NEAR-INFINITY CONCENTRATED NORMS AND THE FIXED POINT PROPERTY FOR NONEXPANSIVE MAPS ON CLOSED, BOUNDED, CONVEX SETS

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Abstract. In this paper we define the concept of a near-infinity concentrated norm on a Banach space X with a boundedly complete Schauder basis. When $\|\cdot\|$ is such a norm, we prove that $(X, \|\cdot\|)$ has the fixed point property (FPP); that is, every nonexpansive self-mapping defined on a closed, bounded, convex subset has a fixed point. In particular, P.K. Lin's norm in ℓ_1 [14] and the norm $\nu_p(\cdot)$ (with $p = (p_n)$ and $\lim_n p_n = 1$) introduced in [3] are examples of near-infinity concentrated norms. When $\nu_p(\cdot)$ is equivalent to the ℓ_1-norm, it was an open problem as to whether $(\ell_1, \nu_p(\cdot))$ had the FPP. We prove that the norm $\nu_p(\cdot)$ always generates a nonreflexive Banach space $X = \mathbb{R} \oplus_{p_1} (\mathbb{R} \oplus_{p_2} (\mathbb{R} \oplus_{p_3} \dots))$ satisfying the FPP, regardless of whether $\nu_p(\cdot)$ is equivalent to the ℓ_1-norm. We also obtain some stability results.

1. Introduction and Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space and C a subset of X. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. The Banach space X endowed with the norm $\|\cdot\|$ has the fixed point property (FPP) if every nonexpansive mapping defined from a closed bounded convex subset C of X into itself has a fixed point. This property is not preserved by isomorphism, that is, it strongly depends on the underlying norm [14] . There is a wide literature relating geometric properties of reflexive Banach spaces with the fulfilment of the fixed point property (see, for instance, the monographs [9], [13] and the references therein).

The Banach space ℓ_1 endowed with its standard norm $\|\cdot\|_1$ is a classical example of a nonreflexive Banach space that fails to have the FPP. It is possible to "perturb" this $(\ell_1, \|\cdot\|_1)$-example to obtain other Banach spaces that fail to have the FPP. One such class of Banach spaces are those that contain asymptotically isometric copies of ℓ_1.

Recall that a Banach space $(X, \|\cdot\|)$ contains an asymptotically isometric copy (a.i.c.) of ℓ_1 if there are a sequence $(x_n) \subset X$ and a decreasing sequence $(\epsilon_n) \subset (0, 1)$ with $\lim_n \epsilon_n = 0$ such that

$$\sum_{n=1}^{\infty} (1 - \epsilon_n)|t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sum_{n=1}^{\infty} |t_n|$$

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for every $(t_n) \in \ell_1$. It was proved in [4] that if a Banach space contains an a.i.c. of ℓ_1 then it fails to have the FPP. It turns out that there exist equivalent norms on ℓ_1 which fail to contain an a.i.c. of ℓ_1. Let us state some examples:

- The so-called P.K. Lin norm, defined as
$$|||x|||_L := \sup_{k\geq 1} \gamma_k \sum_{n=k}^{\infty} |x(n)|; \qquad x = \sum_{n=1}^{\infty} x(n)e_n$$
where (γ_k) is a nondecreasing sequence in $(0,1)$ with $\lim_k \gamma_k = 1$. In [5] it was proved that $(\ell_1, |||\cdot|||_L)$ fails to contain an a.i.c. of ℓ_1. Later on, P.K. Lin [14] proved that $(\ell_1, |||\cdot|||_L)$ has the FPP for $\gamma_k := \frac{8^k}{1+8^k}$. This condition was extended to every sequence (γ_k) with $\lim_k \gamma_k = 1$ (see [7] and [11]). P. K. Lin's result opened new avenues of research in the fixed point theory of nonexpansive mappings, since he settled negatively the long-standing open question: "Does the fixed point property imply reflexivity?" Since then, many other articles have appeared obtaining sufficient conditions that imply the FPP for equivalent norms on ℓ_1 (see for instance [2, 6, 7, 8, 10, 11, 12, 15]).
- Fix a nonincreasing sequence $p = (p_n)_n \subset (1, +\infty)$ with $\lim_n p_n = 1$. In the sequence space c_{00} of all real sequences with finitely many non-null coordinates, we define the norm $\nu_p(x) = \lim_n \nu_n(p, x)$ where
$$\nu_1(p, x) := |x_1|, \qquad \nu_{n+1}(p, x) := \left(|x_1|^{p_1} + \nu_n(Sp, Sx)^{p_1}\right)^{1/p_1},$$
with $x = (x_1, x_2, ...)$ and $Sz := (z_2, z_3, ...)$ when $z = (z_1, z_2, ...)$. The completion of c_{00} with the $\nu_p(\cdot)$ norm gives us a Banach space X with a *boundedly complete* Schauder basis (e_n). Also, X is the set of all real sequences $x = (x_n)$ for which $\nu_p(x) := \sup_n \nu_n(p, x) = \lim_n \nu_n(p, x) < \infty$; which we summarize by writing $X = \mathbb{R} \oplus_{p_1} (\mathbb{R} \oplus_{p_2} (\mathbb{R} \oplus_{p_3} \dots))$.
  Let $q = (q_n)$ be the sequence satisfying $\frac{1}{p_n} + \frac{1}{q_n} = 1$ for every $n \in \mathbb{N}$. Whenever the sequence (p_n) converges to 1 quickly enough, the norm $\nu_p(\cdot)$ provides an equivalent norm in ℓ_1; that is, $(X, \nu_p(\cdot))$ and ℓ_1 are isomorphic Banach spaces. In fact, it was proved in [3, Proposition 1] that $\nu_p(\cdot)$ is equivalent to the ℓ_1 norm if and only if there exists some $\delta > 0$ so that $q_n \geq \delta \log n$ for all $n \in \mathbb{N}$. It is also known that $(\ell_1, \nu_p(\cdot))$ fails to contain asymptotically isometric copies of ℓ_1 [3, Theorem 1]. However, unlike P.K. Lin's norm, it was unknown whether ℓ_1 with the norm $\nu_p(\cdot)$ had the fixed point property.

In what follows, we enlarge the class of norms on ℓ_1 satisfying the FPP and we include, as a particular case, the norm $\nu_p(\cdot)$ defined in [3]. We will extend our result to a more general framework. For instance, we will prove the fulfilment of the FPP for $(X, \nu_p(\cdot))$ even when this norm fails to be an ℓ_1-norm.

Furthermore, we obtain stability of the fixed point property for certain norms along rays emanating from near-infinity concentrated norms.

2. Near-infinity concentrated norms and the FPP

Throughout this paper, let X denote a Banach space with a Schauder basis $\{e_n\}_n$. Given $x = \sum_{n=1}^{\infty} x(n)e_n \in X$, we denote by *supp*$(x) = \{n \in \mathbb{N} : x(n) \neq 0\}$, $Q_k(x) = \sum_{n=k}^{\infty} x(n)e_n$ and $P_k(x) = \sum_{n=1}^{k-1} x(n)e_n$ $(P_1 = 0)$. The basis is said to be premonotone for the norm $\|\cdot\|$ when $\|Q_k\| \leq 1$ for every $k \in \mathbb{N}$.

Given $k \in \mathbb{N}$ and $x \in X$, we write $k \leq x$ whenever $k \leq \min\{supp(x)\}$ and $k < x$ whenever $k < \min\{supp(x)\}$. We say that (y_n) is a block basic sequence for $\{e_n\}_n$ if it is bounded and there exist positive integers $p_1 \leq q_1 < p_2 \leq q_2 < ...$ such that y_n belongs to the span of $\{e_{p_n}, \cdots, e_{q_n}\}$ for every $n \in \mathbb{N}$.

The Schauder basis is said to be boundedly complete if $\sup_n \left\| \sum_{i=1}^n t_i e_i \right\| < +\infty$ implies that $\sum_{i=1}^\infty t_i e_i \in X$. When the Schauder basis (e_n) is boundedly complete, the Banach space X is isomorphic to a dual space Z^*, where Z is the closed subspace spanned by the biorthogonal functionals (e_n^*) in X^*. In this case, we can consider in X the weak* topology $\sigma(X,Z)$, for which the convergence coincides with the coordinate-to-coordinate convergence for norm-bounded sequences. Moreover, the closed unit ball is $\sigma(X,Z)$-sequentially compact and therefore every bounded sequence in X has a subsequence which converges coordinatewise (see for instance Theorem 3.2.10 in [1]). In what follows the weak* topology always refers to the $\sigma(X,Z)$ topology for Banach spaces with boundedly complete Schauder basis. In the case where $X = \ell_1$ endowed with the standard Schauder basis, this w^*-topology coincides with the $\sigma(\ell_1, c_0)$ topology.

Definition 2.1. [2] *A norm $|||\cdot|||$ on a Banach space X with a Schauder basis $\{e_n\}$ is said to be a sequentially separating norm if for every $\epsilon > 0$ there exists some $k \in \mathbb{N}$ such that*

$$|||x||| + \limsup_n |||x_n||| \leq (1+\epsilon) \limsup_n |||x + x_n|||$$

whenever $k \leq x$ and $(x_n)_n$ is a block basic sequence of $\{e_n\}_n$ in X.

Definition 2.2. *Let X be a Banach space with a Schauder basis $\{e_n\}_n$ and let $|||\cdot|||$ be a norm on X. This norm is called near-infinity concentrated (n.i.c.) if it has the following properties:*

(1) *It is a sequentially separating norm.*

(2) *It is premonotone.*

(3) *There exist $R_0 > 5$ and $M \in [0,1)$ such that for every $k \in \mathbb{N}$, there exists a function $F_k : (0,+\infty) \to [0,+\infty)$ satisfying the following conditions:*
   (a) $\lim_{\lambda\to 0^+} \dfrac{F_k(\lambda)}{\lambda} \leq \dfrac{M}{R_0}$.
   (b) *For every bounded pointwise-null sequence (x_n) with $\liminf_n |||x_n||| \geq 1$, for all $\lambda \in (0,+\infty)$, and for every $z \in X$ with $Q_k(z) = 0$ and $|||z||| \leq R_0$,*

$$\limsup_n |||x_n + \lambda z||| \leq \limsup_n |||x_n||| + F_k(\lambda)\, |||z||| .$$

Remark 2.3. *Observe that Property (3) can be re-written as: There exists $K \geq 0$ such that for every $k \in \mathbb{N}$, there exists a function $F_k : (0,+\infty) \to [0,+\infty)$ satisfying (a)' and (b); where condition (a)' is: $\lim_{\lambda\to 0^+} \dfrac{F_k(\lambda)}{\lambda} \leq\ K < \dfrac{1}{5}$.*

Given K, we may take $M := 1 - \frac{1-5K}{K+1} = \frac{6\,K}{K+1}$ and $R_0 := 5 + \frac{1-5K}{K+1} = \frac{6}{K+1}$.

Note that if $|||\cdot|||$ is an equivalent norm on ℓ_1 satisfying

$$a_k \|Q_k(x)\|_1 \leq |||Q_k(x)||| \leq b_k \|Q_k(x)\|_1 \ , \text{ for all } x \in \ell_1 \ ,$$

for every $k \in \mathbb{N}$, with $0 < a_k \le b_k$ and $\lim_k b_k/a_k = 1$, then it is clear that $|||\cdot|||$ is a sequentially separating norm. Nevertheless, there exist some equivalent norms on ℓ_1 which do not satisify this condition but they are still sequentially separating [2, Example 3.2]. Furthermore, there exist Banach spaces with sequentially separating norms that are not isomorphic to ℓ_1, although the existence of such a norm implies that the Banach space X is "similar" to ℓ_1, in the sense that it has the Schur property, and so is hereditarily ℓ_1 [2, Corollary 7.4]. Recall that a Banach space X is hereditarily ℓ_1 if each infinite dimensional closed subspace of X contains a further subspace isomorphic to ℓ_1. This implies, in particular, that if a Banach space with an unconditional Schauder basis has a sequentially separating norm, then the basis is boundedly complete, since otherwise X would contain an isomorphic copy of c_0 (see for instance [1, Theorem 3.3.2]).

Also note that in Definition 2.2, Property (3)(b), if (x_n) is an arbitrary sequence of "bump functions sliding towards infinity", each with their $|||\cdot|||$-norm asymptotically no less than 1, then

$$\frac{\limsup_n |||x_n + \lambda z||| - \limsup_n |||x_n|||}{\lambda}$$

is smaller than one would expect from just the triangle inequality: for all $z = P_k(z)$ with $|||z||| \le R_0$, for all λ positive and very small, the "upper asymptotic value" of the norm of x_n is changed less than expected when we perturb each, x_n by λz, since $F_k(\lambda)\, R_0/\lambda$ is approximately bounded by $M < 1$. In this sense, $|||\cdot|||$ is "near-infinity concentrated". Moreover, this third property prevents X from containing an asymptotically isometric copy of ℓ_1, which we will now prove.

Lemma 2.4. *Let X be a Banach space with a boundedly complete Schauder basis. If $|||\cdot|||$ is an equivalent norm in X satisfying property (3) in Definition 2.2, then $(X, |||\cdot|||)$ fails to have an a.i.c. of ℓ_1.*

Proof. Assume to the contrary that there exists a basic sequence (x_n) in X generating an a.i.c. of ℓ_1, that is, there is a decreasing sequence $(\epsilon_n) \subset (0,1)$ with $\lim_n \epsilon_n = 0$ such that

$$\sum_{n=1}^{\infty}(1-\epsilon_n)|t_n| \le \left|\left|\left|\sum_{n=1}^{\infty} t_n x_n\right|\right|\right| \le \sum_{n=1}^{\infty} |t_n| \ .$$

By extracting a subsequence, we can assume that (x_n) is w^*-convergent and, by replacing (x_n) by $((x_{2n} - x_{2n-1})/2))$, that it is w^*-convergent to the null vector. Finally, using the sliding hump method and the fact that asymptotically isometric copies are stable by adding norm-null sequences, we can assume that the sequence (x_n) generating the a.i.c. of ℓ_1 is a disjointly supported w^*-null sequence.

Take $R_0 > 5$ and $M \in [0,1)$ as in (3) of Definition 2.2. By omitting the first few terms of the sequence $(x_n)_n$, we can also assume that $\epsilon_1 < (R_0 - M)/R_0$.

From the previous inequalities $|||x_n||| \le 1$ for every $n \in \mathbb{N}$ and $\lim_n |||x_n||| = 1$. Let $k := 1 + \max\{supp(x_1)\}$. Since $|||\cdot|||$ satisfies property (3) of a near-infinity concentrated norm, there exists a function $F_k(\lambda)$ such that $\lim_{\lambda\to 0^+} F_k(\lambda)/\lambda \le \frac{M}{R_0}$, and for every $\lambda > 0$

$$\limsup_n |||x_n + \lambda R_0 x_1||| \le \limsup_n |||x_n||| + F_k(\lambda) R_0 |||x_1||| \le 1 + F_k(\lambda) R_0.$$

On the other hand, for every $n \ge 2$,

$$1 - \epsilon_n + \lambda R_0(1-\epsilon_1) \le |||x_n + \lambda R_0 x_1|||.$$

Letting n tend to infinity, we see that

$$1 + \lambda R_0(1-\epsilon_1) \le 1 + F_k(\lambda) R_0$$

and so $\lambda(1-\epsilon_1) \le F_k(\lambda)$ for every $\lambda > 0$. Letting $\lambda \to 0$, we get that $(1-\epsilon_1) \le \lim_{\lambda\to 0^+} \frac{F_k(\lambda)}{\lambda} \le \frac{M}{R_0}$, which implies that $R_0\,(1-\epsilon_1) \le M$, and this is a contradiction. □

Before stating our main result, we recall some standard arguments used to prove the FPP (see for instance [14] or [11]):

Let C be a closed bounded convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be a nonexpansive mapping. Using Banach's Contraction Mapping Theorem, we can always find a sequence $(x_n) \subset C$ such that $\lim_n \|x_n - Tx_n\| = 0$. Such sequences are called approximate fixed point sequences (a.f.p.s.). In fact, if (x_n) is an a.f.p.s. and $r > 0$, the set

$$\{x \in C : \limsup_n \|x_n - x\| \le r\}$$

is either empty, or a non-empty closed convex T-invariant subset of C, in which we can find new approximate fixed point sequences.

In a dual Banach space X with separable predual Y, every a.f.p.s. has a subsequence which is w^*-convergent. For example, if X is a Banach space with a boundedly complete Schauder basis $\{e_n\}_n$, and corresponding biorthogonal functionals $\{f_n\}_n \subset X^*$, then for $Y :=$ the closed linear span of $\{f_n\}_n$ in X^*, Y^* is isomorphic to X and every a.f.p.s. in X has a subsequence that is $\sigma(X,Y)$-convergent. Therefore, we will subsequently assume that approximate fixed point sequences in bounded subsets of X are w^*-convergent.

Using Cantor's theorem (see [14] or [11, Lemma 1]), the above argument lets us deduce that if $T : C \to C$ is a fixed point free nonexpansive mapping, there exist some $a > 0$ and a closed convex T-invariant subset, denoted again by C, such that $\limsup_n \|y_n - y\| > a$ whenever $(y_n) \subset C$ is an a.f.p.s. and $y = w^*\text{-}\lim y_n$.

Note that from Definition 2.1 it is not difficult to check the following [2]:

Lemma 2.5. *A norm $|||\cdot|||$ in a Banach space X with a Schauder basis $\{e_n\}_n$ is sequentially separating if and only if $\lim_k S_k(X, |||\cdot|||) = 1$, with*

$$S_k(X, |||\cdot|||) := \sup\left\{\frac{|||x||| + \limsup_n |||x_n|||}{\limsup_n |||x + x_n|||}\right\},$$

where the supremum is taken over all vectors $x \in X$ with $k \le x$ and all block basic sequences of $\{e_n\}_n$.

If we fix the norm $|||\cdot|||$ in the Banach space X, we will use S_k to denote $S_k(X, |||\cdot|||)$.

Lemma 2.6. *Let $(X, |||\cdot|||)$ be a Banach space with a boundedly complete Schauder basis $\{e_n\}_n$ such that $|||\cdot|||$ is premonotone and sequentially separating. The following holds: if (x_n), (y_n) are two sequences in X that are w^*-convergent to x and y respectively, then*

$$\limsup_m \limsup_n |||x_n - y_m||| \ge \limsup_n |||x_n - x||| + \limsup_m |||y_m - y|||.$$

Proof. Let $k \in \mathbb{N}$ and $\delta > 0$ be given. Choose a subsequence (x_{n_ℓ}) of (x_n) such that

$$\limsup_n |||Q_k(x_n - x)||| = \lim_\ell |||Q_k(x_{n_\ell} - x)|||.$$

Fix $m \in \mathbb{N}$. Then

$$\begin{aligned}\limsup_n |||x_n - y_m||| &\ge \limsup_n |||Q_k(x_n - y_m)||| \\ &= \limsup_n |||Q_k(x_n - x) - Q_k(y_m - x)||| \\ &\ge \limsup_\ell |||Q_k(x_{n_\ell} - x) - Q_k(y_m - x)|||.\end{aligned}$$

By the Bessaga-Pełczynski Selection Principle (see, for example, [1, p. 14]), passing to a further subsequence if necessary, we may choose a block basic sequence (u_ℓ) of (e_n) such that $Q_k(u_\ell) = u_\ell$ and $|||u_\ell - Q_k(x_{n_\ell} - x)||| < \delta$. Then

$$\begin{aligned}
\limsup_n |||x_n - y_m||| &\geq \limsup_\ell |||u_\ell - Q_k(y_m - x)||| - \delta \\
&\geq \frac{1}{S_k}\left(|||Q_k(y_m - x)||| + \limsup_\ell |||u_\ell|||\right) - \delta \\
&\geq \frac{1}{S_k}\left(|||Q_k(y_m - x)||| + \lim_\ell |||Q_k(x_{n_\ell} - x)|||\right) - 2\delta \\
&= \frac{1}{S_k}\left(|||Q_k(y_m - x)||| + \limsup_n |||Q_k(x_n - x)|||\right) - 2\delta\,.
\end{aligned}$$

Letting m tend to ∞; noting that $S_k \geq 1$; and using a perturbation argument similar to the one above then yields:

$$\begin{aligned}
\limsup_m \ \limsup_n |||x_n - y_m||| &\geq \frac{1}{S_k}\left(\limsup_m \ |||Q_k(y_m - x)||| + \limsup_n |||Q_k(x_n - x)|||\right) - 2\delta \\
&= \frac{1}{S_k}\left(\limsup_n |||x_n - x||| + \limsup_m \ |||Q_k(y_m - y) + Q_k(y - x)|||\right) - 2\delta \\
&\geq \frac{1}{S_k}\limsup_n |||x_n - x||| + \frac{1}{S_k^2}\left(|||Q_k(y - x)||| + \limsup_m |||Q_k(y_m - y)|||\right) - 4\delta \\
&\geq \frac{1}{S_k}\limsup_n |||x_n - x||| + \frac{1}{S_k^2}\limsup_m |||Q_k(y_m - y)||| - 4\delta \\
&= \frac{1}{S_k}\limsup_n |||x_n - x||| + \frac{1}{S_k^2}\limsup_m |||y_m - y||| - 4\delta\,.
\end{aligned}$$

In the above calculation, we used the fact that $\limsup_n |||Q_k(x_n - x)||| = \limsup_n |||x_n - x|||$ and $\limsup_m |||Q_k(y_m - y)||| = \limsup_m |||y_m - y|||$ for every $k \in \mathbb{N}$.

Since the above inequalities hold for every $k \in \mathbb{N}$, letting k tend to infinity gives

$$\limsup_m \ \limsup_n |||x_n - y_m||| \geq \limsup_n |||x_n - x||| + \limsup_m |||y_m - y||| - 4\delta\,,$$

for every $\delta > 0$. Since $\delta > 0$ is arbitrary, we obtain the desired inequality.

□

Theorem 2.7. *Let X be a Banach space with a boundedly complete Schauder basis and let $|||\cdot|||$ be a near-infinity concentrated (n.i.c.) norm on X. Then $(X, |||\cdot|||)$ has the FPP, that is, every nonexpansive self-map on a closed bounded convex subset of X has a fixed point.*

Proof. Assume, to the contrary, that there exists a closed bounded convex subset C of X and $T : C \to C$ a nonexpansive mapping such that

$$b = \inf\{\limsup_n |||y_n - y||| : (y_n) \subset C \text{ is an a.f.p.s. and } y_n \xrightarrow{w^*} y\} > 0.$$

Without loss of generality we can assume that $b = 1$. We proceed as follows.

Fix some $0 < \epsilon_1 < \min\{\frac{1}{4}(1 - \frac{M+1}{2}), \frac{1}{10}(R_0 - 5)\}$, where $M \in [0, 1)$ and $R_0 > 5$ are the constants given by condition (3) in Definition 2.2.

Consider an a.f.p.s. (x_n) in C such that $x_n \xrightarrow{w^*} x_0 \in X$ and $\limsup_n |||x_n - x_0||| < 1 + \epsilon_1$. Again, without loss of generality, we can assume that $x_0 = 0$ so that $\limsup_n |||x_n||| < 1 + \epsilon_1$. Define the set

$$D := \left\{z \in C : \limsup_n |||x_n - z||| \leq 2(1 + \epsilon_1)\right\}.$$

Then D is a closed convex T-invariant subset of C. Moreover, using the triangle inequality,

$$\limsup_m \limsup_n |||x_n - x_m||| \leq 2 \limsup_n |||x_n||| < 2(1+\epsilon_1)\,;$$

so D is not empty and we can assume that $x_n \in D$ for n large enough. Define

$$c := \inf \left\{ \limsup_n |||y_n - y||| : (y_n) \subset D \text{ is an a.f.p.s. and } y_n \overset{w^*}{\to} y \right\} .$$

Notice that $1 \leq c$.

To simplify the notation, we define

$$A^*(D) := \left\{ y \in X : \exists\, (y_n) \subset D \text{ an a.f.p.s. such that } w^*\text{-}\lim_n y_n = y \right\}.$$

We now prove that $\sup_{y \in A^*(D)} |||y||| \leq 4 + 4\epsilon_1$:

Indeed, let $(y_n) \subset D$ be an a.f.p.s. with w^*-$\lim y_n = y$. In particular, $\limsup_n |||x_n - y_m||| \leq 2(1+\epsilon_1)$ for every $m \in \mathbb{N}$. Using the triangle inequality and Lemma 2.6 (with $x = 0$):

$$|||y||| \leq \limsup_m |||y - y_m||| + \limsup_n |||x_n||| + \limsup_m \limsup_n |||x_n - y_m||| \leq 4(1+\epsilon_1).$$

Next we show that $\sup_{y \in A^*(D)} |||Q_k(y)||| \leq \mu_k := 2S_k^2(1+\epsilon_1) - S_k - 1$ for every $k \in \mathbb{N}$. Using the proof of Lemma 2.6 we deduce that:

$$2(1+\epsilon_1) \geq \limsup_m \limsup_n |||x_n - y_m||| \geq$$

$$\frac{1}{S_k} \limsup_n |||x_n||| + \frac{1}{S_k^2} \left[\limsup_m |||y_m - y||| + |||Q_k(y)||| \right] \geq \frac{1}{S_k} + \frac{1}{S_k^2} \left[1 + |||Q_k(y)||| \right],$$

which implies that

$$|||Q_k(y)||| \leq 2S_k^2(1+\epsilon_1) - S_k - 1$$

for all $y \in A^*(D)$.

Choose $x := x_\nu$ with $\nu \in \mathbb{N}$ large enough so that $x \in D$ and $|||x||| < 1 + \epsilon_1$. Since the norm satisfies condition (1) in Definition 2.2, we know that $\lim_k S_k = 1$ and therefore $\lim_k \mu_k = 2\epsilon_1$. Take $k_1 \in \mathbb{N}$ so that

$$|||Q_k(x)||| < \epsilon_1, \quad \text{and} \quad \mu_k < 3\epsilon_1$$

if $k \geq k_1$. In particular, this implies that

$$|||Q_{k_1}(y-x)||| \leq |||Q_{k_1}(y)||| + |||Q_{k_1}(x)||| \leq 3\epsilon_1 + \epsilon_1 = 4\epsilon_1 < 1,$$

and

$$\begin{aligned} |||P_{k_1}(y-x)||| &\leq |||x-y||| + |||Q_{k_1}(y-x)||| \\ &\leq |||x||| + |||y||| + 4\epsilon_1 \\ &\leq 1 + \epsilon_1 + 4 + 4\epsilon_1 + 4\epsilon_1 = 5 + 9\epsilon_1 < R_0 \end{aligned}$$

for every $y \in A^*(D)$.

Given $k_1 \in \mathbb{N}$ as before, there exists a corresponding function $F(\lambda) := F_{k_1}(\lambda)$ satisfying property (3) in Definition 2.2. Since $\lim_{\lambda \to 0^+} \frac{F(\lambda)}{\lambda} \leq \frac{M}{R_0}$, take $\lambda \in (0,1)$ such that

$$\frac{F(\lambda)}{\lambda} < \frac{M+1}{2R_0} < \frac{1-4\epsilon_1}{R_0} \leq \frac{c-4\epsilon_1}{R_0},$$

which implies that

$$(2-\lambda)c + F(\lambda)R_0 + \lambda 4\epsilon_1 < 2c.$$

Now, choose $\epsilon_2 > 0$ with

$$(2-\lambda)(c+\epsilon_2) + F(\lambda)R_0 + \lambda 4\epsilon_1 < 2c.$$

Choose $(y_n) \subset D$ an a.f.p.s. with w^*-$\lim_n y_n = y$ and such that

$$\limsup_n |||y_n - y||| \le c + \epsilon_2.$$

By passing to a subsequence, we may also suppose that

$$\liminf_n |||y_n - y||| = \limsup_n |||y_n - y||| \ge c \ge 1.$$

Notice that, for every $m \in \mathbb{N}$, the vectors $(1-\lambda)y_m + \lambda x \in D$. We claim that

$$(**) \qquad \limsup_m \limsup_n |||y_n - [(1-\lambda)y_m + \lambda x]\,||| < 2c.$$

Assume that $(**)$ holds. Then we can find some $m \in \mathbb{N}$ such that

$$\limsup_n |||y_n - [(1-\lambda)y_m + \lambda x]\,||| < 2c.$$

This implies that for some $r \in (0, 2c)$ the set

$$G := \{z \in D : \limsup_n |||y_n - z||| \le r\}$$

is a nonempty closed convex T-invariant subset of D, and therefore it contains an a.f.p.s. (z_s), which tends to some $z \in X$ with respect to the w^*-topology. In this case, using the definition of c, Lemma 2.6, and that each $z_s \in G$, we have

$$2c \le \limsup_s |||z_s - z||| + \limsup_n |||y_n - y||| \le \limsup_s \limsup_n |||y_n - z_s||| \le r\,,$$

which is a contradiction.

We finish by proving the claim $(**)$. Noting that $\liminf_n |||y_n - y||| \ge 1$ and by property (3) in Definition 2.2, we have:

$$\begin{aligned}\limsup_n |||(y_n - y) + \lambda P_{k_1}(y-x)||| &\le \limsup_n |||y_n - y||| + F(\lambda)|||P_{k_1}(y-x)|||\\ &\le c + \epsilon_2 + F(\lambda)R_0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\limsup_m \limsup_n |||y_n - [(1-\lambda)y_m + \lambda x]\,|||\\ &= \limsup_m \limsup_n |||y_n - y + y - (1-\lambda)y_m - \lambda x|||\\ &= \limsup_m \limsup_n |||y_n - y + (1-\lambda)y + \lambda y - (1-\lambda)y_m - \lambda x|||\\ &\le \limsup_m \limsup_n [(1-\lambda)|||y_m - y||| + |||(y_n - y) + \lambda(y-x)|||]\\ &\le (1-\lambda)\limsup_m |||y_m - y||| + \limsup_n |||(y_n - y) + \lambda P_{k_1}(y-x)||| + \lambda|||Q_{k_1}(y-x)|||\\ &\le (1-\lambda)(c+\epsilon_2) + c + \epsilon_2 + F(\lambda)R_0 + \lambda 4\epsilon_1\\ &\le (2-\lambda)(c+\epsilon_2) + F(\lambda)R_0 + \lambda 4\epsilon_1\\ &< 2c\,,\end{aligned}$$

which proves $(**)$, and completes the proof of the theorem.

□

3. Norms with the Fixed Point Property

Throughout this section, we will study several examples of norms which are near-infinity concentrated norms and therefore they satisfy the FPP according to Theorem 2.7. As a particular case of a more general result, we will deduce that $(\ell_1, \nu_p(\cdot))$ has the FPP whenever $\nu_p(\cdot)$ is a renorming of ℓ_1.

We will start by proving that P.K. Lin's norm is an example of a near-infinity concentrated norm. We will deduce this assertion from the following lemma.

Lemma 3.1. *Let $(X, |\cdot|)$ be a Banach space with a Schauder basis and assume that $|\cdot|$ satisfies properties (1) and (2) in Definition 2.1. If (γ_k) is a nondecreasing sequence in $(0,1)$ converging to 1, then the norm defined as*

$$|x|_1 := \sup_k \gamma_k |Q_k(x)|\ , \ \textit{for all } x \in X\ ,$$

is a near-infinity concentrated norm on X that is equivalent to $|\cdot|$.

Proof. Notice that $\gamma_k|Q_k(x)| \le |Q_k(x)|_1 \le |Q_k(x)|$ for every $k \in \mathbb{N}$ and $x \in X$, which implies that $|\cdot|_1$ is a sequentially separating norm whenever $|\cdot|$ satisfies the same property. It is also easy to check that $|\cdot|_1$ satisfies (2) in Definition 2.2. It remains to prove condition (3). Fix some $k \in \mathbb{N}$ and $R > 0$. Let (x_n) be a bounded pointwise-null sequence in X with $\liminf_n |x|_1 \ge 1$. Without loss of generality we can assume that $Q_l(x_n) = Q_k(x_n)$ for every $l \le k$. Moreover, it is not difficult to check that $\limsup_n |x_n| = \limsup_n |x_n|_1$. For every $z \in X$ with $Q_k(z) = 0$, $|z|_1 \le R$ and for every $\lambda > 0$ we have

$$\begin{aligned} |x_n + \lambda z|_1 &= \sup_l \gamma_l |Q_l(x_n + \lambda z)| = \sup_l \gamma_l |Q_l(x_n) + \lambda Q_l(z)| \\ &= \max\left\{ \max_{1 \le l \le k-1} \gamma_l |Q_l(x_n) + \lambda Q_l(z)|, \sup_{l \ge k} \gamma_l |Q_l(x_n)| \right\} \\ &= \max\left\{ \max_{1 \le l \le k-1} \gamma_l |Q_l(x_n) + \lambda Q_l(z)|, |x_n|_1 \right\} \\ &\le \max\{\gamma_{k-1}|x_n| + \lambda |z|_1, |x_n|_1\} \end{aligned}$$

Taking limits when n goes to infinity:

$$\limsup_n |x_n + \lambda z|_1 \le \max\{\gamma_{k-1} \limsup_n |x_n|_1 + \lambda |z|_1, \limsup_n |x_n|_1\}.$$

From above, $\limsup_n |x_n|_1 \ge 1$ and $|z|_1 \le R$; and so

$$\limsup_n |x_n + \lambda z|_1 \le \limsup_n |x_n|_1 + F_k(\lambda)|z|_1 \ ,$$

where $F_k(\lambda) := 0$ if $\lambda \le (1 - \gamma_{k-1})/R$, and $F_k(\lambda) := \lambda$ otherwise. Taking $M = 0$ and any $R > 5$ in Definition 2.2(3), we see that $|\cdot|_1$ is a near infinity concentrated norm. □

If we let $|\cdot| := \|\cdot\|_1$ in ℓ_1 we obtain that $|\cdot|_1$ coincides with P.K. Lin's norm $|||\cdot|||_L$. Moreover, given a norm $|\cdot|_0 := |\cdot|$ satisfying (1) and (2) in Definition 2.2 and defining in a recursive way the equivalent norms

$$|\cdot|_n = \sup_k \gamma_k |Q_k(\cdot)|_{n-1}$$

for every $n \in \mathbb{N}$, we can construct sequences of near-infinity concentrated norms. All of these norms $|\cdot|_n$ $(n \ge 1)$ satisfy the FPP when the basis is boundedly complete, according to Theorem 2.7.

Lemma 3.2. *Assume that $(p_n) \subset (1, +\infty)$ is a nonincreasing sequence with $\lim_n p_n = 1$. Then the norm $\nu_p(\cdot)$ is a near-infinity concentrated norm in the Banach space X, defined as the completion of c_{00} with the norm $\nu_p(\cdot)$.*

Proof. Let us start by proving that $\nu_p(\cdot)$ is a sequentially separating norm, that is, $\nu_p(\cdot)$ satisfies property (1) in Definition 2.2. By Lemma 2.5, it suffices to check that $\lim_k S_k(X, \nu_p(\cdot)) = 1$.

Fix $k \in \mathbb{N}$. First note that if $x = \sum_{i=k}^{l} x(i) e_i$ with $k \le l$ and $l < y$ then

$$\nu_p(x + y) = \nu_p\left(\sum_{i=k}^{l} x(i) e_i + \nu_p(y) e_{l+1} \right) .$$

On the other hand, it is not difficult to check that if $1 < p \le q$ and $a, b, c \ge 0$ then:

$$\|(a, \|(b,c)\|_p)\|_q \ge \|(a, \|(b,c)\|_q)\|_q = \left(a^q + \|(b,c)\|_q^q\right)^{1/q} = (a^q + b^q + c^q)^{1/q} =$$
$$= \|(\|(a,b)\|_q, c)\|_q \, .$$

Using now the definition of the norm $\nu_p(\cdot)$ and taking into account that

$$\cdots p_n \leq p_{n-1} \leq \cdots p_2 \leq p_1,$$

it can recursively be checked that for $x = \sum_{i=k}^{l} x(i)e_i$ with $k \leq l$ and $l < y$

$$\begin{array}{rcl} \nu_p(x+y) & = & \nu_p\left(\sum_{i=k}^{l} x(i)e_i + \nu_p(y)e_{l+1}\right) \\ & \geq & \|(\nu_p(x), \nu_p(y))\|_{p_k} \\ & \geq & C_k(\nu_p(x) + \nu_p(y)) \end{array} \qquad (\dagger)$$

where $C_k := 2^{-1+1/p_k}$.

Let (x_n) be a block basic sequence in X and let $x \in X$ with $Q_k(x) = x$. If $\epsilon > 0$ is given, choose $k' > k$ such that

$$\nu_p(Q_{k'}(x)) < \epsilon\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right).$$

Without loss of generality, we can choose n large enough so that $k' < x_n$. Applying $(\dagger)$ to the vectors $x_0 := x - Q_{k'}(x)$, $x_1 := Q_{k'}(x)$, and x_n, we obtain:

$$\begin{array}{rcl} \limsup_n \nu_p(x + x_n) & \geq & \limsup_n \nu_p(x_0 + x_n) - \nu_p(x_1) \\ & \geq & \limsup_n C_k\left(\nu_p(x_0) + \nu_p(x_n)\right) - \nu_p(x_1) \\ & \geq & C_k\left(\nu_p(x) - \nu_p(x_1) + \limsup_n \nu_p(x_n)\right) - \nu_p(x_1) \\ & = & C_k\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right) - (C_k + 1)\nu_p(x_1) \\ & \geq & C_k\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right) - (C_k + 1)\epsilon\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right) \\ & = & \left(C_k - \epsilon(C_k + 1)\right)\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right) \end{array}$$

Since $\epsilon > 0$ is arbitrary,

$$\limsup_n \nu_p(x + x_n) \geq C_k\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right).$$

Then, by the definition of the coefficient $S_k(X, \nu_p(\cdot))$, we deduce that

$$S_k(X, \nu_p(\cdot)) \leq \frac{1}{C_k}.$$

Taking limits as k goes to infinity and using Lemma 2.5 shows that $\nu_p(\cdot)$ is a sequentially separating norm.

It is clear that $\nu_p(\cdot)$ satisfies (2) in Definition 2.2. Therefore, it remains to prove (3) in Definition 2.2.

Fix $k \in \mathbb{N}$. Consider the equivalent finite dimensional Banach spaces $(\mathbb{R}^k, \|\cdot\|_{p_k})$ and $(\mathbb{R}^k, \nu_p(\cdot))$. Take some constant $L_k > 0$ such that $\|x\|_{p_k} \leq L_k \nu_p(x)$ for every $x \in \mathbb{R}^k$.

Let (x_n) be a bounded pointwise-null sequence with $\liminf_n \nu_p(x_n) \geq 1$, and $z \in X$ with $Q_k(z) = 0$ and $\nu_p(z) \leq R$ for some $R > 0$. We can assume, without loss of generality, that $P_k(x_n) = 0$ for every $n \in \mathbb{N}$.

Having in mind that $(p_n)_n$ is a nonincreasing sequence, it is not difficult to check that for all $\lambda \in (0, +\infty)$,

$$\begin{array}{rcl} \nu_p(x_n + \lambda z) & \leq & \left(\lambda^{p_k}|z(1)|^{p_k} + \cdots \lambda^{p_k}|z(k-1)|^{p_k} + \nu_p(x_n)^{p_k}\right)^{1/p_k} \\ & = & \nu_p(x_n)\left[\dfrac{\lambda^{p_k}}{\nu_p(x_n)^{p_k}}\left(|z(1)|^{p_k} + \cdots |z(k-1)|^{p_k}\right) + 1\right]^{1/p_k} \\ & = & \nu_p(x_n)\left[\dfrac{\lambda^{p_k}}{\nu_p(x_n)^{p_k}}\|z\|_{p_k}^{p_k} + 1\right]^{1/p_k}. \end{array}$$

It is easy to check that $(1+v)^\alpha \le 1+\alpha v$ if $0<v$ and $0<\alpha<1$. Therefore

$$\begin{aligned}\nu_p(x_n+\lambda z) &\le \nu_p(x_n)\left[\frac{1}{p_k}\frac{\lambda^{p_k}}{\nu_p(x_n)^{p_k}}\|z\|_{p_k}^{p_k}+1\right]\\ &= \nu_p(x_n)+\frac{1}{p_k}\frac{\lambda^{p_k}}{\nu_p(x_n)^{p_k-1}}\|z\|_{p_k}^{p_k}\\ &\le \nu_p(x_n)+\frac{1}{p_k}\frac{\lambda^{p_k}}{\nu_p(x_n)^{p_k-1}}L_k^{p_k}\nu_p(z)^{p_k}.\end{aligned}$$

Then, for every $\lambda>0$,

$$\begin{aligned}\limsup_n \nu_p(x_n+\lambda z) &\le \limsup_n \nu_p(x_n)+\frac{1}{p_k}\frac{\lambda^{p_k}}{\liminf_n \nu_p(x_n)^{p_k-1}}L_k^{p_k}\nu_p(z)^{p_k-1}\nu_p(z)\\ &\le \limsup_n \nu_p(x_n)+\frac{1}{p_k}\lambda^{p_k}L_k^{p_k}R^{p_k-1}\nu_p(z).\end{aligned}$$

Define

$$F_k(\lambda):=\frac{1}{p_k}\lambda^{p_k}L_k^{p_k}R^{p_k-1}\ ,$$

for every $\lambda>0$. Since, $\lim_{\lambda\to 0^+}F_k(\lambda)/\lambda=0$, we can take $M=0$ in property (3) of Definition 2.2 and $\nu_p(\cdot)$ is a near-infinity concentrated norm whenever $\lim_n p_n=1$. □

Corollary 3.3. *For every nonincreasing sequence $(p_n)\subset(1,+\infty)$ with $\lim_n p_n=1$, the Banach space $(X,\nu_p(\cdot))$ satisfies the FPP. In particular, if the sequence of conjugates of (p_n) satisfies that $q_n\ge\delta\log n$ for all $n\in\mathbb{N}$ and some $\delta>0$, the norm $\nu_p(\cdot)$ is an equivalent norm in ℓ_1 with the FPP.*

Remark 3.4. We would like to point out that there exist some norms verifying the FPP which are not near-infinity concentrated. For instance we can consider the Banach space ℓ_1 and the equivalent norm $|x|_1=|||P_A(x)|||_L+|||P_B(x)|||_L$ where $A=\{2n:n\ge 1\}$, $B=\{2n-1:n\ge 1\}$, P_A,P_B denote the corresponding projections onto the subspaces $[e_n:n\in A]$ and $[e_n:n\in B]$ respectively and $|||\cdot|||_L$ denotes P.K. Lin's norm as usual. It is easy to check that $(\ell_1,|\cdot|_1)$ is isometric to the space $(\ell_1,|||\cdot|||_L)\oplus_1(\ell_1,|||\cdot|||_L)$, which has the FPP according to [6]. However, $|\cdot|_1$ does not satisfies condition (3) in Definition 2.2.

A similar situation occurs when we consider the ℓ_1-renorming $|x|_1=\|x\|_1+|||x|||_L$. From [10] we know that $(\ell_1,\|x\|_1+|||x|||_L)$ verifies the FPP. However, condition (3) in Definition 2.2 also fails in this example.

Nevertheless, by means of Theorem 2.7, we can obtain a stability result in the following sense.

Theorem 3.5. *Let $|||\cdot|||$ be a near-infinity concentrated norm on a Banach space X with a Schauder basis $\{e_n\}_n$, and let $\|\cdot\|$ be an equivalent norm satisfying conditions (1) and (2) in Definition 2.2. Then there exists $r_0>0$ such that the norm $|\cdot|=|||\cdot|||+r\|\cdot\|$ is also near-infinity concentrated for every $0\le r\le r_0$.*

Moreover, if $\{e_n\}_n$ is boundedly complete, then the spaces $(X,|\cdot|)$ have the FPP for every $0\le r\le r_0$.

Proof. Since $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms, there exist $0<a\le b$ such that $a|||x|||\le\|x\|\le b|||x|||$ for every $x\in X$. Therefore, for all $x\in X$,

$$(1+ar)|||x|||\le|x|\le(1+rb)|||x|||\ \text{and}\ \left(\frac{1}{b}+\frac{ra}{b}\right)\|x\|\le|x|\le\left(\frac{1}{a}+\frac{rb}{a}\right)\|x\|\,.$$

It is easy to check that $|\cdot|$ satisfies conditions (1) and (2). Let us prove (3). By hypotheses, there exists some $R_0>5$ and $M\in[0,1)$ satisfying Definition 2.2.3 for the $|||\cdot|||$ norm. Take some $M'\in(M,1)$. Let (x_n) be a bounded pointwise-null sequence with $\liminf_n|x_n|\ge 1$ and let $z\in X$ with $Q_k(z)=0$ and $|z|\le R_0$. In this case, $\liminf_n |||x_n||| \geq 1/(1+rb)$ and $|||z||| \leq R_0/(1+ar)$. Fix $\lambda \in (0,\infty)$. Then there exists a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$\Gamma := \limsup_n |x_n + \lambda z| = \lim_j |x_{n_j} + \lambda z| \,.$$

Taking further subsequences if necessary, we may assume without loss of generality, that the limits

$$\lim_j |||x_{n_j}||| \ , \ \lim_j \|x_{n_j}\| \ , \ \lim_j |||x_{n_j} + \lambda z||| \text{ and } \lim_j \|x_{n_j} + \lambda z\|$$

all exist in $[0,\infty)$. Therefore,

$$\begin{aligned}
\Gamma &= r\lim_j \|x_{n_j} + \lambda z\| + \lim_j |||x_{n_j} + \lambda z||| \\
&\leq r\lim_j \|x_{n_j}\| + \lambda r\|z\| \\
&\quad + \frac{1}{1+rb}\lim_j |||(1+rb)x_{n_j} + \lambda\frac{1+rb}{1+ar}(1+ar)z||| \\
&\leq r\lim_j \|x_{n_j}\| + \lambda r\|z\| \\
&\quad + \limsup_j |||x_{n_j}||| + F_k\left(\frac{\lambda(1+rb)}{1+ar}\right)\frac{1+ar}{1+rb}|||z||| \\
&= \lim_j |x_{n_j}| + \lambda r\|z\| + F_k\left(\frac{\lambda(1+rb)}{1+ar}\right)\frac{1+ar}{1+rb}|||z||| \\
&\leq \limsup_n |x_n| + \lambda\frac{r}{(\frac{1}{b}+\frac{ra}{b})}|z| + G(\lambda)\frac{1}{1+ar}|z| \\
&= \limsup_n |x_n| + \left[\lambda\frac{r}{(\frac{1}{b}+\frac{ra}{b})} + G(\lambda)\frac{1}{1+ar}\right]|z|,
\end{aligned}$$

where $G(\lambda) := F_k\left(\lambda\frac{(1+rb)}{(1+ar)}\right)\left(\frac{1+rb}{1+ar}\right)^{-1}$. Define the corresponding "F_k-type" function for the $|\cdot|$ norm by

$$F_k'(\lambda) := \lambda\frac{r}{(\frac{1}{b}+\frac{ra}{b})} + G(\lambda)\frac{1}{1+ar} \ , \text{ for all } \lambda \in (0,\infty)\,.$$

Since $\lim_{\lambda\to 0^+} G(\lambda)/\lambda \leq \frac{M}{R_0}$, we have that

$$\lim_{\lambda\to 0^+}\frac{F_k'(\lambda)}{\lambda} \leq \frac{r}{(\frac{1}{b}+\frac{ra}{b})} + \frac{1}{1+ar}\frac{M}{R_0} := g(r)$$

Notice that $\lim_{r\to 0} g(r) = M/R_0$, which implies that there exists some $r_0 > 0$ and $M' \in (M,1)$ (depending on the constants a, b) such that for every $0 \leq r \leq r_0$, $g(r) \leq M'/R_0$; and so $|||\cdot||| + r\|\cdot\|$ is a near-infinity concentrated norm.

The rest of the theorem follows by applying Theorem 2.7. $\square$

If we proceed as in the previous proof, using the above arguments and the fact that $\lim_{\lambda\to 0^+}\frac{F_k(\lambda)}{\lambda} = 0$ for both P.K. Lin's norm and the $\nu_p(\cdot)$ norm, it is not difficult to check that, in the case where $\nu_p(\cdot)$ is equivalent to the $\|\cdot\|_1$-norm, $\nu_p(\cdot) + \lambda|||\cdot|||_L$ is a renorming in ℓ_1 which is also near-infinity concentrated for every $\lambda > 0$. Therefore we can also deduce (see also [2, Section 4]):

Corollary 3.6. *Let (p_n) be a nonincreasing sequence in $(1,+\infty)$ such that there exists some $\delta > 0$ so that $q_n \geq \delta \log n$ for all $n \in \mathbb{N}$. Then $(\ell_1, \nu_p(\cdot) + \lambda|||\cdot|||_L)$ has the FPP for every $\lambda \geq 0$.*

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