PROSPECTS FOR PRACTICE-BASED PHILOSOPHY OF LOGIC*

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ABSTRACT: We explore prospects for practice-based approaches to logical theory, in particular the link between classical and intuitionistic logic and the inferential structure of traditional practices of representation & argument in science and mathematics. After discussing some key notions about practice, we outline the connection between representation practices and classical logic, and then consider a spectrum of actual practices followed or proposed by (real) scientists. Intuitionistic logic helps to clarify the potential of practice-based approaches for understanding pluralism, and to hammer some key points about the general thesis.
KEYWORDS: pragmatism, practice-based philosophy, mathematical practice, view of logic as model, logical constants, classical logic, intuitionistic logic, pluralism in logic.

1. I recently heard a famous logician and philosopher say that "mathematical practice" seems to be today's "holy cow" among certain people. Orientation towards practice-based philosophy of science, of mathematics, of logic, has been a powerful trend in the last few decades. And, at the very least as complement to previous trends, it ought to be welcomed.

2. Readers are likely to be acquainted with the state of academic affairs in relation to logic, and thus to know that the variety of approaches and systems proposed in the field of logic is immensely broader than in mathematics. Just to give an example: a survey of current views about what might be called "basic logic" provides us with many alternatives – traditionalists will still propose first-order logic; more open minded people shall suggest classical logic enriched by modal operators; more radically, an influential group will say IF logic (Hintikka, Sandu); another will insist on epistemic logic (van Benthem).

This would have been extraordinarily surprising for the generation of Frege and Russell, who thought of logic as The One True Logic, and believed it was more solid and strict than mathematics itself. How do we account for that phenomenon?

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It can be understood by placing logic in the context of its associated practices: unicity emerged from systematisation and formalisation of the inferential skeleton of a traditional body of argumentation practices in science and mathematics. It is coherent to think their unicity inherited the homogeneity of a body of practices, far from being *a priori*. Later, when left to itself and to form associations with all kinds of different practices – linguistic, philosophical, computational, etc. –, logic has shown to be highly underdetermined (at least, much more than math).

3. What is a practice-based approach? Simply put: The attempt to ground logical theory on practices of logical inference; e.g. the attempt to solve the vexing problem of the logical constants by arguing from the character of certain (extra-logical) practices and the features of inference and valid argument within them. This means that one argues for the cogency or even necessity of logical features from the problems and goals set for the practice, and considers those logical features as aspects of the proper methodology of the practice.

The main idea in the sequel is simply that classical logic models the inferential structure of traditional practices of representation (and argument) in science and mathematics. I shall speak of mathematico-scientific practices of representation, or even (for brevity) math-scientific practices. But before coming to the details, some preliminaries seems required.

4. It seems necessary to ask twofold: What is a practice? And, how can normative elements emerge from experience within a practice? Here are preliminary answers:

1. A practice can only be discernible – can only exist – when it has an accepted topic and problems (consider e.g. practices of navigation in the Mediterranean by coasting, or in Micronesia by stars, winds, birds;[1] consider the practice of solving practical construction problems in the plane).

2. With an established practice come accepted solutions (examples: to draw a right angle, employ a triangle of sides 3, 4, 5; memorize stars and their positions relative to islands, think of the ship as static and the islands moving towards you).

3. This means that, explicitly of implicitly, conditions of success for problems have been developed. That naturally leads to criteria for success being established, methods being laid out and taught.

4. The initial ingredients for the articulation of normative elements in the practice have been laid. Subsequently may come systematisation, etc., including articulation of criteria as explicit

[1] This famous example of old Micronesian navigation was studied in (Hutchins 1995).

norms.

Note: More than 'normative', perhaps it is reasonable to see logic as being **regulatory**.

5. Attempts to ground logical theory on practices of logical inference (elsewhere) are not very novel. Corcoran in (Corcoran 1973) proposed to understand classical logic as a mathematical model of the inferential structures present "in the practice" of classical mathematics. What may be novel is the meaning we assign to the expression "mathematical practice", the extent to which we mean an idealized form of practice (textbook kind) or else we mean to consider the spectrum of *actual* practices followed or proposed by (real) research mathematicians.

The difference can be crucial, as the case of Corcoran exemplifies. He has a highly specific notion of what mathematical practice is, and his notion is heavily idealized; to put it simply, his math-practice is rigid and also Platonistic. It is easy to show contrary evidence: one can e. g. point to work by individual mathematicians of the first rank that cannot be squared with Corcoran's notion (even if we avoid Brouwer or Martin-Löf, we may consider H. Weyl, W. Lawvere, or T. Gowers). In actual fact, by the mid-20th cent. "mathematical practice" was an expression employed by intuitionists and constructivists above all.

My "mathematical practices" is in the plural, it is not a logician's idealized notion but a historically rich one; it is not rigid but flexible, hence plural.

6. Since my notion of mathematical practice is not rigid, but flexible, it is a problem to explain the incredibly high level of constraint present in mathematical practice as historically given. The mathematics developed by 95% of experts in the last 200 years (notice however the rest 5% includes some of the Abel prizes) can be understood, with some simplification perhaps, to work on the basis of *tertium exclusum* and *reductio ad absurdum*; to be satisfactorily modelled (except perhaps for some traits that one can ignore in good approximation) in deductive reconstructions within the frame of first-order logic; to assume that there is one single structure of the real numbers (an intended model), which in turn can easily be *interpreted* to rest on mathematical Platonism.[2]

Why has that been so? Why such a level of communality? Perhaps it is because the real development of mathematics is an in-time, imperfect expression of ideal mathematics, of Platonic structures that we approximate more and more as we correct methods and results. This is a beautiful perspective that reminds me not only of Gödel, also of French philosopher Lautman, but I find it too difficult to let it go, to be convinced.

[2] Even the celebrated proof of Fermat's last theorem by Wiles, using heavy machinery of algebraic geometry, etc., has been shown to be reducible to axiomatic set theory with minor augmentations; see (McLarty 2010).

There is an alternative: if we forget the 20th century ideology of mathematics for its own sake, math understood as a wholly autonomous discipline,[3] we can bring into the picture the connections between math and physics (let's simplify and forget other disciplines). The argument, presented aphoristically, would be that classical logic is the logic (i.e., the mathematical model of inference structures) of mathematico-scientific representation of phenomena.

7. At a certain level, whose exact boundaries may be difficult to establish, the practices of math-scientific representation share a lot with less specialized practices of representation (like representation of everyday events in the context of legal/practical debates and decisions, or practices of the kind developed by pre-Socratic philosophers). This helps explain how it is possible that Greek authors had already captured the essentials of propositional logic (the Stoics).

At a different level, I believe it is essential for conceptual and historical analysis to insist on the *differences* between Aristotelian (or traditional) logic and the logic of quantifiers. In particular, common representation practices, and even philosophical practices of argument, did not lead to a rich logic of relations and quantifiers. This came from math-scientific practices, and particularly from pure mathematics. De Morgan, around 1860, said: "the algebraist was living in the higher atmosphere of syllogism, the unceasing composition of relations, before it was admitted that such an atmosphere existed." (De Morgan 1966, 241) It was in algebra, he felt, that the general idea of relation emerged, and it was there that "the notions of relation and relation of relation" were first symbolized.

8. How do we get to classical logic from practices of representation? The argument in outline would go this way (please take the sequel as a mere sketch).

Consider **negation**: it seems plausible that all practices of representation, from the common (e.g. involved in navigation or hunting) to the scientific ones (astronomy or geodesy, to name only Ancient sciences), will involve linguistic markers for negation. The relation between proposed models and actually given events (e.g. position of capes in a map vs. actual experience of the capes; orbits of planets vs. their measured positions) will be one of concordance or discordance. This, of course, could and ought to be studied historically, but the goal of representing by means of certain (linguistic or graphic or mental…) representations will make it necessary to speak about *failures* of representation.

However, it seems plausible too that the math-scientific heritage of interest to us has

[3] Perhaps not easy to do, but timely. See for instance the paper: Le retour de Fourier, by J. P. Kahane (2005). URL = http://www.academie-sciences.fr/activite/archive/dossiers/Fourier/Fourier_pdf/Fourier_Kahane.pdf

depended crucially on this: that proposed representations have been subjected to *explicit discursive criticism* (but *if* things are so-and-so, *then*… and this is *not* correct, hence…). We need practices of argumentation, in fact, to have negation inferentially employed in forms like those analysed and systematised by logicians, like those employed by classical scientists and mathematicians. The connection between *modus tollens* and scientific method is too well known to deserve more than a mention.

Take e.g. the Stoics: with practices of representation and argumentation come "assertibles (*axiômata*)", that is to say declarative sentences with a truth-value – at any one time they are either true or false, *tertium exclusum*.[4] And then we have basic inferential schemes like the "five indemonstrables" (modus ponens, modus tollens, hypothetical judgement, etc.), and basic inference rules like the "four *themata*". Thus it seems that one can reconstruct the emergence of **bivalent classical** logic (propositional level) from representation-and-argument practices such as those involved in astronomy or geodesy.

9. It is an interesting historical project to substantiate this concretely; and it seems not only interesting, but also crucial, to inquire the extent to which **realistic assumptions** (of the kind proposed by Plato, Kepler, Gauss) have been influential on the path to these inferential features of the argument practices.

I consider this to be an open question. But notice the issue of realism (or Platonism) is linked with the representation aspect, not primarily with the argumentation. We come back to the topic in connection with intuitionism.

10. I think it will be useful to bring intuitionistic logic (IL) into the discussion. As you all know, intuitionistic propositional logic can be described as classical logic without the Aristotelian law of excluded middle: ($p \lor \neg p$), but with the law of contradiction ($\neg p \rightarrow (p \rightarrow q)$). In this system, double negation does not imply the truth of the proposition: ($\neg\neg p \rightarrow p$) is not generally valid; *reductio ad absurdum* is rejected.

Very often, IL is presented as a system codifying the consequence relation in a language that assigns different meanings to the connectives. The new meaning is understood as directly linked, not to truth *simpliciter*, but to actual proof or to actual verification. ($p \lor \neg p$) would state that either p has been proved or it has been refuted; hence it's easy to see why it fails. ($p \rightarrow q$) would say that we posses a rule for transforming any proof of p into one of q.

4 See e.g. (Bobzien 2008) in the Stanford Enc. of Phil.

For my purposes, it is important to make clear that this account, although useful for teaching IL (and historically linked with expositions by Heyting), methodologically and philosophically distorts the situation. The intuitionistic laws **are** directly linked with truth and falsity. As R. Cook has argued in detail (see his chapter in Shapiro 2006), one cannot interpret the intuitionist as talking about different connectives.

11. What is then the ground for such a strong difference? I will concentrate on the path followed by Brouwer and Weyl, and it is crucial to pay attention not just to what they proposed, but also to what they rejected – and why. Let us assume, for simplicity of the argument, that Brouwer –like Weyl in 1918, or like he himself up to 1912– has no qualms with natural-number arithmetic. If all there was in math were the natural and real numbers, finite or denumerable structures, and the like, there would be no need for changing logic. But mathematics deals with infinite structures going beyond the denumerable: the structure of real numbers is as central to the enterprise as the naturals.

Now, Brouwer and Weyl reject the usual reconstructions of the structure of real numbers (Dedekind, Cantor, Hilbert) because of their realistic assumptions (Platonism). One cannot treat **R** as a fully determined totality: it's individual elements are not fully determined 'in themselves', for each one of them is an actually infinite system; they are only determined insofar as the mathematician is in a position to offer explicit determinations, which perhaps can be done in denumerably many cases; a fortiori, the "set of all" real numbers cannot be treated as a fully determinate mathematical object – to the indeterminacy of individual reals one must add the fact that, apparently, non-denumerably many of them shall remain indeterminate.

12. The failure of some classical logical laws is a consequence, not of a change in the meaning of the connectives, but of realizing the peculiar methodological and (one might add) metaphysical conditions affecting higher mathematics. There is no continuity between finite structures and higher math, no continuity between representations of real world phenomena and higher math, due to the very peculiar, indeterminate nature of what is usually called "the objects" of higher mathematics.

13. The critic could now say, But the structure of real numbers merely represents the structure of time and of geometric lines! And so many physical theories, from mechanics and electromagnetism to relativity and even quantum physics, have verified empirically the adequacy of such representations! Take for instance QED, which is the most successfully corroborated theory in the whole history of science.

The simple answer is that all of these empirical arguments are insufficient to establish the realistic character of the continuum involved in usual physics. In order to jump to the conclusion that the real number system is fully determinate, we should adopt a fully realistic stance regarding the continuum as a means to represent physical phenomena. Neither all physicists (e.g. Heisenberg, Penrose) nor all mathematicians, nor all philosophers adopt such a stance. Again, there is the option of regarding the continuum (via time or space) as given in intuition, and again this is quite controversial. And, shouldn't logic be a neutral system, independent of such strongly contentious questions?

In the case of Brouwer, precisely it was his intention, from the beginning, to develop pure mathematics as an intellectual system, completely independent from applied math or physics. For a number of reasons, some of them strongly philosophical, he wanted mathematics to remain free from any 'contamination' from the goals and biases of science and technology.[5] I know this kind of proposal is normally disregarded with little or no argument, simply because it is so uncommon. Funny that those who so disregard it will immediately present themselves as advocates of pure mathematics!

14. The whole story of intuitionism and intuitionistic logic, that I have presented too quickly here, should do the job of emphasizing some of the key points. The difference between classical and intuitionistic is not a simple question of change of topic: whatever nuanced change of meaning one may find, its source lies in complex methodological and metaphysical issues. Moreover, it should have become clear that classical logic is intimately tied to math-scientific representation understood in the realistic way that has been characteristic of our tradition (meant is the mainstream scientific tradition, from the Greeks all the way until Einstein and present-day theoretical physicists)[6].

The philosopher of logic does not need to solve all of the vexing problems we have discussed, but in my view he or she should be highly conscious of them. At the very least, one ought to accept that intuitionistic mathematics is a highly relevant theoretical practice based on one sound way of approaching the constellation of problems surrounding the continuum (one among several ways, classical Dedekind-Hilbert approaches included). Thus a practice-based approach to the philosophy of logic is congenial to pluralism, but it should also serve to enhance our awareness of the presuppositions behind different logical systems. By doing this, I hope, we shall be

[5] For details, see the excellent biography (van Dalen 2005). Brouwer's early book *Life, Art, and Mysticism* (1905) is highly relevant for the topic.

[6] Notice than even Kantians, for all the sophistication of their standpoint, leave unaltered the realistic interpretation, now regarding the "phenomenal world" (not the "things in themselves").

contributing to better understanding of what logic is, and what it can be.

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