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Research Article

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An inverse problem in elastography involving Lamé systems

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Abstract: This paper deals with some inverse problems for the linear elasticity system with origin in *elastography:* we try to identify the material coefficients from some extra information on (a part of) the boundary. In our main result, we assume that the total variation of the coefficient matrix is a priori bounded. We reformulate the problem as the minimization of a function in an appropriate constraint set. We prove that this extremal problem possesses at least one solution with the help of some regularity results. Two crucial ingredients are a *Meyers-like theorem* that holds in the context of linear elasticity and a nonlinear interpolation result by Luc Tartar. We also perform some numerical experiments that provide satisfactory results. To this end, we apply the *Augmented Lagrangian* algorithm, completed with a limited-memory *BFGS* subalgorithm. Finally, on the basis of these experiments, we illustrate the influence of the starting guess and the errors in the data on the behavior of the iterates.

Keywords: Inverse problems, linear elasticity, Lamé systems, bounded variation coefficients, elastography

MSC 2010: 35L55, 35R30, 49N45, 74B05

1 Introduction

This paper is concerned with some inverse problems for linear elastic materials. These problems are found when we try to apply elastography techniques for instance for breast tumor detection.

Elastography is a technique that intends to detect elastic properties of tissue. The basic motivation is that tumor tissue is considerably stiffer than normal soft tissue and, consequently, its resulting deformation after a mechanical action is much smaller. Elastography can be described by the action of three elements:

- an acoustic wave generator, conceived on the basis of low frequency mechanical excitations,
- a *captor* that detects these waves,
- a mathematical solver, able to identify tissue stiffness from related measurements.

For a more detailed description, see for instance [26, 29, 33, 36].

From the mathematical viewpoint, our main task is to consider and try to solve an inverse problem. The situation is as follows. Let $\Omega \subset \mathbb{R}^N$ be an open, bounded and regular domain which represents an elastic body ($N = 2$ or $N = 3$) and let a positive time T be given. Let us set $Q := \Omega \times (0, T)$ and $\Sigma := \partial\Omega \times (0, T)$. Assume that the external sources f_m, the initial displacement-velocity couples (u_m^0, u_m^1) and the boundary data B_m are given for $m = 1, \dots, M$, where the number M represents the number of measurements. Let us consider

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the following systems, which model the behavior of the body under small displacements:

$$\begin{cases} u_{m,tt} - \nabla \cdot (Ae(u_m)) = f_m, & (x,t) \in Q, \\ u_m = B_m, & (x,t) \in \Sigma, \\ u_m(x,0) = u_{m,0}(x), \quad u_{m,t}(x,0) = u_{m,1}(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, $e(u_m) := \frac{1}{2}(\nabla u_m + \nabla u_m^T)$ and $A = \{A_{ijkl}\}_{1 \le i,j,k,l \le n}$ is a fourth-order tensor-valued function on Ω. Then the inverse problem is to find A such that u solves (1.1) together with the additional boundary condition

$$Ae(u_m) \cdot \nu = \Upsilon_m \quad \text{on } S \times (0,T), \tag{1.2}$$

where $\nu = \nu(x)$ is the outwards directed unit normal vector at points $x \in \partial\Omega$, $S \subset \partial\Omega$ and the Υ_m are prescribed.

In particular, if we assume that $A\xi = 2\mu\,\xi + \lambda\,\mathrm{tr}(\xi)\,\mathrm{Id}$ for some scalar functions μ and λ and all $\xi \in \mathbb{R}^{N \times N}$, then (1.1) reduces to the usual *Lamé system*

$$\begin{cases} u_{m,tt} - \nabla \cdot \big(\mu(x)(\nabla u_m + \nabla u_m^T) + \lambda(x)(\nabla \cdot u_m)\,\mathrm{Id}\big) = f_m, & (x,t) \in Q, \\ u = B_m, & (x,t) \in \Sigma, \\ u(x,0) = u_{m,0}(x), \quad u_t(x,0) = u_{m,1}(x), & x \in \Omega, \end{cases} \tag{1.3}$$

and (1.2) reads

$$(\mu(x)(\nabla u_m + \nabla u_m^T) + \lambda(x)(\nabla \cdot u_m)\,\mathrm{Id}) \cdot \nu = \Upsilon_m \quad \text{on } S \times (0,T) \tag{1.4}$$

and the associated inverse problem is to find μ and λ such that the solution to (1.3) satisfies (1.4).

The literature concerning inverse problems of these kinds is large, in particular in the case of (1.3)–(1.4); see for instance [2, 3, 18, 19, 21, 36]. There are more sophisticated models that take into account visco-elastic effects, porosity, etc. and have been proposed in other papers; see for instance [22, 34, 37].

The main questions concerning these inverse problems are *uniqueness, stability* and *reconstruction*. In this paper, we will focus mainly on the third one. In general terms, the uniqueness problem is as follows:

Problem. Assume that Υ_m and Υ'_m are given and let (μ, λ) and (μ', λ') solve the corresponding associated inverse problems. Do we necessarily have $(\mu, \lambda) = (\mu', \lambda')$?

Here, the number M of measurements or experiments plays a fundamental role. Thus, in [15], the authors established the uniqueness of determining one single coefficient using three measurements ($M = 3$). Later, in [20], with the same amount of measurements, the uniqueness of all the coefficients of the system was established.

On the other hand, the stability of (1.3)–(1.4) has been analyzed recently in several papers. In a typical stability result, it is proved that the "distance" between solutions (μ, λ) and (μ', λ') can be bounded by a function of the "distance" between data $\{\Upsilon_m : 1 \le m \le M\}$ and $\{\Upsilon'_m : 1 \le m \le M\}$ in a neighborhood of a fixed $\{\Upsilon_m^0 : 1 \le m \le M\}$. Since stability implies uniqueness, this is an important question in applications to geophysics, material sciences or medicine.

In [16], conditional stability and uniqueness for all the coefficients of the system with two measurements ($M = 2$) is demonstrated. In [17], the authors proved conditional stability results with one single interior measurement, provided the initial data satisfy some nondegeneracy condition. In [4], the authors proved a logarithmic stability estimate for the Lamé coefficients again with $M = 1$ assuming that the data are known in a neighborhood of the boundary of the spatial domain. See [1, 3, 5, 12, 13, 18, 23, 24, 27] for other results.

We will work with systems of the form (1.1), with

$$M = 2, \quad f_m, f_{m,t} \in L^2(Q)^N, \quad u_{m,0} = 0, \quad u_{m,1} \in H_0^1(\Omega)^N, \quad B = 0, \quad \Upsilon \in L^2(\Sigma)^N \tag{1.5}$$

and

$$A \in \mathbb{M}(\alpha, \beta; \Omega) \cap \mathbb{BV}(\Omega), \tag{1.6}$$

for some $0 < \alpha < \beta$. Here, we have denoted by $\mathbb{M}(\alpha, \beta; \Omega)$ the family of all measurable tensors A satisfying

- $A(x) = A(x)^T$, $A(x)\Lambda \cdot \Lambda \ge \alpha|\Lambda|^2$,
- $|A(x)\Lambda| \le \beta|\Lambda|$ a.e. in Ω for all symmetric $\Lambda \in \mathbb{R}^{N \times N}$;

see the beginning of Section 2 for the notation.

It is known that, under assumptions (1.5)–(1.6), there exists *at most* one solution u to the inverse problem (1.1)–(1.2); see [4, 16]. On the other hand, the direct problem associated to (1.1) is well posed. More precisely, under the same assumptions, for every m there exists a unique solution u_m, with

$$u_m, u_{m,t} \in C^0([0,T]; H_0^1(\Omega)^N), \quad u_{m,tt}, \nabla\cdot(Ae(u_m)) \in C^0([0,T]; L^2(\Omega)^N). \tag{1.7}$$

This is a consequence of the standard semigroup theory. Indeed, it suffices to introduce the Hilbert space $Y := H_0^1(\Omega)^N \times L^2(\Omega)^N$, the unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset Y \mapsto Y$ with

$$\begin{cases} D(\mathcal{A}) := \{(v_0, v_1) \in Y : \nabla\cdot(Ae(v_0)) \in L^2(\Omega)^N,\ v_1 \in H_0^1(\Omega)^N\}, \\ \mathcal{A}(v_0, v_1) := (v_1, \nabla\cdot(Ae(v_0))), \end{cases}$$

the initial data $y_0 := (0, u_{m,1})$ (which belongs to $D(\mathcal{A})$) and the right-hand side $F := (0, f_m)$ and consider the Cauchy problem in Y

$$\begin{cases} ly_t = \mathcal{A}y + F(t),\ t \in (0,T), \\ y(0) = y_0. \end{cases}$$

Moreover, we have

$$(Ae(u_m))\cdot\nu \in L^2(0,T; H^{-1/2}(\partial\Omega)^N), \quad m = 1, 2;$$

we will justify this below, in Section 3.

Therefore, in order to solve the inverse problem associated to (1.1) and (1.2), we can introduce the *cost function*

$$I(A) := \frac{1}{2}\int_0^T \sum_{m=1}^{2} \|Ae(u_m)\cdot\nu|_S - \Upsilon_m\|^2\,dt \tag{1.8}$$

(where A satisfies (1.6), u_m solve the associated system (1.1) and $\|\cdot\|$ is the norm in $H^{-1/2}(S)^N$) and formulate the following (direct) extremal problem:

$$\begin{cases} \text{Minimize } I(A) \\ \text{subject to (1.6) and (1.1).} \end{cases} \tag{1.9}$$

The following assertions are obvious:
- If A is a solution to the inverse problem associated to (1.1)–(1.2), then A also solves (1.9).
- Conversely, if A solves (1.9) and $I(A) = 0$ (which can be expected for realistic data Υ_m), then A is also the unique solution to the original inverse problem.

Therefore, under the uniqueness assumptions (1.5)–(1.6) for the inverse problem, it is completely meaningful to try to solve (1.9). This will be the adopted viewpoint and the goal of this paper.

The rest of this paper is organized as follows. In Section 2, we introduce the appropriate notation, we recall some preliminary results and we state our main result; this is an existence theorem for a modified version of (1.9). Section 3 is devoted to the proof of this result. The main ingredients are a regularity result for the linear elasticity system (1.1) of the Meyers kind and an (abstract) nonlinear interpolation result by Luc Tartar. In Section 4, we present some numerical results. Finally, Section 5 deals with some additional comments and questions.

2 Preliminaries, notation and main result

Let us first recall some definitions and properties needed to analyze the PDEs of linear elasticity.

In the sequel, it will be assumed that $\Omega \subset \mathbb{R}^N$ possesses a regular boundary, at least of class $W^{2,\infty}$. Suppose that $A = (A_{ijkl})_{1\le ijkl\le N}$ is a fourth-order tensor, $\Lambda, \Xi \in \mathbb{R}^{N\times N}$ and $u = (u_1, \dots, u_N)$ is a vector-valued function. The following notation will be used:
- $A\Lambda$ stands for the second-order tensor whose (i,j) component is $(A\Lambda)_{ij} := \sum_{1\le k,\ell\le N} A_{ijk\ell}\Lambda_{k\ell}$.
- $\Xi\cdot\Lambda := \sum_{1\le i,j\le N} \Xi_{ij}\Lambda_{ij}$.

- We set
$$(\nabla u)_{ij} := \frac{\partial u_i}{\partial x_j}, \quad (\nabla \cdot \Xi)_i := \sum_{1\le j\le N} \frac{\partial \Xi_{ij}}{\partial x_j}, \quad (DA)_{ijk\ell m} := \frac{\partial A_{ijk\ell}}{\partial x_m}.$$
- For $1 \le p < +\infty$, we consider the usual Banach spaces $L^p(\Omega)$, $L^p(\Omega)^N$ and $L^p(\Omega)^{N\times N}$, endowed with the norms
$$\|\varphi\|_{L^p(\Omega)} := \Big(\int_\Omega |\varphi|^p\,dt\Big)^{\frac{1}{p}},$$
$$\|u\|_{L^p(\Omega)^N} := \Big(\sum_{1\le i\le N}\int_\Omega |u_i|^p\,dt\Big)^{\frac{1}{p}},$$
$$\|\Xi\|_{L^p(\Omega)^{N\times N}} := \Big(\sum_{1\le i,j\le N}\int_\Omega |\Xi_{ij}|^p\,dt\Big)^{\frac{1}{p}}.$$
We denote by $W^{1,p}(\Omega)$, $W^{1,p}(\Omega)^N$ and $W^{1,p}(\Omega)^{N\times N}$ the similar Sobolev spaces.
- By definition, $\mathbb{L}^p(\Omega)$, $\mathbb{W}^{1,p}(\Omega)$ and $\mathbb{BV}(\Omega)$ are the spaces of x-dependent fourth-order tensors with components in $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $\mathrm{BV}(\Omega)$, respectively. We will use the index S to denote symmetry; thus, for example, $\mathbb{L}^2_S(\Omega)$ denotes the space of symmetric matrix-valued functions whose components belong to $L^2(\Omega)$. For any $A \in \mathbb{BV}(\Omega)$, we denote by $\mathrm{TV}(A)$ the corresponding *total variation* of A.
- Finally, for any Banach space X, any $T > 0$ and any $p \in [1, +\infty)$, $L^p(0, T; X)$ stands for the usual space of measurable and p-integrable (classes of) functions $u : [0, T] \mapsto X$.

In this paper, our main goal is to analyze a problem of the kind (1.9), where I is given by (1.8). Our main result is the following:

Theorem 2.1. *Let us set*
$$\mathbb{K}(R) = \{A \in \mathbb{M}(\alpha, \beta; \Omega) \cap \mathbb{BV}(\Omega) : \mathrm{TV}(A) \le R\} \tag{2.1}$$
and let us assume that the data of the problem satisfy (1.5). *Then, for any $R > 0$, the extremal problem*
$$\begin{cases} \textit{Minimize } I(A) \\ \textit{subject to } A \in \mathbb{K}(R) \textit{ and } (1.1) \end{cases} \tag{2.2}$$
possesses at least one solution A^*_R.

Remark 2.2. Obviously, the crucial assumption in Theorem 2.1 is the uniform bound of $\mathrm{TV}(A)$. This is just what we need to get compactness in the appropriate space (see below). We can consider other similar extremal problems for which the existence of a solution can be established; see Section 5 for more details.

Remark 2.3. In Theorem 2.1, we assert that at least one A^*_R exists, but no uniqueness is ensured. However, if $I(A^*_R) = 0$, in view of the uniqueness of the inverse problem, A^*_R is also the unique minimizer of I in $\mathbb{K}(R)$.

Remark 2.4. Let us consider again the particular case of an isotropic media and the associated inverse problem (1.3)–(1.4). It makes sense to introduce
$$L(\alpha, \beta; \Omega) := \{a \in L^\infty(\Omega) : \alpha \le a(x) \le \beta \text{ a.e.}\},$$
$$\mathbb{K}_0(R) := \{(\mu, \lambda) \in L(\alpha, \beta; \Omega) \times L(\alpha, \beta; \Omega) : \mathrm{TV}(\mu), \mathrm{TV}(\lambda) \le R\}$$
and the corresponding (simplified) extremal problem
$$\begin{cases} \text{Minimize } I_0(\mu, \lambda) \\ \text{subject to } (\mu, \lambda) \in \mathbb{K}_0(R) \text{ and } (1.3). \end{cases} \tag{2.3}$$
The arguments in the proof of Theorem 2.1 can also be used to establish the existence of a minimizer (μ^*_R, λ^*_R). Again, if $I_0(\mu^*_R, \lambda^*_R) = 0$, then (μ^*_R, λ^*_R) is the unique solution to the inverse problem (1.3)–(1.4).

3 Proof of Theorem 2.1

Let us recall that, for any solution to the state equation (1.1), $Ae(u_m) \cdot \nu$ is a well-defined object in $H^{-1/2}(\partial\Omega)^N$ for each $t \in [0, T]$. In fact, $Ae(u_m) \cdot \nu$ is well defined by duality in $H^{-1/2}(\partial\Omega)^N$ through the identities

$$\langle Ae(u_m) \cdot \nu, z \rangle_{H^{-1/2},H^{1/2}} = \langle \nabla \cdot (Ae(u_m)), z \rangle + \int_\Omega Ae(u_m) \cdot \nabla z \quad \text{for all } z \in H^1(\Omega)^N,$$

which are completely meaningful, in view of the regularity of the solutions to (1.1) and, more precisely, (1.7).

Let $\{A^n\}$ be a minimizing sequence for I in $\mathbb{K}(R)$. Then, at least for a subsequence, we have the following for some $A^* \in \mathbb{K}(R)$:

$$\begin{cases} A^n \to A^* & \text{weakly-}* \text{ in } \mathbb{BV}(\Omega), \\ A^n \to A^* & \text{strongly in } \mathbb{L}^p(\Omega) \text{ for all } p \in [1, +\infty) \text{ and a.e.} \end{cases} \tag{3.1}$$

Let us denote by u_m^n the states associated to A^n. Then, from the standard energy estimates, it can also be assumed that

$$\begin{cases} u_m^n \to u_m^* & \text{weakly-}* \text{ in } L^\infty(0, T; H_0^1(\Omega)^N), \\ u_{m,t}^n \to u_{m,t}^* & \text{weakly-}* \text{ in } L^\infty(0, T; L^2(\Omega)^N). \end{cases} \tag{3.2}$$

Let us prove that the u_m^* are the states associated to A^*, i.e.

$$\begin{cases} u_{m,tt}^* - \nabla \cdot (A^* e(u_m^*)) = f_m(x, t), & (x, t) \in Q, \\ u_m^* = 0, & (x, t) \in \Sigma, \\ u_m^*(x, 0) = 0, \quad u_{m,t}^*(x, 0) = u_{m,1}(x), & x \in \Omega. \end{cases} \tag{3.3}$$

From the second assertion in (3.1) and the first one in (3.2), we see that

$$A^n e(u_m^n) \to A^* e(u_m^*) \quad \text{weakly in } L^{p_1}(0, T; L^{p_2}(\Omega)^N) \quad \text{for all } p_1 \in [1, +\infty) \text{ and all } p_2 \in [1, 2).$$

Consequently, we can pass to the limit in the equations and boundary and initial conditions satisfied by u_m^n and deduce that u_m^* is the unique solution to (3.3).

In order to achieve the proof of Theorem 2.1, we have to show that

$$\liminf_{n\to+\infty} I(A^n) \geq I(A^*).$$

Obviously, this will be the case if we are able to prove that

$$A^n e(u_m^n) \cdot \nu \to A^* e(u_m^*) \cdot \nu \quad \text{weakly in } L^2(0, T; H^{-1/2}(\partial\Omega)^N).$$

Taking into account the definition of $A^n e(u_m^n) \cdot \nu$, we have

$$\langle A^n e(u_m^n) \cdot \nu, z \rangle = \langle \nabla \cdot (A^n e(u_m^n)), z \rangle + \iint_Q A^n e(u_m^n) \cdot \nabla z \quad \text{for all } z \in L^1(0, T; H^1(\Omega)^N).$$

Therefore, since $\nabla \cdot (A^n e(u_m^n)) = u_{m,tt}^n - f_m$ converges weakly-$*$ in $L^\infty(0, T; L^2(\Omega)^N)$, it will suffice to show that the ∇u_m^n (and therefore the $e(u_m^n)$) belong to a compact set in $L^2(Q)^{N\times N}$. In fact, we are now going to prove a slightly stronger property: that u_m^n belongs to a compact set in $C^0([0, T]; X)$ for a Hilbert space X that is compactly embedded in $H_0^1(\Omega)^N$. To this end, let us recall some (classical) notation: for any couple of Banach spaces E_0 and E_1, any $\theta \in (0, 1)$ and any $p \in [1, +\infty]$, $[E_0, E_1]_{\theta,p}$ will stand for the usual associated *interpolation space* of Petree, whenever this makes sense.

We will need the following result:

Lemma 3.1. *Assume that $A \in \mathbb{K}(R)$. There exists $\delta \in (0, 1)$, only depending on α, β and R, such that, for any $h \in L^2(\Omega)^N$, the elliptic system*

$$\begin{cases} -\nabla \cdot (Ae(w)) = h, & x \in \Omega, \\ w = 0, & x \in \partial\Omega, \end{cases} \tag{3.4}$$

possesses exactly one solution $w_h \in X_\delta := [D(\Delta), H_0^1(\Omega)^N]_{\delta,\infty}$, *where* $D(\Delta) = H_0^1(\Omega)^N \cap H^2(\Omega)^N$ *is the domain of the Dirichlet Laplacian. Furthermore, the mapping* $h \mapsto w_h$ *is linear and continuous, i.e.*

$$\|w_h\|_{X_\delta} \le C(N, \Omega, \alpha, \beta, R)\|h\|_{L^2} \quad \text{for all } h \in L^2(\Omega)^N. \tag{3.5}$$

In view of this lemma, u_m^n is uniformly bounded in $L^\infty(0, T; X_\delta)$. For any $\rho \in (\delta, 1)$, one has

$$X_\delta \hookrightarrow X_\rho := [D(\Delta), H_0^1(\Omega)^N]_{\rho,\infty} \hookrightarrow H_0^1(\Omega)^N,$$

where the embeddings are compact. On the other hand, $u_{m,t}^n$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega)^N)$. Therefore, from well-known compactness results in spaces of the form $C^0([0, T]; B)$ (see for instance [35]), we deduce that the u_m^n belong to a compact set of $C^0([0, T]; X_\rho)$ for any $\rho \in (\delta, 1)$.

This proves our assertion and ends the proof of Theorem 2.1.

Proof of Lemma 3.1. We will use the following facts:

Proposition. *When* $A \in \mathbb{M}(\alpha, \beta; \Omega)$, *then* (3.4) *is uniquely solvable in* $H_0^1(\Omega)^N$ *for each* $h \in L^2(\Omega)^N$, *with estimates involving only* Ω, α *and* $\|h\|_{L^2(\Omega)}$:

$$\|w_h\|_{H_0^1(\Omega)^N} \le C(\Omega, \alpha)\|h\|_{L^2(\Omega)^N} \quad \text{for all } h \in L^2(\Omega)^N.$$

In fact, from *Meyers-like estimates* for elasticity systems, we can even get something better; see (3.7).

Let us explain this. First, let us recall these estimates:

Theorem 3.2. *Assume that* $A \in \mathbb{M}(\alpha, \beta; \Omega)$ *and* $g \in L^2(\Omega)^{N\times N}$ *and let u be the solution to*

$$\begin{cases} -\nabla \cdot (Ae(u)) = \nabla \cdot g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{3.6}$$

There exists $p_M > 2$, *depending only on* N, Ω, α *and* β, *such that, for all* $p \in [2, p_M)$ *and any* $g \in L_S^p(\Omega)^{N\times N}$, *the corresponding solution to* (3.6) *belongs to* $W_0^{1,p}(\Omega)^N$ *and satisfies*

$$\|u\|_{W^{1,p}(\Omega)^N} \le C(p, N, \Omega, \alpha, \beta)\|g\|_{L^p(\Omega)^{N\times N}}.$$

Recall that the original *Meyers' Theorem* deals with scalar elliptic problems, see [28]. It has been established in the context of linear elasticity in [6] and [10]. The scalar version was used in [14] to prove a result similar to Theorem 2.1, concerning an inverse problem for the wave equation.

In order to use this result in the context of (3.4), it will be enough to check that, for each $h \in L^2(\Omega)^N$, there exist tensor-valued functions $g \in L_S^p(\Omega)^{N\times N}$ with $p > 2$ such that

$$\nabla \cdot g = h, \quad \|g\|_{L^p(\Omega)^{N\times N}} \le C(p, N, \Omega)\|h\|_{L^2(\Omega)^N}.$$

But this is true. Indeed, it suffices to take $g = e(\phi)$, where ϕ is, for instance, the unique solution to

$$\begin{cases} -\nabla \cdot (e(\phi)) = h, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases}$$

Thus, it is clear that there exists $p > 2$ such that, for all $h \in L^2(\Omega)^N$, the associated solution to (3.4) satisfies $w_h \in W_0^{1,p}(\Omega)^N$ and

$$\|w_h\|_{W^{1,p}(\Omega)^N} \le C(p, N, \Omega, \alpha, \beta)\|h\|_{L^2(\Omega)^N}. \tag{3.7}$$

Now, let us introduce

$$r = \frac{2p}{p-2} \tag{3.8}$$

and let us fix $h \in L^2(\Omega)^N$. Assume that A and A' are given in $\mathbb{M}(\alpha, \beta; \Omega)$ and let us denote by w_h (respectively, w_h') the solution to the elliptic problem (3.4) corresponding to A (respectively, A'). Then the following holds:

$$\begin{aligned} \alpha\|e(w_h - w_h')\|_{L^2(\Omega)^N}^2 &\le \int_\Omega A'e(w_h - w_h') \cdot e(w_h - w_h') \\ &= \int_\Omega (A' - A)e(w_h) \cdot e(w_h - w_h') \\ &\le \|A' - A\|_{\mathbb{L}^r(\Omega)}\|e(w_h)\|_{L^p(\Omega)^N}\|e(w_h - w_h')\|_{L^2(\Omega)^N}. \end{aligned}$$

From *Korn's inequality*, we also get

$$\|w_h' - w_h\|_{H_0^1(\Omega)^N} \le C(p, N, \Omega, \alpha, \beta)\|A' - A\|_{\mathbb{L}^r(\Omega)}\|h\|_{L^2(\Omega)^N}. \tag{3.9}$$

This estimate will be used below.

Proposition. *When r is given by* (3.8) *and $A \in \mathbb{M}(\alpha, \beta; \Omega) \cap \mathbb{W}^{1,r}(\Omega)$, then system* (3.4) *is uniquely solvable in $D(\Delta) = H^2(\Omega)^N \cap H_0^1(\Omega)^N$ for all $h \in L^2(\Omega)$, with estimates involving only $p, N, \Omega, \alpha, \beta, \|DA\|_{\mathbb{L}^r(\Omega)^N}$ and $\|h\|_{L^2(\Omega)^N}$:*

$$\|w_h\|_{H^2(\Omega)^N} \le C(p, N, \Omega, \alpha, \beta)(1 + \|DA\|_{\mathbb{L}^r(\Omega)^N})\|h\|_{L^2(\Omega)^N}. \tag{3.10}$$

This is a consequence of (3.7) and the standard linear elasticity regularity theory, see for instance [11, 32]. The basic ideas of the argument are the following.

After introducing a partition of unit, the first task is to prove that any difference quotient $\tau_{i,\varepsilon}(\phi w_h)$ with

$$(\tau_{i,\varepsilon} w_h)(x) := \frac{1}{\varepsilon}(w_h(x + \varepsilon e_i) - w_h(x))$$

and $\phi \in \mathcal{D}(\Omega)$ satisfies an estimate of the form

$$\|\nabla \tau_{i,\varepsilon}(\phi w_h)\|_{L^2(\Omega)^{N\times N}} \le C(p, N, \Omega, \alpha, \beta)\,(1 + \|DA\|_{\mathbb{L}^r(\Omega)^N})\|h\|_{L^2(\Omega)^N}$$

for any sufficiently small $\varepsilon > 0$. Notice that

$$-\nabla \cdot (A(x)\, e(\tau_{i,\varepsilon}(\phi w_h))) = \tau_{i,\varepsilon}(\phi h) + \nabla \cdot (\tau_{i,\varepsilon} A\, e(\tau_{i,\varepsilon}(\phi w_h)) + \cdots,$$

where the dots contain lower order terms. Consequently,

$$\begin{aligned}\alpha\|e(\tau_{i,\varepsilon}(\phi w_h))\|^2_{H_0^1(\Omega)^{N\times N}} &\le C\|\phi h\|_{L^2(\Omega)^N}\|e(\tau_{i,\varepsilon}(\phi w_h))\|_{H_0^1(\Omega)^{N\times N}} \\ &\quad + \|\tau_{i,\varepsilon} A\|_{\mathbb{L}^r(\Omega)}\|e(\tau_{i,\varepsilon}(\phi w_h))\|_{W^{1,p}(\Omega)^{N\times N}}\|e(\tau_{i,\varepsilon}(\phi w_h))\|_{H_0^1(\Omega)^{N\times N}} + \cdots \\ &\le C(p, N, \Omega, \alpha, \beta)(1 + \|DA\|_{\mathbb{L}^r(\Omega)})\|h\|_{L^2(\Omega)^N}\|e(\tau_{i,\varepsilon}(\phi w_h))\|_{H_0^1(\Omega)^{N\times N}} + \cdots.\end{aligned}$$

This, used together with the standard Korn's inequality, furnishes a satisfactory estimate of $\|\phi w_h\|_{H^2(\Omega)^N}$. Similar estimates can be obtained near the boundary using that $\partial\Omega \in W^{2,\infty}$. Collecting all together, we easily get (3.10).

In the sequel, we will take r as in (3.8), where p is the exponent found above.

Proposition. *The assumptions in Lemma 3.1 imply in particular that A belongs to a (Besov) space that can be obtained by applying the real interpolation method of Peetre to $\mathbb{W}^{1,r}(\Omega)$ and $\mathbb{L}^r(\Omega)$. More precisely, we have*

$$A \in \mathbb{BV}(\Omega) \cap \mathbb{L}^\infty(\Omega) \subset [\mathbb{W}^{1,p}(\Omega), \mathbb{L}^p(\Omega)]_{1/p',\infty} \cap \mathbb{L}^\infty(\Omega)$$

for any $p \in [1, +\infty)$; see for instance [7, 9, 39]. *In particular,*

$$A \in [\mathbb{W}^{1,r}(\Omega), \mathbb{L}^r(\Omega)]_{1/r',\infty} \cap \mathbb{L}^\infty(\Omega).$$

Thus, it is reasonable to expect that the associated solutions belong to X_δ.

In order to prove this, let us recall the following nonlinear interpolation result by Luc Tartar [38] (see also [39]):

Theorem 3.3. *Let E_i and F_i be Banach spaces, with $E_0 \subset E_1$ and $F_0 \subset F_1$. Let $U \subset E_1$ be a nonempty open set and let $S : U \mapsto F_1$ be a mapping. We will assume that:*

(1) *S maps $U \cap E_0$ into F_0.*
(2) *There exist $\lambda_* \in (0, 1]$ and $\mu_* > 0$ such that, for any $A \in U$, we can find a neighborhood of $V_A \subset U$ and a constant C_A with the following properties:*
   (a) *$\|S(A') - S(A)\|_{F_1} \le C_A\|A' - A\|^{\lambda_*}_{E_1}$ for all $A' \in V_A$.*
   (b) *$\|S(A')\|_{F_0} \le C_A(1 + \|A'\|^{\mu_*}_{E_0})$ for all $A' \in V_A \cap E_0$.*

Then, for any $\theta \in (0, 1)$, any $p \in [1, +\infty]$ and any $A \in [E_0, E_1]_{\theta,p} \cap U$, one has that $S(A) \in [F_0, F_1]_{\eta,q}$, where

$$\frac{1-\eta}{\eta} = \frac{1-\theta}{\theta}\frac{\lambda_*}{\mu_*}, \qquad q = \max\Big(1, \Big(\frac{1-\theta}{\mu_*} + \frac{\theta}{\lambda_*}\Big)p\Big).$$

Let us fix $h \in L^2(\Omega)^N$, let us take in this result $E_0 = \mathbb{W}^{1,r}(\Omega)$, $E_1 = \mathbb{L}^r(\Omega)$, $F_0 = D(\Delta)$, $F_1 = H_0^1(\Omega)$ and $U = \mathbb{L}^r(\Omega)$ and let us introduce the mapping S, with

$$S(A) = w \iff \begin{cases} -\nabla\cdot(T_{\alpha,\beta}(A)e(w)) = h, & x \in \Omega, \\ w = 0, & x \in \partial\Omega. \end{cases}$$

Here, $T_{\alpha,\beta}(A)$ stands for the following function:

$$(T_{\alpha,\beta}(A))_{ijkl} = \begin{cases} \alpha & \text{if } A_{ijkl} < \alpha, \\ A_{ijkl} & \text{if } \alpha \le A_{ijkl} \le \beta, \\ \beta & \text{if } A_{ijkl} > \beta, \end{cases} \qquad 1 \le i, j, k, l \le N.$$

In view of (3.9) and (3.10), we see that all the hypotheses of Theorem 3.3 are satisfied with $\lambda_* = \mu_* = 1$, $V_A = U = \mathbb{L}^r(\Omega)$ and $C_A = C(\Omega, N, \alpha, \beta)\|h\|_{L^2(\Omega)^N}$ for any $A \in \mathbb{L}^r(\Omega)$. Indeed, for any $h \in L^2(\Omega)^N$ one has

$$\begin{aligned} \|S(A') - S(A)\|_{H_0^1(\Omega)^N} &\le C(\Omega, N, \alpha, \beta)\|h\|_{L^2(\Omega)^N}\|T_{\alpha,\beta}(A') - T_{\alpha,\beta}(A)\|_{\mathbb{L}^r(\Omega)} \\ &\le C(\Omega, N, \alpha, \beta)\|h\|_{L^2(\Omega)^N}\|A' - A\|_{\mathbb{L}^r(\Omega)} \end{aligned}$$

and

$$\begin{aligned} \|S(A')\|_{H^2(\Omega)^N} &\le C(\Omega, N, \alpha, \beta)\|h\|_{L^2(\Omega)^N}(1 + \|DT_{\alpha,\beta}(A')\|_{\mathbb{L}^r(\Omega)}) \\ &\le C(\Omega, N, \alpha, \beta)\|h\|_{L^2(\Omega)^N}(1 + \|DA'\|_{\mathbb{L}^r(\Omega)}) \end{aligned}$$

for all $A, A' \in \mathbb{L}^r(\Omega)$. Therefore, for any $A \in \mathbb{BV}(\Omega) \cap \mathbb{M}(\alpha, \beta; \Omega)$, the associated solution to (3.4) satisfies

$$w_h = S(A) \in [D(\Delta), H_0^1(\Omega)^N]_{1/r',\infty},$$

i.e.

$$w_h \in X_\delta \quad \text{for } \delta = \frac{1}{r'} = \frac{p-2}{2p}.$$

Recall that

$$\|w_h\|_{X_\delta} = \|\varsigma^{-\delta}K(\varsigma; w_h)\|_{L^\infty(\mathbb{R}_+)},$$

where for any $\varsigma \ge 0$ and $w \in H_0^1(\Omega)^N$ one has

$$K(\varsigma; w) := \inf\{\|w_0\|_{H^2(\Omega)^N} + \varsigma\|w_1\|_{H_0^1(\Omega)^N} : w = w_0 + w_1,\ w_0 \in D(\Delta),\ w_1 \in H_0^1(\Omega)^N\}.$$

On the other hand, for any $A \in E_\delta := [\mathbb{W}^{1,r}(\Omega), \mathbb{L}^r(\Omega)]_{\delta,\infty}$, we have

$$\|A\|_{E_\delta} = \|\varsigma^{-\delta}K_0(t; A)\|_{L^\infty(\mathbb{R}_+)}$$

with

$$K_0(\varsigma; A) := \inf\{\|A_0\|_{\mathbb{W}^{1,r}(\Omega)} + \varsigma\|A_1\|_{\mathbb{L}^r(\Omega)} : A = A_0 + A_1,\ A_0 \in \mathbb{W}^{1,r}(\Omega),\ A_1 \in \mathbb{L}^r(\Omega)\}.$$

Taking into account that, for any $A = A_0 + A_1 \in \mathbb{BV}(\Omega) \cap \mathbb{M}(\alpha, \beta; \Omega)$, one has

$$w_h = S(A) = S(A_0) + [S(A) - S(A_0)]$$

and, using again (3.10) and (3.9), we see that

$$K(\varsigma; w_h) = K(\varsigma; S(A)) \le C(\Omega, N, \alpha, \beta)\,\|h\|_{L^2(\Omega)^N}\,(1 + K_0(\varsigma; A)).$$

Therefore,

$$\|\varsigma^{-\delta}K(\varsigma; w_h)\|_{L^\infty(\mathbb{R}_+)} \le C(\Omega, N, \alpha, \beta)\,\|h\|_{L^2(\Omega)^N}(1 + \|\varsigma^{-\delta}K_0(\varsigma; A)\|_{L^\infty(\mathbb{R}_+)}),$$

which yields (3.5).

This ends the proof of Lemma 3.1. □

4 Some numerical results

In this section, we present and apply a numerical method for the solution of (2.2).

First, we will indicate the way we can compute the gradient of the cost function. Secondly, we will describe briefly an iterative algorithm that leads to good numerical results. This will be illustrated with some numerical experiments in the particular case of an elastic body governed by the Lamé system, i.e. for tensors A satisfying $A\xi = 2\mu\xi + \lambda ontr(\xi)\,\mathrm{Id}$ for some real λ and μ and for all $\xi \in \mathbb{R}^{N\times N}$.

4.1 The computation of the gradient

Let $R > 0$ be given. Our aim is to solve numerically the extremal problem (2.2):

$$\begin{cases} \text{Minimize } I(A) \\ \text{subject to } A \in \mathbb{K}(R) \text{ and (1.1).} \end{cases}$$

Recall (1.8) and (2.1) for the definitions of $I(A)$ and $\mathbb{K}(R)$, respectively.

Let us assume that $A \in \mathbb{K}(R)$ and $A' \in \mathbb{L}^\infty_S(\Omega) \cap \mathbb{BV}(\Omega)$ and let us compute *formally* the derivative of the cost function at A in the direction A'. More precisely, let us see that, under appropriate regularity hypotheses on A and the associated states u_m, the derivative of I with respect to A in the direction A' exists and takes the form

$$\frac{dI(A)}{dA} \cdot A' = \sum_{m=1}^{2} \int\int_Q A' e(u_m) \cdot e(p_m), \tag{4.1}$$

where, for each m, p_m is the unique solution to the adjoint system

$$\begin{cases} p_{m.tt} - \nabla \cdot (Ae(p_m)) = 0, & (x,t) \in Q, \\ p_m = \chi^{-1}((Ae(u_m)\cdot \nu - \Upsilon_m)1_S), & (x,t) \in \Sigma, \\ p_m(x,0) = 0, \quad p_{m,t}(x,0) = 0, & x \in \Omega. \end{cases} \tag{4.2}$$

Here, we have denoted by χ the canonical isomorphism from $H^{1/2}(\partial\Omega)^N$ onto $H^{-1/2}(\partial\Omega)^N$.

Indeed, for any small $s \neq 0$, let us set $A^s := A + sA'$, let us denote by u^s_m the solutions to the associated system (1.1) and let us put $u^s_m = u_m + sz^s_m$. Note that z^s_m satisfies

$$\begin{cases} z^s_{m,tt} - \nabla \cdot (A^s e(z^s_m)) = \nabla \cdot (A' e(u_m)), & (x,t) \in Q, \\ z^s_m = 0, & (x,t) \in \Sigma, \\ z^s_m(x,0) = 0, \quad z^s_{m,t}(x,0) = 0, & x \in \Omega. \end{cases}$$

Consequently, z^s_m and $z^s_{m,tt}$ are uniformly bounded in $C^0([0,T]; H^1_0(\Omega)^N)$ and $C^0([0,T]; H^{-1}(\Omega)^N)$, respectively.

We will assume that $\|u^s_{m,tt} - u_{m,tt}\|_{C^0([0,T];L^2(\Omega)^N)} \le Cs$ for some constant $C > 0$. Note that this is equivalent to supposing that $z^s_{m,tt}$ is uniformly bounded in the space $C^0([0,T]; L^2(\Omega)^N)$. As a consequence, $\nabla \cdot (A'e(u_m) + A^s e(z^s_m))$ is bounded in this space and the normal trace $(A'e(u_m) + A^s e(z^s_m)) \cdot \nu$ is uniformly bounded in $C^0([0,T]; H^{-1/2}(\partial\Omega)^N)$. Observe that

$$\begin{aligned} \frac{1}{s}(I(A+sA') - I(A)) &= \frac{1}{2s} \sum_{m=1}^{2} \int_0^T [\|A^s e(u^s_m)\cdot \nu|_S - \Upsilon_m\|^2 - \|Ae(u_m)\cdot\nu|_S - \Upsilon_m\|^2]\,dt \\ &= \sum_{m=1}^{2} \int_0^T ((A'e(u_m) + A^s e(z^s_m))\cdot \nu|_S, Ae(u_m)\cdot \nu|_S - \Upsilon_m)_{H^{-1/2}}\,dt \\ &\qquad + \frac{s}{2} \sum_{m=1}^{2} \int_0^T \|(A'e(u_m) + A^s e(z^s_m))\cdot\nu|_S\|^2\,dt \\ &= B_1(s) + B_2(s). \end{aligned} \tag{4.3}$$

In the second equality, we are using that

$$A^s e(u_m^s)\cdot \nu - Ae(u_m)\cdot \nu = s(A'e(u_m) + A^s e(z_m^s))\cdot \nu,$$

where both sides belong to $H^{-1/2}(\partial\Omega)^N$. If we denote by p_m the solution to (4.2), we get

$$\begin{aligned}
B_1(s) &= \sum_{m=1}^{2}\int_0^T \langle (A'e(u_m) + A^s e(z_m^s))\cdot \nu, p_m\rangle_{H^{-1/2},H^{1/2}}\,dt\\
&= \sum_{m=1}^{2}\iint_Q [(A'e(u_m) + A^s e(z_m^s))\cdot e(p_m) + (\nabla\cdot(A'e(u_m) + A^s e(z_m^s)))\cdot p_m]\\
&= \sum_{m=1}^{2}\iint_Q [(A'e(u_m) + Ae(z_m^s))\cdot e(p_m) + sA'e(z_m^s)\cdot e(p_m) + z_{m,tt}^s\cdot p_m]\\
&= \sum_{m=1}^{2}\iint_Q A'e(u_m)\cdot e(p_m) + s\sum_{m=1}^{2}\iint_Q A'e(z_m^s)\cdot e(p_m) + \sum_{m=1}^{2}\Big[-\int_0^T \langle p_{m,tt}, z_m^s\rangle_{H^{-1},H_0^1}\,dt + \iint_Q z_{m,tt}^s\cdot p_m\Big].
\end{aligned}$$

The terms in the last sum vanish since, after integration by parts, one has

$$\int_0^T \langle p_{m,tt}, z_m^s\rangle_{H^{-1},H_0^1}\,dt = -\iint_Q z_{m,t}^s\cdot p_{m,t} = \iint_Q z_{m,tt}^s\cdot p_m.$$

Consequently,

$$\lim_{s\to 0} B_1(s) = \sum_{m=1}^{2}\iint_Q A'e(u_m)\cdot e(p_m). \tag{4.4}$$

On the other hand, one has

$$B_2(s) \le \frac{Ts}{2}\sum_{m=1}^{2}\|(A'e(u_m) + A^s e(z_m^s))\cdot \nu\|^2_{C^0([0,T];H^{-1/2}(\partial\Omega)^N)}$$

and $\lim_{s\to 0} B_2(s) = 0$. From (4.3), (4.4) and this property of $B_2(s)$, we deduce (4.1).

In the context of a gradient method for the solution of (2.2), it is natural to choose the directions A' such that the derivative of I be nonpositive. In view of (4.1), we can take

$$A'_{ijk\ell}(x) = -\sum_{m=1}^{2}\int_0^T e_{k\ell}(u_m)\cdot e_{ij}(p_m)\,dt, \quad x\in\Omega,\ 1\le i,j,k,\ell\le N. \tag{4.5}$$

Thus, a general gradient algorithm to solve numerically (2.2) is the following:

- Initialization: Choose $A_0 \in \mathbb{K}(R)$.
- For $n \ge 0$, iterate until convergence as follows:
  (1) With $A = A_n$, compute the solutions $u_{m,n}$ to system (1.1) and then the solutions $p_{m,n}$ to system (4.2).
  (2) Compute the associated descent direction A'_n, given by (4.5) with $u_m = u_{m,n}$ and $p_m = p_{m,n}$.
  (3) Update A_n:

$$A_{n+1} = A_n + s_n A'_n, \tag{4.6}$$

  with $s_n \in \mathbb{R}$ appropriate, in order to ensure a significant decrease of the cost function and the constraint $A_{n+1} \in \mathbb{K}(R)$. A quite natural way to choose s_n is to minimize I along the line $s \mapsto A_n + sA'_n$.

Let us now consider the particular case of the inverse problem for the Lamé system (1.3)–(1.4). As before, our aim is to minimize a functional; see (2.3). Now, if we set

$$\begin{aligned}
A_{ijk\ell} &= 2\mu\delta_{ik}\delta j\ell + \lambda\delta_{ij}\delta_{k\ell},\\
A'_{ijk\ell} &= 2\mu'\delta_{ik}\delta j\ell + \lambda'\delta_{ij}\delta_{k\ell}
\end{aligned}$$

for all $i, j, k, \ell = 1, \dots, N$, we easily find that

$$\begin{aligned} \frac{dI}{dA}(A) \cdot A' &= \frac{dI_0}{dI_0} d(\mu, \lambda)(A) \cdot (\mu', \lambda') \\ &= \sum_{m=1}^{2} \iint_Q \Big[2\mu'(x) \sum_{i,j=1}^{N} e_{ij}(u_{m,n}) e_{ij}(p_{m,n}) + \lambda'(x) \Big(\sum_{i=1}^{N} e_{ii}(u_{m,n}) \Big) \Big(\sum_{j=1}^{N} e_{jj}(p_{m,n}) \Big) \Big]. \end{aligned} \tag{4.7}$$

Consequently, if we apply a gradient algorithm, in the n-th step we must take A_{n+1} as in (4.6), with

$$(A'_n)_{ijk\ell} = 2\mu'_n \delta_{ik} \delta j\ell + \lambda'_n \delta_{ij} \delta_{k\ell}$$

and

$$\begin{aligned} \mu'_n &= -\sum_{m=1}^{2} \int_0^T \Big(\sum_{i,j=1}^{N} e_{ij}(u_{m,n}) e_{ij}(p_{m,n}) \Big) \, dt, \\ \lambda'_n &= -\sum_{m=1}^{2} \int_0^T \Big(\sum_{i=1}^{N} e_{ii}(u_{m,n}) \Big) \Big(\sum_{j=1}^{N} e_{jj}(p_{m,n}) \Big) \, dt. \end{aligned}$$

4.2 The optimization strategy

Our aim is to solve numerically the extremal problem (2.3). Obviously, it is interesting to take R as large as possible.

As explained below, at the numerical level, we will introduce a mesh of Ω, we will consider associated piecewise constant Lamé coefficients μ and λ and we will use standard finite difference and finite element approximation schemes for the computation of the states. In view of the particular structure of μ and λ, if

$$\alpha \le \mu(x), \lambda(x) \le \beta \quad \text{a.e.},$$

then the total variations of μ and λ are automatically bounded by a constant only depending on $\beta - \alpha$ and the size of the mesh and we get $(\mu, \lambda) \in \mathbb{K}_0(R)$ for all large $R > 0$. Therefore, for numerical purposes, it makes sense to forget the constraints on $\mathrm{TV}(\mu)$ and $\mathrm{TV}(\lambda)$.

For the tests, among other possibilities, we have decided to use the *Augmented Lagrangian* algorithm, completed with the *L-BFGS* subalgorithm. This gives reasonably good results.

The idea of the Augmented Lagrangian method is to integrate the objective function and the inequality constraints in a single function which penalizes any violated constraint. To this purpose, suitable multipliers and new (slack) variables are introduced. The original problem is then decomposed in a family of unconstrained problems that must be solved sequentially, by specifying a second (sub)algorithm; see [8] and [31] for more details.

On the other hand, the limited-memory BFGS (or L-BFGS) algorithm is a quasi-Newton method close to the so-called BFGS algorithm (by Broyden, Fletcher, Goldfarb and Shanno) that only needs a limited amount of computer memory. It was introduced by J. Nocedal [30] and is very well suited for extremal problems with a large amount of variables.

The idea is the following. In the original BFGS algorithm, for an optimization problem in N_{tot} variables, one stores a full $N_{\mathrm{tot}} \times N_{\mathrm{tot}}$ approximation to the inverse Hessian; here, one constructs a different approximation that only needs a reduced number of vectors (more precisely, the "history" of the last p computed variables and gradients; typically, p can take values from 5 to 10; see [25, 30] for details).

Our numerical experiments have been implemented using the free software FreeFem++ v 3.44 (see http://www.freefem.org/), complemented with the library `NLopt` (see http://ab-initio.mit.edu/wiki/index.php/NLopt). The main required input data are the initial A_0, routines furnishing the values of the cost function and the associated gradient through the adjoint state (see (4.1) and (4.7)), the lower and upper bounds of μ and the stopping criteria.

4.3 Numerical experiments for the Lamé system

Let us present the results of some experiments.

4.3.1 First tests: Solving (2.3)

We take $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and we consider the particular case of Lamé systems, i.e. with $A\xi = 2\mu\xi + \lambda\,\mathrm{tr}(\xi)\,\mathrm{Id}$ for all $\xi \in \mathbb{R}^{N\times N}$. We take the following initial condition u^1 and right-hand side:

$$u^1 \equiv (\sin(\pi x_1)\sin(\pi x_2), x_1(x_1-1)x_2(x_2-1)), \quad f \equiv (1,1).$$

In order to show the efficiency of our approach, we first fix

$$\overline{\mu}(x) \equiv \overline{\lambda}(x) \equiv \beta\chi_D + \alpha(1-\chi_D),$$

where $D \subset \Omega$ and χ_D is the associated characteristic function.

The domain Ω may be viewed as a region that contains healthy and tumoral cells. The set D can be interpreted as the tumoral area and β and α respectively represent the stiffness levels of the tumor and healthy cells. Having in mind the properties of the tumor and healthy tissues, we take in our numerical experiments $\alpha = 5$ and $\beta = 10$ (roughly speaking, we assume that stiffness is twice higher within the tumor tissue).

We use P_1-Lagrange finite element approximations in space and centered finite difference approximations in time in both systems (1.1) and (4.2). The triangulation is shown in Figure 1.

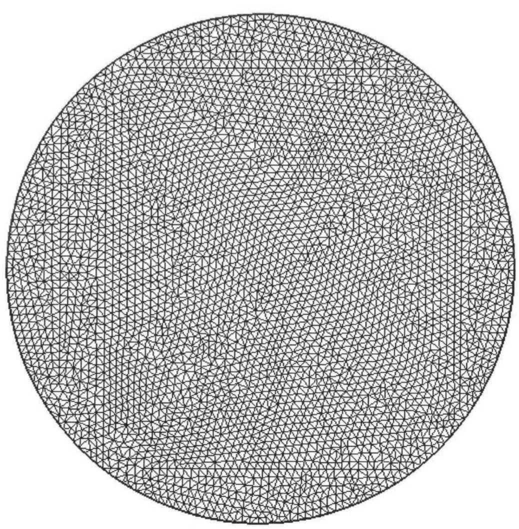

Figure 1: The domain Ω and its triangulation. Number of nodes: 3,629. Number of triangles: 7,056.

We have solved (1.1) and we have computed the associated boundary data

$$\Upsilon := (\overline{\mu}(x)(\nabla u + \nabla u^T) + \overline{\lambda}(x)(\nabla\cdot u)\,\mathrm{Id})\cdot \nu|_{S\times(0,T)},$$

where $S \subset \partial\Omega$ is the whole upper half-circle. Thus, it is ensured that at least one solution to the extremal problem (2.2) exists. It is given by $\overline{\mu}$ and $\overline{\lambda}$ and the associated cost is zero.

We choose $\mu_0 \equiv \lambda_0 \equiv \alpha$ (that is, we start from initial healthy tissue) and we make the numerical simulations in two different cases.

Case 1: The "one isolated tumor" case. We take

$$D = \{(x,y) \in \mathbb{R}^2 : (x+0.3)^2 + y^2 < 0.0625\}.$$

Figure 2 shows the computed optimal value of μ on the left and the target on the right. The results concerning λ

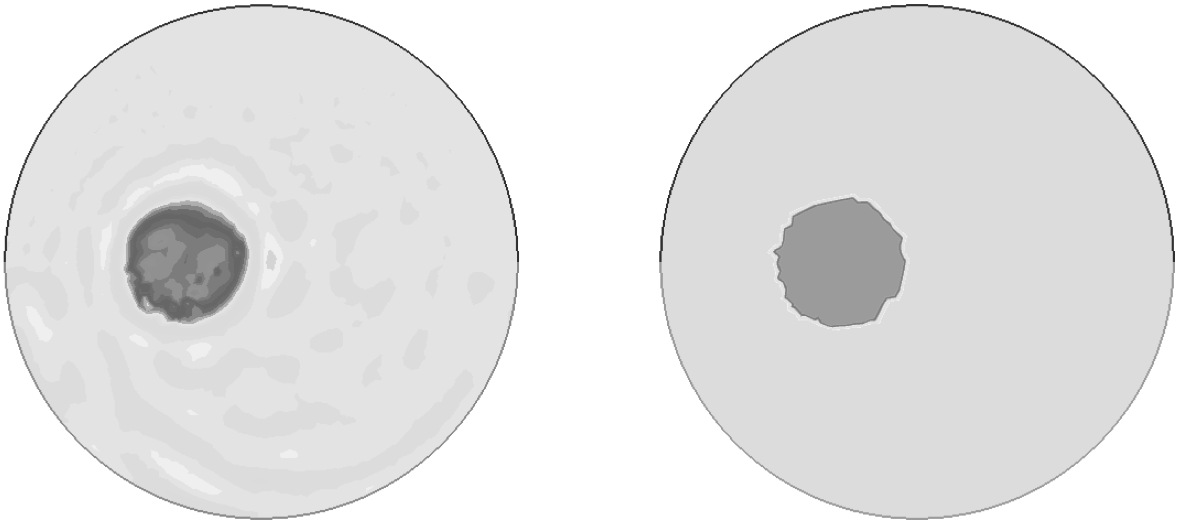

Figure 2: Case 1 – The best computed μ (left) and the target $\overline{\mu}$ (right).

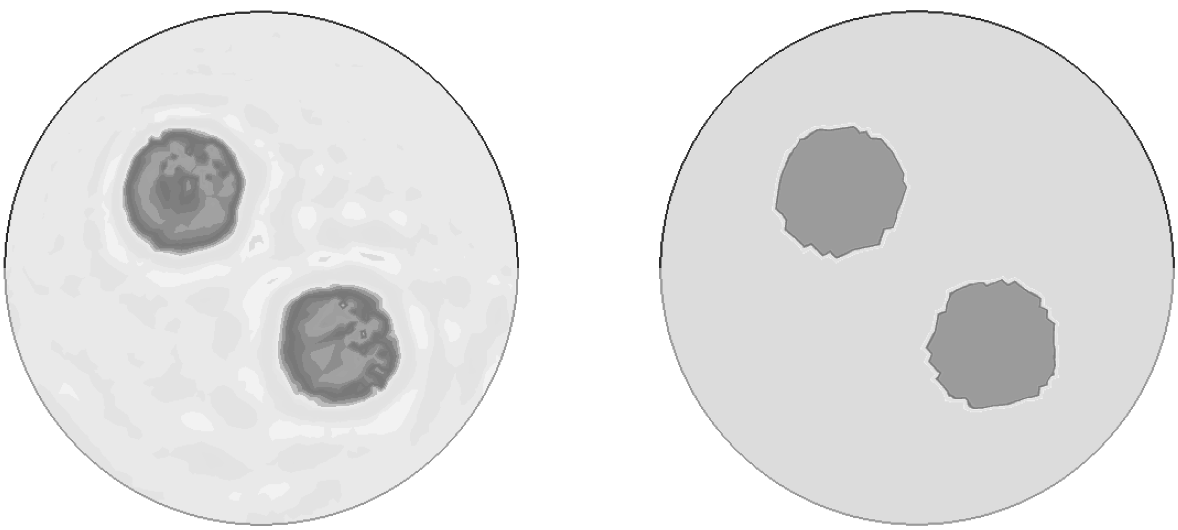

Figure 3: Case 2 – The best computed μ (left) and the target $\overline{\mu}$ (right).

are completely similar. It is thus clear that the previous numerical algorithm detects the tumoral cells; in this simulation, the final value of the cost function is $\sim 9.50 \times 10^{-8}$ after 148 computations of the cost and 78 computations of the gradient. This shows that the computed solution to (2.2) also solves the original inverse problem.

Case 2: The "two isolated tumors" case. Now, we take

$$D = \{(x,y) \in \mathbb{R}^2 : (x+0.3)^2 + (y-0.3)^2 < 0.0625\} \cup \{(x,y) \in \mathbb{R}^2 : (x-0.3)^2 + (y+0.3)^2 < 0.0625\};$$

see Figure 3. The cost function corresponding to the optimal computed μ (depicted in the left) is $\sim 9.88 \times 10^{-8}$ after 176 computations of the cost and 80 computations of the gradient. Again, the computed λ is completely similar and, therefore, we see that we have again solved numerically the original inverse problem.

For completeness, we present in both cases the evolution of the cost versus the number of iterates at logarithmic scale in Figure 4.

Let us mention that the numerical solution of optimization problems of this kind, where the cost function depends on the normal derivative of a function, needs in practice a very sharp approximation of this value, in order to ensure the convergence of the method.

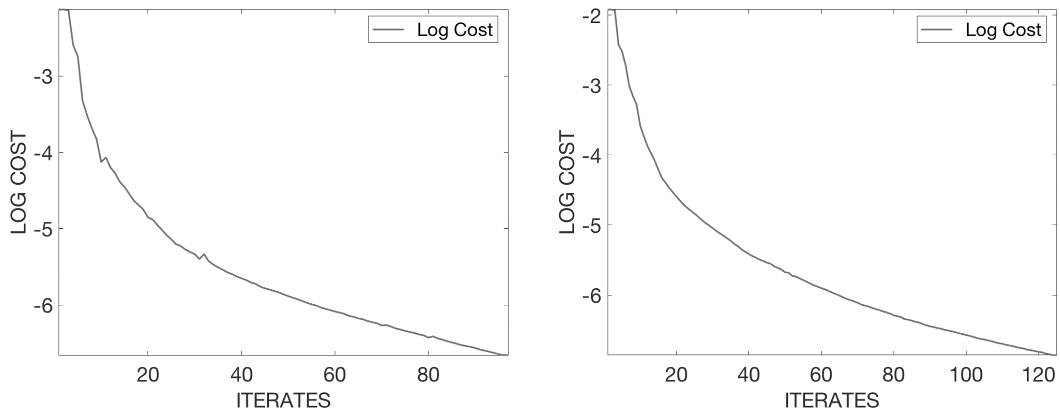

Figure 4: The decimal logarithm of the cost versus the number of iterates. Case 1 (left) and Case 2 (right).

μ_0	λ_0: α	$\frac{1}{2}(\alpha+\beta)$	β
α	148	196	314
$\frac{1}{2}(\alpha+\beta)$	162	184	330
β	204	260	436

Table 1: Case 1 – The behavior of the algorithm for various starting μ^0 and λ^0.

μ_0	λ_0: α	$\frac{1}{2}(\alpha+\beta)$	β
α	176	205	285
$\frac{1}{2}(\alpha+\beta)$	184	229	308
β	266	280	304

Table 2: Case 2 – The behavior of the algorithm for various starting μ^0 and λ^0.

4.3.2 The role of the starting (μ^0, λ^0)

In order to investigate the influence of the choice of μ^0 and λ^0 on the behavior of the L-BFGS algorithm, we have re-started the iterates from various constant coefficients. In all the experiments, we stopped the iterates as soon as the cost was $\leq 10^{-8}$. The computational cost corresponding to each case, measured as the number of times that a new value of I is computed, is indicated in Tables 1 and 2. We see that, as expected, better results are obtained by starting from the lowest values of μ and λ, that is, from the configuration corresponding to healthy tissue. However, the results in the Tables show that the algorithm also converges for other initial coefficients.

4.3.3 Checking robustness

We have also tried to illustrate the influence of numerical errors in the data. To this purpose, we have introduced perturbations in the Υ_m of orders 20 %, 10 %, 5 %, etc. and we have performed the same number of iterates needed to get a cost $\leq 10^{-8}$ with exact data. This way, we have been able to compare the accuracy of the algorithm with and without data errors. The perturbations have been obtained by modifying randomly the values of the Υ_m at the nodes. The results of these experiments are depicted in Table 3. We see that in both cases, the chosen algorithm is reasonably robust. Indeed, for instance, in Case 1, a 10 % (respectively, 5 % or 1 %) noise on the observed Υ_m leads to an increase of order 70 % (respectively, 50 % or 30 %) of the logarithm of the cost.

5 Some additional comments

The constraint $\mathrm{TV}(A) \leq R$ in (2.2) seems artificial at first sight. However, it is satisfied in practice in some particular realistic situations.

	Cost	
Perturbation size	Case 1	Case 2
0 %	$9.5020 \cdot 10^{-8}$	$9.8828 \cdot 10^{-8}$
0.1 %	$2.1481 \cdot 10^{-7}$	$2.4437 \cdot 10^{-7}$
1 %	$1.3877 \cdot 10^{-5}$	$1.3369 \cdot 10^{-5}$
5 %	$3.4841 \cdot 10^{-4}$	$3.5258 \cdot 10^{-4}$
10 %	$1.4731 \cdot 10^{-3}$	$1.4268 \cdot 10^{-3}$
20 %	$5.8029 \cdot 10^{-2}$	$5.9114 \cdot 10^{-2}$

Table 3: The behavior of the algorithm for randomly perturbed data.
The number of iterates is 90 in Case 1 and 119 in Case 2.

Obviously, it would be interesting to know what happens to the solutions A_R^* furnished by Theorem 2.1 as $R \to +\infty$. Unfortunately, by analogy with many other related problems, it is reasonable to expect that in general A_R^* oscillates. In the limit, we are led to a *relaxed* and more general problem.

As indicated above, we can consider other extremal problems similar to (2.2) for which the existence of a solution can be established. For instance, we can introduce for each $\varepsilon > 0$ the following penalized problem:

$$\begin{cases} \text{Minimize } I_\varepsilon(A) = I(A) + \frac{\varepsilon}{2}\mathrm{TV}(A)^2 \\ \text{subject to (1.6) and (1.1).} \end{cases} \tag{5.1}$$

For each $\varepsilon > 0$, there exists at least one solution A^ε to (5.1). This can be easily established arguing as in Section 3. However, the behavior of A^ε as $\varepsilon \to 0^+$ can be again oscillatory and, in the limit, we find once more a *relaxed* problem.

Finally, let us consider the extremal problem

$$\begin{cases} \text{Minimize } J(A, u, e(u), u_t) \\ \text{subject to } A \in \mathbb{K}(R) \text{ and (1.1),} \end{cases} \tag{5.2}$$

where J is assumed to satisfy the following property:

Property. If the A^n belong to $\mathbb{K}(R)$ and satisfy (3.1) and the u^n are the associated states, then, at least for a subsequence,

$$\liminf_{n\to+\infty} J(A^n, u^n, e(u^n), u_t^n) \geq J(A^*, u^*, e(u^*), u_t^*),$$

where u^* is the state associated to A^*.

Then the argument in the proof of Theorem 2.1 can also be used to show that (5.2) possesses at least one solution.

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