IRREDUCIBLE TRIANGULATIONS
OF THE ONCE-PUNCTURED TORUS

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Abstract. A triangulation of a surface with fixed topological type is called irreducible if no edge can be contracted to a vertex while remaining in the category of simplicial complexes and preserving the topology of the surface. A complete list of combinatorial structures of irreducible triangulations is made by hand for the once-punctured torus, consisting of exactly 297 non-isomorphic triangulations.

Keywords: triangulation of 2-manifold, irreducible triangulation, 2-manifold with boundary, punctured torus.

1. Introduction

Let $S \in \{S_h, N_k\}$ be a *closed surface*—that is, the closed orientable (connected compact) 2-manifold S_h of genus h or the closed nonorientable 2-manifold N_k of nonorientable genus k. Using this terminology, S_0 is the sphere, S_1 is the torus, N_1 is the projective plane and N_2 is the Klein bottle. Let D be an open disk in S, with boundary $\partial D = \partial(S - D)$ homeomorphic to a circle. In particular, $S_0 - D$ is a disk, $N_1 - D$ is the Möbius band, and $S_1 - D$ is the punctured torus. We use the notation "Σ" whenever we assume the general case in which Σ is meant to be either S or $S - D$.

If a graph G is 2-cell embedded in Σ, the components of $\Sigma - G$ are called *faces*. A *triangulation* of Σ with a simple graph G (where "simple" means "without loops

and without parallel edges") is a 2-cell embedding $T : G \rightarrow \Sigma$ in which each face is bounded by a *3-cycle*, that is, a cycle of length 3 made up of 3 vertices connected by 3 edges of G; moreover, we demand that the closures of any two faces are either disjoint, share a single vertex, or share a single edge. We denote by $V = V(T)$, $E = E(T)$, and $F = F(T)$ the sets of vertices, edges, and faces of T, respectively. The cardinality $| V(T) |$ is called the *order* of T. By $G(T)$ we denote the graph $(V(T), E(T))$ of T. Two triangulations T_1 and T_2 are called *isomorphic* (denoted $T_1 \cong T_2$) when there exists a bijection, called an *isomorphism*, $\varphi : V(T_1) \rightarrow V(T_2)$, such that $[u, v, w] \in F(T_1)$ if and only if $[\varphi(u), \varphi(v), \varphi(w)] \in F(T_2)$. Throughout this paper we distinguish between triangulations only up to isomorphism. For $\Sigma = S - D$, let ∂T ($= \partial D$) denote the boundary cycle of T. The vertices and edges of ∂T are called *boundary vertices* and *boundary edges* of T.

A triangulation of a 2-manifold with fixed topological type is viewed as a member of the category of simplicial complexes. A triangulation is called *irreducible* if no edge can be contracted (to a vertex) without vacating the category of simplicial complexes or changing the topology of the underlying 2-manifold. Obstacles for edge contraction are studied in Section 3; one typical obstacle is the creation of parallel edges (forbidden in a simplicial complex). The term "irreducible triangulation" is more accurately introduced in Section 3. For the sake of brevity we abbreviate "irreducible triangulation" as "IT", and "irreducible triangulation of the (once-) punctured torus" as "ITPT".
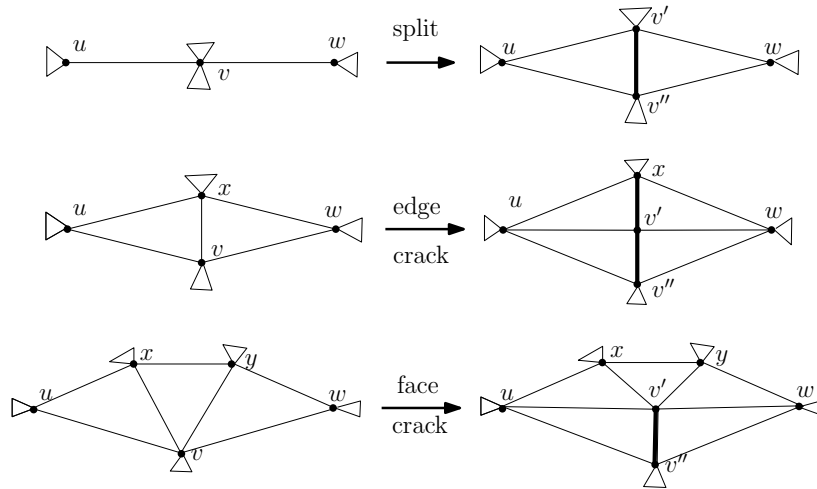
Figure 1. Splitting (top), edge cracking (middle), face cracking (bottom).

The collection of all ITs of Σ is a basis for the family of all triangulations of Σ, in the sense that any triangulation of Σ can be obtained from a member of the basis by repeating the *splitting* operation introduced in Section 3 and illustrated by Figure 1 (top) along with two special cases in the middle and bottom. Barnette and Edelson [2] and independently Negami [23] proved that for every closed 2-manifold S the basis of ITs is finite.

At present, bases of ITs are known for seven closed 2-manifolds: the sphere (Steinitz [24], Bowen and Fisk [4]), projective plane (Barnette [1]), torus (Lawrencenko [11, 12]), Klein bottle (Lawrencenko and Negami [20], and Sulanke [25]) as well as

S_2, N_3, and N_4 (Sulanke [26, 27, 28]). In Section 2, we briefly consider the case of the sphere in the historical retrospect.

In 2012 the first author (Lawrencenko) proposed [15] the problem of determining all ITs of a given 2-manifold with boundary, that is, all irreducible "punctured" triangulations with given genus and number of punctures (the latter is equal to the number of boundary components). This problem is motivated by the enormous growth of the number of ITs on closed 2-manifolds as the genus grows: already $396,784$ ITs on S_2, and $6,297,982$ ITs on N_4 (see [27, p. 3]). Firstly, Boulch, Colin de Verdière, and Nakamoto [3] gave upper bounds on the order of an IT of the punctured 2-manifold with given genus and number of punctures; their bounds are asymptotically tight but loose for low genus. Then, three of the authors of the present paper with participation of Quintero [7, 8] produced a complete list of the 6 ITs on the Möbius band.

Before the current study, a nearly comprehensive list of 293 non-isomorphic ITPTs was already obtained with the aid of a computer program in [5] and was announced in [18]. That list missed 4 ITPTs and was corrected in [6]. In this paper we develop a fully comprehensive list of 297 ITPTs (Theorem 12.1) independently and without using a computer. (The terminology used in [5, 6] is slightly different from that used in this paper.) Throughout this paper we intentionally avoid the use of a computer and all work can be checked tediously by hand.

2. The sphere

In this section we examine more closely the case of the sphere; it is a very instructive one. It is not very hard to prove the following theorem; this is the content of Exercise 1 [10, p. 243] and Exercise 6 [9, p. 59].

Theorem 2.1. ([24, 4]). *There is only one irreducible triangulation of the sphere: the boundary of a tetrahedron.*

In the historical setting, Bowen and Fisk [4] were the first who brought a modern version of Theorem 2.1; in fact, [4] contains a stronger result. However, Theorem 2.1 can be derived from the considerations in the book [24, pp. 227-229] of Steinitz and Rademacher (1934), although the authors use a different terminology. In that book it is shown that (after appropriate translation and interpretation) every 3-regular polyhedron can be obtained from a tetrahedron by a succession of the face-splitting operations. This means in dual form that the tetrahedral triangulation is a unique spherical IT.

The shortest proof [17] of Theorem 2.1 uses a corollary of [21, Theorem 2, p. 264]. In fact, that corollary is a characteristic property of the sphere; it distinguishes the sphere from the rest of the 2-manifolds. The corollary states that every triangulation of the sphere contains a *clean* vertex—that is, a vertex v whose link is chordless; the *link* of v consists of the cycle of vertices and edges surrounding v. Then v is not incident with any *non-facial* 3-cycle—that is, a 3-cycle that does not bound a face of the triangulation. Thus any edge incident with v can be contracted without creating parallel edges. One can keep iterating the edge contraction process until it terminates at some triangulation with only four faces. Although all vertices are still clean in such a tetrahedral triangulation, no edge is contractible anymore, but for another reason: after having attempted to contract an edge in the tetrahedral triangulation, one gets to a doubly covered triangle, which is not a simplicial complex.

3. Preliminaries

Let T be a triangulation of Σ. An unordered pair of distinct adjacent edges $[v, u], [v, w]$ of T is called a *divider* of T (*centered*) *at vertex* v, denoted by $\langle u, v, w \rangle$ $(= \langle w, v, u \rangle)$. The *splitting* of $\langle u, v, w \rangle$, denoted $\mathrm{sp}\langle u, v, w \rangle$, is the operation which consists of deleting $\langle u, v, w \rangle$ from T and closing the resulting hole with two new triangular faces, $[v', v'', u]$ and $[v', v'', w]$, where v' and v'' denote the two split vertices; see Figure 1 (top). Under this operation, the vertex v is extended to the edge $[v,' v'']$ and the two faces incident with this edge are inserted into the triangulation. Specifically in the case in which $(\Sigma = S - D)$ *AND* $([u, v] \in E(T))$ *AND* $(v \in V(\partial T))$ *AND* $(u \notin V(\partial T))$, the operation $\mathrm{sp}\langle u, v]$ of *splitting a truncated divider* $\langle u, v]$ produces a single new triangular face $[u, v', v'']$, where $[v', v''] \in E\big(\, \partial\,(\mathrm{sp}\langle u, v](T))\,\big)$.

Under the inverse operation, which is called *contracting* the edge $[v', v'']$, this edge collapses into a single vertex v, the faces $[v', v'', u]$ and $[v', v'', w]$ collapse into single edges $[v, u]$ and $[v, w]$, respectively. This operation is denoted by $\mathrm{sh}\rangle v', v''\langle$, which comes from the word "shrinking", a synonym for "contracting". Therefore, $\mathrm{sh}\rangle v', v''\langle\big(\,\mathrm{sp}\,\langle u, v, w \rangle (T)\,\big) = T$. It should be noticed that in the case $(\Sigma = S - D)$ *AND* $([v', v''] \in E(\partial T))$, there is only one face incident with $[v', v'']$ and that face collapses to a single edge under $\mathrm{sh}\rangle v', v''\langle$. Clearly, the operation of splitting never changes the topology of Σ. We demand that the contraction operation must preserve the topology of Σ as well; moreover, parallel edges must not be created in a triangulation. In the case in which an edge $\varepsilon \in E(T)$ occurs in some non-facial 3-cycle, if we still insist on contracting ε, parallel edges would be produced, which would exclude $\mathrm{sh}\rangle\varepsilon\langle(T)$ from the category of triangulations. An edge ε is called *contractible* or a *rope* (or a *cable*) if $\mathrm{sh}\rangle\varepsilon\langle(T)$ is still a triangulation of Σ; otherwise ε is called *noncontractible* or a *rod*. Therefore, one can contract ropes but not rods. The subgraph of $G(T)$ made up of all ropes is called the *rope subgraph* of $G(T)$.

The only constraint to edge-contractibility in a non-tetrahedral triangulation T of a *closed* 2-manifold S is determined in [1, 2, 11, 12]: an edge $\varepsilon \in E(T)$ is a rod if and only if ε satisfies the following condition:

(3.1) ε is in a non-facial 3-cycle of $G(T)$.

That is, one cannot contract the edges of a non-facial 3-cycle.

There are, in all, three constraints to edge-contractibility in a triangulation T of a *punctured* 2-manifold $S - D$. Two of them are determined in [3]: an edge $\varepsilon \in E(T)$ is a rod if and only if ε satisfies either condition (3.1) or the following condition:

(3.2) ε is a *chord* of D—that is, the end vertices of ε are in $V(\partial D)$ but $\varepsilon \notin E(\partial D)$.

That is, one cannot contract chords.

A triangulation is said to be *irreducible* if it has no ropes, or in other words, each edge is a rod. For instance, a single triangle is the only IT of the disk $S_0 - D$ although its edges don't meet either of conditions (3.1) and (3.2). Thus, the following is yet one more constraint to edge-contractibility of ε:

(3.3) ε is a boundary edge in the case the boundary cycle is a 3-cycle.

Although condition (3.3) is a specific case of condition (3.1) unless $S = S_0$ and is not explicitly stated in [3], this condition deserves special mention. In the remainder of this paper we assume that $S \neq S_0$.

4. THE STRUCTURE OF IRREDUCIBLE PUNCTURED TRIANGULATIONS

As an energy metaphor (with ropes thought of as high voltage cables), a vertex of a triangulation R of a closed 2-manifold S is called a *pylonic vertex* or a *∗-vertex* if that vertex is incident with all ropes of R. A triangulation that has at least one rope and at least one ∗-vertex is called a *pylonic triangulation* or a *∗-triangulation*. In other words, R is pylonic if and only if the rope subgraph of R is isomorphic to the complete bipartite graph $K_{1,n}$ for some n.

Let T be a triangulation of $S - D$. Let us restore the disk D in T, add a vertex p in D and join p to the vertices in ∂D. We thus obtain a triangulation, $T^* = T \cup D$, of S. We call D the *patch*, call p the *central vertex of the patch*, and call T^* a *parent triangulation* for T.

It will be shown in Section 10 that there exist single-roped triangulations of S_1— that is, triangulations that have only one rope and thus two ∗-vertices. However, if a ∗-triangulation R has at least two ropes, R has a unique ∗-vertex. It is to be noted that if T is an IT of $S - D$, then T^* may turn out to be an IT of S, but not necessarily.

Lemma 4.1. *If T is an IT of $S - D$, then unless T^* is irreducible T^* is pylonic.*

Proof. Let ε be an edge of T, necessarily a rod. Then ε satisfies either (3.1) or (3.2). If ε satisfies (3.1), the corresponding edge ε^* in T^* also satisfies (3.1). If ε satisfies (3.2), ε^* still satisfies (3.1). In either case, ε^* is a rod. Thus, all ropes of T^* (if any) are not edges of T and therefore are incident with the central vertex of the patch. □

Corollary 4.2. *A triangulation T of $S - D$ is irreducible if and only if T is obtained from a parent triangulation T^* of S either by deleting a vertex when T^* is irreducible, or by deleting a ∗-vertex p when T^* is pylonic.*

Proof. The "only if" part follows from Lemma 4.1. The "if" part is trivial: any edge that is not incident with p but occurs in some non-facial 3-cycle through p is a chord of the link of p and therefore is still a rod after the deletion of p. □

It is not a trivial question as to how the rope subgraph evolves under successively performed splittings. However, the following observation is easy to see: it is hard to make a rod out of a rope.

Lemma 4.3. *The only situation in which a rope in a triangulation of S changes into a rod under a single application of the splitting operation is when the splitting is equivalent to the stellar subdivision of a face having that rope as a boundary edge.*

Proof. In such a situation, the edge contraction inverse to the splitting changes some rod ε into a rope. Since the contraction does not change the homotopy type of the cycles in $\pi_1(S)$, it follows by condition (3.1) above that ε occurs in a non-facial, null-homotopic 3-cycle, which is possible if and only if the disk bounded by that 3-cycle is stellar subdivided. □

Corollary 4.4. *If a triangulation of S is neither irreducible nor pylonic, it can never become pylonic after any sequence of splittings, except the case in which the rope subgraph is a 3-cycle forming the boundary of a face.*

We refer to the exceptional case of Corollary 4.4 as a "Δ". It is easy to see that a Δ cannot occur in a once-split IT, but may arise in a twice-split IT of S. By the way, this is a good exercise for the reader to find an example of a twice-split IT $K_{12} \to S_6$ in which a Δ occurs. Interestingly, Δ does not occur for the Klein bottle but does occur for S_2 and N_3, which is obtained by searching ITs with the second author's computer program *surftri* [28]. The example with Δ for S_2 can be joined to an IT of S_1 to produce an example with Δ for S_3, etc. So Δ does occur for S_h ($h \geq 2$) and, similarly, for N_k ($k \geq 3$).

Lemma 4.5. *No Δ can ever be on the torus.*

The proof of Lemma 4.5 is postponed until the end of Section 11 when we will have more factual material to draw upon. By Lemma 4.5, we can restate Corollary 4.4 as follows.

Corollary 4.6. *If a triangulation of S_1 is neither irreducible nor pylonic, it can never become pylonic after any sequence of splittings.*

5. THE TORUS

Throughout the remainder of this paper, we only consider triangulations of S_1 or $S_1 - D$.

A theorem of the first author [11, 12] states that for S_1 there exist, in all, twenty-one non-isomorphic ITs: $T_1, T_2, \ldots, T_{21}$. They are represented in each of Figures 2, 3, and 4 with their vertices numbered by $1, 2, \ldots, 10$; each T_i is identical for the three figures, with fixed vertex numbering. For each rectangle identify each pair of opposite sides to obtain an actual triangulation of S_1.

An *automorphism* of a triangulation T is an isomorphism of T with itself. The set of dividers of T as well as the sets $V(T), E(T), F(T)$ naturally fall into disjoint orbits under the action of the automorphism group $\mathrm{Aut}(T)$. The groups $\mathrm{Aut}(T_i)$ are determined explicitly for each $i = 1, 2, \ldots, 21$ in [13] and are reproduced here in Table I in the form of generating sets. In particular, the generating set for $\mathrm{Aut}(T_2)$ was found in [14, p. 544]. Originally, the technique used in [13] is based on a computer program, but it is a good exercise for the reader to verify without a computer that the results in Table I are correct. (Interestingly, the reader may notice from Figure 2 that for $i = 6$ to 17, by appropriately dissecting the quadrilaterals into triangles, each T_i can be obtained from the Cartesian product of two 3-cycles quadrangularly embedded in S_1; the quadrangulation itself has a flag-transitive automorphism group of order $8 \times 3^2 = 72$; see [16, 19, 22].)

Based on Table I, we have identified the vertex, face, and edge orbits in each T_i ($i = 1, 2, \ldots, 21$). Elements in the same orbit are called *similar*. Figure 2 shows the vertex orbits in each T_i, where two vertices are marked by the same letter provided that the vertices are similar. Analogously, Figures 3 and 4 show the face and edge orbits, respectively. The same set of letters $\{a, b, c, d, \ldots\}$ is used for marking Figures 2, 3, and 4 in which, for each i, the three sets $V(T_i)$, $E(T_i)$, and $F(T_i)$, respectively, are marked up independently of each other.
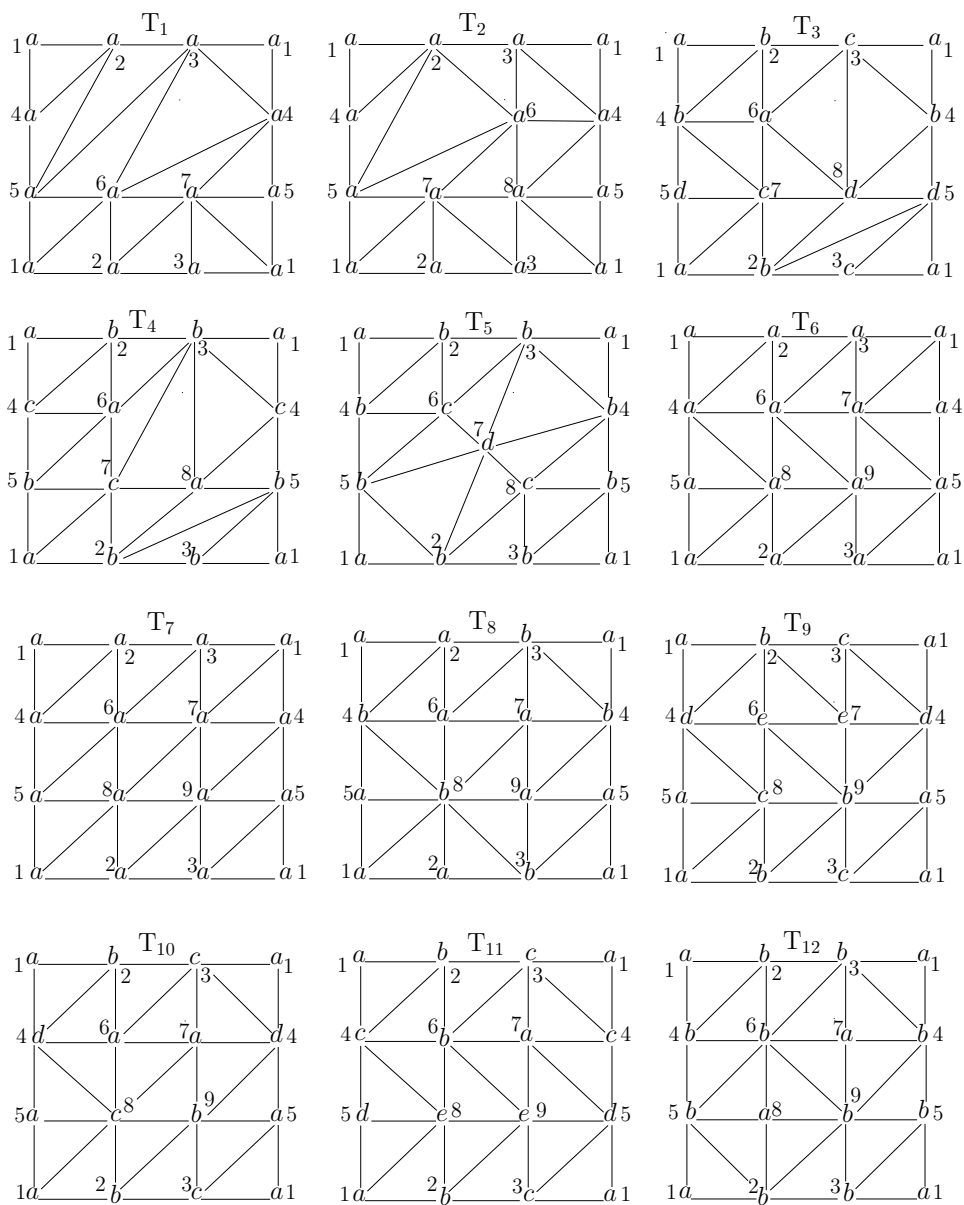
Figure 2. The vertex orbits (to be continued).

To count the number of vertex, face, or edge orbits in a specified T_i, we just count the number of distinct letters used for marking T_i in Figures 2, 3, or 4, respectively.

In what follows we implicitly use the obvious fact that if two dividers $\langle u_1, v_1, w_1 \rangle$ and $\langle u_2, v_2, w_2 \rangle$ of T_i are similar, then the triangulations $\mathrm{sp}\langle u_1, v_1, w_1 \rangle(T_i)$ and $\mathrm{sp}\langle u_2, v_2, w_2 \rangle(T_i)$ are isomorphic. However, the converse is not generally true, as

can be seen in the forthcoming sections; in particular, many counterexamples can be found in Table II.
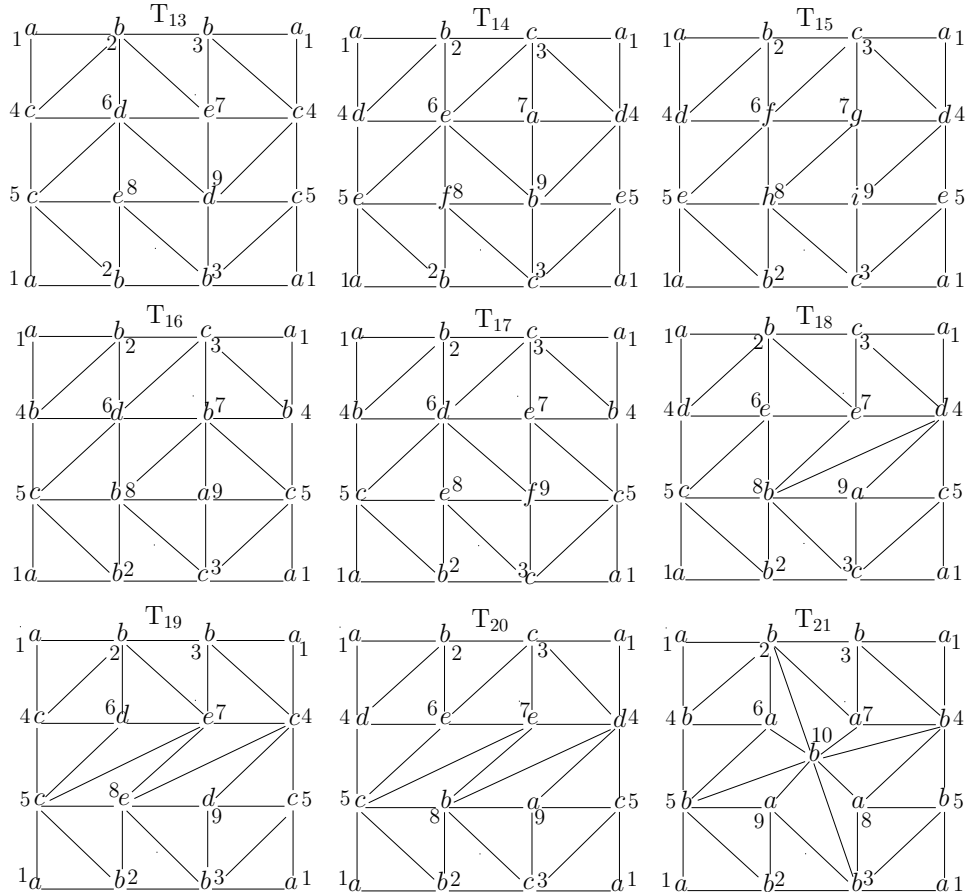
Figure 2. The vertex orbits (contd.)

Let T be an arbitrary triangulation of S_1. We define the *spread of a divider* $\langle u, v, w \rangle$ in T, denoted $|\, u, v, w\,|$, to be the *least* distance between u and w in the link of v. A divider with spread k is referred to as a k-divider. We observe from Figure 2 that the largest degree of a vertex in any T_i is 8, thus the largest spread of a divider in any T_i is 4, and thus the set of all triangulations obtainable by single splitting from toroidal ITs can be written as $\Lambda = \bigcup_{k=1}^{4} \Lambda_k$, where

$$(1) \qquad \Lambda_k = \bigcup_{i=1}^{21} \{ \mathrm{sp}\langle u, v, w \rangle (\mathrm{T}_i) \,|\, |\, u, v, w \,| = k \}$$

The spread of the divider used in generating the splitting is not an invariant of the triangulation obtained because $\Lambda_3 \cap \Lambda_4 \neq \emptyset$; for example, $\mathrm{sp}\langle 2, 6, 9 \rangle (\mathrm{T}_6) \cong \mathrm{sp}\langle 1, 4, 5 \rangle (\mathrm{T}_9)$ with $|\, 2, 6, 9\,| = 3$ in T_6 and $|\, 1, 4, 5\,| = 4$ in T_9. The bottom line, however, is that we generate *all* triangulations of S_1 by repeatedly applying the splitting operation to the basis triangulations T_i.

Lemma 5.1. $\Lambda_1 \cap \Lambda_2 = \emptyset$, $\Lambda_2 \cap \Lambda_3 = \emptyset$, and $\Lambda_3 \cap \Lambda_1 = \emptyset$.

Proof. For $k = 1$ or 2, the $*$-vertex of any $*$-triangulation T in Λ_k is incident with at least two ropes and has degree 3 or 4, respectively, but any $*$-triangulation in Λ_3 has either only one rope, or at least two ropes with the degree of the $*$-vertex of at least 5, and the statement follows immediately. $\square$

Corollary 5.2. *Any two ITPTs obtained by deleting a $*$-vertex from parent $*$-triangulations in different sets Λ_1, Λ_2, or Λ_3, are non-isomorphic.*

Proof. We assume to the contrary that two such ITPTs T and R are isomorphic. Then any isomorphism from T to R takes ∂T onto ∂R and naturally extends to an isomorphism between the corresponding toroidal $*$-triangulations T^* and R^*, which contradicts Lemma 5.1. $\square$

Table I. The automorphism groups of the toroidal ITs.

T_i	Set of permutations generating $\mathrm{Aut}(T_i)$	$|\mathrm{Aut}(T_i)|$
T_1	(1 5 3 4 7 2), (1 2 3 4 5 6 7)	42
T_2	(3 5)(4 7), (1 6)(3 7)(4 5), (1 5 2 7 6 3 8 4)	32
T_3	(2 4)(3 7), (1 6)(3 7)(5 8)	4
T_4	(3 5)(4 7)(6 8), (1 6 8)(5 3 2)	6
T_5	(2 3)(4 5)(6 8), (2 5)(3 4)(6 8)	4
T_6	(2 3)(4 5)(6 9)(7 8), (1 6 5 2 7 8 3 4 9)	18
T_7	(2 8 5 3 7 4)(6 9), (4 8)(5 7)(6 9), (1 2 3)(4 6 7)(5 8 9), (1 5 4)(2 8 6)(3 9 7)	108
T_8	(2 5)(3 4)(6 9), (1 5)(2 9)(3 8)(6 7), (1 9 6)(2 5 7)(3 8 4)	12
T_9	(1 5)(2 9)(3 8)(6 7)	2
T_{10}	(1 5)(2 9)(3 8)(6 7), (1 6)(3 8)(5 7)	4
T_{11}	(1 7)(2 6)(3 4)(8 9)	2
T_{12}	(1 7 8)(2 4 9)(3 6 5), (2 3)(4 5)(6 9)(7 8), (2 4)(3 5)(7 8)	12
T_{13}	(2 3)(4 5)(6 9)(7 8)	2
T_{14}	(1 7)(2 9)(5 6)	2
T_{15}	The trivial automorphism group	1
T_{16}	(2 4)(3 5)(7 8), (1 9)(2 7)(4 8)	4
T_{17}	(2 4)(3 5)(7 8)	2
T_{18}	(1 9)(2 8)(3 5)(6 7)	2
T_{19}	(2 3)(4 5)(6 9)(7 8)	2
T_{20}	(1 9)(2 8)(3 5)(6 7)	2
T_{21}	(2 4 3 5)(6 7 8 9), (1 6 7 9)(3 5 4 10), (1 8 6 9)(2 3 4 10)	20

Thanks to Corollary 4.2, the search for ITPTs is reduced to the search for vertex orbits in toroidal ITs and the search for toroidal $*$-triangulations, both in and out of Λ. This task naturally splits into five cases depending on the origin of the parent triangulation T in Corollary 4.2.

6. The search for ITPTs: Case 0: Series 1

Case 0. Parent triangulation T^* is an irreducible triangulation of S_1.

With help from Figure 2, we can count a total of 80 vertex orbits in $T^* = \mathrm{T}_i$, where i runs over the set $\{1, \ldots, 21\}$. By deleting an arbitrary vertex in each of the orbits, we obtain Series 1 of 80 non-isomorphic ITPTs, thanks to Corollary 4.2. This Series contains no isomorphic pairs as can be proved using an argument similar to the one used in Corollary 5.2 by reduction to a contradiction due to the non-similarity of the vertices deleted.

7. The search for ITPTs: Case 1: Series 2

Case 1. Parent triangulation T^* is in Λ_1 or can be obtained from a member of Λ_1 by a sequence of splittings.

By Eq. (1), if $T^* = \mathrm{sp}\langle u, v, w \rangle(\mathrm{T}_i) \in \Lambda_1$, then $\langle u, v, w \rangle$ is a 1-divider and $\mathrm{sp}\langle u, v, w \rangle$ is equivalent to the stellar subdivision of the face $[u, v, w]$. Thus, such a triangulation T^* is pylonic with the only (necessarily 3-valent) $*$-vertex—either v' or v'' as a matter of notation. It can be easily seen that there are not any other $*$-triangulations that belong to Λ_1 and that any further splitting of T^* would lead to a triangulation that is no longer pylonic. By Corollary 4.6, there are not any other $*$-triangulations obtainable from a member of Λ_1 by any sequence of splittings.

The deletion of the 3-valent $*$-vertex from T^* is equivalent to the deletion of the corresponding face from T_i. Figure 3 shows the face orbits—there are totally 129 non-similar faces in T_i ($i = 1, \ldots, 21$). By deleting an arbitrary face in each of the 129 orbits from T_i, we obtain Series 2 of 129 non-isomorphic ITPTs thanks to Corollary 4.2; this Series is complete by Corollary 4.6.

Just as in Section 6 it can be shown that the set of the 129 triangulations contains no isomorphic pairs.

8. The search for ITPTs: Case 2: Series 3

Case 2. Parent triangulation T^* is in Λ_2 or can be obtained from a member of Λ_2 by a sequence of splittings.

By Eq. (1), if $T^* = \mathrm{sp}\langle u, v, w \rangle(\mathrm{T}_i) \in \Lambda_2$, then $\langle u, v, w \rangle$ is a 2-divider. Denote by x the vertex determining a path of length 2 in the link of v together with vertices u and w; see middle left of Figure 1. This specific type of splitting is equivalent to the *cracking of the edge* $[v, x]$—that is, adding a vertex v' to $[v, x]$ and connecting v' to the apices u and w of the triangular faces incident with $[v, x]$. This transformation always leads to a triangulation with a new 4-valent vertex v' at which the two ropes $[v', v'']$ and $[v', x]$ form a 2-divider shown in bold in the middle right of Figure 1. Sometimes v' may turn out to be pylonic, in which event it can be easily seen that any further splitting of T^* would lead to a triangulation that is no longer pylonic. Thus, by Corollaries 4.2 and 4.6, our first goal is to find all $*$-triangulations T^* in Λ_2. Figure 4 shows the edge orbits in T_i. There are totally 203 non-similar edges, but only 89 of them, when cracked, actually produce $*$-triangulations, as we have checked by direct inspection; the 89 edges can be seen in Table II.

The removal of the 4-valent $*$-vertex is equivalent to the removal of the corresponding edge (the one being cracked) from T_i which produces a quadrilateral hole. It is easy to see that the removal of similar edges gives isomorphic triangulations

(of $S_1 - D$), but it may happen that the removal of non-similar edges produces isomorphic triangulations. It can be verified straightforwardly that there are exactly 27 isomorphic pairs among the 89 triangulations, and there are, in all, $89 - 27 = 62$ non-isomorphic ITPTs obtained by deleting an edge from T_i. As a result, we get Series 3 of ITPTs. This Series is provided in Table II with isomorphic pairs placed in one row; as a matter of notation, we write, for instance, $T_2 - a$ to denote the triangulation obtained from T_2 by deleting an arbitrary edge in orbit a.
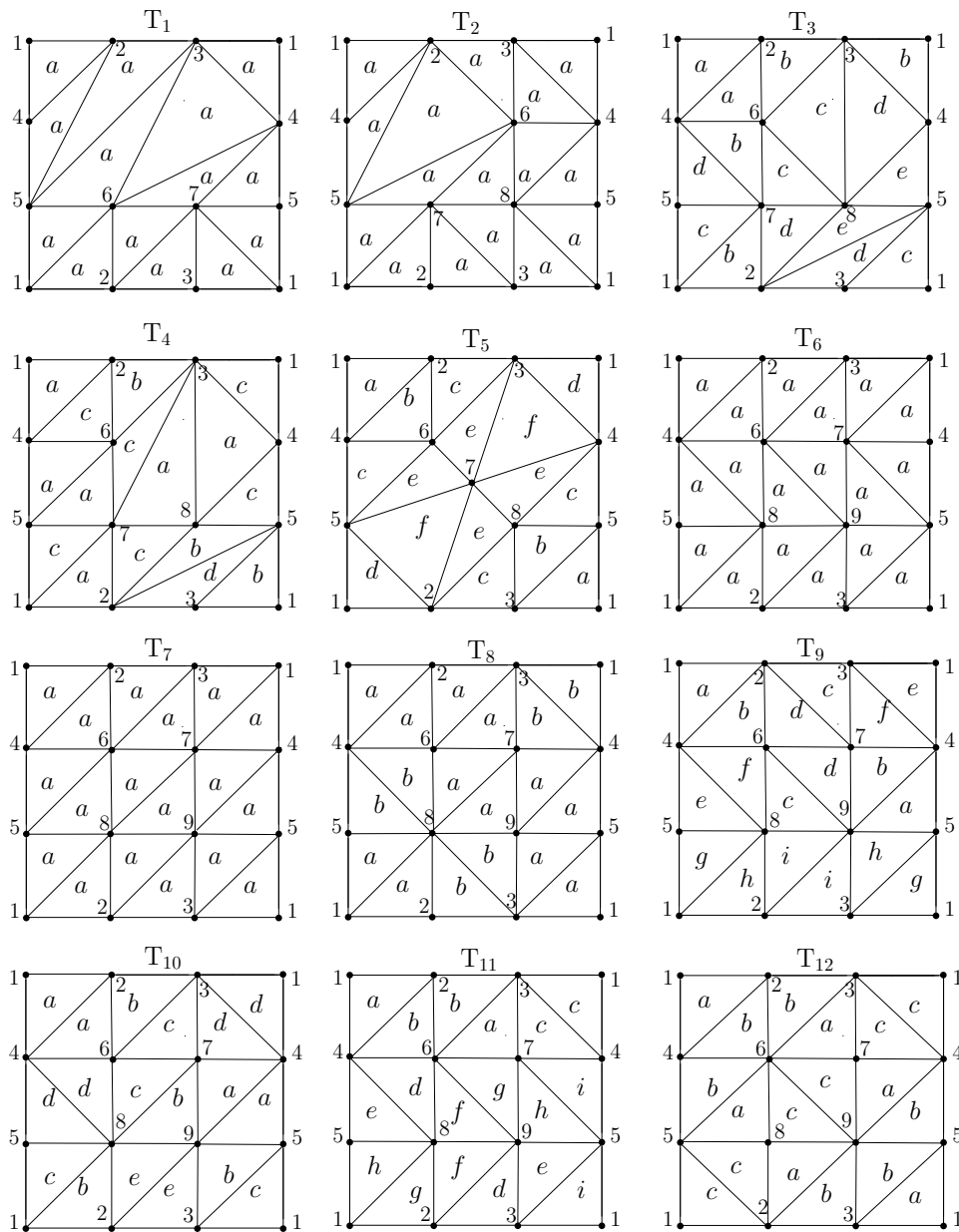
Figure 3. The face orbits (to be continued).
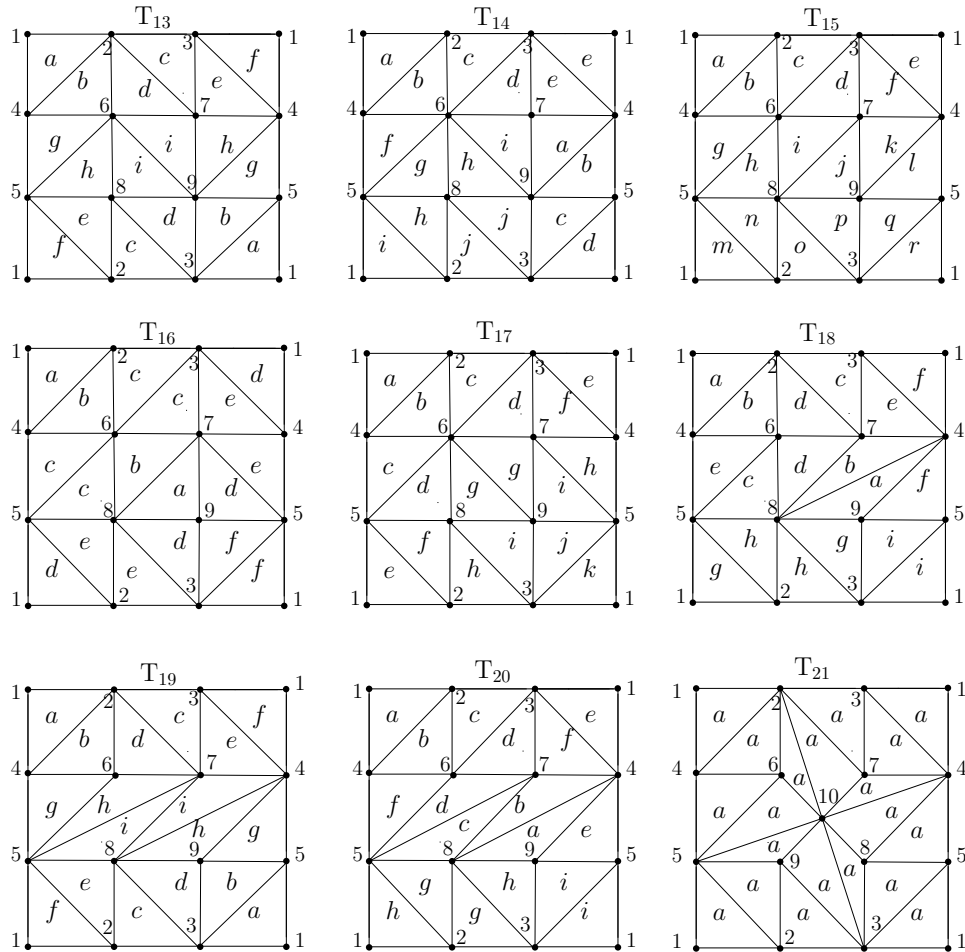
Figure 3. The face orbits (contd.)

It remains to check that the 62 triangulations in Series 3 (Table II) are pairwise non-isomorphic. The d-*vector* (*degree vector*) of a triangulation T is defined to be $d(T) = (n_3, n_4, \ldots, n_{|V(T)|-1})$, where n_r is the number of r-valent vertices. The bd-*sequence* (*boundary degree sequence*) is the cyclic sequence of the degrees of boundary vertices.

The triangulations in Table II either have differing d-vectors or bd-sequences except for the following three non-isomorphic pairs: (i) $T_2 - a \not\cong T_2 - b$, (ii) $T_9 - f \not\cong T_{10} - e$, (iii) $T_9 - k \not\cong T_{10} - f$. Proofs of their non-isomorphism are provided in the next couple of paragraphs.

To show that the triangulations in pair (i) are non-isomorphic, we pick $T_2 - [6, 8]$ as a representative of $T_2 - a$, and pick $T_2 - [6, 7]$ as $T_2 - b$. Since 6 and 8 are the only two 5-valent vertices in $T_2 - [6, 8]$, and 6 and 7 are the only two such vertices in $T_2 - [6, 7]$, and since the edges $[6, 8]$ and $[6, 7]$ are non-similar in T_2, it follows that no isomorphism is possible between $T_2 - [6, 8]$ and $T_2 - [6, 7]$.
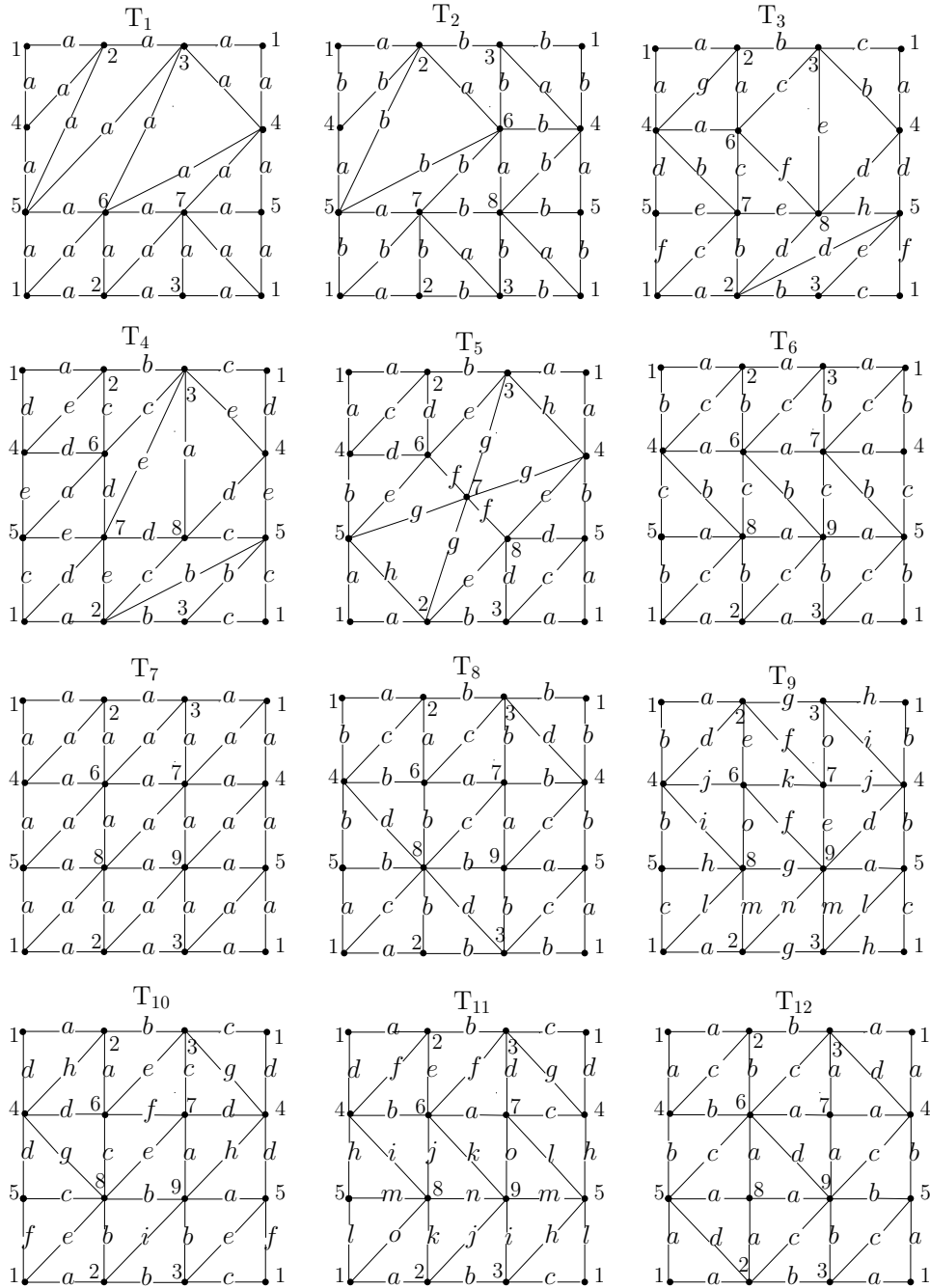
Figure 4. The edge orbits (to be continued).

The triangulations in pair (ii) are non-isomorphic because only $T_{10} - e$ has a face with all vertices 6-valent. Finally, the ones in pair (iii) are non-isomorphic because the only two 6-valent vertices are adjacent in $T_{10} - f$ but non-adjacent in $T_9 - k$.

9. The search for ITPTs: Case 3: Identifying non-similar 3-dividers

Case 3. Parent triangulation T^* is in Λ_3 or can be obtained from a member of Λ_3 by a sequence of splittings.

By Eq. (1), if $T^* = \text{sp}\langle u, v, w\rangle(\text{T}_i) \in \Lambda_3$, then $\langle u, v, w\rangle$ is a 3-divider which divides the link of v into two edge-disjoint paths, *sublinks*, one of which—u, x, y, w— has length 3 and the other has length at least 3; see bottom left of Figure 1. This type of splitting can be thought of as the *cracking of the face* $[x, v, y]$. In this context, we regard the edge $\varepsilon = [x, y]$ as the *base* and the vertex v as one of the two *apices* opposite to the base.
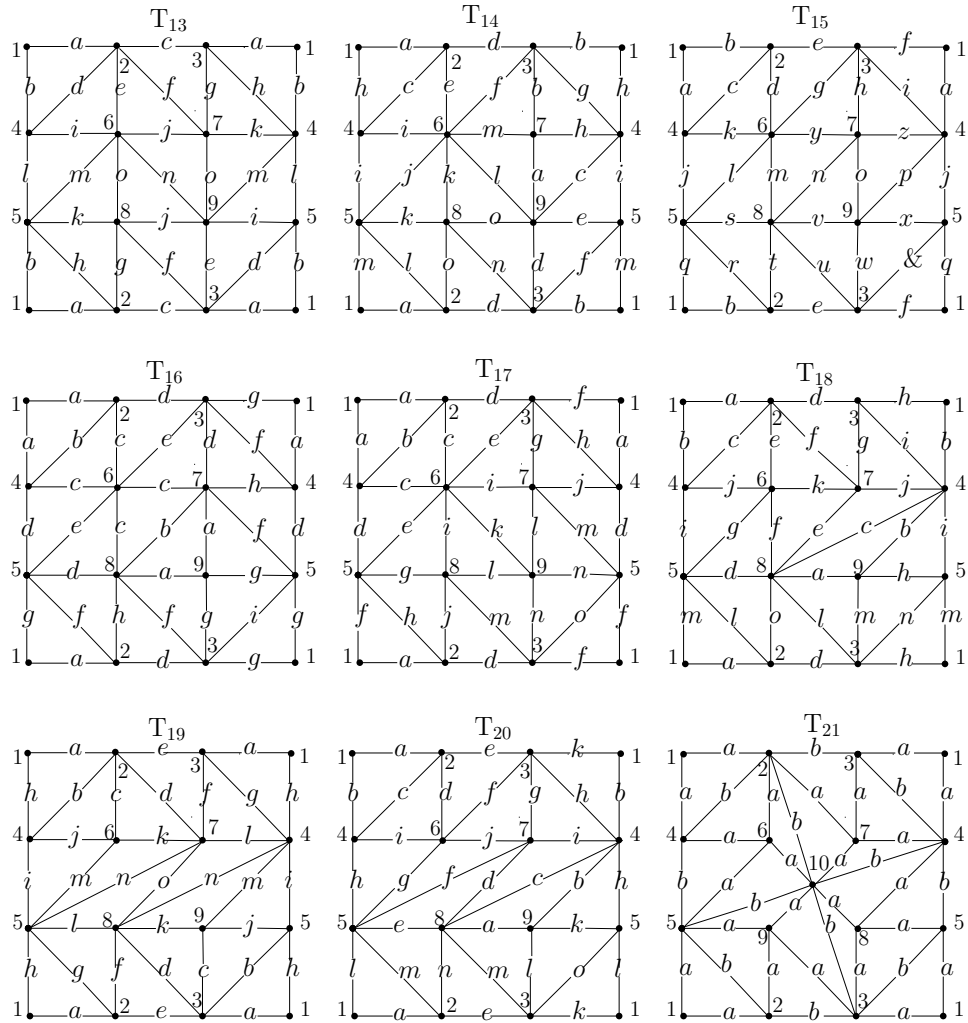
Figure 4. The edge orbits (contd.)

Table II. ITPTs: Series 3 (to be continued).

No.	Triangulation	d-vector	bd-sequence
1	$T_1 - a$	(0,0,2,5)	(5,6,5,6)
2	$T_2 - a \cong T_3 - g$	(0,0,2,6,0)	(5,6,5,6)
3	$T_2 - b$	(0,0,2,6,0)	(5,6,5,6)
4	$T_3 - c$	(0,1,2,3,2)	(4,6,5,7)
5	$T_3 - d$	(0,0,3,4,1)	(5,6,6,6)
6	$T_3 - e$	(0,0,4,2,2)	(5,5,5,7)
7	$T_3 - f \cong T_5 - c$	(0,1,2,3,2)	(4,6,5,6)
8	$T_3 - h$	(0,0,4,2,2)	(5,7,5,7)
9	$T_4 - a \cong T_5 - h$	(0,1,2,3,2)	(4,6,6,6)
10	$T_4 - b$	(0,0,3,4,1)	(5,6,7,6)
11	$T_4 - d$	(0,1,3,1,3)	(4,7,5,7)
12	$T_5 - d$	(0,2,1,2,3)	(4,7,6,7)
13	$T_5 - e$	(0,2,1,2,3)	(4,6,6,7)
14	$T_5 - f$	(0,2,2,0,4)	(4,7,5,7)
15	$T_5 - g$	(0,1,3,1,3)	(5,5,6,7)
16	$T_6 - c \cong T_{11} - g$	(0,0,2,7,0,0)	(5,6,5,6)
17	$T_8 - a$	(0,2,4,0,0,3)	(4,8,4,8)
18	$T_8 - c \cong T_{17} - b$	(0,1,5,0,1,2)	(4,5,7,5)
19	$T_8 - d \cong T_{10} - i$	(0,0,6,0,2,1)	(5,7,5,7)
20	$T_9 - c \cong T_{18} - n$	(0,2,2,2,2,1)	(4,6,4,6)
21	$T_9 - d \cong T_{11} - j$	(0,0,4,3,2,0)	(5,6,5,7)
22	$T_9 - f \cong T_{15} - c$	(0,1,3,3,1,1)	(4,5,6,6)
23	$T_9 - k$	(0,2,2,2,2,1)	(4,7,4,7)
24	$T_9 - l \cong T_{17} - h$	(0,1,4,1,2,1)	(4,5,5,7)
25	$T_9 - n$	(0,0,4,4,1)	(6,6,6,6)

In this section, we identify all non-similar 3-dividers in each T_i by using inclusion-exclusion. The idea behind this is that instead of counting non-similar 3-dividers, we judiciously count the base edges ε that give rise to them. For this, we associate with each 3-divider $\langle u, v, w \rangle$ the edge $\varepsilon = [x, y]$ as indicated on the bottom left of Figure 1 in which the degree of the apex v is assumed to be at least 6 (in T_i). We say that ε *gives rise to the* 3-*divider* $\langle u, v, w \rangle$. Each edge (taken as the base) in a triangulation of a closed 2-manifold gives rise to at most two 3-dividers centered at the apices.

Let fidx(ε) denote the f-*index* (*face-orbit index*) of an edge ε in a given triangulation, defined as follows: fidx(ε) = 1 if the incident faces are in the same orbit (that is, the faces are marked by the same letter in Figure 3), fidx(ε) = 2 if the incident faces are in different orbits (marked by two different letters).

Let us call an edge ε of a triangulation T *dually reversible* if there is an involutory automorphism of T that fixes the base ε and swaps the two apices (such an automorphism reverses the edge dual to ε). Of course, ε is (dually) irreversible whenever fidx(ε) = 2, but may be either reversible or irreversible when fidx(ε) = 1.

We first study T_i for $i = 1, 2, \ldots, 20$, and postpone T_{21} until Lemma 9.5. By direct inspection (with help from Figures 3, 4 along with Table I), we have verified the following lemma.

Table II. ITPTs: Series 3 (contd.)

No.	Triangulation	d-vector	bd-sequence
26	$T_9 - o$	(0,1,4,1,2,1)	(4,7,5,8)
27	$T_{10} - e \cong T_{15} - r$	(0,1,3,3,1,1)	(4,5,6,6)
28	$T_{10} - f \cong T_{20} - o$	(0,2,2,2,2,1)	(4,7,4,7)
29	$T_{11} - f$	(0,0,3,5,1,0)	(5,5,6,6)
30	$T_{11} - l \cong T_{15} - \&$	(0,1,2,4,2,0)	(4,6,5,7)
31	$T_{11} - o \cong T_{13} - h$	(0,1,2,4,2,0)	(4,6,5,6)
32	$T_{12} - c \cong T_{14} - n$	(0,3,0,2,4,0)	(4,6,7,6)
33	$T_{12} - d$	(0,3,0,2,4,0)	(4,6,4,6)
34	$T_{13} - f \cong T_{14} - f$	(0,2,1,3,3,0)	(4,6,6,7)
35	$T_{13} - j \cong T_{18} - c$	(0,2,2,1,4,0)	(4,6,5,7)
36	$T_{13} - m \cong T_{15} - g$	(0,1,3,2,3,0)	(5,5,6,7)
37	$T_{13} - n$	(0,1,4,0,4,0)	(5,5,5,5)
38	$T_{14} - c \cong T_{17} - m$	(0,2,2,2,2,1)	(4,5,7,6)
39	$T_{14} - g$	(0,2,1,3,3,0)	(4,6,4,7)
40	$T_{14} - j \cong T_{16} - e$	(0,2,1,4,1,1)	(5,6,7,6)
41	$T_{14} - l \cong T_{15} - n$	(0,2,2,2,2,1)	(4,5,5,6)
42	$T_{14} - o$	(0,3,1,1,3,1)	(4,7,5,8)
43	$T_{15} - l \cong T_{17} - e$	(0,1,3,3,1,1)	(5,6,6,7)
44	$T_{15} - o \cong T_{19} - b$	(0,3,0,3,2,1)	(4,6,4,7)
45	$T_{15} - p \cong T_{16} - f$	(0,2,1,4,1,1)	(4,5,6,7)
46	$T_{15} - t$	(0,1,4,1,2,1)	(5,7,5,8)
47	$T_{16} - b \cong T_{17} - k$	(0,2,2,3,0,2)	(4,5,6,5)
48	$T_{16} - h$	(0,2,2,3,0,2)	(5,8,5,8)
49	$T_{16} - i$	(0,2,0,5,2,0)	(4,7,4,7)
50	$T_{17} - j$	(0,2,3,1,1,2)	(4,8,5,8)
51	$T_{17} - l$	(0,3,1,2,1,2)	(4,7,4,8)
52	$T_{17} - o$	(0,1,3,2,3,0)	(4,7,5,7)
53	$T_{18} - f \cong T_{19} - n$	(0,3,1,1,3,1)	(4,5,6,7)
54	$T_{18} - g$	(0,3,1,1,3,1)	(4,7,6,8)
55	$T_{18} - k$	(0,4,0,0,4,1)	(4,7,4,7)
56	$T_{18} - o$	(0,2,2,2,2,1)	(6,7,6,7)
57	$T_{19} - d \cong T_{20} - f$	(0,3,1,2,1,2)	(4,5,7,6)
58	$T_{19} - f$	(0,3,1,2,1,2)	(5,7,6,8)
59	$T_{19} - o$	(0,3,2,0,2,2)	(5,8,5,8)
60	$T_{20} - d$	(0,3,2,1,0,3)	(4,8,5,8)
61	$T_{20} - j$	(0,4,0,2,0,3)	(4,8,4,8)
62	$T_{20} - n$	(0,2,4,0,0,3)	(5,8,5,8)

Lemma 9.1. *Each edge of* T_i *(*$i = 1, 2, \ldots, 20$*) with* f*-index 1 is dually reversible.*

Lemma 9.2. *Let* $i \in \{1, 2, \ldots, 20\}$ *and let* $\varepsilon \in E(T_i)$*. Assume that* ε *gives rise to two 3-dividers. Then the two 3-dividers are similar if and only if* $\mathrm{fidx}(\varepsilon) = 1$*. (Equivalently, they are non-similar if and only if* $\mathrm{fidx}(\varepsilon) = 2$*.)*

Proof. To prove the "if" part, assume $\mathrm{fidx}(\varepsilon) = 1$. Then the two 3-dividers are similar by Lemma 9.1.

To prove the "only if" part, assume that the two 3-dividers are similar. We have two cases to consider: (i) the two faces incident with ε are similar, and (ii) the two incident faces are non-similar. In case (i), we immediately come to the desired conclusion: $\mathrm{fidx}(\varepsilon) = 1$. In case (ii), denote by v and z the two apices corresponding to ε. Clearly, any automorphism that takes the 3-divider at v onto the 3-divider at z necessarily sends the sublink of v that contains ε onto the sublink of z that does not contain ε, which requires that both sublinks have the same length, so the apices are both necessarily 6-valent. On the other hand, by direct inspection, we have verified that for each edge of T_i ($i = 1, 2, \ldots, 20$) with f-index 2, the apices are either non-similar or non-6-valent. Thereby, we still come to the ultimate equality: $\mathrm{fidx}(\varepsilon) = 1$. $\square$

Given a triangulation T, we define the v-*index* (*vertex-degree index*) of a base $\varepsilon_m \in E(T)$ to be the number of its apices with degree at least 6, and denote it by $\mathrm{vidx}(\varepsilon_m)$. Clearly, $\mathrm{vidx}(\varepsilon_m) = 0, 1$, or 2. We define the s_1-*invariant* as follows:

$$(2) \qquad s_1 = s_1(T) := \sum_m \min(\mathrm{fidx}(\varepsilon_m), \mathrm{vidx}(\varepsilon_m)),$$

where the sum is taken over all edge orbits in T. The general (mth) term in Eq. (2) has value 0 if and only if vidx=0, has value 2 if and only if fidx=vidx=2, and has value 1 in all other cases. To construct such a function, we use min(fidx, vidx).

By Lemma 9.2, $s_1(\mathrm{T}_i)$ is equal to the number of non-similar 3-dividers in T_i ($i = 1, 2, \ldots, 20$) with some of them counted twice, as explained in the next paragraph. The bases ε_m in Eq. (2) give rise to the counted 3-dividers.

In the specific case in which the degree of a vertex, v_n, is precisely 6, any two edges opposite to each other in the link of v_n, when taken as bases, give rise to the same 3-divider, centered at v_n, regardless of whether the two edges are similar or not. Let $\mathrm{kidx}(v_n)$ be the χ-*index* (*link-chirality index*) of a 6-valent vertex $v_n \in V(T)$, that is, the number of distinct pairs of non-similar opposite edges in the link of v_n (distinct as unordered pairs of corresponding letters). Clearly, $\mathrm{kidx}(v_n) \in \{0, 1, 2, 3\}$. We define the s_2-*invariant* as follows:

$$s_2 = s_2(T) := \sum_n \mathrm{kidx}(v_n),$$

where the sum is taken over all 6-valent vertex orbits in T; if there are no 6-valent vertices, $s_2 := 0$. It is not hard to observe that s_2 is equal to the number of 3-dividers doubly counted in Eq. (2). We finally come to the following simple formula:

Lemma 9.3. *The number of non-similar 3-dividers in* T_i ($i = 1, 2, \ldots, 20$) *is equal to the difference* $s_1(\mathrm{T}_i) - s_2(\mathrm{T}_i)$.

The following is an example of how to intelligently obtain a complete list of pairwise non-similar 3-dividers in a given triangulation without going through a tedious check. The example addresses the "hardest triangulation" T_3.

Example 9.4. To illustrate the proof of Lemma 9.3, which was given prior to its statement, we consider an example of its use in determining all (pairwise) non-similar 3-dividers in T_3. Under the action of $\mathrm{Aut}(\mathrm{T}_3)$, the vertices of T_3 fall into four orbits marked by the letters a–d in Figure 2, the edges fall into eight orbits marked by a–h in Figure 4, and the faces fall into five orbits a–e in Figure 3. We

choose, arbitrarily, the following eight edges as representatives of the edge orbits: edge $[2, 6]$ in orbit a, $[3, 4]$ in b, $[6, 7]$ in c, $[4, 5]$ in d, $[3, 8]$ in e, $[1, 5]$ in f, $[2, 4]$ in g, $[5, 8]$ in h. We calculate the s_1-invariant by writing the edge orbits in alphabetical order and checking with Figure 3—see Table III.

Table III. Calculation of the s_1-invariant.

edge	[2,6]	[3,4]	[6,7]	[4,5]	[3,8]	[1,5]	[2,4]	[5,8]	
orbit	a	b	c	d	e	f	g	h	
f-index	2	2	2	2	2	1	1	1	
v-index	2	1	2	2	1	2	0	2	
min(fidx,vidx)	2	1	2	2	1	1	0	1	$s_1 = \sum$ $= 10$

$$s_1(T_3) = \Sigma_m \min(\mathrm{fidx}(\varepsilon_m), \mathrm{vidx}(\varepsilon_m))$$
$$= \min(\mathrm{fidx}([2, 6]), \mathrm{vidx}([2, 6])) + \min(\mathrm{fidx}([3, 4]), \mathrm{vidx}([3, 4]))$$
$$+ \min(\mathrm{fidx}([6, 7]), \mathrm{vidx}([6, 7])) + \min(\mathrm{fidx}([4, 5]), \mathrm{vidx}([4, 5]))$$
$$+ \min(\mathrm{fidx}([3, 8]), \mathrm{vidx}([3, 8])) + \min(\mathrm{fidx}([1, 5]), \mathrm{vidx}([1, 5]))$$
$$+ \min(\mathrm{fidx}([2, 4]), \mathrm{vidx}([2, 4])) + \min(\mathrm{fidx}([5, 8]), \mathrm{vidx}([5, 8]))$$
$$= \min(2, 2) + \min(2, 1) + \min(2, 2) + \min(2, 2) + \min(2, 1)$$
$$+ \min(1, 2) + \min(1, 0) + \min(1, 2)$$
$$= 2 + 1 + 2 + 2 + 1 + 1 + 0 + 1 = 10.$$

Now we calculate the s_2-invariant. There are four orbits into which the vertices of T_3 fall: $a, b, c,$ and d; see Figure 2. However, only the vertices in orbits c and d have degree 6; we pick vertices 7 and 8 as their representatives (respectively). The opposite pairs of edges in the link of vertex 7 are marked by letters (Figure 4) as follows: $\{a, a\}$, $\{f, f\}$, and $\{d, d\}$; here kidx$(7) = 0$ because there is no pair of different letters. The opposite pairs of edges in the link of vertex 8 are: $\{c, d\}$, $\{c, d\}$, and $\{b, b\}$; here kidx$(8) = 1$ because there is only one pair of different letters, $\{c, d\}$, distinct from the other pairs. Therefore,

$$s_2(T_3) = \sum_n \mathrm{kidx}(v_n) = \mathrm{kidx}(7) + \mathrm{kidx}(8) = 0 + 1 = 1.$$

Thus, T_3 has exactly $s_1 - s_2 = 10 - 1 = 9$ non-similar 3-dividers. In fact, we can go even further. The proposed approach allows *explicitly* identifying all non-similar 3-dividers in T_3; see below and check with Table III.

Firstly, we inspect the edges with min(fidx,vidx) $= 0$ which is equivalent to vidx $= 0$. We discard such edges because the 3-dividers that they give rise to are, in fact, 2- or 1-dividers. In the example under consideration, we discard the edge $[2, 4]$.

Secondly, we inspect the edges with min(fidx, vidx) $= 1$ which is equivalent to either condition (i) (fidx $= 2$) *AND* (vidx $= 1$), or (ii) (fidx $= 1$) *AND* (vidx $= 2$), or (iii) (fidx $= 1$) *AND* (vidx $= 1$). In condition (i), since vidx $= 1$, there is only one 3-divider each such edge gives rise to; in the example under consideration, we have two edges in this case: $[3, 4]$ and $[3, 8]$; the 3-dividers that these edges give rise to are $\langle 5, 8, 6 \rangle$ and $\langle 1, 4, 5 \rangle$, respectively (Figure 3). In condition (ii), since fidx $= 1$, then there is up to similarity just one 3-divider that each such edge gives rise to, by Lemma 9.2; in the example under consideration, we have two edges in this case: $[1, 5]$ and $[5, 8]$; the 3-dividers that the edges give rise to are $\langle 2, 7, 4 \rangle$ and $\langle 7, 4, 3 \rangle$, respectively (Figure 3). In condition (iii), there is only one 3-divider that each such

edge gives rise to, by Lemma 9.2; however, in the example under consideration, we have no edges in this case.

Thirdly, among the remaining edges, we inspect the ones with $\min(\text{fidx}, \text{vidx}) = 2$ which is equivalent to condition (fidx $= 2$) AND (vidx $= 2$). By Lemma 9.2, each such edge gives rise to two non-similar 3-dividers—namely, as seen in Figure 3, edge $[2, 6]$ gives rise to $\langle 1, 4, 7 \rangle$ and $\langle 8, 3, 5 \rangle$ (the latter is similar to $\langle 8, 7, 5 \rangle$—Table I), edge $[6, 7]$ gives rise to $\langle 2, 4, 5 \rangle$ and $\langle 3, 8, 2 \rangle$ (similar to $\langle 7, 8, 4 \rangle$), edge $[4, 5]$ gives rise to $\langle 6, 7, 1 \rangle$ and $\langle 3, 8, 2 \rangle$ (similar to $\langle 7, 8, 4 \rangle$); check with Figure 3. Observe that we have a duplication: the 3-divider $\langle 3, 8, 2 \rangle$ is doubly counted because on one hand it is given rise to by two non-similar edges ($[6, 7]$ and $[4, 5]$), but on the other hand, since vertex 8 is 6-valent, these opposite (in the link of 8) edges give rise to the *same* 3-divider—$\langle 3, 8, 2 \rangle$.

Table IV. Calculation of the number of 3-dividers.

T_i	s_1	s_2	The # of 3-dividers
T_1	1	0	1
T_2	2	0	2
T_3	10	1	9
T_4	6	1	5
T_5	9	0	9
T_6	3	0	3
T_7	1	0	1
T_8	2	0	2
T_9	17	3	14
T_{10}	9	1	8
T_{11}	22	6	16
T_{12}	4	0	4
T_{13}	20	3	17
T_{14}	21	3	18
T_{15}	40	9	31
T_{16}	12	3	9
T_{17}	18	3	15
T_{18}	18	0	18
T_{19}	21	3	18
T_{20}	18	3	15
T_{21}	1	0	2*
			$\sum = 217$

It is easy to count $s_1 - s_2$ with help from Figures 3, 4. The results are collected in Table IV; this table can be regarded as a corollary of Lemma 9.3; the asterisk in the last row indicates that the formula in Lemma 9.3 does not apply for T_{21} addressed in Lemma 9.5. Moreover, similarly to Example 9.4, we have produced a complete list of non-similar 3-dividers in each T_i for $i = 1, 2, \ldots, 20$; there are totally 215 3-dividers. Check with Table IV. In addition, two more are found in T_{21}:

Lemma 9.5. *If an edge ε of T_{21} gives rise to at least one 3-divider, then ε is similar to the edge $[2, 6]$ (Figure 4) which, in fact, gives rise to two non-similar 3-dividers: $\langle 5, 10, 7 \rangle$ and $\langle 5, 4, 1 \rangle$ (the latter is similar to $\langle 5, 10, 8 \rangle$—Table I).*

Proof. All edges of T_{21} split into two orbits—a and b (see Figure 4). For each edge in orbit b as the base, both apices are 4-valent. Therefore, no edge in b gives rise to a 3-divider. On the other hand, each edge in orbit a is dually irreversible (check with Figure 4 and Table I) and thereby gives rise to two non-similar 3-dividers even though it has f-index 1 (Figure 3).   □

Table V. Λ_3 with isomorphic duplications (to be continued).

Triangulations					Total
sp⟨1,7,6⟩(T₁)					1
sp⟨7,8,5⟩(T₂)	sp⟨6,8,1⟩(T₂)				2
sp⟨1,4,7⟩(T₃)	sp⟨1,4,5⟩(T₃)	sp⟨2,4,5⟩(T₃)	sp⟨7,4,3⟩(T₃)	sp⟨6,7,1⟩(T₃)	
sp⟨8,7,5⟩(T₃)	sp⟨2,7,4⟩(T₃)	sp⟨7,8,4⟩(T₃)	sp⟨5,8,6⟩(T₃)		9
sp⟨4,3,2⟩(T₄)	sp⟨4,3,6⟩(T₄)	sp⟨8,3,2⟩(T₄)	sp⟨6,3,1⟩(T₄)	sp⟨8,7,5⟩(T₄)	5
sp⟨6,2,5⟩(T₅)	sp⟨6,2,7⟩(T₅)	sp⟨3,2,1⟩(T₅)	sp⟨3,2,5⟩(T₅)	sp⟨8,2,4⟩(T₅)	
sp⟨8,2,1⟩(T₅)	sp⟨7,2,4⟩(T₅)	sp⟨3,7,2⟩(T₅)	sp⟨8,7,6⟩(T₅)		9
sp⟨8,6,3⟩(T₆)	sp⟨7,6,4⟩(T₆)	sp⟨9,6,2⟩(T₆)			3
sp⟨4,6,7⟩(T₇)					1
sp⟨1,8,9⟩(T₈)	sp⟨2,8,4⟩(T₈)				2
sp⟨2,9,4⟩(T₉)	sp⟨2,9,7⟩(T₉)	sp⟨3,9,6⟩(T₉)	sp⟨3,9,7⟩(T₉)	sp⟨5,9,8⟩(T₉)	
sp⟨5,9,6⟩(T₉)	sp⟨4,9,8⟩(T₉)	sp⟨1,8,6⟩(T₉)	sp⟨5,8,9⟩(T₉)	sp⟨4,8,2⟩(T₉)	
sp⟨3,4,5⟩(T₉)	sp⟨5,4,2⟩(T₉)	sp⟨2,4,7⟩(T₉)	sp⟨6,4,3⟩(T₉)		14
sp⟨5,4,2⟩(T₁₀)	sp⟨5,4,3⟩(T₁₀)	sp⟨8,9,5⟩(T₁₀)	sp⟨2,9,4⟩(T₁₀)	sp⟨7,8,5⟩(T₁₀)	
sp⟨1,8,7⟩(T₁₀)	sp⟨9,8,4⟩(T₁₀)	sp⟨9,8,5⟩(T₁₀)			8
sp⟨3,6,8⟩(T₁₁)	sp⟨7,6,4⟩(T₁₁)	sp⟨9,6,2⟩(T₁₁)	sp⟨7,3,5⟩(T₁₁)	sp⟨7,3,9⟩(T₁₁)	
sp⟨6,3,5⟩(T₁₁)	sp⟨6,3,1⟩(T₁₁)	sp⟨2,3,1⟩(T₁₁)	sp⟨2,3,4⟩(T₁₁)	sp⟨9,3,4⟩(T₁₁)	
sp⟨4,5,3⟩(T₁₁)	sp⟨8,5,9⟩(T₁₁)	sp⟨1,5,7⟩(T₁₁)	sp⟨6,8,1⟩(T₁₁)	sp⟨9,8,5⟩(T₁₁)	
sp⟨2,8,4⟩(T₁₁)					16
sp⟨9,6,4⟩(T₁₂)	sp⟨4,6,7⟩(T₁₂)	sp⟨8,6,3⟩(T₁₂)	sp⟨5,6,3⟩(T₁₂)		4
sp⟨1,2,7⟩(T₁₃)	sp⟨1,2,3⟩(T₁₃)	sp⟨5,2,7⟩(T₁₃)	sp⟨5,2,6⟩(T₁₃)	sp⟨8,2,4⟩(T₁₃)	
sp⟨8,2,6⟩(T₁₃)	sp⟨3,2,4⟩(T₁₃)	sp⟨9,4,1⟩(T₁₃)	sp⟨9,4,2⟩(T₁₃)	sp⟨7,4,2⟩(T₁₃)	
sp⟨7,4,6⟩(T₁₃)	sp⟨3,4,6⟩(T₁₃)	sp⟨3,4,5⟩(T₁₃)	sp⟨1,4,5⟩(T₁₃)	sp⟨5,6,7⟩(T₁₃)	
sp⟨4,6,9⟩(T₁₃)	sp⟨2,6,8⟩(T₁₃)				17
sp⟨5,6,3⟩(T₁₄)	sp⟨5,6,7⟩(T₁₄)	sp⟨4,6,7⟩(T₁₄)	sp⟨4,6,9⟩(T₁₄)	sp⟨2,6,8⟩(T₁₄)	
sp⟨2,6,9⟩(T₁₄)	sp⟨3,6,8⟩(T₁₄)	sp⟨2,3,5⟩(T₁₄)	sp⟨2,3,4⟩(T₁₄)	sp⟨8,3,1⟩(T₁₄)	
sp⟨1,3,6⟩(T₁₄)	sp⟨3,9,7⟩(T₁₄)	sp⟨8,9,4⟩(T₁₄)	sp⟨6,9,5⟩(T₁₄)	sp⟨9,4,1⟩(T₁₄)	
sp⟨9,4,2⟩(T₁₄)	sp⟨3,4,5⟩(T₁₄)	sp⟨1,4,5⟩(T₁₄)			18
sp⟨1,2,3⟩(T₁₅)	sp⟨5,2,6⟩(T₁₅)	sp⟨8,2,4⟩(T₁₅)	sp⟨9,4,1⟩(T₁₅)	sp⟨9,4,2⟩(T₁₅)	
sp⟨7,4,2⟩(T₁₅)	sp⟨7,4,6⟩(T₁₅)	sp⟨3,4,6⟩(T₁₅)	sp⟨3,4,5⟩(T₁₅)	sp⟨1,4,5⟩(T₁₅)	
sp⟨6,8,3⟩(T₁₅)	sp⟨7,8,2⟩(T₁₅)	sp⟨9,8,5⟩(T₁₅)	sp⟨9,3,4⟩(T₁₅)	sp⟨9,3,6⟩(T₁₅)	
sp⟨5,3,7⟩(T₁₅)	sp⟨5,3,2⟩(T₁₅)	sp⟨1,3,8⟩(T₁₅)	sp⟨1,3,6⟩(T₁₅)	sp⟨4,3,2⟩(T₁₅)	
sp⟨7,3,8⟩(T₁₅)	sp⟨2,6,8⟩(T₁₅)	sp⟨3,6,5⟩(T₁₅)	sp⟨7,6,4⟩(T₁₅)	sp⟨9,5,8⟩(T₁₅)	
sp⟨9,5,2⟩(T₁₅)	sp⟨4,5,1⟩(T₁₅)	sp⟨4,5,2⟩(T₁₅)	sp⟨6,5,3⟩(T₁₅)	sp⟨6,5,1⟩(T₁₅)	
sp⟨8,5,3⟩(T₁₅)					31
sp⟨5,8,9⟩(T₁₆)	sp⟨6,8,3⟩(T₁₆)	sp⟨7,8,2⟩(T₁₆)	sp⟨4,6,7⟩(T₁₆)	sp⟨3,6,5⟩(T₁₆)	
sp⟨2,3,5⟩(T₁₆)	sp⟨2,3,4⟩(T₁₆)	sp⟨8,3,1⟩(T₁₆)	sp⟨1,3,6⟩(T₁₆)		9
sp⟨9,3,4⟩(T₁₇)	sp⟨9,3,6⟩(T₁₇)	sp⟨5,3,2⟩(T₁₇)	sp⟨5,3,7⟩(T₁₇)	sp⟨1,3,8⟩(T₁₇)	
sp⟨1,3,6⟩(T₁₇)	sp⟨4,3,2⟩(T₁₇)	sp⟨7,3,8⟩(T₁₇)	sp⟨1,2,3⟩(T₁₇)	sp⟨5,2,6⟩(T₁₇)	
sp⟨8,2,4⟩(T₁₇)	sp⟨2,6,9⟩(T₁₇)	sp⟨3,6,5⟩(T₁₇)	sp⟨7,6,4⟩(T₁₇)	sp⟨7,6,5⟩(T₁₇)	15

Table V. Λ_3 with isomorphic duplications (contnd.)

Triangulations					Total
$\mathrm{sp}\langle 2,8,7\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 2,8,4\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 5,8,4\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 5,8,9\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 6,8,3\rangle(\mathrm{T}_{18})$	
$\mathrm{sp}\langle 6,8,9\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 7,8,3\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 2,3,5\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 2,3,1\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 8,3,1\rangle(\mathrm{T}_{18})$	
$\mathrm{sp}\langle 8,3,4\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 9,3,4\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 9,3,7\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 5,3,7\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 7,4,2\rangle(\mathrm{T}_{18})$	
$\mathrm{sp}\langle 3,4,6\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 3,4,9\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 2,4,9\rangle(\mathrm{T}_{18})$			18
$\mathrm{sp}\langle 5,7,3\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 6,7,4\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 2,7,8\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 4,2,8\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 4,2,3\rangle(\mathrm{T}_{19})$	
$\mathrm{sp}\langle 1,2,3\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 1,2,7\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 5,2,6\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 5,2,7\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 8,2,6\rangle(\mathrm{T}_{19})$	
$\mathrm{sp}\langle 1,4,8\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 1,4,5\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 2,4,7\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 2,4,9\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 6,4,3\rangle(\mathrm{T}_{19})$	
$\mathrm{sp}\langle 6,4,8\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 5,4,7\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 9,4,3\rangle(\mathrm{T}_{19})$			18
$\mathrm{sp}\langle 5,8,9\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 7,8,3\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 4,8,2\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 2,3,5\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 2,3,4\rangle(\mathrm{T}_{20})$	
$\mathrm{sp}\langle 8,3,1\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 8,3,7\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 9,3,6\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 9,3,4\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 5,3,7\rangle(\mathrm{T}_{20})$	
$\mathrm{sp}\langle 1,3,6\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 7,4,5\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 3,4,9\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 1,4,8\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 6,4,8\rangle(\mathrm{T}_{20})$	15
$\mathrm{sp}\langle 5,10,7\rangle(\mathrm{T}_{21})$	$\mathrm{sp}\langle 5,10,8\rangle(\mathrm{T}_{21})$				2

Using Lemma 9.3 (as in Example 9.4) and Lemma 9.5, we collect in Table V the 217 triangulations produced by splitting all the 217 non-similar 3-dividers in T_i ($i = 1, 2, \ldots, 21$), which provides the whole set Λ_3 (with isomorphic duplications).

10. The search for ITPTs: Case 3: Series 4-6

In order to apply Corollary 4.2, we identify $*$-triangulations from the 217 triangulations in Table V. Moreover, we retain only non-isomorphic triangulations. It is a matter of mere inspection to obtain the following lemma.

Lemma 10.1. *There are precisely eleven $*$-triangulations in Λ_3, as follows:*
$\mathrm{sp}\langle 1,7,6\rangle(\mathrm{T}_1), \mathrm{sp}\langle 6,8,1\rangle(\mathrm{T}_2), \mathrm{sp}\langle 7,4,3\rangle(\mathrm{T}_3), \mathrm{sp}\langle 2,7,4\rangle(\mathrm{T}_3), \mathrm{sp}\langle 7,8,4\rangle(\mathrm{T}_3),$
$\mathrm{sp}\langle 5,8,6\rangle(\mathrm{T}_3), \mathrm{sp}\langle 6,3,1\rangle(\mathrm{T}_4), \mathrm{sp}\langle 3,2,1\rangle(\mathrm{T}_5), \mathrm{sp}\langle 3,2,5\rangle(\mathrm{T}_5), \mathrm{sp}\langle 3,7,2\rangle(\mathrm{T}_5),$
and $\mathrm{sp}\langle 8,7,6\rangle(\mathrm{T}_5)$.
These triangulations are boxed in Table V and are also presented in Figure 5 with all ropes in bold type.

It should be noticed that eight of the eleven $*$-triangulations in Lemma 10.1 have only one rope and two $*$-vertices. The removal of a $*$-vertex in each of the eleven triangulations creates the following 19 ITPTs:

$\mathrm{sp}\langle 1,7,6\rangle(\mathrm{T}_1) - 7', \mathrm{sp}\langle 1,7,6\rangle(\mathrm{T}_1) - 7'', \mathrm{sp}\langle 6,8,1\rangle(\mathrm{T}_2) - 8',$
$\mathrm{sp}\langle 6,8,1\rangle(\mathrm{T}_2) - 8'', \mathrm{sp}\langle 7,4,3\rangle(\mathrm{T}_3) - 4', \mathrm{sp}\langle 7,4,3\rangle(\mathrm{T}_3) - 4'',$
$\mathrm{sp}\langle 2,7,4\rangle(\mathrm{T}_3) - 7', \mathrm{sp}\langle 2,7,4\rangle(\mathrm{T}_3) - 7'', \mathrm{sp}\langle 7,8,4\rangle(\mathrm{T}_3) - 8',$
$\mathrm{sp}\langle 5,8,6\rangle(\mathrm{T}_3) - 8', \mathrm{sp}\langle 5,8,6\rangle(\mathrm{T}_3) - 8'', \mathrm{sp}\langle 6,3,1\rangle(\mathrm{T}_4) - 3'',$
$\mathrm{sp}\langle 3,2,1\rangle(\mathrm{T}_5) - 2', \mathrm{sp}\langle 3,2,5\rangle(\mathrm{T}_5) - 2', \mathrm{sp}\langle 3,2,5\rangle(\mathrm{T}_5) - 2'',$
$\mathrm{sp}\langle 3,7,2\rangle(\mathrm{T}_5) - 7', \mathrm{sp}\langle 3,7,2\rangle(\mathrm{T}_5) - 7'', \mathrm{sp}\langle 8,7,6\rangle(\mathrm{T}_5) - 7',$
$\mathrm{sp}\langle 8,7,6\rangle(\mathrm{T}_5) - 7''$.

However, fourteen of these are isomorphic in pairs as follows:

$\mathrm{sp}\langle 1,7,6\rangle(\mathrm{T}_1) - 7' \cong \mathrm{sp}\langle 1,7,6\rangle(\mathrm{T}_1) - 7'', \mathrm{sp}\langle 6,8,1\rangle(\mathrm{T}_2) - 8' \cong \mathrm{sp}\langle 6,8,1\rangle(\mathrm{T}_2) - 8'',$
$\mathrm{sp}\langle 2,7,4\rangle(\mathrm{T}_3) - 7' \cong \mathrm{sp}\langle 2,7,4\rangle(\mathrm{T}_3) - 7'', \mathrm{sp}\langle 7,8,4\rangle(\mathrm{T}_3) - 8' \cong \mathrm{sp}\langle 3,2,1\rangle(\mathrm{T}_5) - 2',$
$\mathrm{sp}\langle 5,8,6\rangle(\mathrm{T}_3) - 8' \cong \mathrm{sp}\langle 5,8,6\rangle(\mathrm{T}_3) - 8'', \mathrm{sp}\langle 3,7,2\rangle(\mathrm{T}_5) - 7' \cong \mathrm{sp}\langle 3,7,2\rangle(\mathrm{T}_5) - 7'',$
$\mathrm{sp}\langle 8,7,6\rangle(\mathrm{T}_5) - 7' \cong \mathrm{sp}\langle 8,7,6\rangle(\mathrm{T}_5) - 7''$.

$\mathrm{sp}\langle 1, 7, 6\rangle(\mathrm{T}_1)$ $\qquad$ $\mathrm{sp}\langle 6, 8, 1\rangle(\mathrm{T}_2)$ $\qquad$ $\mathrm{sp}\langle 7, 4, 3\rangle(\mathrm{T}_3)$

$\mathrm{sp}\langle 2, 7, 4\rangle(\mathrm{T}_3)$ $\qquad$ $\mathrm{sp}\langle 7, 8, 4\rangle(\mathrm{T}_3)$ $\qquad$ $\mathrm{sp}\langle 5, 8, 6\rangle(\mathrm{T}_3)$

$\mathrm{sp}\langle 6, 3, 1\rangle(\mathrm{T}_4)$ $\qquad$ $\mathrm{sp}\langle 3, 2, 1\rangle(\mathrm{T}_5)$ $\qquad$ $\mathrm{sp}\langle 3, 2, 5\rangle(\mathrm{T}_5)$

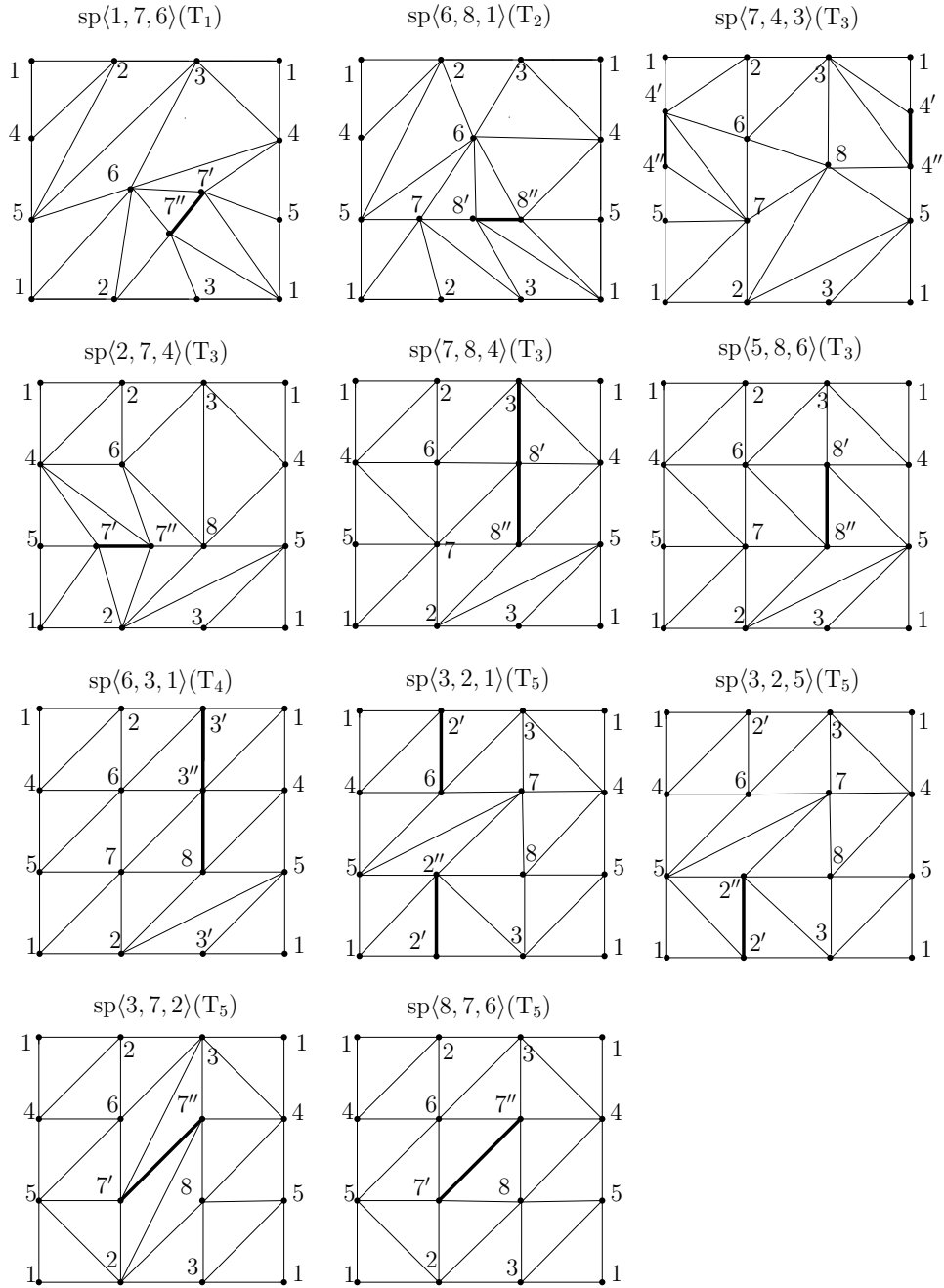$\mathrm{sp}\langle 3, 7, 2\rangle(\mathrm{T}_5)$ $\qquad$ $\mathrm{sp}\langle 8, 7, 6\rangle(\mathrm{T}_5)$

Figure 5. Pylonic triangulations in Λ_3.

We thus get Series 4 amounting to $19 - 14/2 = 12$ non-isomorphic ITPTs:

Lemma 10.2. *There are precisely 12 ITPTs obtainable by deleting a $*$-vertex from a $*$-triangulation in Λ_3. The twelve triangulations form Series 4 and are collected in Table VI.*

Proof. Actually, it remains to verify that these twelve triangulations are non-isomorphic. This follows from the fact that they have either differing d-vectors or bd-sequences; check with Table VI. □

Table VI. ITPTs: Series 4.

No.	Triangulation	d-vector	bd-sequence
1	$\mathrm{sp}\langle 1,7,6\rangle(\mathrm{T}_1)-7'$	(0,1,2,4)	(4,6,5,5,6)
2	$\mathrm{sp}\langle 6,8,1\rangle(\mathrm{T}_2)-8'$	(0,1,2,5,0)	(4,6,5,5,6)
3	$\mathrm{sp}\langle 7,4,3\rangle(\mathrm{T}_3)-4'$	(0,3,0,5,0)	(4,6,4,6,4,6)
4	$\mathrm{sp}\langle 7,4,3\rangle(\mathrm{T}_3)-4''$	(0,0,5,2,1)	(5,6,5,5,6)
5	$\mathrm{sp}\langle 2,7,4\rangle(\mathrm{T}_3)-7'$	(0,2,2,2,2)	(4,7,5,4,7)
6	$\mathrm{sp}\langle 7,8,4\rangle(\mathrm{T}_3)-8'$	(0,2,2,2,2)	(4,6,4,5,7)
7	$\mathrm{sp}\langle 5,8,6\rangle(\mathrm{T}_3)-8'$	(0,1,3,3,1)	(4,5,5,6,6)
8	$\mathrm{sp}\langle 6,3,1\rangle(\mathrm{T}_4)-3''$	(0,2,4,0,2)	(4,5,5,4,5,5)
9	$\mathrm{sp}\langle 3,2,5\rangle(\mathrm{T}_5)-2'$	(1,2,1,2,2)	(3,6,4,7,4,7)
10	$\mathrm{sp}\langle 3,2,5\rangle(\mathrm{T}_5)-2''$	(0,2,3,0,3)	(4,5,7,5,7)
11	$\mathrm{sp}\langle 3,7,2\rangle(\mathrm{T}_5)-7'$	(0,3,1,1,3)	(4,6,7,4,7)
12	$\mathrm{sp}\langle 8,7,6\rangle(\mathrm{T}_5)-7'$	(0,2,2,2,2)	(4,5,6,6,5)

It is possible to produce more $*$-triangulations by further splitting the $*$-triangulations in Lemma 10.1 that have a unique rope, such as $\mathrm{sp}\langle 6,8,1\rangle(\mathrm{T}_2)$ (Figure 5). Some of the twice-split triangulations are out of Λ, but some of them may have a rope whose contraction yields an IT; for instance,

$$\mathrm{sh}\rangle 6, p\langle\big(\ \mathrm{sp}\langle 6,8',1\rangle\ \big(\ \mathrm{sp}\langle 6,8,1\rangle(\mathrm{T}_2)\ \big)\ \big)\cong \mathrm{T}_{16},$$

where p stands for the $*$-vertex. Thus the latter twice-split triangulations remain in Λ!

Lemma 10.3. *Splitting a $*$-triangulation T^* in Lemma 10.1 (Figure 5) still produces a $*$-triangulation if and only if the splitting is equivalent to: (a) the stellar subdivision of either of the two faces incident with a rope provided that rope is a unique rope of T^*, or (b) the cracking of a rope provided that rope is a unique rope of T^*.*

Proof. The "if" part is obvious. In proving the "only if" part, observe from Figure 5 that there are two cases to consider as follows.

Case "T^* has only one rope". Observe from Figure 5 that the degrees of the end vertices of the rope are 5 or 6. Furthermore, splitting any divider centered at a 5 or 6-valent end vertex certainly destroys the pylonicity of T^* unless it meets condition (a) or (b); the only questionable situation is if the center is 6-valent and the divider has spread 3; then an additional consideration is as follows. There are precisely two 6-valent end vertices of a rope—vertex $4'$ in $\mathrm{sp}\langle 7,4,3\rangle(\mathrm{T}_3)$ and vertex $2'$ in $\mathrm{sp}\langle 3,2,5\rangle(\mathrm{T}_5)$. By inspection, we verify that the following six triangulations produced by splitting a 3-divider are not pylonic:

$$\mathrm{sp}\langle 1,4',7\rangle\big(\ \mathrm{sp}\langle 7,4,3\rangle(\mathrm{T}_3)\ \big),$$

$$\mathrm{sp}\langle 2, 4', 4''\rangle \left(\ \mathrm{sp}\langle 7, 4, 3\rangle(\mathrm{T}_3) \ \right),$$
$$\mathrm{sp}\langle 6, 4', 3\rangle \left(\ \mathrm{sp}\langle 7, 4, 3\rangle(\mathrm{T}_3) \ \right),$$
$$\mathrm{sp}\langle 6, 2', 5\rangle \left(\ \mathrm{sp}\langle 3, 2, 5\rangle(\mathrm{T}_5) \ \right),$$
$$\mathrm{sp}\langle 4, 2', 2''\rangle \left(\ \mathrm{sp}\langle 3, 2, 5\rangle(\mathrm{T}_5) \ \right),$$
$$\mathrm{sp}\langle 1, 2', 3\rangle \left(\ \mathrm{sp}\langle 3, 2, 5\rangle(\mathrm{T}_5) \ \right).$$

Case "T^* has precisely two ropes". In fact, there are only three such triangulations in Figure 5: $T^* \in \{\mathrm{sp}\langle 7, 8, 4\rangle(\mathrm{T}_3), \mathrm{sp}\langle 6, 3, 1\rangle(\mathrm{T}_4), \mathrm{sp}\langle 3, 2, 1\rangle(\mathrm{T}_5)\}$. Observe from Figure 5 that each T^* contains precisely two ropes—which we denote by $[p, y]$ and $[p, z]$—and also observe that the degree of the central $*$-vertex p is 5 or 6, and that $\mid y, p, z \mid \geq 2$. Assume that splitting $\mathrm{sp}\langle u, v, w\rangle$ of T^* produces a $*$-triangulation. Then, necessarily, $v = p$, since otherwise the newly produced edge $[v', v'']$ would not be incident with the $*$-vertex p. Furthermore, it is not hard to prove that for preserving the pylonicity property, it is necessary that $\langle u, v, w\rangle = \langle u, p, w\rangle$ is a 3-divider that does not cross $\langle y, p, z\rangle$ at p and is edge disjoint from $\langle y, p, z\rangle$. Since $\mid y, p, z \mid \geq 2$, such a situation is theoretically possible but requires the degree of p to be at least 7.   □

By Corollary 4.6, there are not any other $*$-triangulations that belong to Λ_3 or can be obtained from a member of Λ_3 by splitting. By Corollary 4.2, the corresponding ITPTs are obtained from Figure 5: (a) by the removal of either of the two faces incident with a single rope, and (b) by the removal of a single rope. In case (a), we have to inspect the sixteen triangulations obtained by the face removal from the eight single-roped triangulations in Lemma 10.1. These sixteen are naturally paired with each other in Table VII. The triangulations in each pair $\sharp 1 - 6$ are isomorphic, which is verified straightforwardly. The ones in pair $\sharp 7$ are not isomorphic because they have differing bd-sequences (check with Table VII). Moreover, as shown in the next paragraph, the ones in pair $\sharp 8$ are not isomorphic even though they have the same d-vector and the same bd-sequence.

To show that the triangulations $\mathrm{sp}\langle 3, 2, 5\rangle(\mathrm{T}_5) - [5, 2', 2'']$ and $\mathrm{sp}\langle 3, 2, 5\rangle(\mathrm{T}_5) - [3, 2', 2'']$ in pair $\sharp 8$ of Table VII are not isomorphic, we observe on one hand that any such isomorphism would fix the single rope $[2', 2'']$, swapping the apices 5 and 3. On the other hand, the degrees of the neighboring vertices are ordered differently in the links of vertices 5 and 3, and hence vertices 5 and 3 are non-similar in $\mathrm{sp}\langle 3, 2, 5\rangle(\mathrm{T}_5)$ and hence no such isomorphism is possible.

There are no more isomorphic pairs in Table VII except the above-mentioned six pairs because the rest of the pairs have differing d-vectors or bd-sequences (except pair $\sharp 8$). Therefore, Table VII provides Series 5 of 10 non-isomorphic ITPTs.

In case (b) we obtain eight more ITPTs from the single-roped triangulations in Figure 5 by the rope removal. However, four of them are isomorphic to the ones originated from Λ_2 and already present in Series 3 (Table II):

$$\mathrm{sp}\langle 6, 8, 1\rangle(\mathrm{T}_2) - [8', 8''] \cong \mathrm{T}_{16} - i,$$
$$\mathrm{sp}\langle 7, 4, 3\rangle(\mathrm{T}_3) - [4', 4''] \cong \mathrm{T}_{17} - o,$$
$$\mathrm{sp}\langle 5, 8, 6\rangle(\mathrm{T}_3) - [8', 8''] \cong \mathrm{T}_{14} - g,$$
$$\mathrm{sp}\langle 8, 7, 6\rangle(\mathrm{T}_5) - [7', 7''] \cong \mathrm{T}_{12} - d.$$

The remaining four form Series 6 of ITPTs, collected in Table VIII; those four are pairwise non-isomorphic because they have differing d-vectors.

Case 4. Parent triangulation T^* is in Λ_4 or can be obtained from a member of Λ_4 by a sequence of splittings.

By Eq. (1), if $T^* = \mathrm{sp}\langle u, v, w\rangle(\mathrm{T}_i) \in \Lambda_4$, then $\langle u, v, w\rangle$ is a 4-divider. A comprehensive list of non-similar 4-dividers can be generated by a sort of inclusion-exclusion technique like that introduced in Section 9. We omit the details here since it is not very hard to determine such a list directly, using Table I. There are, in total, 42 non-similar 4-dividers in the list. They are collected in Table IX in the form of the corresponding splittings of the corresponding ITs. The resulting split-up triangulations collectively form the whole set Λ_4 (with isomorphic duplications). It is a matter of a routine inspection to verify the following.

Table VII. ITPTs: Series 5.

No. of ITPT	No. of pair	Triangulation	d-vector	bd-sequence
1	1	$\mathrm{sp}\langle 1,7,6\rangle(\mathrm{T}_1) - [6,7',7''] \cong$ $\cong \mathrm{sp}\langle 1,7,6\rangle(\mathrm{T}_1) - [1,7',7'']$	(0,0,2,4,2)	(5,5,7)
2	2	$\mathrm{sp}\langle 6,8,1\rangle(\mathrm{T}_2) - [6,8',8''] \cong$ $\cong \mathrm{sp}\langle 6,8,1\rangle(\mathrm{T}_2) - [1,8',8'']$	(0,0,2,5,2,0)	(5,5,7)
3	3	$\mathrm{sp}\langle 7,4,3\rangle(\mathrm{T}_3) - [7,4',4''] \cong$ $\cong \mathrm{sp}\langle 7,4,3\rangle(\mathrm{T}_3) - [3,4',4'']$	(0,0,3,3,3,0)	(5,6,7)
4	4	$\mathrm{sp}\langle 2,7,4\rangle(\mathrm{T}_3) - [4,7',7''] \cong$ $\cong \mathrm{sp}\langle 2,7,4\rangle(\mathrm{T}_3) - [2,7',7'']$	(0,0,4,3,0,2)	(5,5,8)
5	5	$\mathrm{sp}\langle 3,7,2\rangle(\mathrm{T}_5) - [3,7',7''] \cong$ $\cong \mathrm{sp}\langle 3,7,2\rangle(\mathrm{T}_5) - [2,7',7'']$	(0,1,4,0,2,2)	(5,5,8)
6	6	$\mathrm{sp}\langle 8,7,6\rangle(\mathrm{T}_5) - [6,7',7''] \cong$ $\cong \mathrm{sp}\langle 8,7,6\rangle(\mathrm{T}_5) - [8,7',7'']$	(0,1,2,2,4,0)	(5,5,6)
7	7	$\mathrm{sp}\langle 5,8,6\rangle(\mathrm{T}_3) - [6,8',8'']$	(0,0,3,3,3,0)	(5,5,6)
8	7	$\mathrm{sp}\langle 5,8,6\rangle(\mathrm{T}_3) - [5,8',8'']$	(0,0,3,3,3,0)	(5,5,7)
9	8	$\mathrm{sp}\langle 3,2,5\rangle(\mathrm{T}_5) - [5,2',2'']$	(0,1,3,2,1,2)	(5,8,6)
10	8	$\mathrm{sp}\langle 3,2,5\rangle(\mathrm{T}_5) - [3,2',2'']$	(0,1,3,2,1,2)	(5,8,6)

Table VIII. ITPTs: Series 6.

No.	Triangulation	d-vector	bd-sequence
1	$\mathrm{sp}\langle 1,7,6\rangle(\mathrm{T}_1) - [7',7'']$	(0,2,0,4,2)	(7,4,7,4)
2	$\mathrm{sp}\langle 2,7,4\rangle(\mathrm{T}_3) - [7',7'']$	(0,2,2,3,0,2)	(4,8,4,8)
3	$\mathrm{sp}\langle 3,2,5\rangle(\mathrm{T}_5) - [2',2'']$	(0,2,3,1,1,2)	(4,8,5,8)
4	$\mathrm{sp}\langle 3,7,2\rangle(\mathrm{T}_5) - [7',7'']$	(0,3,2,0,2,2)	(4,8,4,8)

11. The search for ITPTs: Case 4

Lemma 11.1. *None of the triangulations in Λ_4 are pylonic.*

Thus, by Lemma 11.1 and Corollaries 4.2, 4.6, there are no ITPTs which can be derived in Case 4.

Proof of Lemma 4.5: It is not hard to prove that a Δ may occur only after the second consecutive splitting of an IT and only if the first applied splitting produces a single rope, ε. We know from the above that a single rope may only occur as described above in the proof of Lemma 10.3 (case "T^* has only one rope"), in

which event the degrees of the end vertices of ε are necessarily equal to 5 or 6, but not both equal to 6. Then, it can be easily seen that the second splitting may lead to a Δ only if it is the splitting of the 3-divider that contains ε and is centered at the 6-valent end vertex of ε. There are only two such second splittings (check with Figure 5), $\mathrm{sp}\langle 2, 4', 4''\rangle$ ($\mathrm{sp}\langle 7, 4, 3\rangle(\mathrm{T}_3)$) and $\mathrm{sp}\langle 4, 2', 2''\rangle$ ($\mathrm{sp}\langle 3, 2, 5\rangle(\mathrm{T}_5)$), but neither has a Δ.

Table IX. Λ_4 with isomorphic duplications.

Triangulations				Total
$\mathrm{sp}\langle 4, 8, 3\rangle(\mathrm{T}_8)$	$\mathrm{sp}\langle 5, 8, 9\rangle(\mathrm{T}_8)$	$\mathrm{sp}\langle 1, 8, 7\rangle(\mathrm{T}_8)$		3
$\mathrm{sp}\langle 1, 4, 5\rangle(\mathrm{T}_9)$	$\mathrm{sp}\langle 3, 4, 8\rangle(\mathrm{T}_9)$	$\mathrm{sp}\langle 7, 4, 6\rangle(\mathrm{T}_9)$	$\mathrm{sp}\langle 9, 4, 2\rangle(\mathrm{T}_9)$	4
$\mathrm{sp}\langle 1, 4, 5\rangle(\mathrm{T}_{10})$	$\mathrm{sp}\langle 2, 4, 9\rangle(\mathrm{T}_{10})$	$\mathrm{sp}\langle 8, 4, 3\rangle(\mathrm{T}_{10})$		3
$\mathrm{sp}\langle 1, 3, 2\rangle(\mathrm{T}_{14})$	$\mathrm{sp}\langle 4, 3, 8\rangle(\mathrm{T}_{14})$	$\mathrm{sp}\langle 6, 3, 5\rangle(\mathrm{T}_{14})$		3
$\mathrm{sp}\langle 2, 3, 1\rangle(\mathrm{T}_{15})$	$\mathrm{sp}\langle 8, 3, 4\rangle(\mathrm{T}_{15})$	$\mathrm{sp}\langle 9, 3, 7\rangle(\mathrm{T}_{15})$	$\mathrm{sp}\langle 5, 3, 6\rangle(\mathrm{T}_{15})$	4
$\mathrm{sp}\langle 2, 3, 1\rangle(\mathrm{T}_{16})$	$\mathrm{sp}\langle 8, 3, 4\rangle(\mathrm{T}_{16})$	$\mathrm{sp}\langle 5, 3, 6\rangle(\mathrm{T}_{16})$		3
$\mathrm{sp}\langle 1, 3, 2\rangle(\mathrm{T}_{17})$	$\mathrm{sp}\langle 4, 3, 8\rangle(\mathrm{T}_{17})$	$\mathrm{sp}\langle 7, 3, 9\rangle(\mathrm{T}_{17})$	$\mathrm{sp}\langle 6, 3, 5\rangle(\mathrm{T}_{17})$	4
$\mathrm{sp}\langle 1, 4, 9\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 2, 4, 8\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 6, 4, 7\rangle(\mathrm{T}_{18})$	$\mathrm{sp}\langle 5, 4, 3\rangle(\mathrm{T}_{18})$	4
$\mathrm{sp}\langle 1, 4, 9\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 2, 4, 8\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 6, 4, 7\rangle(\mathrm{T}_{19})$	$\mathrm{sp}\langle 5, 4, 3\rangle(\mathrm{T}_{19})$	4
$\mathrm{sp}\langle 1, 4, 9\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 2, 4, 8\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 6, 4, 7\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 5, 4, 3\rangle(\mathrm{T}_{20})$	
$\mathrm{sp}\langle 9, 3, 7\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 5, 3, 6\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 1, 3, 2\rangle(\mathrm{T}_{20})$	$\mathrm{sp}\langle 4, 3, 8\rangle(\mathrm{T}_{20})$	8
$\mathrm{sp}\langle 2, 10, 3\rangle(\mathrm{T}_{21})$	$\mathrm{sp}\langle 6, 10, 8\rangle(\mathrm{T}_{21})$			2

12. Concluding theorem

Combining the results in the previous sections, we have identified a total of 297 non-isomorphic ITPTs as detailed in the following summarizing theorem.

Theorem 12.1. *Up to isomorphism, there are totally 297 irreducible triangulations of the once-punctured torus. They are presented in six series as follows: 80 triangulations in Series 1 (Section 6), 129 triangulations in Series 2 (Section 7), 62 triangulations in Series 3 (Section 8), and 12, 10, and 4 triangulations in (respectively) Series 4 in Table VI, Series 5 in Table VII, and Series 6 in Table VIII (all in Section 10).*

Proof. The considerations of Sections 6–11, along with Corollaries 4.2 and 4.6, guarantee that we have not missed any ITPT in the search. It remains to show that all the ITPTs we have found are pairwise non-isomorphic.

We have verified in Section 6 that Series 1 contains no isomorphic pairs, nor do Series 2 (Section 7), Series 3 (Section 8), nor any of Series 4–6 (Section 10). It remains to prove that Series 1–6 are pairwise disjoint from each other.

By Corollary 5.2, Series 2 (obtained by deleting the $*$-vertex from the members of Λ_1), Series 3 (from Λ_2), and Series 4 (from Λ_3) are pairwise disjoint. Clearly, each of these three Series is disjoint from Series 1 (produced immediately from the toroidal ITs). Furthermore, each of the triangulations in Series 5 and 6, with the patch restored, can be produced from some IT by two, but not by one, consecutive splittings, hence the triangulations resulting from these splittings collectively are out of Λ, and hence Series 5 and 6 are disjoint from the other Series. Finally, the triangulations in the union of Series 5 and 6 have pairwise differing d-vectors or bd-sequences (except the non-isomorphic pair $\sharp 8$ in Series 5 considered in Section 10) and thus are all non-isomorphic.   □

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