamma_{\bar{x}y} \in \Gamma^Q_{\bar{x}y}.$$

The set of all solutions of MVIP (5.7) is denoted by $\mathrm{MVIP}(V, f, Q)$. The MVIP has been studied extensively in the linear space setting; see, e.g., [17, 19, 44]; and it seems that [11] is the first paper to explore the MVIP in the Hadamard manifold setting, where only the existence issue of the solution for the MVIP is concerned with.

To reformulate the MVIP as an EP considered in the previous sections, we define $F : M \times M \to (-\infty, +\infty]$ as follows:

$$(5.8) \qquad F(x, y) := \sup_{u \in \exp_x^{-1} y} \langle V(x), u \rangle_x + f(y) - f(x) \quad \text{for any } (x, y) \in M \times M,$$

where $\exp_x^{-1} y$ is defined by (2.9) and we adopt the convention that $a - (+\infty) = +\infty$ for any $a \in \mathbb{R}$.

PROPOSITION 5.3. *Let $F : M \times M \to (-\infty, +\infty]$ be defined by (5.8). Suppose that f is convex. Then we have*

$$(5.9) \qquad\qquad\qquad \mathrm{MVIP}(V, f, Q) = \mathrm{EP}(F, Q).$$

Proof. It is evident that $\mathrm{MVIP}(V, f, Q) \subseteq \mathrm{EP}(F, Q)$. To show the converse inclusion, let $\bar{x} \in \mathrm{EP}(F, Q)$ and it suffices to prove that (5.7) holds. To this end, let $y \in Q$ and $\gamma_{\bar{x}y} \in \Gamma^Q_{\bar{x}y}$. We have to show that

$$(5.10) \qquad\qquad\qquad \langle V(\bar{x}), \dot{\gamma}_{\bar{x}y}(0) \rangle + f(y) - f(\bar{x}) \geq 0.$$

Take $\bar{t} \in (0, 1]$ such that $\mathrm{d}(\bar{x}, \gamma_{\bar{x}y}(\bar{t})) \leq r_{\bar{x}}$ (note that $r_{\bar{x}} > 0$ by (2.1)). Denote $\bar{y} := \gamma_{\bar{x}y}(\bar{t})$. Then $\bar{y} \in Q$ and $\Gamma^Q_{\bar{x}\bar{y}} = \{\gamma_{\bar{x}\bar{y}}\}$ is a singleton, where $\gamma_{\bar{x}\bar{y}} : [0, 1] \to M$ is defined by

$$\gamma_{\bar{x}\bar{y}}(s) := \gamma_{\bar{x}y}(\bar{t}s) \quad \text{for any } s \in [0, 1].$$

Then $\dot{\gamma}_{\bar{x}\bar{y}}(0) = \bar{t}\dot{\gamma}_{\bar{x}y}(0)$. In view of $\bar{x} \in \mathrm{EP}(F, Q)$ and $\bar{y} \in Q$, we see that

$$\langle V(\bar{x}), \dot{\gamma}_{\bar{x}\bar{y}}(0) \rangle + f(\bar{y}) - f(\bar{x}) = \langle V(\bar{x}), \bar{t}\dot{\gamma}_{\bar{x}y}(0) \rangle + f(\bar{y}) - f(\bar{x}) \geq 0.$$

Noting that $\frac{f(\bar{y}) - f(\bar{x})}{\bar{t}} \leq f(y) - f(\bar{x})$ by the convexity of $f \circ \gamma_{\bar{x}y}$ (as f is convex), we conclude that (5.10) holds, which completes the proof. □

We assume in the present subsection that

(H$_\mathrm{M}$-a) f is convex and $Q \subseteq \mathrm{int}\mathcal{D}(f)$ is closed weakly convex.

(H$_\mathrm{M}$-b) V is continuous on Q.

The following theorem gives the existence, the uniqueness and the convexity property about the solution set $\mathrm{MVIP}(V, f, Q)$.

THEOREM 5.4. *The following assertions hold:*

(i) *Suppose that Q contains a weak pole $o \in \mathrm{int}_R Q$. Then $\mathrm{MVIP}(V, f, Q) \neq \emptyset$ provided that Q is compact, or Q has the BCC property and there exists a compact subset $L \subseteq M$ such that*

$$(5.11) \quad x \in Q \setminus L \Rightarrow [\forall v \in \partial f(x), \exists y \in Q \cap L, \gamma_{xy} \in \min{-}\Gamma_{xy}^Q \ s.t. \ \langle V(x) + v, \dot{\gamma}_{xy}(0) \rangle < 0].$$

(ii) *If $V + \partial f$ is strictly monotone on Q, then $\mathrm{MVIP}(V, f, Q)$ is at most a singleton.*

(iii) *If $V + \partial f$ is monotone on Q and $\mathrm{MVIP}(V, f, Q) \neq \emptyset$, then $\mathrm{MVIP}(V, f, Q)$ is locally convex, and is D_κ-convex if M is additionally assumed to be of the sectional curvatures bounded above by some $\kappa \geq 0$.*

Proof. We first show that F satisfies hypotheses (H1)-(H4) made in Section 3. To do this, let $G_V : M \times M \to \mathbb{R}$ be defined by (3.2), and let $G : M \times M \to \mathbb{R}$ be defined by

$$(5.12) \qquad\qquad G(x, y) := f(y) - f(x) \quad \text{for any } (x, y) \in Q \times M.$$

Then $F = G_V + G$. Noting by assumption $(\mathrm{H_M\text{-}a})$ that both G and $G + \delta_{Q \times Q}$ are point-wise weakly convex on Q, we see from Proposition 3.2(iii) that $F = G_V + G$ and $F + \delta_{Q \times Q} = G_V + (G + \delta_{Q \times Q})$ are point-wise weakly convex on Q. This particularly means that F satisfies (H2). To show (H1) and (H4), recalling (3.3) in Proposition 3.2(i), one checks that

$$\mathcal{D}(F(x, \cdot)) = \mathcal{D}(G(x, \cdot)) \bigcap \mathcal{D}(G_V(x, \cdot)) = \mathcal{D}(f) \quad \text{for any } x \in Q.$$

In view of assumption $(\mathrm{H_M\text{-}a})$, (H1) is checked; while (H4) is trivial since, by (3.3), $F(x, x) = 0$ for any $x \in Q$. Thus it remains to check (H3). Since by assumption $(\mathrm{H_M\text{-}a})$, the function $x \mapsto G(x, y)$ is continuous on Q (see Lemma 2.6 (i)). In view of assumption $(\mathrm{H_M\text{-}b})$, Proposition 3.2(v) is applicable to getting that $x \mapsto G_V(x, y)$ is usc on Q and so is F. Thus, (H3) is checked. Next, we check that

$$(5.13) \qquad\qquad A_F(x) = V(x) + \partial f(x) \quad \text{for each } x \in Q,$$

where A_F is defined by (3.8). Granting this, one verifies that conditions of Theorems 3.6 and 3.7 are satisfied, and then assertions (i)-(iii) follow by (5.9) (which is valid by assumption $(\mathrm{H_M\text{-}a})$). To show (5.13), let $x \in Q$. Then $\partial G_V(x, \cdot)(x) = V(x)$ by Proposition 3.2(iv). Thus, by assumption $(\mathrm{H_M\text{-}a})$, one applies Lemma 2.7 to obtain (5.13) and the proof is complete. $\square$

With a similar argument that we did for Corollary 5.2, but using Theorem 5.4 in place of Theorem 5.1, we have the following corollary. In particular, assertion (i) was claimed in [11, Theorem 3.5] with its proof being incorrect; while assertion (ii) is new even in the Hadamard manifold setting.

COROLLARY 5.5. *Suppose that M is a Hadamard manifold. Then, the following assertions hold:*

(i) *The solution set* $\mathrm{MVIP}(V, f, Q) \neq \emptyset$ *provided Q is compact, or there exists a compact subset $L \subseteq M$ such that (5.11) holds.*

(ii) *If $V + \partial f$ is monotone on Q with $\mathrm{MVIP}(V, f, Q) \neq \emptyset$, then $\mathrm{MVIP}(V, f, Q)$ is convex.*

REMARK 5.3. *Under one of the following assumptions, a compact subset $L \subseteq M$ exists such that (5.11) holds:*

(a) *V satisfies the coerciveness condition on Q;*

(b) *∂f satisfies the coerciveness condition on Q and V is monotone on Q.*

Indeed, in view of assumption (H$_\mathrm{M}$-a), we see that ∂f is monotone on Q by definition of the subdifferential of f. Assuming (b) or (c), it is easy to verify by definition that $V + \partial f$ satisfies the coerciveness condition on Q. Thus, one checks that (5.11) is satisfied as we have explained in Remark 2.3 with $V + \partial f$ in place of A.

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