Soliton dynamics in optical fibers using the generalized traveling-wave method

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The generalized traveling wave method (GTWM) is applied to the nonlinear Schrödinger (NLS) equation with general perturbations in order to obtain the equations of motion for an *ansatz* with six collective coordinates, namely the soliton position, the amplitude, the inverse of the soliton width, the velocity, the chirp, and the phase. The advantage of the new ansatz is that it yields three pairs of canonically conjugated coordinates and momenta that all are well-behaved. The new ansatz is applied to model the dynamics of a soliton in a dispersion-shifted optical fiber described by the generalized NLS, including dissipation, higher-order dispersion, Raman scattering, and self-steepening perturbations. It is shown that the GTWM is equivalent to the modified method of moments, which considers the time variation of the norm, the first and the second moment of the norm, the momentum, the first moment of the momentum, and the energy for the perturbed NLS equation.

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I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation is one of the paradigms of soliton physics because it represents a completely integrable system and has numerous applications in practically all fields of physics, which include charge density waves [1], long Josephson junctions [2], optical fibers [3–5], plasmas driven by RF fields [6], and Bose-Einstein condensates [7,8] (see also review articles [9–11]). In particular, in optical fibers, the perturbed NLS equation,

$$i u_t + \sigma u_{xx} + \gamma_0 |u|^2 u = R[u(x,t);x,t], \tag{1}$$

is investigated [12–14], where $\beta_2 = -2\sigma$ is the group velocity dispersion and γ_0 is the nonlinear parameter responsible for self-phase modulation. The complex function R represents many different kinds of perturbations and may also depend on $u^\star$ and the spatial derivatives of u and $u^\star$. For sufficiently small perturbations it is usually assumed that the dynamics of a single soliton can be approximately described by an ansatz in the form of the 1-soliton solution of the unperturbed NLS equation, where the parameters of that solution become time-dependent unknown variables, the so-called collective coordinates (CCs). This ansatz is given by [9]

$$\begin{aligned} u(x,t) &= 2\, i\, \eta \sqrt{\frac{2\sigma}{\gamma_0}}\, \mathrm{sech}[2\eta(x-\zeta)] \exp(-i\Theta), \\ \Theta &= 2\xi x + \phi, \end{aligned} \tag{2}$$

with the four CCs: $\eta(t)$, $\zeta(t)$, $\xi(t)$, and $\phi(t)$. By specifying perturbation $R = a \exp[i K(t) x]$ without dissipation, and using the Lagrangian approach, a set of ordinary differential equations (ODEs) for the four CCs was developed [15]. This driving term was already used in the discrete form to model an array of coupled nonlinear optical wave guides, in which discrete cavity solitons can be excited [16]. Although in the case of this external driving force, the numerical solution of the CC equations predicted soliton dynamics that was confirmed by simulations of the perturbed NLS equation [15], the ansatz Eq. (2), from the physical point of view, presents certain disadvantages: First, when the ansatz Eq. (2) is inserted into the Lagrangian density and an integration over x is performed, the Lagrangian (L) is obtained as a function of the CCs and the time derivatives $\dot\phi$ and $\dot\xi$. The canonical momentum $dL/d\dot\phi$ can be identified with the norm $N = \int dx |u|^2$. However, the canonical momentum $dL/d\dot\xi$ does not have any obvious physical interpretation. Second, the Hamiltonian as a function of canonically conjugated variables can be obtained only after a complicated transformation. Finally, when the forcing is time-independent, that is, $K(t) =$ constant, then the CCs $\eta(t)$ and $\xi(t)$ perform periodic oscillations, and $\zeta(t)$ has a linear term and oscillations around it. However, $\phi(t)$ exhibits oscillations with a *growing* amplitude around a linear term.

The above disadvantages can be avoided through the use of a slightly different ansatz [17]: Θ in Eq. (2) is replaced by $\Theta = 2\xi(x-\zeta) + \Phi$. By using this modified ansatz, the new Lagrangian depends on $\dot\Phi$ and $\dot\zeta$. The canonical momentum $dL/d\dot\Phi$ can again be identified with the norm. The second canonical momentum $dL/d\dot\zeta$ now has a physical interpretation: namely, it can be identified with the field momentum. The Hamiltonian is obtained by a simple Legendre transformation and $\Phi(t)$ performs oscillations with *constant* amplitude, as do the other CCs. Remarkably, this new ansatz allows one to calculate a so-called "phase portrait" in which the soliton dynamics is described by a point moving on a curve in the complex plane. The "phase portrait" makes sense only when this curve is closed, then the point moves on the same orbit in each period. Interestingly, the shape of the orbit allows a prediction about the stability of the soliton, which is indeed confirmed by simulations [17].

One of the main goals of our paper is to show that similar disadvantages to those mentioned above appear in collective-coordinate theories for optical solitons and can also be avoided in a similar way as above. Indeed, in the literature on optical solitons the following ansatz with five CCs has been widely

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used [12–14]:

$$w(z,T) = \sqrt{\frac{E_p}{2T_p}}\,\mathrm{sech}\frac{T-q_p}{T_p}\exp\left\{-i\left[\phi_p + \Omega_p(T-q_p) + C_p\frac{(T-q_p)^2}{2T_p^2}\right]\right\}. \quad (3)$$

Here we use the notation of Refs. [12,14], where the propagation distance z corresponds to the time t in our NLS equation, and the time T corresponds to our spatial variable x. The phase ϕ_p is assumed to be constant. The CCs depend on z and are the energy E_p, the temporal shift q_p, the frequency shift from the original carrier frequency, Ω_p, the soliton duration T_p, and the time-domain chirp C_p. The chirp term is introduced as a second-order perturbation to the phase of the soliton.

We show that the ansatz Eq. (3) also produces problems with the definition of canonical momenta. In particular, $dL/d\dot{T}_p$ yields an expression that contains T_p, which means that $dL/d\dot{T}_p$ cannot be a canonical momentum. In contrast to this, an improved new ansatz with a new chirp term and the phase as a sixth CC produces three pairs of canonically conjugated variables. As the Lagrange formalism generally works only for systems without dissipation, and as a dissipative optical fiber is considered, we present the so-called generalized traveling wave method (GTWM), which works for arbitrary perturbations R. The method was introduced in a general way in Ref. [18]: only the Hamilton equations of the unperturbed system must be known and the unperturbed system need not be integrable. The method was already applied to the zero-temperature dynamics [18] and thermal diffusion [19,20] of magnetic vortices in the two-dimensional anisotropic Heisenberg model, and to the dynamics of topological solitons in nonlinear Klein-Gordon equations [21].

In particular for four CCs [see Eq. (2)], in Ref. [22] it is shown that the GTWM is equivalent to the time variation: of the norm, of the first moment of the norm, of the momentum, and of the energy. This approach has successfully been applied to the nonlinear Schrödinger equation in higher spatial dimension [23]. In the optical interpretation, this technique is known as the method of moments (MoM), also termed as modified conservation laws, which also work for arbitrary perturbations R. This is the method that is mostly used in the literature on optical solitons with five collective variables [12–14,24]. Therefore, in order to obtain the equations of motion, five moments (the norm N, its first moment N_1, its second moment N_2, the momentum P, and its first moment P_1) are used [12–14]. Notice that a direct comparison between the ansatz Eqs. (2) and (3) shows three major differences. First, in the ansatz Eq. (3) with five CCs, the amplitude and the width of the soliton are two independent variables. Second, in the ansatz Eq. (3), the phase is no longer an independent variable. Finally, the chirp term is introduced in Eq. (3) as the fifth collective coordinate.

In the current work, we use six independent CCs, namely the soliton position, the amplitude, the inverse of the soliton width, the velocity, the chirp, and the phase. It can be shown that the MoM yields six CC equations, identical to those from GTWM, if the phase in Eq. (3) is introduced as a sixth variable and the energy as the sixth moment. In Ref. [25], a different identity instead of a sixth moment was used. However, the resulting CC equations differ from the CC equations obtained by the GTWM.

The presentation of the above results is organized as follows: In the following section, the physical interpretation of canonical momenta is given using an ansatz with six CCs. It is also shown that an inconsistency is obtained when, instead of six CCs, five CCs are used. In Sec. III, the GTWM is developed for a general perturbation and its equivalence with the modified method of moments is shown. In Sec. IV, we apply the above methods to soliton dynamics in a dispersion-shifted fiber. The dynamics is modeled by a perturbed NLS equation, where the perturbation R consists of several terms [12–14], which account for dissipation, higher-order dispersion, delayed Raman response, energy loss through intrapulse Raman scattering, and self-steepening. We express R in our notation, calculate the relevant integrals, and obtain six CC equations. The sixth ODE is an equation for $\Phi(t)$. The soliton dynamics is studied through the numerical solutions of the equations of motion for the collective coordinates. To conclude the paper, in Sec. V our main findings are summarized.

II. PHYSICAL INTERPRETATION OF CANONICAL MOMENTA: ANSATZ WITH SIX COLLECTIVE VARIABLES

In this section it is shown that the ansatz that is used in the following sections possesses certain advantages compared to the ansatz that has been widely used in the literature [12–14]. This concerns the form of the chirp term and the introduction of a phase as a sixth collective coordinate (in the above literature only five collective coordinates were used). It is shown that our ansatz allows the definition of three canonical momenta, thereby providing three pairs of canonically conjugated variables.

In order to achieve our goal, we focus on the perturbed NLS Eq. (1) with the driving term $R = a\exp[iK(t)x]$ [15], which is obtained from the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}_{\rm tot}}{\partial u_t^\star} + \frac{d}{dx}\frac{\partial\mathcal{L}_{\rm tot}}{\partial u_x^\star} - \frac{\partial\mathcal{L}_{\rm tot}}{\partial u^\star} = 0,$$

with the Lagrangian density $\mathcal{L}_{\rm tot} = \mathcal{L}_{\rm kin} - \mathcal{L}_{\rm pot} + \mathcal{L}_{\rm pert}$,

$$\mathcal{L}_{\rm kin} = \frac{i}{2}(u_t u^\star - u_t^\star u), \quad \mathcal{L}_{\rm pot} = \sigma|u_x|^2 - \frac{\gamma_0}{2}|u|^4,$$
$$\mathcal{L}_{\rm pert} = -a\{\exp[iK(t)x]u^\star + \exp[-iK(t)x]u\}.$$

Our ansatz,

$$u(x,t) = 2iA\,\mathrm{sech}[2\eta(x-\zeta)]\exp(-i\Theta),$$
$$\Theta = \Phi + 2\xi(x-\zeta) + C(x-\zeta)^2, \quad (4)$$

contains six collective variables: soliton position $\zeta(t)$, amplitude $A(t)$, inverse width $2\eta(t)$, velocity $\xi(t)$, chirp $C(t)$, and phase $\Phi(t)$ [25].

By inserting the ansatz into $\mathcal{L}_{\rm kin}$ and integrating over x, the kinetic part of the Lagrangian is obtained

$$L_{\rm kin} = 4\frac{A^2}{\eta}\dot{\Phi} - 8\frac{A^2}{\eta}\xi\dot{\zeta} + \frac{\pi^2}{12}\frac{A^2}{\eta^3}\dot{C}. \quad (5)$$

From this, we obtain

$$\frac{\partial L_{\text{kin}}}{\partial \dot{\Phi}} = 4\frac{A^2}{\eta} = N,$$

where the canonical momentum N is conjugated to the phase Φ and is identified as the norm $N = \int dx\,|u|^2$. Furthermore,

$$\frac{\partial L_{\text{kin}}}{\partial \dot{\zeta}} = -8\frac{A^2}{\eta}\xi = P,$$

where the canonical momentum P (field momentum),

$$P = \frac{i}{2}\int_{-\infty}^{+\infty} dx\,(uu_x^\star - u^\star u_x), \tag{6}$$

is conjugated to the soliton position ζ. Moreover, $P = MV$, where $M = N/2$ is the mass and $V = -4\xi$ is the soliton velocity. Finally, we obtain

$$\frac{\partial L_{\text{kin}}}{\partial \dot{C}} = \frac{\pi^2}{12}\frac{A^2}{\eta^3} = D,$$

where the canonical momentum D is conjugated to the chirp C and can be written in the form $D = \pi^2 N B^2/12$, with $B = 1/(2\eta)$ as the soliton width.

Now we want to show that the ansatz Eq. (3) that has often been used in the literature on optical solitons [12–14] does not allow the definition of three canonical momenta.

The CCs in Eq. (3) for $\omega(z,T)$ have the following relations to the CCs in Eq. (4),

$$\begin{aligned} &q_p \equiv \zeta, \qquad \frac{1}{T_p} \equiv 2\eta, \qquad \sqrt{\frac{E_p}{2T_p}} \equiv 2A,\\ &\Omega_p \equiv 2\xi, \qquad \frac{C_p}{2T_p^2} \equiv C, \end{aligned} \tag{7}$$

and constant phase ϕ_p. The factor i in our ansatz is equivalent to a constant phase $\phi_0 = \pi/2$, because $i = \exp(-i\phi_0)$.

Using the above variables, we obtain

$$L_{\text{kin}} = -E_p\Omega_p\dot{q}_p + \frac{\pi^2}{24}E_p\dot{C}_p - \frac{\pi^2}{12}\frac{E_pC_p}{T_p}\dot{T}_p.$$

For the first canonical momentum we get

$$\frac{\partial L_{\text{kin}}}{\partial \dot{C}_p} = \frac{\pi^2}{24}E_p,$$

where the energy $E_p = \int d\tau\,|w|^2$ is conjugated to the chirp. For the second canonical momentum, we obtain

$$\frac{\partial L_{\text{kin}}}{\partial \dot{q}_p} = -E_p\Omega_p, \tag{8}$$

and this is conjugated to the temporal shift q_p, where $\Omega_p = (i/2E_p)\int d\tau(ww_T^\star - w^\star w_T)$ is identified as the frequency shift. The righthand side of Eq. (8) is equivalent to the canonical momentum P in Eq. (6). Finally, we obtain

$$\frac{\partial L_{\text{kin}}}{\partial \dot{T}_p} = -\frac{\pi^2}{12}\frac{E_pC_p}{T_p}. \tag{9}$$

Here the chirp should be conjugated to the soliton duration T_p. However, T_p also appears on the righthand side of Eq. (9), which must not be the case. Thus the righthand side of Eq. (9) cannot be interpreted as a canonical momentum.

Going back to our ansatz Eq. (4), $\mathcal{L}_{\text{pot}}$ is integrated over x and we obtain for the potential part of the Lagrangian,

$$L_{\text{pot}} = \frac{16\sigma}{3}A^2\eta + 16\sigma\frac{A^2\xi^2}{\eta} + \frac{\pi^2\sigma}{3}\frac{A^2C^2}{\eta^3} - \frac{16\gamma_0}{3}\frac{A^4}{\eta},$$

and for the perturbative part $L_{\text{pert}} = \int dx\,\mathcal{L}_{\text{pert}}$. By denoting the nonperturbative part as $L = L_{\text{kin}} - L_{\text{pot}}$, the six Lagrange equations become

$$\frac{d}{dt}\frac{\partial L}{\partial \psi_t} - \frac{\partial L}{\partial \psi} = \frac{\partial L_{\text{pert}}}{\partial \psi},$$

where ψ stands for the six CCs: $\Phi(t)$, $\zeta(t)$, $\eta(t)$, $\xi(t)$, $A(t)$, and $C(t)$.

However, the Lagrange formalism only works for Hamiltonian systems, that is, the perturbation R does not contain dissipative terms. If the damping is very simple, for example, $R = -i\beta u(x,t)$ with $\beta > 0$, then the Euler-Lagrange equation can be generalized through the introduction of a dissipation function; for the above example, see Refs. [15,17].

For this reason, in the next section two methods are used which both work for *arbitrary* perturbation R: For the generalized traveling wave method only the Hamiltonian of the *unperturbed* system must be known. Using our ansatz Eq. (4), six equations of motion are obtained. The same CC equations are derived by using the method of moments, if a sixth moment is introduced.

III. GENERALIZED TRAVELING WAVE METHOD AND METHOD OF MOMENTS WITH SIX COLLECTIVE COORDINATES

The perturbed NLS Eq. (1) actually consists of two equations for the real and imaginary parts, for $u(x,t)$ and $u^*(x,t)$. For our purpose, these equations can be rewritten as

$$i\,u_t = \frac{\delta H_0}{\delta u^*} + R[u(x,t);x,t], \tag{10}$$

$$-\,i\,u_t^* = \frac{\delta H_0}{\delta u} + R^*[u(x,t);x,t], \tag{11}$$

where

$$H_0 = \int_{-\infty}^{+\infty} dx\,\mathcal{H}_0 = \int_{-\infty}^{+\infty} dx\left(\sigma u_x u_x^* - \frac{\gamma_0}{2}u^2u^{*2}\right). \tag{12}$$

For the following, only this Hamiltonian of the *unperturbed* system must be known. We now assume that the time dependence of $u(x,t)$ and $u^*(x,t)$ in Eqs. (10) and (11) only appears via a set of m real collective coordinates $\{Y_1(t),Y_2(t),\ldots Y_m(t)\} := \vec{Y}(t)$, that is, $u(x,t) = u(x,\vec{Y}(t))$ and $u^*(x,t) = u^*(x,\vec{Y}(t))$. It is then necessary to multiply Eq. (10) by $\partial u^*/\partial Y_n$, and Eq. (11) by $\partial u/\partial Y_n$, add the resulting equations and integrate over the system, which yields

$$\sum_{j=1}^{m} I_{Y_nY_j}\dot{Y}_j = F_n(\vec{Y}) + R_n(\vec{Y}), \qquad n = 1,2,\ldots m, \tag{13}$$

with

$$I_{Y_nY_j} = \int_{-\infty}^{+\infty} dx\, i\left[\frac{\partial u}{\partial Y_n}\frac{\partial u^*}{\partial Y_j} - \frac{\partial u^*}{\partial Y_n}\frac{\partial u}{\partial Y_j}\right], \tag{14}$$

$$\begin{aligned} F_n(\vec{Y}) &= -\int_{-\infty}^{+\infty} dx\left[\frac{\delta H_0}{\delta u^*}\frac{\partial u^*}{\partial Y_n} + \frac{\delta H_0}{\delta u}\frac{\partial u}{\partial Y_n}\right] \\ &= -\int_{-\infty}^{+\infty} dx\,\frac{\partial \mathcal{H}_0}{\partial Y_n} = -\frac{\partial H_0}{\partial Y_n}, \end{aligned} \tag{15}$$

$$R_{Y_n}(\vec{Y}) = -\int_{-\infty}^{+\infty} dx\left(R\frac{\partial u^*}{\partial Y_n} + R^*\frac{\partial u}{\partial Y_n}\right), \tag{16}$$

where the overdot in Eq. (13) denotes the derivative with respect to time. Equations (13)–(16) represent a set of m first-order ODEs for our m CCs.

In order to evaluate the integrals, Eq. (14), which appear in the ODEs, Eq. (13), we now take the one-soliton solution of the unperturbed NLS equation [9] and make the specific ansatz Eq. (4) for $u(x,\vec{Y}(t))$ [25]. The soliton energy Eq. (12) is obtained:

$$H_0 = \frac{16\sigma}{3}A^2\eta + 16\sigma\frac{A^2\xi^2}{\eta} + \frac{\pi^2\sigma}{3}\frac{A^2C^2}{\eta^3} - \frac{16\gamma_0}{3}\frac{A^4}{\eta}. \tag{17}$$

Setting $Y_1=\zeta, Y_2=\Phi, Y_3=\eta, Y_4=\xi, Y_5=A$, and $Y_6=C$, in Eqs. (13)–(16), we obtain for $n=1,\ldots,6$,

$$8\frac{A^2\xi}{\eta^2}\dot{\eta} - 8\frac{A^2}{\eta}\dot{\xi} - 16\frac{A\xi}{\eta}\dot{A} = -R_\zeta, \tag{18}$$

$$-4\frac{A^2}{\eta^2}\dot{\eta} + 8\frac{A}{\eta}\dot{A} = -R_\Phi, \tag{19}$$

$$\begin{aligned} &-8\frac{A^2\xi}{\eta^2}\dot{\zeta} + 4\frac{A^2}{\eta^2}\dot{\Phi} + \frac{\pi^2}{4}\frac{A^2}{\eta^4}\dot{C} \\ &= -\frac{16\sigma}{3}A^2 + 16\sigma\frac{A^2\xi^2}{\eta^2} + \pi^2\sigma\frac{A^2C^2}{\eta^4} - \frac{16\gamma_0}{3}\frac{A^4}{\eta^2} - R_\eta, \end{aligned} \tag{20}$$

$$8\frac{A^2}{\eta}\dot{\zeta} = -32\sigma\frac{A^2}{\eta}\xi - R_\xi, \tag{21}$$

$$\begin{aligned} &16\frac{A\xi}{\eta}\dot{\zeta} - 8\frac{A}{\eta}\dot{\Phi} - \frac{\pi^2}{6}\frac{A}{\eta^3}\dot{C} \\ &= -\frac{32\sigma}{3}A\eta - 32\sigma\frac{A\xi^2}{\eta} - \frac{2\pi^2\sigma}{3}\frac{AC^2}{\eta^3} + \frac{64\sigma}{3}\frac{A^3}{\eta} - R_A, \end{aligned} \tag{22}$$

$$-\frac{\pi^2}{4}\frac{A^2}{\eta^4}\dot{\eta} + \frac{\pi^2}{6}\frac{A}{\eta^3}\dot{A} = -\frac{2\pi^2\sigma}{3}\frac{A^2C}{\eta^3} - R_C, \tag{23}$$

respectively. By setting $A=\eta$ and $C=0$, Eqs. (18)–(23) reduce to the equations of motions for ζ, Φ, η, and ξ obtained in Ref. [22] for a four-CC ansatz (see Eqs. (11)–(14) of Ref. [22]).

A very particular property of GTWM is related to its relationship with the so-called modified conservation laws, also called method of moments (the time evolution of the quantities which are conserved for the unperturbed system) [25,26].

We define the following moments in a similar way to that in Ref. [25]. For the ansatz Eq. (4) they read

$$N = \int_{-\infty}^{+\infty} dx\,|u|^2 = \frac{4A^2}{\eta}, \tag{24}$$

$$N_1 = \int_{-\infty}^{+\infty} dx\,x|u|^2 = \frac{4A^2}{\eta}\zeta, \tag{25}$$

$$N_2 = \int_{-\infty}^{+\infty} dx\,(x-\zeta)^2|u|^2 = \frac{\pi^2}{12}\frac{A^2}{\eta^3}, \tag{26}$$

$$P = \int_{-\infty}^{+\infty} dx\,\frac{i}{2}[uu_x^* - u^*u_x] = -8\frac{A^2\xi}{\eta}, \tag{27}$$

$$P_1 = \int_{-\infty}^{+\infty} dx\,\frac{i}{2}(x-\zeta)[uu_x^* - u^*u_x] = -\frac{\pi^2}{6}\frac{A^2C}{\eta^3}, \tag{28}$$

where N is the norm, N_1 is the first moment of the norm, N_2 is the second moment of the norm, P is the momentum, and P_1 is the first moment of the momentum. In addition, the energy H_0 is used as the sixth moment given by Eq. (12). Notice that in Ref. [25] only five moments namely P, N, P_1, N_1, and N_2 were defined. These moments yield five equations of motion for five CCs. In order to obtain the sixth equation of motion, Ref. [25] used a certain identity. However, we show that if the ansatz Eq. (4) is used, then the correct sixth equation of motion can be obtained from the time variation of the energy H_0 defined by Eq. (12) and given by Eq. (17).

From Eqs. (18) and (27), Eqs. (19) and (24), and Eqs. (23) and (26), it can be shown that Eqs. (18), (19), (21), and (23) can be rewritten in the following way:

$$\frac{dP}{dt} = -R_\zeta, \tag{29}$$

$$\frac{dN}{dt} = -R_\Phi, \tag{30}$$

$$\frac{d\tilde{N}_1}{dt} = \dot{\zeta} = -4\sigma\xi - \frac{\eta}{8A^2}R_\xi, \tag{31}$$

$$\frac{dN_2}{dt} = -\frac{2\pi^2\sigma}{3}\frac{A^2C}{\eta^3} - R_C, \tag{32}$$

where $\tilde{N}_1 = N_1/N$. In other words, four of the six equation of motions are directly related with the time variation of the momentum, the norm, and the first and second moments of the norm. In Ref. [22] for an arbitrary number of CCs and arbitrary ansatz it was shown that the time variation of the energy reads

$$\frac{dH_0}{dt} = -R_t. \tag{33}$$

Finally, by multiplying Eq. (20) by $-\eta$, Eq. (22) by $-A/2$, and Eq. (23) by $-2C$, and by adding the resulting equations, we obtain

$$\begin{aligned} \frac{dP_1}{dt} &= \frac{32}{3}\frac{A^2}{\eta}\left(\sigma\eta^2 - \frac{\gamma_0}{2}A^2\right) + \frac{2\pi^2\sigma}{3}\frac{A^2C^2}{\eta^3} + \frac{A}{2}R_A \\ &\quad + \eta R_\eta + 2CR_C. \end{aligned} \tag{34}$$

Hence, it is shown that, for a general perturbation and assuming the ansatz Eq. (4), the GTWM is equivalent to the time variation of the norm, the momentum, the energy, the first and second moments of the norm, and the first moment of the momentum.

Clearly, the equation of motion for ζ is precisely Eq. (31). By multiplying Eq. (19) by 2ξ and adding it to Eq. (18),

$$\dot{\xi} = \frac{\eta}{8A^2}R_\zeta + \frac{\eta\xi}{4A^2}R_\Phi. \tag{35}$$

By multiplying Eq. (23) by $-48\eta^2/\pi^2$ and adding it to Eq. (19),

$$\dot{\eta} = 4\sigma\eta C + \frac{6}{\pi^2}\frac{\eta^4}{A^2}R_C - \frac{\eta^2}{8A^2}R_\Phi. \tag{36}$$

Subsequent insertion of this expression for $\dot{\eta}$ in Eq. (19) gives

$$\dot{A} = 2\sigma AC + \frac{3}{\pi^2}\frac{\eta^3}{A}R_C - \frac{3\eta}{16A}R_\Phi. \tag{37}$$

Multiplying Eq. (22) by $A/(2\eta)$ and adding Eq. (20), yields

$$\dot{C} = 4\sigma C^2 - \frac{64}{\pi^2}\eta^2\left(\sigma\eta^2 - \frac{\gamma_0}{2}A^2\right) - \frac{3\eta^3}{\pi^2 A}R_A - \frac{6\eta^4}{\pi^2 A^2}R_\eta. \tag{38}$$

Equations (32) and (35)–(38) agree with those obtained in Ref. [25]. However, we now show that the equation for the phase Φ can be directly obtained from the GTWM or equivalently for the time derivative of the energy. From Eq. (20) we obtain

$$\dot{\Phi} = 2\xi\dot{\zeta} - \frac{\pi^2\dot{C}}{16\eta^2} - \frac{4\sigma}{3}\eta^2 - \frac{4\gamma_0}{3}A^2 + 4\sigma\xi^2 + \frac{\pi^2C^2}{4\eta^2} - \frac{\eta^2}{4A^2}R_\eta. \tag{39}$$

It is interesting to note that this equation does *not* agree with that obtained in Ref. [25]. Notice that, in Ref. [25], the equation for Φ was obtained from a certain identity, which is not related to the time variation of modified conserved quantities.

IV. PROPAGATION OF SOLITONS IN OPTICAL FIBERS

The nonlinear propagation of a Raman soliton in an optical fiber can be modeled by using the generalized NLS [27]. Under certain approximations this equation becomes the perturbed NLS Eq. (1), with

$$R = -i\beta u + i\beta_1 u_{xxx} - i\gamma_1\frac{\partial}{\partial x}(u|u|^2) + \gamma_0 T_R u\frac{\partial|u|^2}{\partial x} + i\gamma_1 T_R\frac{\partial}{\partial x}\left(u\frac{\partial|u|^2}{\partial x}\right), \tag{40}$$

where the first term on the righthand side is the dissipation, the second term accounts for the third-order dispersion, and the third describes the influence of self-steepening. Moreover, the term proportional to $\gamma_0 T_R$ represents the intrapulse Raman scattering, while the last term is related with the energy loss through intrapulse Raman scattering [14]. By using the ansatz Eq. (4) with six CCs, from Eqs. (31) and (35)–(39), and after a number of straightforward calculations, we obtain

$$\dot{\zeta} = -4\sigma\xi + 4\beta_1\eta^2 + 12\beta_1\xi^2 + \frac{\pi^2}{4}\beta_1\frac{C^2}{\eta^2} + 4\gamma_1 A^2, \tag{41}$$

$$\dot{\xi} = -\frac{64}{15}\gamma_0 T_R A^2\eta^2 + \frac{8}{3}\gamma_1 A^2 C - \frac{128}{15}\gamma_1 T_R\eta^2 A^2\xi, \tag{42}$$

$$\dot{\eta} = 4\sigma\eta C - 24\beta_1\eta\xi C - \frac{128}{\pi^2}\gamma_1 T_R A^2\eta^3, \tag{43}$$

$$\dot{A} = -8\beta A + 2\sigma AC - 12\beta_1 AC\xi - \gamma_1 T_R A^3\eta^2\left(\frac{64}{\pi^2} + \frac{64}{15}\right), \tag{44}$$

$$\dot{C} = 4C^2(\sigma - 6\beta_1\xi) + \frac{64}{\pi^2}\eta^2\left(\frac{\gamma_0}{2}A^2 - \sigma\eta^2 + \gamma_1 A^2\xi + 6\beta_1\xi\eta^2\right) + \left(\frac{64}{\pi^2} - \frac{128}{15}\right)\gamma_1 T_R A^2\eta^2 C, \tag{45}$$

$$\dot{\Phi} = 2\xi\dot{\zeta} + 4\sigma\xi^2 + \frac{8\sigma}{3}\eta^2 - \frac{10\gamma_0}{3}A^2 - 8\beta_1\xi^3 - 16\beta_1\eta^2\xi - \frac{20}{3}\gamma_1 A^2\xi + \gamma_1 T_R A^2 C\left(\frac{8\pi^2}{45} - 4\right). \tag{46}$$

It is interesting to note that an ansatz with five CCs [12] was used in Ref. [14], which is essentially the ansatz Eq. (4) with $\Phi = \pi/2$. Using this five-CC ansatz, the chirp term displays oscillations, which grow in amplitude. In contrast, the ansatz with six collective coordinates, which includes the new chirp term and the time-dependent phase, has, at least, two advantages: first, the oscillations of the chirp do not grow and second, the extra equation for the phase is obtained, which is crucial for a correct physical interpretation of the canonical momentum associated with certain problems.

We now define

$$\sigma = -\beta_2/2, \qquad \beta = \alpha/2, \qquad \beta_1 = \beta_3/6, \tag{47}$$

using the parameters of the optical fiber α, β_1, and β_2. The values for the required fiber parameters are given in Table I. By using the initial conditions shown in Table II, the equations of motion are numerically solved for the six collective coordinates, Eqs. (41)–(45) and $A(t)$, $\zeta(t)$, $\xi(t)$, $\eta(t)$, $C(t)$, and $\Phi(t)$ are obtained (see Fig. 1). In Fig. 1, the soliton amplitude $A(t)$ and the inverse of the soliton width $1/\eta(t)$ decay with a propagation distance while the soliton is accelerated. The chirp $C(t)$ oscillates with a spatial period approximately equal to 0.09 m, which is much smaller than the fiber length. Interestingly, the phase $\Phi(t)$ no longer has linear behavior as a function of the propagation distance t and, due to the perturbations, nonlinear terms appear in t.

TABLE I. Parameters of the optical fiber.

Parameter	Values
Speed of light	$c = 2.99792458 \times 10^8$ m/s
Center wavelength	$\lambda_0 = 1.55 \times 10^{-6}$ m
Fiber length	200 m
Fiber linear loss	$\alpha = 4.6 \times 10^{-5}$ m^{-1}
Group velocity dispersion (GVD)	$\beta_2 = -5.1 \times 10^{-27}$ s^2/m
Third-order dispersion	$\beta_3 = 10^{-40}$ s^3/m
Nonlinearity	$\gamma_0 = 2 \times 10^{-3}$ W^{-1} m^{-1}
Self-steepening	$\gamma_1 = \gamma_0\lambda_0/(2\pi c)$ s/(m W)
Raman parameter	$T_R = 2.5 \times 10^{-15}$ s

TABLE II. Initial conditions.

Collective coordinates at $t = 0$	Values
A(0)	$15.97\ \sqrt{J/s}$
$\zeta(0)$	0 m
$\xi(0)$	$0\ s^{-1}$
$\eta(0)$	$10^{13}\ s^{-1}$
C(0)	$0\ s^{-2}$
$\Phi(0)$	$\pi/2$

By using the change of variables given by Eqs. (7), the energy E_p, the temporal shift q_p, the frequency Ω_p, the soliton duration T_p, and the chirp C_p are obtained as functions of the propagation distance t (see Fig. 2). As $q_p = \zeta$, q_p is not plotted in Fig. 2.

In Fig. 2, the dynamics of the soliton show the expected behavior in energy E_p, frequency shift Ω_p, and soliton duration T_p as a function of propagation distance. In particular, the energy is decreasing due to fiber loss and intrapulse Raman scattering. The Raman scattering also leads to continuous downshift in soliton center frequency. This effect is rather strong for femtosecond solitons. As the soliton downshifts, it experiences increasing dispersion, which leads to increasing duration. In Fig. 2, the normalized new chirp C/N (black curve) is shown together with the normalized old chirp C_p/N_p

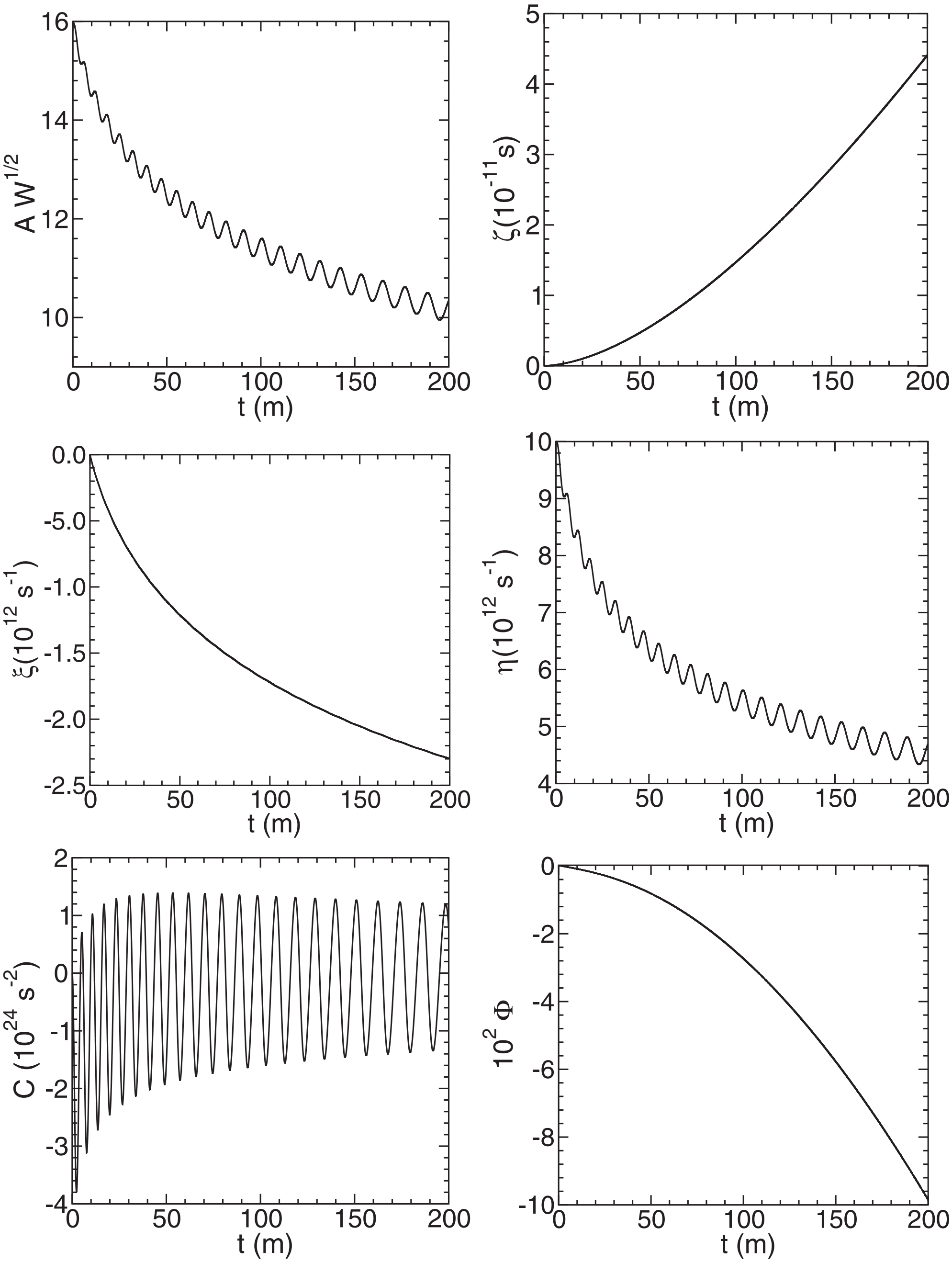

FIG. 1. A, ζ, ξ, η, C, and Φ vs. propagation distance t for a large propagation distance.

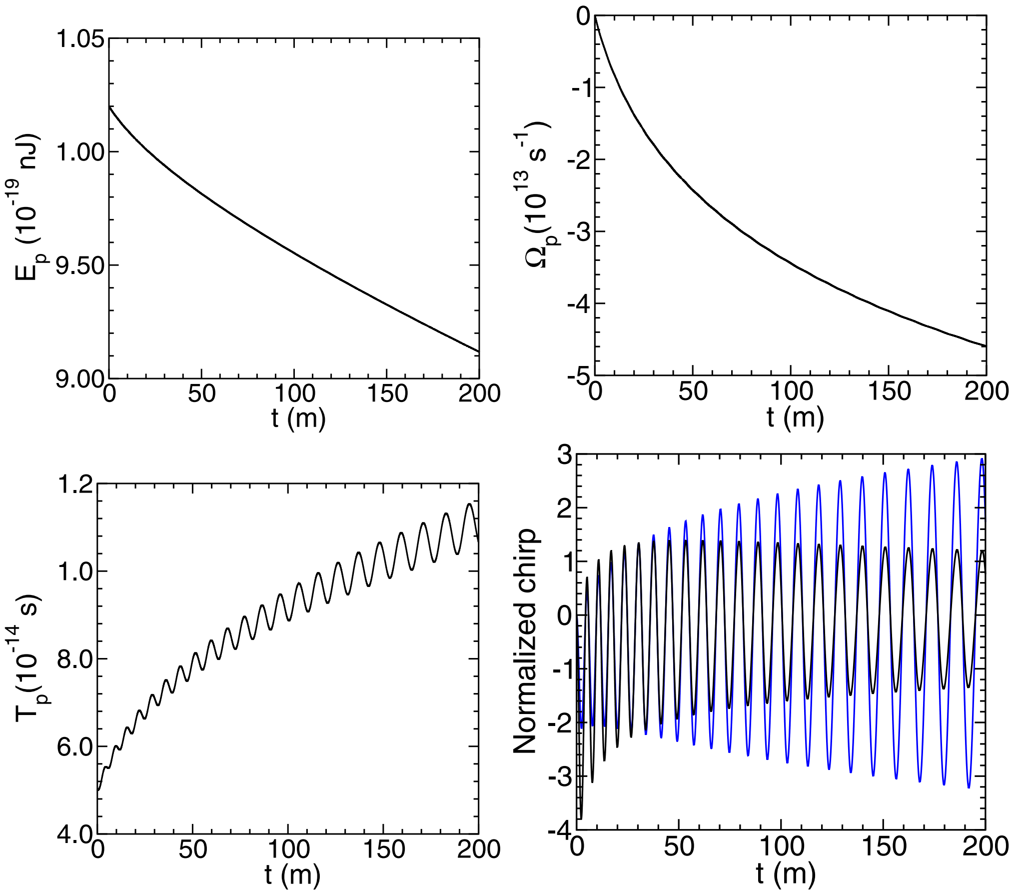

FIG. 2. E_p, Ω_p, T_p, and the normalized chirp parameter vs. propagation distance t for a larger fiber length. For the normalized chirp (lower right panel), an old definition C_p/N_p (blue curve) displays growing oscillations, while the new definition C/N (black curve) results in more subdued oscillations. Factors $N_p = 10^{-2}$ and $N = 10^{24}$ s^{-2} are used to display the curves on comparable vertical scales.

(blue curve), where the factors $N = 10^{24}$ s^{-2} and $N_p = 10^{-2}$ are used to display the curves on comparable vertical scales. Clearly, the old chirp oscillations grow as a function of propagation distance, whereas the new chirp term oscillates with a decreasing amplitude.

V. CONCLUSIONS

For the description of NLS solitons, in particular in the application to optical solitons, we have introduced a new ansatz with six CCs, which avoids certain disadvantages of the five-CC ansatz that is widely used in the literature. The six independent CCs (the soliton position, the amplitude, the inverse of the soliton width, the velocity, the chirp, and the phase) are unknown functions of time. In optical notation, these collective coordinates correspond to the following magnitudes: the temporal shift, the energy, the soliton duration, the carrier frequency, the chirp, and phase. In contrast to a five-CC ansatz, with six independent CCs we obtain three pairs of canonically conjugated variables, and hence three canonical momenta can be defined and have physical interpretations. To this end we have given a new form to the chirp term in the six-CC ansatz.

As the NLS equations that describe optical solitons have several complicated perturbation terms, such as higher-order dispersion, delayed Raman response, energy loss through intra-pulse Raman scattering, and self-steepening, we apply two methods which both work for *arbitrary* perturbations: The GTWM and the MoM. Indeed, it is shown that in the case of the perturbed NLS Eq. (1), the GTWM with the ansatz Eq. (4) yields six modified conservation laws, namely the time variation of the norm, the first and second moment of the norm, the momentum, the first moment of the momentum, and the energy. Therefore, both methods yield six identical ODEs, which are the equations of motion for the collective coordinates.

We numerically solve the resulting ODEs for the six CCs, using the parameters for a femtosecond soliton propagating in a typical dispersion-shifted fiber. We also show that the six-CC ansatz results in a better-behaved chirp with reduced oscillations.

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