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CR-PRODUCTS OF S-MANIFOLDS

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Abstract: In this paper certain CR-submanifolds of S-manifolds, ([3]), namely the CR-products, are studied. A characterization theorem is given and the case of CR-products of S-manifolds whose invariant f-sectional curvature is constant, is investigated.

0 – Introduction

For manifolds with an f-structure, D.E. Blair, ([3]), has introduced the analogue of the Kaehler structure in the almost complex case and of quasi-Sasakian structure in the almost contact case, by defining S-manifolds.

On the other hand, A. Bejancu, ([1]), has initiated to study CR-submanifolds of Kaehlerian manifolds. The concept of CR-submanifold has been also considered in other structures, like Sasakian structures, ([2]). I. Mihai, ([7]), and L. Ornea, ([8]), have studied CR-submanifolds of S-manifolds.

The purpose of this paper is to study certain CR-submanifolds of S-manifolds, namely the CR-products. In section 1, we review basic formulas for submanifolds in Riemannian manifolds. In section 2, we give several results on CR-submanifolds of S-manifolds, for later use. In section 3, we study CR-products of S-manifolds and give a characterization theorem. In the last section, we consider CR-products of S-manifolds whose invariant f-sectional curvature is constant.

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1 – Preliminaries

Let N^n be a Riemannian manifold of dimension n and M^m an m-dimensional submanifold of N^n. Let g be the metric tensor field on N^n as well as the induced metric on M^m. We denote by $\tilde{\nabla}$ the covariant differentiation in N^n and by ∇ the covariant differentiation in M^m determined by the induced metric. Let $T(N)$ (resp. $T(M)$) be the Lie Algebra of vector fields in N^n (resp. in M^m) and $T(M)^\perp$ the set of all vector fields normal to M^m.

The Gauss–Weingarten formulas are given by:

$$\begin{aligned} &\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) , \\ &\tilde{\nabla}_X V = -A_V X + D_X V , \qquad X,Y \in T(M), \quad V \in T(M)^\perp , \end{aligned} \tag{1.1}$$

where D is the connection in the normal bundle, σ is the second fundamental form of M^m, A_V the Weingarten Endomorphism associated with V and satisfy:

$$g(A_V X, Y) = g\Big(\sigma(X,Y), V\Big) .$$

We denote by $\tilde{R}$ and R the curvature tensor fields associated with $\tilde{\nabla}$ and ∇, respectively. The Gauss equation is given by:

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = {}& R(X,Y,Z,W) + g\Big(\sigma(X,Z),\sigma(Y,W)\Big) \\ & - g\Big(\sigma(X,W),\sigma(Y,Z)\Big), \quad X,Y,Z,W \in T(M). \end{aligned} \tag{1.2}$$

Moreover, we have the following Codazzi equation:

$$\tilde{R}(X,Y,Z,V) = g\Big((\nabla'_X\sigma)(Y,Z),V\Big) - g\Big((\nabla'_Y\sigma)(X,Z),V\Big) , \tag{1.3}$$

for any $X,Y,Z \in T(M)$ and $V \in T(M)^\perp$, where $\nabla'\sigma$ is the covariant derivative of the second fundamental form given by:

$$(\nabla'_X\sigma)(Y,Z) = D_X\sigma(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) , \tag{1.4}$$

for any $X,Y,Z \in T(M)$. Finally, the submanifold M^m is said to be totally geodesic in N^n if its second fundamental form is identically zero.

2 – CR-submanifolds of S-manifolds

Let N^{2n+s} be an S-manifold of dimension $2n+s$, with structure tensors $(f, \xi_1, ..., \xi_s, \eta_1, ..., \eta_s, g)$. Then, these structure tensors satisfy the following equations, ([3]):

$$\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}; \quad f\,\xi_\alpha = 0; \quad \eta_\alpha(fX) = 0; \tag{2.1}$$
$$f^2 = -I + \sum \xi_\alpha \otimes \eta_\alpha\,, \qquad X \in T(N)\,.$$

Thus, the tensor f is an f-structure, ([9]), of rank $2n$. Furthermore, the metric g is compatible with f, that is:

$$g(X,Y) = g(fX, fY) + \Phi(X,Y)\,, \tag{2.2}$$

for any $X, Y \in T(N)$, where $\Phi(X,Y) = \sum \eta_\gamma(X)\,\eta_\gamma(Y)$. Moreover, f is normal:

$$[f,f] + 2\sum \xi_\alpha \otimes d\eta_\alpha = 0\,,$$

where $[f,f]$ is the Nijenhuis torsion of f. The covariant differentiation $\tilde{\nabla}$ of N^{2n+s} satisfies, ([3]), for any $X, Y \in T(N)$, $\alpha = 1, ..., s$:

$$\tilde{\nabla}_X \xi_\alpha = -fX\,, \tag{2.3}$$

$$(\tilde{\nabla}_X f)Y = \sum \Big[g(fX, fY)\,\xi_\alpha + \eta_\alpha(Y)\,f^2X\Big]\,. \tag{2.4}$$

Furthermore, in an S-manifold, we have $F = d\eta_\alpha$, $\alpha = 1, ..., s$, where F is the fundamental 2-form defined by $F(X,Y) = g(X, fY)$, $X, Y \in T(N)$. We denote by $\mathcal{L}$ the distribution determined by $-f^2$ and by $\mathcal{M}$ the complementary distribution. Then $\mathcal{M}$ is determined by $f^2 + I$ and spanned by $\xi_1, ..., \xi_s$. If $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for all α, and if $X \in \mathcal{M}$, $fX = 0$.

In the case $s = 1$, an S-manifold is a Sasakian manifold. For $s \geq 2$, examples of S-manifolds are given in [3], [4] and [5]. Thus, the bundle space of a principal toroidal bundle over a Kaehler manifold, with certain conditions, is an S-manifold.

Now, let M^m be an m-dimensional submanifold immersed in N^{2n+s}. For any $X \in T(M)$, we write:

$$fX = TX + NX\,, \tag{2.5}$$

where TX is the tangential component of fX and NX is the normal component of fX. Then, T is an endomorphism of the tangent bundle and N is a normal-bundle valued 1-form on the tangent bundle. It is easy to show that if T does not vanish, it defines an f-structure in the tangent bundle. On the other hand, the submanifold M^m is said to be invariant if $N \equiv 0$ and anti-invariant if $T \equiv 0$.

In the same way, for any vector field $V \in T(M)^\perp$, we write:

$$fV = tV + nV \ , \tag{2.6}$$

where tV (resp. nV) is the tangential component (resp. normal component) of fV. Then, t is a tangent-bundle valued 1-form on the normal bundle and n is an endomorphism of the normal bundle. Moreover, if n does not vanish, it is an f-structure.

Now, we assume that the structure vector fields $\xi_1, \ldots, \xi_s$ are tangent to M^m (and so, $m \geq s$). Then, M^m is called a CR-submanifold of N^{2n+s} if there exist two differentiable distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M^m satisfying:

i) $T(M) = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{M}$, where $\mathcal{D}$, $\mathcal{D}^\perp$ and $\mathcal{M}$ are mutually orthogonal to each other;

ii) The distribution $\mathcal{D}$ is invariant by f, that is, $f\mathcal{D}_x = \mathcal{D}_x$, for any $x \in M^m$;

iii) The distribution $\mathcal{D}^\perp$ is anti-invariant by f, that is $f\mathcal{D}^\perp \subseteq T_x(M)^\perp$, for any $x \in M^m$.

We denote by $2p$ and q the real dimensions of $\mathcal{D}_x$ and $\mathcal{D}_x^\perp$ respectively, $x \in M^m$. Then, we see that for $p = 0$ we obtain an anti-invariant submanifold tangent to $\xi_1, \ldots, \xi_s$ and for $q = 0$ we obtain an invariant submanifold.

As an example, it is easy to prove that each hypersurface of N^{2n+s} which is tangent to $\xi_1, \ldots, \xi_s$ inherits the structure of CR-submanifold of N^{2n+s}.

We denote by P and Q the projection morphisms of $T(M)$ on $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. Then, for any $X \in T(M)$, we have that $X = PX + QX + \sum \eta_\alpha(X)\,\xi_\alpha$. Now, for later use, we shall prove some lemmas.

Lemma 2.1. *Let M^m be a CR-submanifold of an S-manifold N^{2n+s}. The following assertions are satisfied:*

$$\nabla_X \xi_\alpha = -fPX \ , \tag{2.7}$$

(2.8) $$\sigma(X,\xi_\alpha) = -f\,QX\ ,$$

(2.9) $$A_V\,\xi_\alpha \in \mathcal{D}^\perp\ ,$$

for any $X \in T(M)$, $V \in T(M)^\perp$ *and* $\alpha \in (1,\dots,s)$.

Proof: (2.7) and (2.8) follow immediately from (2.3) and (1.1). Now, if $Y \in \mathcal{D}$, we have

$$g(A_V\,\xi_\alpha, Y) = g\Big(\sigma(Y,\xi_\alpha), V\Big) = 0$$

and so, (2.9) holds. ∎

Lemma 2.2. *Let M^m be a CR-submanifold of an S-manifold N^{2n+s}. If $X, Y \in T(M)$, then:*

(2.10) $$P\,\nabla_X f\,PY - P\,A_{f\,QY}X$$
$$= f\,P\,\nabla_X Y + \sum g(fX, fY)\,\xi_\alpha - \sum \eta_\alpha(Y)\,PX + \sum_{\alpha,\beta} \eta_\alpha(X)\,\eta_\beta(Y)\,\xi_\alpha\ ,$$

(2.11) $$Q\,\nabla_X f\,PY - Q\,A_{f\,QY}X = t\,\sigma(X,Y) - \sum \eta_\alpha(Y)\,QX\ ,$$

(2.12) $$\sigma(X, f\,PY) + D_X f\,QY = f\,Q\,\nabla_X Y + n\,\sigma(X,Y)\ .$$

Proof: From (1.1), we get, by a direct expansion:

$$\begin{aligned}(\tilde\nabla_X f)Y &= \tilde\nabla_X fY - f\,\tilde\nabla_X Y\\ &= \tilde\nabla_X PY + \tilde\nabla_X QY - f\,\nabla_X Y - f\,\sigma(X,Y)\\ &= P\,\nabla_X f\,PY + Q\,\nabla_X f\,PY + \sigma(X, f\,PY) - P\,A_{f\,QY}X\\ &\quad - Q\,A_{f\,QY}X + D_X f\,QY - f\,P\,\nabla_X Y - f\,Q\,\nabla_X Y\\ &\quad - t\,\sigma(X,Y) - n\,\sigma(X,Y)\ .\end{aligned}$$

Now, using (2.4) and comparing the components in $\mathcal{D} \oplus \mathcal{M}$, $\mathcal{D}^\perp$ and $T(M)^\perp$, we complete the proof. ∎

Lemma 2.3. *Let M^m be a CR-submanifold of an S-manifold. Then:*

(2.13) $g(\nabla_X Z, Y) = g(T A_{fZ} X, Y)$, $X \in T(M)$, $Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

Proof: A direct expansion, using (2.4), gives:

$$(\nabla_X T)Z = \nabla_X TZ - T\,\nabla_X Z = -T\,\nabla_X Z = \sum g(X,Z)\,\xi_\alpha + A_{fZ}X + t\,\sigma(X,Z)\ , \tag{2.14}$$

for any $X \in T(M)$, $Z \in \mathcal{D}^\perp$. Then, since $fY = TY$ for any $Y \in \mathcal{D}$, we have:

$$g(\nabla_X Z, Y) = g(f\,\nabla_X Z, fY) = g(T\,\nabla_X Z, TY) = -g(A_{fZ}X, fY) = g(f\,A_{fZ}X, Y)$$

and the proof is complete. ∎

With regard to the integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$, I. Mihai, ([7]), has proved that the distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\perp \oplus \mathcal{M}$ are always integrable. On the other hand, if $p > 0$, the distributions $\mathcal{D}$ and $\mathcal{D} \oplus \mathcal{D}^\perp$ are not integrable. Finally, the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable if and only if:

$$\sigma(X, fY) = \sigma(fX, Y)\ , \tag{2.15}$$

for any $X, Y \in \mathcal{D}$. From these results, we see that it might be interesting to study the geometry of the leaves of the distributions $\mathcal{D} \oplus \mathcal{M}$ and $\mathcal{D}^\perp$. Then, we can prove:

Theorem 2.4. *Let M^m be a CR-submanifold of an S-manifold N^{2n+s}. Then:*

i) *The distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable and its leaves are totally geodesic in M^m if and only if:*

$$g\big(\sigma(\mathcal{D},\mathcal{D}), f\,\mathcal{D}^\perp\big) = 0\ . \tag{2.16}$$

ii) *The distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable and its leaves are totally geodesic in N^{2n+s} if and only if:*

$$\sigma(\mathcal{D},\mathcal{D}) = 0\ . \tag{2.17}$$

Proof: Let $X, Y \in \mathcal{D} \oplus \mathcal{M}$ and $Z \in \mathcal{D}^\perp$. If the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable and its leaves are totally geodesic in M^m, then, $\nabla_X fY \in \mathcal{D} \oplus \mathcal{M}$. From (2.14), we get:

$$0 = g(\nabla_X fY, Z) = -g(\nabla_X Z, fY) = g(T\,\nabla_X Z, Y) = g\big(\sigma(X,Y), fZ\big).$$

Conversely, if (2.16) holds, then the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable by virtue of (2.15). Moreover, we have, using (2.4):

$$\begin{aligned} 0 = g\Big(\sigma(X, fY), fZ\Big) &= g(\tilde{\nabla}_X fY, fZ) \\ &= g\Big((\tilde{\nabla}_X f)Y, fZ\Big) + g(f\,\tilde{\nabla}_X Y, fZ) \\ &= g(\tilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z)\ . \end{aligned}$$

Thus, $\nabla_X Y \in \mathcal{D} \oplus \mathcal{M}$ and the leaves of $\mathcal{D} \oplus \mathcal{M}$ are totally geodesic in M^m.

Now, we suppose that $\mathcal{D} \oplus \mathcal{M}$ is integrable and its leaves are totally geodesic in N^{2n+s}. Then, we have $\tilde{\nabla}_X Y \in \mathcal{D} \oplus \mathcal{M}$ for any $X, Y \in \mathcal{D} \oplus \mathcal{M}$. By using the Gauss formula we obtain:

$$g\Big(\sigma(X,Y), V\Big) = g(\tilde{\nabla}_X Y, V) = 0\ ,$$

for any $V \in T(M)^{\perp}$, that is, (2.17) is satisfied, where we have used (2.8). Conversely, if (2.17) is satisfied, then, by (2.15), $\mathcal{D} \oplus \mathcal{M}$ is integrable. Let M_1 a leaf of $\mathcal{D} \oplus \mathcal{M}$ and denote by σ_1 (resp. by σ') the second fundamental form of the immersion of M_1 in N^{2n+s} (resp. in M^m). Then, from (2.17), we have $\sigma_1 = \sigma'$. But, from (2.16), M_1 is totally geodesic in M^m. Hence, M_1 is totally geodesic in N^{2n+s}. ∎

Theorem 2.5. *Let M^m be a CR-submanifold of an S-manifold N^{2n+s}. Then, any leaf of $\mathcal{D}^{\perp}$ is totally geodesic in M^m if and only if:*

$$g\Big(\sigma(\mathcal{D}, \mathcal{D}^{\perp}), f\,\mathcal{D}^{\perp}\Big) = 0\ . \tag{2.18}$$

Proof: From (2.14) we get:

$$\begin{aligned} g(T\,\nabla_W Z, Y) &= -g(A_{fZ}W, Y) - g\Big(t\,\sigma(Z,W), Y\Big) \\ &= -g\Big(\sigma(Y,W), fZ\Big)\ , \end{aligned}$$

for any $Y \in \mathcal{D}$, $Z, W \in \mathcal{D}^{\perp}$. Then:

$$g(\nabla_W Z, TY) = g\Big(\sigma(Y,W), fZ\Big)\ .$$

Consequently, $\nabla_W Z \in \mathcal{D}^{\perp}$ if and only if (2.18) holds. ∎

3 – CR-products in S-manifolds

Let M^m be a CR-submanifold of an S-manifold N^{2n+s}, $m \geq s$. We say that M^m is a CR-product if the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable and locally M^m is a Riemannian product $M_1 \times M_2$, where M_1 (resp. M_2) is a leaf of $\mathcal{D} \oplus \mathcal{M}$ (resp. $\mathcal{D}^\perp$). If we have $p\,q \neq 0$, we say that M^m is a proper CR-product.

Theorem 3.1. *Let M^m be a CR-submanifold of an S-manifold N^{2n+s}. Then, the following assertions are equivalent to each other:*

i) *M^m is a CR-product;*

ii) $A_{f\,\mathcal{D}^\perp} f\,\mathcal{D} = 0$;

iii) $(\nabla_X T)\mathcal{D} \subseteq \mathcal{D}$, $X \in T(M)$;

iv) $(\nabla_X T)\mathcal{D}^\perp \subseteq \mathcal{D}^\perp$, $X \in T(M)$.

Proof: **i)⇒ii):** We suppose that M^m is a CR-product locally represented by $M_1 \times M_2$. Then M_1 and M_2 are totally geodesic in M^m. Thus, the Gauss formula implies:

$$\nabla_X Y \in \mathcal{D} \oplus \mathcal{M} \;, \tag{3.1}$$

for any $X, Y \in \mathcal{D} \oplus \mathcal{M}$ and

$$\nabla_Z W \in \mathcal{D}^\perp \;, \tag{3.2}$$

for any $Z, W \in \mathcal{D}^\perp$. Now, using (3.2) and (2.7), we obtain:

$$\begin{aligned} &g(\nabla_Z Y, W) = -g(Y, \nabla_Z W) = 0 \;, \\ &g(\nabla_{\xi_\alpha} Y, W) = -g\Big(-fY + [\xi_\alpha, Y], W\Big) = 0 \;, \end{aligned} \tag{3.3}$$

for any $Y \in \mathcal{D}$, $Z, W \in \mathcal{D}^\perp$ and $\alpha \in \{1, ..., s\}$. Thus, from (3.1), we get that $\nabla_X Y \in \mathcal{D} \oplus \mathcal{M}$, $X \in T(M)$ and $Y \in \mathcal{D} \oplus \mathcal{M}$. In the same way, $\nabla_X Z \in \mathcal{D}^\perp$, $X \in T(M)$, $Z \in \mathcal{D}^\perp$. Then, using (2.13):

$$\begin{aligned} 0 = g(\nabla_X Y, Z) &= -g(Y, \nabla_X Z) = -g(T\,A_{fZ} X, Y) \\ &= g(A_{fZ} X, fY) = g(A_{fZ} fY, X) \;. \end{aligned}$$

ii)⇒i): We know that $\mathcal{D}^\perp$ is integrable. Moreover, if $X \in T(M)$, $Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$, we have, from (2.8) and ii):

$$g\Big(\sigma(X,Y), fZ\Big) = g\Big(\sigma(X, -f^2Y), fZ\Big) = -g(A_{fZ} f^2 Y, X) = 0 \; . \tag{3.4}$$

If $X \in \mathcal{D}^\perp$, from Theorem 2.5, the leaves of $\mathcal{D}^\perp$ are totally geodesic in M^m. On the other hand, $\mathcal{D} \oplus \mathcal{M}$ is integrable. Moreover, from (3.4), we get:

$$g\Big(\sigma(X,Y), fZ\Big) = 0 \; ,$$

for any $X, Y \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$ and so, the leaves of $\mathcal{D} \oplus \mathcal{M}$ are totally geodesic in M^m, by virtue of (2.16). Then, M^m is a CR-product.

ii)⇔iii): From (2.4), a direct expansion gives:

$$(\nabla_X T)Y = \sum g(X,Y)\, \xi_\alpha + t\,\sigma(X,Y) \; ,$$

for any $X \in T(M)$, $Y \in \mathcal{D}$. Then, if $Z \in \mathcal{D}^\perp$, we have:

$$g\Big((\nabla_X T)Y, Z\Big) = g\Big(t\,\sigma(X,Y), Z\Big) = -g\Big(\sigma(X,Y), fZ\Big) = -g(A_{fZ}Y, X) \; . \tag{3.5}$$

Thus, since $Y = -f^2 Y$, the result is immediate from (3.5).

iii)⇔iv): It is obvious because $\nabla_X T$ is an antisymmetric operator. ∎

Proposition 3.2. *Let M^m be a CR-submanifold of an S-manifold N^{2n+s}. Then, M^m is a CR-product if and only if one of the following assertions is satisfied:*

$$\nabla_X Y \in \mathcal{D} \oplus \mathcal{M} \, , \quad X \in T(M), \;\; Y \in \mathcal{D} \; , \tag{3.6}$$

$$t\,\sigma(X,Y) = 0 \, , \quad X \in T(M), \;\; Y \in \mathcal{D} \; , \tag{3.7}$$

$$\sigma(X, fY) = n\,\sigma(X,Y) \, , \quad X \in T(M), \;\; Y \in \mathcal{D} \; . \tag{3.8}$$

Proof: If M^m is a CR-product, we have seen in the proof of the above theorem that (3.6) is satisfied. Conversely, we suppose that

(3.6) is satisfied. Thus, the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable, since we have:

$$[X, Y] = \nabla_X Y - \nabla_Y X \in \mathcal{D} \oplus \mathcal{M} ,$$
$$[X, \xi_\alpha] = -fX - \nabla_{\xi_\alpha} X \in \mathcal{D} \oplus \mathcal{M} ,$$

for any $X, Y \in \mathcal{D}$. Moreover, if M_1 is a leaf of $\mathcal{D} \oplus \mathcal{M}$, then, from (3.6) and the Gauss formula for the immersion of M_1 in M^m, it follows that M_1 is totally geodesic in M^m. Now, from (3.6) we get that $\nabla_X Z \in \mathcal{D}^\perp$ for any $X \in T(M)$ and $Z \in \mathcal{D}^\perp$. Using again the Gauss formula for a leaf M_2 of $\mathcal{D}^\perp$ we obtain that M_2 is totally geodesic in M^m. So M^m is a CR-product.

Next, from (2.10), (2.11) and (2.12) it follows that:

$$\nabla_X fY = f\,P\,\nabla_X Y + \sum g(fX, fY)\,\xi_\alpha + t\,\sigma(X, Y)$$

and:

$$\sigma(X, fY) = f\,Q\,\nabla_X Y + n\,\sigma(X, Y) ,$$

for any $X \in T(M)$, $Y \in \mathcal{D}$. Thus, from (3.6) we get that assertions (3.7) and (3.8) are satisfied. ∎

Let M^m be a CR-submanifold of an S-manifold N^{2n+s}. Then, we say that M^m is totally f-umbilical, ([8]), if there exists a normal vector field H such that:

$$\sigma(X, Y) = g(fX, fY)\,H + \sum\Big[\eta_\alpha(Y)\,\sigma(X, \xi_\alpha) + \eta_\alpha(X)\,\sigma(Y, \xi_\alpha)\Big] , \tag{3.9}$$

for any $X, Y \in T(M)$. If $H = 0$ in (3.9), that is, if:

$$\sigma(X, Y) = \sum\Big[\eta_\alpha(Y)\,\sigma(X, \xi_\alpha) + \eta_\alpha(X)\,\sigma(Y, \xi_\alpha)\Big] , \tag{3.10}$$

then we say that M^m is totally f-geodesic. We know:

Proposition 3.3 ([8]). *Any proper totally f-umbilical CR-submanifold of an S-manifold with $q > 1$ is a totally f-geodesic submanifold.* ∎

Now, we can prove:

Theorem 3.4. *Any totally f-geodesic CR-submanifold M^m of an S-manifold N^{2n+s} is locally a Riemannian product $M_1 \times M_2$ where*

M_1 is a totally geodesic invariant submanifold of N^{2n+s} and M_2 is a totally geodesic anti-invariant submanifold of N^{2n+s} and $\xi_1, \ldots, \xi_s$ are normal to M_2.

Proof: From (3.10), we get that $\sigma(X,Y) = 0$ for any $X \in T(M)$, $Y \in \mathcal{D}$. Then, by Proposition 3.2, M^m is a CR-product locally represented by $M_1 \times M_2$. Now, using (3.10) we obtain that M_1 and M_2 are totally geodesic in N^{2n+s} and the proof is complete. ∎

Corollary 3.5. *Any proper totally f-umbilical CR-submanifold M^m of an S-manifold N^{2n+s} with $q > 1$ is locally the Riemannian product $M_1 \times M_2$, where M_1 and M_2 have the properties of Theorem 3.4.* ∎

To close this section, we recall that the bisectional curvature of an S-manifold N^{2n+s} is defined by

$$B(X,Y) = \tilde{R}(X, fX, fY, Y) , \tag{3.11}$$

where X and Y are unit vector fields in $\mathcal{L}$ and such that $g(fX,Y)=0$. Now, we have:

Lemma 3.6. *Let M^m be a CR-product of an S-manifold N^{2n+s}. If $Y \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$ are unit vector fields, then:*

$$g\Big(\sigma(\nabla_Y fY, Z), fZ\Big) = -s , \tag{3.12}$$

$$g\Big(\sigma(\nabla_{fY} Y, Z), fZ\Big) = s , \tag{3.13}$$

$$g\Big(\sigma(fY, \nabla_Y Z), fZ\Big) = g\Big(\sigma(Y, \nabla_{fY} Z), fZ\Big) = 0 . \tag{3.14}$$

Proof: From (3.4)

$$\begin{aligned} g\Big(\sigma(\nabla_Y fY, Z), fZ\Big) &= -\sum \eta_\alpha(\nabla_Y fY)\, g(Z,Z) \\ &= \sum g(\nabla_Y \xi_\alpha, fY) = -\sum g(fY, fY) = -s , \end{aligned}$$

which is (3.12). We also have:

$$g\Big(\sigma(\nabla_{fY} Y, Z), fZ\Big) = \sum g(\nabla_{fY} \xi_\alpha, Y) = -\sum g(f^2 Y, Y) = s ,$$

which is (3.13). Finally:

$$g\Big(\sigma(fY,\nabla_Y Z), fZ\Big) = -\sum \eta_\alpha(fY)\, g(\nabla_Y Z, Z) = 0\ ,$$

$$g\Big(\sigma(Y,\nabla_{fY} Z), fZ\Big) = -\sum \eta_\alpha(Y)\, g(\nabla_{fY} Z, Z) = 0\ ,$$

which are (3.14). ∎

Theorem 3.7. *Let M^m be a proper CR-product of an S-manifold N^{2n+s}. Then:*

$$B(Y,Z) = 2\,\|\sigma(Y,Z)\|^2 - 2\,s\ , \tag{3.15}$$

for any unit vector fields $Y \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$. Consequently, if N^{2n+s} is of bisectional curvature $B < -2\,s$, any CR-product in N^{2n+s} is either an invariant submanifold or an anti-invariant submanifold of N^{2n+s} tangent to $\xi_1, \ldots, \xi_s$.

Proof: From the Codazzi equation (1.3), the Weingarten formula, Theorem 3.1, (2.4) and the above lemma, we have, for any unit vector fields $Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$:

$$\begin{aligned}
\widetilde{R}(Y,fY,Z,fZ) &= g\Big((\nabla'_Y\sigma)(fY,Z), fZ\Big) - g\Big((\nabla'_{fY}\sigma)(Y,Z), fZ\Big) \\
&= 2\,s + g\Big(\widetilde{\nabla}_Y \sigma(fY,Z), fZ\Big) - g\Big(\widetilde{\nabla}_{fY}\sigma(Y,Z), fZ\Big) \\
&= 2\,s - g\Big(\sigma(fY,Z), f\,\nabla_Y Z\Big) - g\Big(\sigma(fY,Z), f\,\sigma(Y,Z)\Big) \\
&\quad + g\Big(\sigma(Y,Z), f\,\nabla_{fY} Z\Big) + g\Big(\sigma(Y,Z), f\,\sigma(fY,Z)\Big)\ .
\end{aligned}$$

Now, using (3.6), we get that $\nabla_Y Z$ and $\nabla_{fY} Z$ belong to $\mathcal{D}^\perp$ and by Theorem 3.1 again:

$$g\Big(\sigma(fY,Z), f\,\nabla_Y Z\Big) = g(A_{f\,\nabla_Y Z} fY, Z) = 0\ ,$$

$$g\Big(\sigma(Y,Z), f\,\nabla_{fY} Z\Big) = g(A_{f\,\nabla_{fY} Z} Y, Z) = 0\ .$$

Thus, from (3.8):

$$B(Y,Z) = -\widetilde{R}(Y,fY,Z,fZ) = -2\,s + 2\,\|\sigma(Y,Z)\|^2$$

and we complete the proof. ∎

4 – *CR*-products of *S*-manifolds with constant invariant *f*-sectional curvature

In this section, let $N^{2n+s}(k)$ be an S-manifold of constant invariant f-sectional curvature k. Then, it is well known that its curvature tensor has the form, ([6]):

$$\begin{aligned}(4.1)\quad \tilde{R}(X,Y,Z,W) &= \sum_{\alpha,\beta}\Big[g(fX,fW)\,\eta_\alpha(Y)\,\eta_\beta(Z) - \\ &- g(fX,fZ)\,\eta_\alpha(Y)\,\eta_\beta(W) - g(fY,fW)\,\eta_\alpha(X)\,\eta_\beta(Z) \\ &+ g(fY,fZ)\,\eta_\alpha(X)\,\eta_\beta(W)\Big] + \left(\frac{1}{4}\right)(k+3\,s)\Big[g(X,W)\,g(fY,fZ) \\ &- g(X,Z)\,g(fY,fW) + g(fY,fW)\,\Phi(X,Z) - g(fY,fZ)\,\Phi(X,W)\Big] \\ &+ \left(\frac{1}{4}\right)(k-s)\Big[F(X,W)\,F(Y,W) - F(X,Z)\,F(Y,W) \\ &- 2\,F(X,Y)\,F(Z,W)\Big]\,,\end{aligned}$$

for any $X,Y,Z,W \in T(M)$.

Theorem 4.1. *There exist no proper CR-products in S-manifolds $N^{2n+s}(k)$ with $k < -3\,s$.*

Proof: From (4.1) we obtain:

$$(4.2)\qquad B(Y,Z) = \left(\frac{1}{2}\right)(k-s)\,,$$

for any unit vector fields $Y \in D$, $Z \in D^\perp$. Now, using (3.15), we get $k+3\,s \geq 0$ and this completes the proof. ∎

Theorem 4.2. *Let M^{m+s} be a proper CR-product of an S-manifold $N^{2n+s}(k)$, with $k \geq -3\,s$. Then*

$$(4.3)\qquad \|\sigma\|^2 \geq \Big((k+3\,s)\,p + 2\,s\Big)\,q\,.$$

If the equality holds, then M_1 is an S-manifold of constant invariant f-sectional curvature k and M_2 is of constant curvature $(\frac{1}{4})\,(k+3\,s)$.

Proof: Let $Y \in D$, $Z \in D^\perp$ unit vector fields. Then, from (3.15) and (4.2):

$$\|\sigma(Y,Z)\|^2 = \left(\frac{1}{4}\right)(k+3\,s)\,.$$

Now, we choose a local field of orthonormal frames

$$\Big\{E_1, ..., E_p, E_{p+1} = f\, E_1, ..., E_{2p} = f\, E_p, E_{2p+1} = F_1, ..., E_m = F_q, \\ E_{m+1} = f\, F_1, ..., E_{m+q} = f\, F_q, E_{m+q+1}, ..., E_{2n}, \xi_1, ..., \xi_s\Big\}$$

on $N^{2n+s}(k)$ in such a way that $\{E_1, ..., E_{2p}\}$ is a local frame field on $\mathcal{D}$ and $\{F_1, ..., F_p\}$ is a local frame field on $\mathcal{D}^\perp$. Since $\sigma(E_i, \xi_\alpha) = 0$, $i = 1, ..., 2p$, $\alpha = 1, ..., s$ and $\sigma(\xi_\alpha, \xi_\beta) = 0$, $\alpha, \beta = 1, ..., s$, we obtain, using (2.8):

$$\begin{aligned} \|\sigma\|^2 &= \sum_{i,j} \|\sigma(E_i, E_j)\|^2 + \sum_{a,b} \|\sigma(F_a, F_b)\|^2 \\ &\quad + 2 \sum_{a,\alpha} \|\sigma(F_a, \xi_\alpha)\|^2 + 2 \sum_{a,i} \|\sigma(F_a, E_i)\|^2 \\ &= 2\, s\, q + p\, q\, (k + 3\, s) + \sum_{i,j} \|\sigma(E_i, E_j)\|^2 + \sum_{a,b} \|\sigma(F_a, F_b)\|^2 \end{aligned}$$

and so, we have (4.3). Now, if the equality holds, then $\sigma(\mathcal{D}, \mathcal{D}) = 0$ and $\sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0$. Since M_1 and M_2 are totally geodesic in M^{m+s}, then they are totally geodesic in $N^{2n+s}(k)$. The Gauss equation (1.2) and (4.1) complete the proof. ∎

Let M^m be a CR-product in $N^{2n+s}(k)$. If $q = \dim T_x^\perp(M)$, for any $x \in M^m$, we say that M^m is a generic CR-product. Then, using Theorem 4.1, we have:

Corollary 4.3. *There exist no proper generic CR-products in an S-manifold $N^{2n+s}(k)$ with $k \neq -3\, s$.*

Proof: Suppose M^m is a proper generic CR-product in $N^{2n+s}(k)$. Since we have $f = t$ and, from (3.7), $\sigma(Y, Z) = 0$, for any $Y \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$, thus, from (3.15) and (4.2), we get $k = -3\, s$. ∎

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