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Some remarks on the comparison principle in Kirchhoff equations

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Abstract. In this paper we study the validity of the comparison principle and the sub-supersolution method for Kirchhoff type equations. We show that these principles do not work when the Kirchhoff function is increasing, contradicting some previous results. We give an alternative sub-supersolution method and apply it to some models.

1. Introduction

In the last years the nonlinear elliptic Kirchhoff equation has attracted much attention, see for instance [12], [13], [15], [16], and references therein. The equation has the following general form:

$$\text{(1.1)} \qquad \begin{cases} -M(\|u\|^2)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded and regular domain,

$$\|u\|^2 := \int_\Omega |\nabla u|^2 \, dx,$$

where M is a continuous function verifying

$$(M_0) \qquad \begin{array}{l} M : [0,+\infty) \mapsto [0,+\infty) \\ \text{and } \exists m_0 > 0 \text{ such that } M(t) \geq m_0 > 0 \; \forall t \in [0,+\infty), \end{array}$$

and where $f \in C(\overline{\Omega} \times \mathbb{R})$. We assume (M_0) along the paper.

Problem (1.1) models small vertical vibrations of an elastic string with fixed ends when the density of the material is homogeneous and there is a external force, see [9] for a explication of the model. To study this problem different methods

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have been used, mainly variational methods and fixed point arguments, and also bifurcation and sub-supersolution.

In this note, we have two main objectives. On the one hand, we present some examples demonstrating that some comparison results and those based on the sub-supersolution method appearing in the literature are not correct. On the other hand, we prove a sub-supersolution method that includes the above ones, specifically those in [3], [4], and [14].

An outline of the paper is as follows: in Section 2 we recall the previous results related to comparison and sub-supersolution method, and we present our main result. In Section 3 we give some counterexamples showing that some comparison results are not correct. Section 4 is devoted to prove our main result, and in Section 5 we apply our result to some specific examples.

2. Previous and main results

To our knowledge, there are basically three results concerning the comparison and sub-supersolution results related to (1.1). Let us recall them. In [3] (Theorems 2 and 3) the following result was proved:

Theorem 2.1. *Assume that:*

(M_1) *M is non-increasing in $[0,+\infty)$.*

(H) *Define the function*

$$H(t) := M(t^2)\,t\,,$$

and assume that H is increasing and $H(\mathbb{R}) = \mathbb{R}$.

Then:

a) *If there exist two non-negative functions $\underline{u}, \overline{u} \in C^2(\overline{\Omega})$ such that $\underline{u} = \overline{u} = 0$ on $\partial\Omega$ and*

$$-M(\|\underline{u}\|^2)\Delta\underline{u} \le -M(\|\overline{u}\|^2)\Delta\overline{u} \quad \text{in } \Omega, \tag{2.1}$$

then (comparison principle)

$$\underline{u} \le \overline{u} \quad \text{in } \overline{\Omega}.$$

b) *If*

(f_1) *f is increasing in the variable u for each $x \in \Omega$ fixed,*

and there exist two regular functions $0 \le \underline{u} \le \overline{u}$ in Ω, $\underline{u} = \overline{u} = 0$ on $\partial\Omega$ satisfying

$$-M(\|\overline{u}\|^2)\Delta\overline{u} \ge f(x,\overline{u}), \quad -M(\|\underline{u}\|^2)\Delta\underline{u} \le f(x,\underline{u}), \quad \text{in } \Omega. \tag{2.2}$$

then (sub-supersolution method) there exists a solution u of (1.1) such that $\underline{u} \le u \le \overline{u}$ in Ω.

In [14] (Theorems 3.2 and 3.3), se also [8], the authors proved a similar result to Theorem 2.1 in the case that M is increasing.

Theorem 2.2. *Assume that*

(M_2) *M is increasing.*

Then, the comparison principle holds. Moreover, if f satisfies (f_1), the sub-supersolution method also works.

Finally, in [4] the following result is shown:

Theorem 2.3. *Assume that M satisfies (M_2) and*

(f_2) *f is a positive function.*

If there exist $\overline{u} \in W^{1,\infty}(\Omega)$, $\overline{u} \geq 0$ on $\partial\Omega$, and a family $(\underline{u}_\delta) \subset W_0^{1,\infty}(\Omega)$ such that

$$-m_0\, \Delta\overline{u} \geq f(x, \overline{u}), \tag{2.3}$$

$\|\underline{u}_\delta\|_{1,\infty} \to 0$ as $\delta \to 0$, $\underline{u}_\delta \leq \overline{u}$ in Ω for δ small enough, and given $\alpha > 0$, there is δ_0 such that

$$-\Delta\underline{u}_\delta \leq \frac{1}{\alpha} f(x, \underline{u}_\delta), \quad \text{for } \delta \leq \delta_0, \tag{2.4}$$

then there is a small enough $\delta > 0$ such that there exists a solution u of (1.1) such that $\underline{u}_\delta \leq u \leq \overline{u}$ in Ω.

Of course, the above inequalities (2.3) and (2.4) are considered in the weak sense.

Our main result reads as follows:

Theorem 2.4. *Assume (M_0) and*

(M_3) *$G(t) := M(t)\, t$ is invertible and denote by $R(t) := G^{-1}(t)$.*

Define now the non-local operator

$$\mathcal{R}(w) := R\Big(\int_\Omega f(x, w)\, w\, dx\Big).$$

If there exist $\underline{u}, \overline{u} \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $\underline{u} \leq \overline{u}$ in Ω, $\underline{u} \leq 0 \leq \overline{u}$ on $\partial\Omega$ satisfying

$$-M(\mathcal{R}(w))\Delta\overline{u} \geq f(x, \overline{u}), \quad -M(\mathcal{R}(w))\Delta\underline{u} \leq f(x, \underline{u}), \quad \forall w \in [\underline{u}, \overline{u}], \tag{2.5}$$

then there exists a solution u of (1.1) such that $\underline{u} \leq u \leq \overline{u}$ in Ω.

Remark 2.5. (1) We would like to remark that Theorem 2.4 does not assume (f_1).

(2) Observe that condition (2.5) involves all the functions w in the interval $[\underline{u}, \overline{u}]$. So, in order to verify for instance the first inequality in (2.5), we have to take into account the sign of $-\Delta\overline{u}$ and the monotonicity of the map $w \mapsto M(\mathcal{R}(w))$. See Section 5 for applications of this result.

In Section 3, we show that Theorem 2.2 is not correct. Now, we deduce Theorems 2.1 b) and 2.3 from Theorem 2.4.

Corollary 2.6. *Assume that M is smooth, (M_1), (H), (f_1) and that there exist a sub-supersolution in the sense of Theorem 2.1. Then, there exists a solution u of (1.1) such that $\underline{u} \le u \le \overline{u}$ in Ω.*

Proof. Observe that if M is non-increasing and H is increasing, then G is increasing. Indeed, using that $M' \le 0$ we get that

$$G'(t) = M'(t)t + M(t) \ge 2t\,M'(t) + M(t) = H'(t^{1/2}) > 0, \quad t > 0,$$

and then, $w \mapsto \mathcal{R}(w)$ is increasing because f is also increasing.

Consider now that $\underline{u}$, $\overline{u}$ is a sub-supersolution in the sense of Theorem 2.1. We are going to show that it is also sub-supersolution in the sense of Theorem 2.4. We show this fact for $\underline{u}$; for $\overline{u}$ we can apply an analogous reasoning. Since f is increasing and $0 \le \underline{u} \le \overline{u}$, we have that

$$M(\|\underline{u}\|^2)\|\underline{u}\|^2 \le \int_\Omega f(x,\underline{u})\underline{u} \le \int_\Omega f(x,w)w \quad \forall w \in [\underline{u},\overline{u}],$$

and so,

$$\|\underline{u}\|^2 \le \mathcal{R}(w) \Longrightarrow M(\|\underline{u}\|^2) \ge M(\mathcal{R}(w)) \quad \forall w \in [\underline{u},\overline{u}].$$

Hence,

$$-\Delta \underline{u} \le \frac{f(x,\underline{u})}{M(\|\underline{u}\|^2)} \le \frac{f(x,\underline{u})}{M(\mathcal{R}(w))} \quad \forall w \in [\underline{u},\overline{u}].$$

Then, $(\underline{u},\overline{u})$ satisfies the hypotheses of Theorem 2.4 and we can conclude the existence of a solution u of (1.1) and $u \in [\underline{u},\overline{u}]$. □

Corollary 2.7. *Assume (M_2), (H), (f_2) and that there exists a sub-supersolution in the sense of Theorem 2.3. Then, there exists a solution u of (1.1) such that $\underline{u} \le u \le \overline{u}$ in Ω.*

Proof. Observe that if M is increasing, then G is increasing. Assume now the existence of a supersolution $\overline{u}$ and family of sub-solution $\underline{u}_\delta$ in the sense of Theorem 2.3. Then,

$$-\Delta \overline{u} \ge \frac{f(x,\overline{u})}{m_0} \ge \frac{f(x,\overline{u})}{M(\mathcal{R}(w))} \quad \forall w \in [\underline{u},\overline{u}].$$

Consider now

$$\alpha = \max_{0 \le w \le \overline{u}} M(\mathcal{R}(w)),$$

and take $\underline{u} = \underline{u}_\delta$ for some $\delta \le \delta_0$ given by Theorem 2.3. Then, using that $f \ge 0$,

$$-\Delta \underline{u} \le \frac{1}{\alpha} f(x,\underline{u}) \le \frac{1}{M(\mathcal{R}(w))} f(x,\underline{u}) \quad \forall w \in [\underline{u},\overline{u}],$$

and so, $\underline{u}$, $\overline{u}$ is sub-supersolution in the sense of Theorem 2.4. Theorem 2.4 concludes the result. □

3. Counterexamples

In this section we have two objectives: to show that when M satisfies only (M_2), or M satisfies only (M_1) (Cases 1 and 2 below, respectively), the comparison principle and the sub-supersolutions fail.

For that consider

$$\Omega = (0, \pi), \quad u_1 := \sin(x), \quad u_2 := x(\pi - x),$$

and

$$M(t) := a + b(t + c)^p, \quad c \geq 0, a, b > 0, p \in \mathbb{R}. \tag{3.1}$$

Observe that

$$-\Delta u_1 = u_1, \quad -\Delta u_2 = 2, \quad \|u_1\|^2 = \frac{\pi}{2}, \quad \|u_2\|^2 = \frac{\pi^3}{3},$$

and

$$\max_{x \in [0,\pi]} \frac{u_1(x)}{u_2(x)} = \max_{x \in [0,\pi]} \frac{\sin(x)}{x(\pi - x)} = \frac{4}{\pi^2} := \rho^* \simeq 0.4083$$

In order to prove that the comparison principle fails, we take $\overline{u} = \rho u_2$, $\rho > 0$ and $\underline{u} = u_1$. So,

$$-M(\|\underline{u}\|^2)\Delta \underline{u} \leq -M(\|\overline{u}\|^2)\Delta \overline{u} \quad \text{for all } x \in (0, \pi)$$

if and only if

$$M(\pi/2) \leq 2\rho\, M\Big(\rho^2\, \frac{\pi^3}{3}\Big). \tag{3.2}$$

Hence, if

$$\rho < \rho^* \quad \text{and} \quad M(\pi/2) \leq 2\rho\, M\Big(\rho^2\, \frac{\pi^3}{3}\Big), \tag{3.3}$$

then the comparison principle fails.

To prove that the sub-supersolutions method fails, we consider the problem

$$\begin{cases} -M(\|u\|^2)\Delta u = f(x) = \sin(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Observe that the solutions of (3.4) are

$$u_0 = k \sin(x),$$

with k satisfying

$$kM\left(k^2\, \frac{\pi}{2}\right) = 1. \tag{3.5}$$

Consider the pair $(\overline{u}, \underline{u}) = (\rho u_2, 0)$. Then there is a pair of sub-supersolution of (3.4) in the sense of Theorem 2.2 provided that

$$2\rho\, M\left(\rho^2\,\frac{\pi}{2}\right) \geq 1. \tag{3.6}$$

Then, if the method is applicable, then there exists a solution u_0 of (3.4) such that

$$0 \leq u_0 \leq \overline{u} = \rho u_2 \quad \text{in } \Omega.$$

Hence, if

$$\rho < k\rho^* \quad \text{and} \quad 2\rho M\left(\rho^2\,\frac{\pi}{2}\right) \geq 1, \tag{3.7}$$

where k satisfies (3.5), then the sub-supersolution method fails.

Case 1. *M is increasing.* Consider in this case $c = 0$, $a > 0$ and $p > 0$ in (3.1). Then, (3.3) is equivalent to

$$a + b\left(\frac{\pi}{2}\right)^p \leq 2\rho\Big(a + b\Big(\rho^2\,\frac{\pi^3}{3}\Big)^p\Big) \quad \text{and} \quad \rho < \rho^*.$$

Taking b large, we need that for some $\rho < \rho^*$,

$$\frac{1}{2}\frac{1}{\left(\pi^2\frac{2}{3}\right)^p} < \rho^{1+2p}.$$

By continuity, it is enough that the above inequality holds for $\rho = \rho^*$, that is,

$$\frac{1}{2}\frac{1}{\left(\pi^2\frac{2}{3}\right)^p} < \left(\frac{4}{\pi^2}\right)^{1+2p} \Longleftrightarrow 1 < \frac{8}{\pi^2}\left(\frac{32}{3\pi^2}\right)^p,$$

which is true for p large.

Now, we analyze the sub-supersolution method. First, observe that since M is increasing, (3.4) possesses a unique solution, $u_0 = k\sin(x)$, where k satisfies

$$k\Big(a + b\Big(k^2\frac{\pi}{2}\Big)^p\Big) = 1. \tag{3.8}$$

Observe that if $a \to 0$, then

$$k(a) \to \left(\frac{1}{b}\right)^{1/(2p+1)}\left(\frac{2}{\pi}\right)^{p/(2p+1)}.$$

Take

$$p > \frac{\ln(\pi^2/8)}{\ln(32/(3\pi^2))}.$$

Then there exists $a_0 > 0$ such that for all $a \in (0, a_0)$ we have

$$\frac{a + b\left(\frac{16k^2}{3\pi}\right)^p}{a + b\left(\frac{k^2\pi}{2}\right)^p} > \frac{\pi^2}{8}. \tag{3.9}$$

Hence, $\overline{u}$ is supersolution if (see (3.6))

$$2\rho\Big(a+b\Big(\frac{\rho^2\pi^3}{3}\Big)^p\Big)\geq 1.$$

By continuity, (3.7) is verified for some $\rho<\rho^*$ if

$$2\,\frac{4}{\pi^2}\,k\Big(a+b\Big(\frac{(\frac{4}{\pi^2}\,k)^2\,\pi^3}{3}\Big)^p\Big)>1,$$

which is equivalent, using (3.8), to (3.9).

Case 2. *M is decreasing.* Take in this case $a,c>0$ and $p<0$. In this case, (3.3) is equivalent to

$$a(1-2\rho)\leq b\left(2\rho\left(\rho^2\,\frac{\pi^3}{3}+c\right)^p-\left(\frac{\pi}{2}+c\right)^p\right)\quad\text{and}\quad\rho<\rho^*. \tag{3.10}$$

Take $\rho^2<\frac{3}{2\pi^2}$. Then

$$\frac{\pi/2+c}{\rho^2\,\pi^3/3+c}>1. \tag{3.11}$$

For this ρ fixed, take p such that

$$2\rho>\Big(\frac{\pi/2+c}{\rho^2\,\pi^3/3+c}\Big)^p, \tag{3.12}$$

or equivalently,

$$2\rho\Big(\rho^2\,\frac{\pi^3}{3}+c\Big)^p>\Big(\frac{\pi}{2}+c\Big)^p.$$

Now, take b large enough to have (3.10).

Now, we show that the sub-supersolution method does not work for M only satisfying (M_1). Take ρ and p such that (3.11) and (3.12) are satisfied. On the other hand, take b such that $b(\pi/2+c)^p>1$, and then for all $a>0$ we have

$$a+b\left(\frac{\pi}{2}+c\right)^p>1. \tag{3.13}$$

Then, thanks to (3.13), there exists at least a positive solution of (3.5), that is,

$$k\left(a+b\left(k^2\,\frac{\pi}{2}+c\right)^p\right)=1,$$

and then, there exists at least a positive solution of (3.4). Observe that in this case, we can not assure that (3.4) has a unique solution. On the other hand, since $p<0$, observe that

$$k\geq\frac{1}{a+b\,c^p}. \tag{3.14}$$

In order to verify (3.7) we need

$$2\rho\Big(a+b\Big(\rho^2\,\frac{\pi^3}{3}+c\Big)^p\Big)\geq 1 \quad\text{and}\quad \rho<k\,\frac{4}{\pi^2}.$$

Now, we prove that

$$2\rho\Big(a+b\Big(\rho^2\,\frac{\pi^3}{3}+c\Big)^p\Big)>a+b\left(\frac{\pi}{2}+c\right)^p$$

whence we conclude the result from (3.13). This is equivalent to

$$b\left[2\rho\left(\rho^2\,\frac{\pi^3}{3}+c\right)^p-\left(\frac{\pi}{2}+c\right)^p\right]>a(1-2\rho),$$

which is true taking a small and positive. Finally, we need to verify $\rho<4k/\pi^2$. But, observe that by (3.14), we have that, for a small, the above inequality is true.

Remark 3.1. The above example shows that Theorem 3.3 in [14] (which assures that the sub-supersolution method and the comparison principle work), see also Theorem 2.3 in [11], seems not correct. Hence, the existence results of, for instance, [1], [2], [6], have been obtained using a method that fails.

4. Proof of Theorem 2.4

First, we are going to transform our equation (1.1) into another non-local elliptic equation. Indeed, multiplying (1.1) by u and integrating, we get

$$M(\|u\|^2)\|u\|^2=\int_\Omega f(x,u)\,u\,dx.$$

By (H_3), G is invertible, and so

$$\|u\|^2=R\Big(\int_\Omega f(x,u)\,u\,dx\Big)=\mathcal{R}(u).$$

Then, (1.1) is equivalent to the problem

$$\begin{cases} -\Delta u=\dfrac{f(x,u)}{M(\mathcal{R}(u))} & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Observe that (4.1) is a non-local elliptic equation, without terms in $\|u\|$, and so it suffices to apply Theorem 3.2 in [7]. This completes the proof.

In the following result, we prove a specific comparison principle which is valid when M only satisfies hypothesis (H) and the second member of the equation is constant. Define e to be the unique positive solution of the equation

$$\begin{cases} -\Delta e=1 & \text{in } \Omega,\\ e=0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Lemma 4.1. *Assume that M satisfies (H) and let $u_i \in H_0^1(\Omega)$, $i = 1, 2$, be functions such that*

$$-M(\|u_i\|^2)\Delta u_i = f_i \in \mathbb{R}_+$$

and $f_1 \le f_2$. Then, $u_1 \le u_2$ in Ω.

Proof. Observe that

$$M(\|u_i\|^2)u_i = f_i\, e, \tag{4.3}$$

and then $u_1 \le u_2$ if and only if

$$\frac{f_1}{M(\|u_1\|^2)} \le \frac{f_2}{M(\|u_2\|^2)}. \tag{4.4}$$

But observe that, from (4.3),

$$f_i = \frac{M(\|u_i\|^2)\|u_i\|}{\|e\|},$$

and then (4.4) is equivalent to

$$\|u_1\| \le \|u_2\|. \tag{4.5}$$

Since $M(\|u_i\|^2)\|u_i\| = f_i\|e\|$ and due to (H), (4.5) follows. □

Remark 4.2. We would like to emphasize that Lemma 4.1 is only true when f_i, $i = 1, 2$ are real numbers.

5. Applications

In this section we apply our result to some models. We only assume that M satisfies (M_3). Denote by $\lambda_1 > 0$ the principal eigenvalue of the Laplacian and $\varphi_1 > 0$ the eigenfunction associated to it with $\|\varphi_1\|_\infty = 1$.

Example 1. Consider the equation

$$\begin{cases} -M(\|u\|^2)\Delta u = \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\lambda \in \mathbb{R}$ and $0 < q < 1$. This problem was analyzed in [3] when M satisfies (M_1) and (H). We are going to show that (5.1) possesses a positive solution if and only if $\lambda > 0$. From the maximum principle, if $\lambda \le 0$ problem (5.1) does not have any positive solution. Assume $\lambda > 0$ and take as sub-supersolutions $\underline{u} = \varepsilon\varphi_1$ and $\overline{u} = Ke$ with $\varepsilon, K > 0$ to be chosen. Then, $\overline{u}$ is supersolution if

$$K^{1-q} \ge \frac{1}{m_0}\lambda\,\|e\|_\infty^q.$$

Fix such K. Then, $\underline{u}$ is subsolution if

$$M(\mathcal{R}(w))\,\varepsilon^{1-q} \le \frac{\lambda}{\lambda_1}, \quad \forall w \in [\underline{u}, \overline{u}].$$

It is enough to take ε small such that the above inequality holds and that $\underline{u} \le \overline{u}$.

Example 2. Consider the classical concave-convex equation

$$\begin{cases} -M(\|u\|^2)\Delta u = \lambda u^q + u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where $\lambda \in \mathbb{R}$ and $0 < q < 1 < p$. Again we assume only that M satisfies (M_3). We show that there exists at least a positive solution for λ small and positive. For that, again take the same sub-supersolution of the above example. We can show that $\overline{u} = Ke$ es supersolution provided that

$$m_0\, K^{1-q} \ge \lambda\, \|e\|_\infty^q + K^{p-q}\, \|e\|_\infty^p.$$

Then there exists $\lambda_0 > 0$ such that, for $\lambda \in (0, \lambda_0)$, there exists K_0 such that $\overline{u} = K_0 e$ is supersolution.

Now, $\underline{u} = \varepsilon\varphi_1$ is subsolution provided that

$$M(\mathcal{R}(w))\,\varepsilon^{1-q}\,\lambda_1 \le \lambda + \varepsilon^{p-q}\,\varphi_1^{p-q}, \quad \forall w \in [\underline{u}, \overline{u}].$$

It suffices again to take ε small.

Example 3. Consider now the logistic equation

$$\begin{cases} -M(\|u\|^2)\Delta u = \lambda u - u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

where $\lambda \in \mathbb{R}$ and $1 < p$. In this example we assume, in addition to (M_3), that there exist two positive constants m_0, m_∞ such that

$$m_0 \le M \le m_\infty. \tag{5.4}$$

This equation was studied in [5] when $M = M(u)$ is a continuous function from $L^p(\Omega)$ into $\mathbb{R}$ that satisfies (5.4); however, they do not assume (M_3). They used a fixed point argument and showed the existence of positive solution for $\lambda > \lambda_1 m_\infty$ (Theorem 2.1 in [5]).

We obtain a similar result for the Kirchhoff equation (5.3) by the sub-supersolution method. Observe that in this case we can not apply Theorem 2.1. Take $\overline{u} = \lambda$. It is clear that $\overline{u}$ is supersolution. As subsolution, take $\underline{u} = \varepsilon\varphi_1$. Then we need that

$$M(\mathcal{R}(w))\,\lambda_1 + (\varepsilon\varphi_1)^{p-1} \le \lambda, \quad \forall w \in [\underline{u}, \overline{u}].$$

Then there exists at least a positive solution for $\lambda > \lambda_1 m_\infty$.

Remark 5.1. After we have finished and revised the paper, we have known the paper [10], where the authors give some counterexamples showing that equation (1.1) does not enjoy the comparison principle nor the sub supersolutions method.

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