Uniform boundedness of the attractor in H^2 of a non-autonomous epidemiological system

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Abstract

In this paper, we prove the uniform boundedness of the pullback attractor of a non-autonomous SIR (susceptible, infected, recovered) model from epidemiology considered in Anguiano and Kloeden [2]. We prove two uniform boundedness of this pullback attractor, firstly in the norm H_0^1, and later, under appropriate additional assumptions, in the norm H^2.

Keywords: SIR epidemic model with diffusion; invariant sets; uniform boundedness in H^2
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1 Introduction and setting of the problem

Let us consider the following problem for a temporally-forced SIR (susceptible, infected, recovered) model with diffusion

$$\left.\begin{aligned} &\frac{\partial S}{\partial t}-\Delta S=aq(t)-aS+bI-\gamma\frac{SI}{N},\\ &\frac{\partial I}{\partial t}-\Delta I=-(a+b+c)I+\gamma\frac{SI}{N},\\ &\frac{\partial R}{\partial t}-\Delta R=cI-aR,\end{aligned}\right\} \tag{1}$$

with the Dirichlet boundary condition

$$S(x,t)=I(x,t)=R(x,t)=0 \text{ on } \quad \partial\Omega\times(t_0,+\infty), \tag{2}$$

and initial condition

$$S(x,t_0)=S_0(x), \quad I(x,t_0)=I_0(x), \quad R(x,t_0)=R_0(x) \text{ for } x\in\Omega, \tag{3}$$

where $\Omega\subset\mathbb{R}^d$, $d\geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$, $N(t)=S(t)+I(t)+R(t)$ and $t_0\in\mathbb{R}$. We assume that the parameters a, b, c and γ are positive constants such that $\gamma+\frac{b}{2}+\frac{c}{2}<\lambda_1$, where $\lambda_1>0$ is the first eigenvalue of the negative Laplacian with zero Dirichlet boundary condition in Ω. The temporal forcing term is given by a continuous function $q:\mathbb{R}\to\mathbb{R}$ taking positive bounded values, i.e. $q(t)\in[q^-,q^+]$ for all $t\in\mathbb{R}$ where $0<q^-\leq q^+$, such that $q'\in L^2_{loc}\left(\mathbb{R};L^2(\Omega)\right)$ and satisfies

$$\sup_{t_0\in\mathbb{R}}\int_{t_0}^{t_0+1}|q'(s)|^2_{L^2(\Omega)}\,ds<\infty. \tag{4}$$

Epidemiology is the study of the spread of diseases with the objective of tracing factors that are responsible for or contribute to their occurrence. Mathematical models are used extensively in the study of epidemiological phenomena. Most models for the transmission of infectious diseases (see for instance [1, 4]) descend from the classical SIR model of Kermack and McKendrick [7] established in 1927. Its classical form involves a system of autonomous ordinary differential equations for three classes, the susceptibles S, infectives I and recovereds R, of a constant total population.

There is a strong biological motivation to include time-dependent terms into epidemiological models, for instance temporally varying forcing is typical of seasonal variation of a disease [6, 9].

Our model (1)-(3) is a classical and well-known model from mathematical epidemiology in the form of the SIR equations, with diffusion, in which a temporal forcing term is considered.

Several studies on this model have already been published. More precisely, in [2] we prove the existence and uniqueness of positive solutions of (1)-(3) for initial data in $L^2(\Omega)^3$, and we establish that, if q takes positive bounded values, the process associated to (1)-(3) has a unique pullback attractor $\mathcal{A}$.

In [3], we establish a regularity result for the unique positive solution to problem (1)-(3), and we prove some regularity results for the pullback attractor $\mathcal{A}$. This study has motived the fact of investigating the problem considering in this paper. Moreover, as far as we know, there are no results in the literature concerning the uniform boundedness of the pullback attractor $\mathcal{A}$ as we will consider in the present paper.

The structure of the paper is as follows. In Section 2, we prove the uniform boundedness of the attractor $\mathcal{A}$ in $H_0^1(\Omega)^3$. Then, under appropriate additional assumptions, the uniform boundedness in $H^2(\Omega)^3$ of $\mathcal{A}$ is proved in Section 3.

2 Uniform boundedness of the pullback attractor in $H_0^1(\Omega)^3$

We denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$, and by $|\cdot|_{L^2(\Omega)}$ the associated norm. By $||\cdot||$ we denote the norm in $H_0^1(\Omega)$, which is associated to the inner product $((\cdot,\cdot)) := (\nabla\cdot,\nabla\cdot)$. We will denote by $\langle\cdot,\cdot\rangle$ the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

In addition, X_3 denotes the space of functions $(u_1,u_2,u_3) \in L^2(\Omega)^3$ with the scalar product

$$((u_1,u_2,u_3),(v_1,v_2,v_3)) = (u_1,v_1) + (u_2,v_2) + (u_3,v_3)\,,$$

and norm

$$|(u_1,u_2,u_3)|_{L^2(\Omega)} = |u_1|_{L^2(\Omega)} + |u_2|_{L^2(\Omega)} + |u_3|_{L^2(\Omega)}\,,$$

for all (u_1,u_2,u_3),$(v_1,v_2,v_3) \in X_3$, while Y_3 denotes the space of functions $(u_1,u_2,u_3) \in H_0^1(\Omega)^3$ with the scalar product

$$(((u_1,u_2,u_3),(v_1,v_2,v_3))) = ((u_1,v_1)) + ((u_2,v_2)) + ((u_3,v_3))\,,$$

and norm

$$||(u_1,u_2,u_3)|| = ||u_1|| + ||u_2|| + ||u_3||\,,$$

for all (u_1,u_2,u_3),$(v_1,v_2,v_3) \in Y_3$. Finally, let X_3^+ be the subspace of non-negative functions in X_3 and Y_3^+ be the subspace of non-negative functions in Y_3.

The globally defined nonnegative solutions of (1)–(3) generate a process in the Banach space X_3^+ (see [2]), i.e., a family of mappings $U_{t,t_0} : X_3^+ \to X_3^+$ with $t \geq t_0$ in $\mathbb{R}$ satisfying

$$U_{t_0,t_0}x = x, \quad U_{t,t_0}x = U_{t,r} \circ U_{r,t_0}x,$$

for all $t_0 \leq r \leq t$ and $x \in X_3^+$. In [2, Proposition 1] we established that the 2-parameter family of mappings $U_{t,t_0} : X_3^+ \to X_3^+$, $t_0 \leq t$, given by

$$U_{t,t_0}(S_0,I_0,R_0) = (S(t),I(t),R(t)), \tag{5}$$

where $(S(t),I(t),R(t))$ is the unique positive solution of (1)–(3) with the initial value (S_0,I_0,R_0), defines a continuous process on X_3^+.

Recall that a pullback attractor for the process U_{t,t_0} (e.g., cf. [5]) in the space X_3^+ is a family $\mathcal{A} = \{\mathcal{A}(t), t \in \mathbb{R}\}$ of nonempty compact subsets of X_3^+, which is invariant in the sense that

$$U_{t,t_0}\mathcal{A}(t_0) = \mathcal{A}(t), \qquad \text{for all } t \geq t_0,$$

and pullback attracts bounded subsets D of X_3^+, i.e.,

$$\operatorname{dist}_{X_3^+}\left(U_{t,t_0}D, \mathcal{A}(t)\right) \to 0 \quad \text{as} \quad t_0 \to -\infty,$$

where we denote by $\operatorname{dist}_{X_3^+}(\cdot,\cdot)$ the Hausdorff semi-distance in X_3^+.

In [2, Theorem 6.2, Remark 6] we establish that the process associated to (1)–(3) has a unique pullback attractor $\mathcal{A}$, which satisfies

$$\mathcal{A}(t) \subset \Sigma_3^+, \quad t \in \mathbb{R}, \tag{6}$$

where Σ_3^+ is a closed and bounded subset of X_3^+.

We recall a lemma (see [8]) which is necessary for the proof of our results.

Lemma 1 *Let X, Y be Banach spaces such that X is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty(t_0,T;X)$ such that $u_n \rightharpoonup u$ weakly in $L^q(t_0,T;X)$ for some $q \in [1,+\infty)$ and $u \in C^0([t_0,T];Y)$. Then, $u(t) \in X$ for all $t \in [t_0,T]$ and*

$$\|u(t)\|_X \leq \sup_{n\geq 1} \|u_n\|_{L^\infty(t_0,T;X)} \qquad \forall t \in [t_0,T].$$

Let $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the linear operator associated with the negative Laplacian. The operator A is symmetric, coercive and continuous.

Since the space $H_0^1(\Omega)$ is included in $L^2(\Omega)$ with compact injection, as a consequence of the Hilbert-Schmidt Theorem there exists a nondecreasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ of eigenvalues of A with zero Dirichlet boundary condition in Ω, with $\lim_{j\to\infty} \lambda_j = +\infty$ and there exists an orthonormal basis of Hilbert $\{w_j : \ j \geq 1\}$ of $L^2(\Omega)$ and orthogonal in $H_0^1(\Omega)$ with $V_n := \mathit{span}\,\{w_j : \ 1 \leq j \leq n\}$ and $\{V_n : n \in \mathbb{N}\}$ densely embedded in $H_0^1(\Omega)$, such that

$$Aw_j = \lambda_j w_j \text{ for all } j \geq 1.$$

For each integer $n \geq 1$, we denote by $(S_n(t), I_n(t), R_n(t)) = (S_n(t;t_0,S_0), I_n(t;t_0,I_0), R_n(t;t_0,R_0))$ the Galerkin approximation of the solution $(S(t;t_0,S_0), I(t;t_0,I_0), R(t;t_0,R_0))$ of (1)-(3), which is given by

$$S_n(t) = \sum_{j=1}^{n} \gamma_{nj}^1(t) w_j, \quad I_n(t) = \sum_{j=1}^{n} \gamma_{nj}^2(t) w_j, \quad R_n(t) = \sum_{j=1}^{n} \gamma_{nj}^3(t) w_j,$$

and is the solution of

$$\begin{aligned}
&\frac{d}{dt}\left(S_n(t), w_j\right) = \langle \Delta S_n(t), w_j \rangle + \left(f_1(S_n(t), I_n(t), R_n(t), t), w_j\right), \\
&\frac{d}{dt}\left(I_n(t), w_j\right) = \langle \Delta I_n(t), w_j \rangle + \left(f_2(S_n(t), I_n(t), R_n(t)), w_j\right), \\
&\frac{d}{dt}\left(R_n(t), w_j\right) = \langle \Delta R_n(t), w_j \rangle + \left(f_3(S_n(t), I_n(t), R_n(t)), w_j\right),
\end{aligned}$$

with initial data

$$\left(S_n(t_0), w_j\right) = \left(S_0, w_j\right), \left(I_n(t_0), w_j\right) = \left(I_0, w_j\right), \left(R_n(t_0), w_j\right) = \left(R_0, w_j\right),$$

for all $w_j \in V_n$, where

$$\gamma_{nj}^1(t) = (S_n(t), w_j), \quad \gamma_{nj}^2(t) = (I_n(t), w_j), \quad \gamma_{nj}^3(t) = (R_n(t), w_j).$$

We denote

$$\begin{aligned}
f_1(S_n(t), I_n(t), R_n(t), t) &:= aq(t) - aS_n(t) + bI_n(t) - \gamma \frac{S_n(t)I_n(t)}{N_n(t)}, \\
f_2(S_n(t), I_n(t), R_n(t)) &:= -(a+b+c)I_n(t) + \gamma \frac{S_n(t)I_n(t)}{N_n(t)}, \\
f_3(S_n(t), I_n(t), R_n(t)) &:= cI_n(t) - aR_n(t),
\end{aligned}$$

where

$$N_n(t) = S_n(t) + I_n(t) + R_n(t).$$

On the other hand, if we denote

$$D(A) = \left\{ v \in H_0^1(\Omega) : Av \in L^2(\Omega) \right\},$$

with the scalar product

$$(v, w)_{D(A)} = (Av, Aw) \quad \forall v, w \in D(A),$$

then $D(A)$ is a Hilbert space, and $D(A)$ is included in $H_0^1(\Omega)$ with continuous and dense injection. Let $D(A)^+$ be the subspace of non-negative functions in $D(A)$.

Remark 2 *We note that if $\Omega \subset \mathbb{R}^d$ is a bounded C^2 domain, then we have that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, and moreover the norm induced by $(\cdot,\cdot)_{D(A)}$ in $D(A)$ and the norm of $H^2(\Omega)$ are equivalent.*

Now, in our first main result, we prove the uniform boundedness of the attractor $\mathcal{A}(t)$ in $H_0^1(\Omega)^3$.

Theorem 3 *Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded C^2 domain and assume that $\gamma + \frac{b}{2} + \frac{c}{2} < \lambda_1$ where λ_1 is the first eigenvalue of the operator A on the domain Ω with Dirichlet boundary condition. Then $\mathcal{A}(t)$ is uniformly bounded in t in $H_0^1(\Omega)^3$.*

Proof. From the inequality (27) of [3], for any $t \geq t_0$ we have

$$\begin{aligned}
&|S_n(r)|^2_{L^2(\Omega)} + |I_n(r)|^2_{L^2(\Omega)} + |R_n(r)|^2_{L^2(\Omega)} + \int_{t_0}^{r} \left(\|S_n(s)\|^2 + \|I_n(s)\|^2 + \|R_n(s)\|^2 \right) ds \\
\leq\; & C_1 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + (t - t_0) \right), \qquad (7)
\end{aligned}$$

for all $r \in [t_0, t]$, and all $n \geq 1$, where $C_1 := \dfrac{\max\left\{1, \frac{a}{2}(q^+)^2 |\Omega|\right\}}{\min\left\{1, 2 - \lambda_1^{-1}(b + c + 2\gamma)\right\}}$.

From (7) and (26) in [3] we now obtain that

$$\begin{aligned}
& (r - t_0)\left(\|S_n(r)\|^2 + \|I_n(r)\|^2 + \|R_n(r)\|^2 \right) \qquad (8) \\
\leq\; & C_1 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + (t - t_0) \right) \\
+\; & (q^+)^2 |\Omega| (t - t_0)^2 (2a^2 + \frac{a}{2} k_1 C) \\
+\; & k_1 C \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} \right)(t - t_0),
\end{aligned}$$

for any $t \geq t_0$, all $r \in [t_0, t]$, and all $n \geq 1$, where $C := (2\lambda_1 - b - c - 2\gamma)^{-1}$ and k_1 is a positive constant.

In particular, from (8) we deduce

$$\|S_n(r)\|^2 + \|I_n(r)\|^2 + \|R_n(r)\|^2 \leq C_2 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + 1 \right), \qquad (9)$$

for all $r \in [t_0 + 1, t_0 + 2]$, and any $n \geq 1$, where

$$C_2 := \max\left\{ C_1 + 2k_1 C, 2C_1 + 4(q^+)^2 |\Omega| \left(2a^2 + \frac{a}{2} k_1 C \right) \right\}.$$

Using Lemma 3 in [3], we have that $(S_n(\cdot), I_n(\cdot), R_n(\cdot)) = (S_n(\cdot; t_0, S_0), I_n(\cdot; t_0, I_0), R_n(\cdot; t_0, R_0))$ converges weakly to the unique solution to (1)-(3) $(S(\cdot), I(\cdot), R(\cdot)) = (S(\cdot; t_0, S_0), I(\cdot; t_0, I_0), R(\cdot; t_0, R_0))$ in $L^2(t_0, t; (Y^+)^3)$, for all $t > t_0$. Thus, from (9) and Lemma 1, we in particular obtain

$$\|S(t_0+1)\|^2+\|I(t_0+1)\|^2+\|R(t_0+1)\|^2\leq C_2\left(|S_0|^2_{L^2(\Omega)}+|I_0|^2_{L^2(\Omega)}+|R_0|^2_{L^2(\Omega)}+1\right),$$

which together with (6) imply that $\mathcal{A}(t)$ is uniformly bounded in t in $H_0^1(\Omega)^3$. ■

3 Uniform boundedness of the pullback attractor in $H^2(\Omega)^3$

The aim of this section is to continue with the analysis of the model in the sense of proving that the attractor $\mathcal{A}(t)$ satisfies that is uniformly bounded in the space $H^2(\Omega)^3$ provided some additional assumptions are fulfilled. Our second main result is the following.

Theorem 4 *In addition to the assumptions in Theorem 3, assume moreover that $q' \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, and satisfies (4). Then $\mathcal{A}(t)$ is uniformly bounded in t in $H^2(\Omega)^3$.*

Proof. From inequality (35) in [3], taking $t = t_0 + 3$ and $\varepsilon = 2$, we have

$$\begin{aligned} & |S_n'(r)|^2_{L^2(\Omega)} + |I_n'(r)|^2_{L^2(\Omega)} + |R_n'(r)|^2_{L^2(\Omega)} \\ \leq\; & (4k_3+1)\int_{t_0+1}^{t_0+3}\left(|S_n'(\theta)|^2_{L^2(\Omega)} + |I_n'(\theta)|^2_{L^2(\Omega)} + |R_n'(\theta)|^2_{L^2(\Omega)}\right) d\theta \\ +\; & a\int_{t_0+1}^{t_0+3} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta, \end{aligned} \tag{10}$$

for all $r \in [t_0+2, t_0+3]$, and any $n \geq 1$, where k_3 is a positive constant.

Analogously, and if we take $s = t_0 + 1$ and $r = t = t_0 + 3$ in inequality (25) of [3], we, in particular, have

$$\begin{aligned} & \int_{t_0+1}^{t_0+3}\left(|S_n'(\theta)|^2_{L^2(\Omega)} + |I_n'(\theta)|^2_{L^2(\Omega)} + |R_n'(\theta)|^2_{L^2(\Omega)}\right) d\theta \\ \leq\; & \|S_n(t_0+1)\|^2 + \|I_n(t_0+1)\|^2 + \|R_n(t_0+1)\|^2 \\ +\; & 3(q^+)^2\,|\Omega|\,(2a^2 + \frac{a}{2}k_1 C) \\ +\; & k_1 C\left(|S_0|^2_{L^2(\Omega)}+|I_0|^2_{L^2(\Omega)}+|R_0|^2_{L^2(\Omega)}\right), \end{aligned} \tag{11}$$

for all $n \geq 1$, where k_1 is a positive constant and $C := (2\lambda_1 - b - c - 2\gamma)^{-1}$.

From (10) and (11), we obtain

$$\begin{aligned} & |S_n'(r)|^2_{L^2(\Omega)} + |I_n'(r)|^2_{L^2(\Omega)} + |R_n'(r)|^2_{L^2(\Omega)} \\ \leq\; & (4k_3+1)\left(\|S_n(t_0+1)\|^2 + \|I_n(t_0+1)\|^2 + \|R_n(t_0+1)\|^2\right) \\ +\; & (4k_3+1)\,3(q^+)^2\,|\Omega|\,(2a^2 + \frac{a}{2}k_1 C) + a\int_{t_0+1}^{t_0+3} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta \\ +\; & (4k_3+1)\,k_1 C\left(|S_0|^2_{L^2(\Omega)}+|I_0|^2_{L^2(\Omega)}+|R_0|^2_{L^2(\Omega)}\right), \end{aligned}$$

for all $r \in [t_0+2, t_0+3]$, and any $n \geq 1$.

Owing to this inequality and (9), there exists a constant $\widetilde{C}_1 > 0$ such that

$$\begin{aligned} & |S_n'(r)|^2_{L^2(\Omega)} + |I_n'(r)|^2_{L^2(\Omega)} + |R_n'(r)|^2_{L^2(\Omega)} \\ \leq\; & \widetilde{C}_1\left(|S_0|^2_{L^2(\Omega)}+|I_0|^2_{L^2(\Omega)}+|R_0|^2_{L^2(\Omega)} + \int_{t_0+1}^{t_0+3} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta + 1\right), \end{aligned} \tag{12}$$

for all $r \in [t_0+2, t_0+3]$, and any $n \geq 1$.

From inequality (36) of [3], and thanks to (12), we have

$$\begin{aligned} & |\Delta S_n(r)|^2_{L^2(\Omega)} + |\Delta I_n(r)|^2_{L^2(\Omega)} + |\Delta R_n(r)|^2_{L^2(\Omega)} \\ \leq \;& 4\widetilde{C}_1 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + \int_{t_0+1}^{t_0+3} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta + 1 \right) \\ + \;& 8a^2(q^+)^2\,|\Omega| + 4k_2 \left(|S_n(r)|^2_{L^2(\Omega)} + |I_n(r)|^2_{L^2(\Omega)} + |R_n(r)|^2_{L^2(\Omega)} \right), \end{aligned}$$

for all $r \in [t_0+2, t_0+3]$, and any $n \geq 1$, where k_2 is a positive constant.

Therefore, by (7) we obtain that there exists a constant $\widetilde{C}_2 > 0$ such that

$$\begin{aligned} & |\Delta S_n(r)|^2_{L^2(\Omega)} + |\Delta I_n(r)|^2_{L^2(\Omega)} + |\Delta R_n(r)|^2_{L^2(\Omega)} \\ & \leq \widetilde{C}_2 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + \int_{t_0+1}^{t_0+3} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta + 1 \right), \end{aligned} \tag{13}$$

for all $r \in [t_0+2, t_0+3]$, and any $n \geq 1$.

By Theorem 6 in [3], we have that $(S(\cdot; t_0, S_0), I(\cdot; t_0, I_0), R(\cdot; t_0, R_0)) \in C\left([t_0+2, t_0+3]; Y_3^+\right)$. On the other hand, in the proof of Theorem 4 in [3], we proved that $\{(S_n(\cdot; t_0, S_0), I_n(\cdot; t_0, I_0), R_n(\cdot; t_0, R_0))\}$ is bounded in $L^2(t_0, t; (D(A)^+)^3)$ for all $t > t_0$. Then, in particular, we have that $(S_n(\cdot), I_n(\cdot), R_n(\cdot)) = (S_n(\cdot; t_0, S_0), I_n(\cdot; t_0, I_0),$ $R_n(\cdot; t_0, R_0))$ converges weakly to the unique solution, $(S(\cdot), I(\cdot), R(\cdot)) = (S(\cdot; t_0, S_0), I(\cdot; t_0, I_0), R(\cdot; t_0, R_0))$, to (1)-(3) in $L^2(t_0+2, t_0+3; (D(A)^+)^3)$.

Then, by Lemma 1, inequality (13) and the equivalence of the norms $|\Delta v|_{L^2(\Omega)}$ and $\|v\|_{H^2(\Omega)}$, we have that there exists a constant $\widetilde{C}_3 > 0$ such that

$$\begin{aligned} & \|(S(r; t_0, S_0), I(r; t_0, I_0), R(r; t_0, R_0))\|^2_{H^2(\Omega)^3} \\ & \leq \widetilde{C}_3 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + \int_{t_0+1}^{t_0+3} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta + 1 \right), \end{aligned} \tag{14}$$

for all $r \in [t_0+2, t_0+3]$, any $t_0 \in \mathbb{R}$, and $(S_0, I_0, R_0) \in X_3^+$.

Thus, from (14), and using (5), we deduce that there exists a constant $\widetilde{C}_4 > 0$ such that

$$\|U_{t_0+2,t_0}(S_0, I_0, R_0)\|^2_{H^2(\Omega)^3} \leq \widetilde{C}_4 \left(|(S_0, I_0, R_0)|^2_{L^2(\Omega)} + \int_{t_0+1}^{t_0+3} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta + 1 \right),$$

for all $t_0 \in \mathbb{R}$, $(S_0, I_0, R_0) \in X_3^+$.

From this inequality, and the fact that $\mathcal{A}(t_0) = U_{t_0,t_0-2}\mathcal{A}(t_0-2)$, we obtain

$$\begin{aligned} & \|(v_1, v_2, v_3)\|^2_{H^2(\Omega)^3} \\ \leq \;& \widetilde{C}_4 \left(\sup_{(w_1,w_2,w_3) \in \mathcal{A}(t_0-2)} |(w_1, w_2, w_3)|^2_{L^2(\Omega)} + \int_{t_0-1}^{t_0+1} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta + 1 \right), \end{aligned} \tag{15}$$

for all $(v_1, v_2, v_3) \in \mathcal{A}(t_0)$, and any $t_0 \in \mathbb{R}$.

Now, from (6) and (15), we have that there exists $M > 0$ such that

$$\left(\sup_{(v_1,v_2,v_3) \in \mathcal{A}(t_0)} \|(v_1, v_2, v_3)\|_{H^2(\Omega)^3} \right)^2 \leq M + \int_{t_0-1}^{t_0+1} |q'(\theta)|^2_{L^2(\Omega)}\, d\theta,$$

for any $t_0 \in \mathbb{R}$. Finally, the assumption (4) implies the uniform boundedness of $\mathcal{A}(t)$ in $H^2(\Omega)^3$.

■

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