

# Abductive Reasoning Through $\delta$ -Resolution

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**Abstract:** In order to explore the duality between deductive and abductive reasoning, we present a propositional calculus, named  $\delta$ -resolution, dual to the resolution one. We say it is an abductive calculus because every formula we obtain is not a logical consequence of the premises, but an hypothesis of them. Within this calculus we define an abductive process which produces, for a given abductive problem, every minimal abductive hypothesis.

**Keywords:**

**MSC code:**

## 1 Introduction

Abduction is a way of reasoning which goes in search of a plausible explanation  $\alpha$  for a given fact  $\phi$ , within a background theory  $\Theta$ . When an abductive problem arises,  $\Theta \not\models \phi$ ; then  $\alpha$  is an abductive solution if (among other requisites that will be introduced later)  $\Theta, \alpha \models \phi$ . Many logical and computational approaches have been proposed to formalize the abductive reasoning. But most of them [1, 3] lie in *abductive uses* of *abductive calculi*. That is, those approaches do not define an abductive calculus, but they establish techniques to use existing deductive calculi in an abductive way, so the products (formulas) they obtain are correct abductive solutions. In this paper we present the  $\delta$ -resolution calculus, dual to standard resolution. Within this calculus, given a set of formulas  $A$  (we call this set a  $\delta$ -clausal form), we obtain formulas  $\Sigma$  ( $\delta$ -clauses) such that  $\Sigma \models A$ , so the obtained formulas entail the original set, not the reverse (as in standard deductive calculi). For this reason, we consider  $\delta$ -resolution a proper abductive calculus. In section 2 we introduce the

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basic definitions of *abductive problem* and *abductive solution*. In section 3 the  $\delta$ -resolution calculus and its main properties are presented. Then, section 4 defines an abductive process by using only the  $\delta$ -resolution calculus, which produces all the minimal abductive solutions for a given abductive problem.

## 2 Preliminaries

We center our analysis in classical propositional logic. We take  $\mathcal{L}_p$  as a propositional language, defined with the habitual connectives  $\vee$ ,  $\wedge$ ,  $\rightarrow$  and  $\neg$ . We use lower-case Greek letters for formulas and upper-case Greek letters for sets of formulas. If  $\lambda$  is a literal (positive or negative),  $\bar{\lambda}$  represents its complementary (negative or positive, respectively). The logical consequence relation,  $\models$ , is taken as usual.

**Definition 1 (Abductive problem)** *Given  $\Theta \subset \mathcal{L}_p$  and  $\phi \in \mathcal{L}_p$ ,  $\langle \Theta, \phi \rangle$  is an abductive problem iff*

$$\Theta \not\models \phi \quad (1)$$

$$\Theta \not\models \neg\phi \quad (2)$$

**Definition 2 (Abductive solution)** *The set of literals  $\Sigma \subset \mathcal{L}_p$  is an abductive solution for the abductive problem  $\langle \Theta, \phi \rangle$  iff*

$$\Theta \cup \Sigma \models \phi \quad (3)$$

$$\Theta \cup \Sigma \not\models \perp \quad (4)$$

$$\Sigma \not\models \phi \quad (5)$$

$$\text{For every } \Sigma' \subset \Sigma, \Theta \cup \Sigma' \not\models \phi \quad (6)$$

*We represent as  $Abd(\Theta, \phi)$  the set of abductive solutions for the abductive problem  $\langle \Theta, \phi \rangle$ .*

**Definition 3 (Minimal model)** *A satisfiable set of literals  $\Sigma \subset \mathcal{L}_p$  is a minimal model of  $\alpha \in \mathcal{L}_p$  iff*

$$\Sigma \models \alpha \quad (7)$$

$$\text{For every } \Sigma' \subset \Sigma, \Sigma' \not\models \alpha \quad (8)$$

**Observation 4** *A direct conclusion from definition 3 is that for every satisfiable set of literals  $\Sigma \subset \mathcal{L}_p$  and every formula  $\alpha \in \mathcal{L}_p$ , if  $\Sigma \models \alpha$ , then there is a  $\Sigma' \subseteq \Sigma$  which is a minimal model of  $\alpha$ .*

**Observation 5** *From definition 3, and using the deduction theorem, we can transform the requisites (3) and (6) of definition 2 (provided that, by (4),  $\Sigma$  must be satisfiable) in*

$$\Sigma \text{ is a minimal model of } \neg\theta \vee \phi \quad (9)$$

where  $\theta$  is  $\bigwedge_{\eta \in \Theta} \eta$ , since  $\Theta \cup \Sigma \models \phi$ ,  $\Sigma \models \theta \rightarrow \phi$ , and  $\Sigma \models \neg\theta \vee \phi$  are equivalent. Also, requisites (4) and (5) of definition 2 can be transformed, respectively, in

$$\Sigma \not\models \neg\theta \quad (10)$$

$$\Sigma \not\models \phi \quad (11)$$

Then,  $\Sigma$  is an abductive solution for the abductive problem  $\langle \Theta, \phi \rangle$  iff (9)–(11).

### 3 The $\delta$ -resolution calculus

**Definition 6 ( $\delta$ -clause)** A  $\delta$ -clause  $\Sigma = \{\lambda_1, \dots, \lambda_n\}$  is a finite set of literals of  $\mathcal{L}_p$ . Given a boolean valuation  $v$ ,  $v \models \Sigma$  iff  $v \models \lambda_i$ , for every  $\lambda_i$ ,  $1 \leq i \leq n$ . With  $\diamond$  we represent the empty  $\delta$ -clause, which is universally valid.

**Definition 7 ( $\delta$ -clausal form)** A  $\delta$ -clausal form  $A = \{\Sigma_1, \dots, \Sigma_n\}$  is a finite set of  $\delta$ -clauses. Given a boolean valuation  $v$ ,  $v \models A$  iff  $v \models \Sigma_i$  for at least one of the  $\Sigma_i$ ,  $1 \leq i \leq n$ . The empty  $\delta$ -clausal form is not satisfiable. Given  $\alpha \in \mathcal{L}_p$ , if  $A$  is equivalent to  $\alpha$ , we say it is the  $\delta$ -clausal form of  $\alpha$ .

**Observation 8** For a given formula  $\alpha \in \mathcal{L}_p$  there are many  $\delta$ -clausal forms equivalent to it. But we can refer to  $A$  as the  $\delta$ -clausal form of  $\alpha$  because for every other  $\delta$ -clausal form  $B$  equivalent to  $\alpha$ , it is obvious that  $\models A \leftrightarrow B$ .

**Theorem 9 (Conversion to  $\delta$ -clausal form)** For each  $\alpha \in \mathcal{L}_p$ , there is a  $\delta$ -clausal form  $A$  such that  $\alpha$  and  $A$  are equivalent.

**Proof:** Let  $(\lambda_1^1 \wedge \dots \wedge \lambda_1^{j_1}) \vee \dots \vee (\lambda_n^1 \wedge \dots \wedge \lambda_n^{j_n})$  be the disjunctive normal form of  $\alpha$ . Then,  $A = \{\{\lambda_1^1, \dots, \lambda_1^{j_1}\}, \dots, \{\lambda_n^1, \dots, \lambda_n^{j_n}\}\}$  is equivalent to  $\alpha$ . ■

**Observation 10** Given the  $\delta$ -clausal form  $A = \{\Sigma_1, \dots, \Sigma_n\}$ , if each  $\Sigma_i = \{\lambda_i^1, \dots, \lambda_i^{j_i}\}$ , for any  $\Sigma_i$ ,  $1 \leq i \leq n$ , then every formula  $\alpha \in \mathcal{L}_p$  equivalent to  $(\lambda_1^1 \wedge \dots \wedge \lambda_1^{j_1}) \vee \dots \vee (\lambda_n^1 \wedge \dots \wedge \lambda_n^{j_n})$  has  $A$  as its  $\delta$ -clausal form. In this way, we can obtain, for each  $\delta$ -clausal form  $A$ , a formula  $\alpha$  such that  $A$  is its  $\delta$ -clausal form.

**Definition 11 ( $\delta$ -resolution rule)** Given two  $\delta$ -clauses  $\Sigma_1 \cup \{\lambda\}$  and  $\Sigma_2 \cup \{\neg\lambda\}$ , the  $\delta$ -resolution rule produces their  $\delta$ -resolvent  $\Sigma_1 \cup \Sigma_2$ :

$$\frac{\Sigma_1 \cup \{\lambda\} \quad \Sigma_2 \cup \{\neg\lambda\}}{\Sigma_1 \cup \Sigma_2}$$

Though this rule is presented with the same format than the standard resolution rule, they are different since now we are working with  $\delta$ -clauses. In the standard resolution calculus [6], every obtained clause is a logical consequence of the original set. Now, as we prove in theorem 13, any  $\delta$ -clausal form which contains  $\Sigma_1 \cup \{\lambda\}$  and  $\Sigma_2 \cup \{\neg\lambda\}$  is a logical consequence of  $\Sigma_1 \cup \Sigma_2$ .

**Definition 12 (Proof by  $\delta$ -resolution)** *The  $\delta$ -clause  $\Sigma$  is provable by  $\delta$ -resolution from the  $\delta$ -clausal form  $A$ , what we express with  $A \vdash_\delta \Sigma$ , iff there is a sequence of  $\delta$ -clauses such that:*

- *Each  $\delta$ -clause of the sequence is either a member of  $A$  or a  $\delta$ -resolvent of previous  $\delta$ -clauses.*
- *The last  $\delta$ -clause of the sequence is  $\Sigma$ .*

**Theorem 13 (Soundness)** *For every  $\delta$ -clausal form  $A$  and  $\delta$ -clause  $\Sigma$ , if  $A \vdash_\delta \Sigma$ , then  $\Sigma \models A$ .*

**Proof:** The proof is made by an induction on the number of applications of the  $\delta$ -resolution rule. In the base case (0 applications),  $\Sigma \in A$ , and by definition 7,  $\Sigma \models A$ . For the induction step, let us consider that until the  $n^{\text{th}}$  application, every resulting  $\delta$ -clause entails  $A$ . Then, let  $\Sigma = \Sigma_1 \cup \Sigma_2$  be the  $n+1^{\text{th}}$   $\delta$ -clause obtained, a  $\delta$ -resolvent of  $\Sigma_1 \cup \{\lambda\}$  and  $\Sigma_2 \cup \{\neg\lambda\}$ . Then, every boolean valuation  $v$  which satisfies  $\Sigma$  satisfies one of the previous  $\delta$ -clauses, because  $v$  should satisfy every literal of  $\Sigma_1 \cup \Sigma_2$  and  $v \models \lambda$  or  $v \models \neg\lambda$ . Finally, by using the induction hypothesis,  $v$  satisfies  $A$ . So,  $\Sigma \models A$ . ■

**Theorem 14 (Completeness)** *If  $A$  is an universally valid  $\delta$ -clausal form, then  $A \vdash_\delta \diamond$ .*

**Proof:** Let  $s$  be the cardinality of  $A$  and  $t$  the sum of the cardinalities of all the  $\delta$ -clauses of  $A$ . Take  $k = t - s$ . We proceed by induction on the value of  $k$ , by considering that  $\diamond \notin A$  (to avoid the trivial case, where it is obvious that  $A \vdash_\delta \diamond$ ). When  $k = 0$ , the only possibility is that  $A$  is a set of unary  $\delta$ -clauses. Then, if  $A$  is universally valid, it should contain two  $\delta$ -clauses  $\{\lambda\}$  and  $\{\neg\lambda\}$ . With only one application of the  $\delta$ -resolution rule, we get  $\diamond$ .

Suppose that the theorem is valid for  $k \leq n$ , we will prove it for  $k = n + 1$ . In this case, there is in  $A$  one  $\delta$ -clause  $\Sigma = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ , where  $m \geq 2$ . Then, we define  $\Sigma' = \{\lambda_1\}$ , and  $\Sigma'' = \{\lambda_2, \dots, \lambda_m\}$ . But, if  $A$  is universally valid, then  $(A - \Sigma) \cup \{\Sigma'\}$  and  $(A - \Sigma) \cup \{\Sigma''\}$  are also universally valid (it is a direct conclusion from definitions 6 and 7), and for these sets,  $k \leq n$ , so we can get  $\diamond$  from them, through the proofs (sequences of  $\delta$ -clauses)  $\mathcal{D}em'$  and  $\mathcal{D}em''$ , respectively. Let  $\mathcal{D}em$  be a proof from  $A$ , constructed in a similar way that  $\mathcal{D}em''$ , but each time  $\Sigma''$  is used in  $\mathcal{D}em''$ ,  $\Sigma$  is used in  $\mathcal{D}em$ . Then, as

the only difference between  $\Sigma''$  and  $\Sigma$  is that  $\Sigma$  contains  $\lambda_1$ , the last  $\delta$ -clause in  $\mathcal{D}em$  is either  $\diamond$  (in that case  $A \vdash_\delta \diamond$ , and the proof finishes) or  $\{\lambda_1\}$ . In the last case, we have on the proof all the  $\delta$ -clauses of  $(A - \Sigma) \cup \{\Sigma'\}$  (as  $\Sigma'$  is  $\{\lambda_1\}$ ), and we can complete it as  $\mathcal{D}em'$  to get  $\diamond$ . ■

**Theorem 15 (Abductive Completeness)** *Let  $A$  be the  $\delta$ -clausal form of  $\alpha \in \mathcal{L}_p$ . Then,  $A \vdash_\delta \Sigma$  for each  $\Sigma$  which is a minimal model of  $\alpha$ .*

**Proof:** If  $\Sigma = \{\lambda_1, \dots, \lambda_n\}$  is a minimal model of  $\alpha$ , then, by definition 3,  $\Sigma \models \alpha$ , so applying the deduction theorem,  $\models (\lambda_1 \wedge \dots \wedge \lambda_n) \rightarrow \alpha$ , and by evaluation of  $\rightarrow$  and  $\neg$ ,  $\models \overline{\lambda_1} \vee \dots \vee \overline{\lambda_n} \vee \alpha$ . As  $A \cup \{\{\overline{\lambda_1}\}, \dots, \{\overline{\lambda_n}\}\}$  is the  $\delta$ -clausal form of the latter formula, we have that, for being universally valid, it is possible to get  $\diamond$  from it (theorem 14), through a proof we call  $\mathcal{D}em$ . In a similar way that we did when proving the completeness theorem, we construct a proof parallel to  $\mathcal{D}em$ , called  $\mathcal{D}em'$ , which only uses  $\delta$ -clauses of  $A$ . Now, when a  $\delta$ -clause of type  $\{\overline{\lambda_i}\}$ ,  $1 \leq i \leq n$ , is used in  $\mathcal{D}em$ , nothing is done in  $\mathcal{D}em'$ . It is easy to observe that the last  $\delta$ -clause of  $\mathcal{D}em'$  should be a  $\Sigma' \subseteq \Sigma$ , because, each  $\delta$ -clause  $\{\overline{\lambda_i}\}$  can only eliminate the literal  $\lambda_i$  in the proof. But if  $\Sigma' \subset \Sigma$ , as by theorem 13,  $\Sigma' \models A$ , then  $\Sigma' \models \alpha$ , what is contrary to the supposition that  $\Sigma$  is a minimal model of  $\alpha$ . So,  $\Sigma' = \Sigma$ . ■

**Definition 16 (Subsumption)** *We say that the  $\delta$ -clause  $\Sigma'$  subsumes the  $\delta$ -clause  $\Sigma$  iff  $\Sigma' \subset \Sigma$ .*

**Corollary 17 (Subsumption rule)** *For  $\delta$ -clauses  $\Gamma$ ,  $\Sigma$  and  $\Lambda$ , and each  $\delta$ -clausal form  $A$ , if  $A \cup \{\Sigma\} \cup \{\Sigma \cup \Lambda\} \vdash_\delta \Gamma$  and  $\Gamma$  is satisfiable, then there is a  $\delta$ -clause  $\Gamma'$  such that  $A \cup \{\Sigma\} \vdash_\delta \Gamma'$  and  $\Gamma' \subseteq \Gamma$ .*

**Proof:** If we eliminate  $\Sigma \cup \Lambda$  from  $A \cup \{\Sigma\} \cup \{\Sigma \cup \Lambda\}$ , the resulting  $\delta$ -clausal form,  $A \cup \{\Sigma\}$ , is equivalent to the former, as one can easily observe by a simple semantical reasoning, based on definitions 6 and 7. For theorem 13,  $\Gamma \models A \cup \{\Sigma\} \cup \{\Sigma \cup \Lambda\}$ , and using the above mentioned equivalence,  $\Gamma \models A \cup \{\Sigma\}$ . But, let  $\alpha \in \mathcal{L}_p$  be a formula with  $A \cup \{\Sigma\}$  as its  $\delta$ -clausal form (observation 10). Then,  $\Gamma \models \alpha$  and, by observation 4, there is a  $\Gamma' \subseteq \Gamma$  such that  $\Gamma'$  is a minimal model of  $\alpha$ . Then, for theorem 15,  $A \cup \{\Sigma\} \vdash_\delta \Gamma'$ . ■

**Corollary 18 (Elimination of contradictory  $\delta$ -clauses)** *For each satisfiable  $\delta$ -clause  $\Sigma$ , each  $\delta$ -clausal form  $A$  and each literals  $\lambda, \neg\lambda, \gamma_1, \dots, \gamma_n$ , if  $A \cup \{\{\lambda, \neg\lambda, \gamma_1, \dots, \gamma_n\}\} \vdash_\delta \Sigma$ , then there is a  $\Sigma' \subseteq \Sigma$  such that  $A \vdash_\delta \Sigma'$ .*

**Proof:** As  $\{\lambda, \neg\lambda, \gamma_1, \dots, \gamma_n\}$  is not satisfiable,  $A \cup \{\{\lambda, \neg\lambda, \gamma_1, \dots, \gamma_n\}\}$  is equivalent to  $A$  (definitions 6 and 7). So, for theorem 13,  $\Sigma \models A$ . Let  $\alpha$

be a formula with  $A$  as its  $\delta$ -clausal form (observation 10). Then,  $\Sigma \models \alpha$  and, following the observation 4,  $\Sigma' \subseteq \Sigma$  is a minimal model of  $\alpha$ . So, by theorem 15,  $A \vdash_{\delta} \Sigma'$ . ■

**Definition 19 (Saturation)** *Given the  $\delta$ -clausal form  $A$ , the set saturation by  $\delta$ -resolution from  $A$ , that we represent as  $A^{\delta}$ , is the minimal set which contains every  $\delta$ -clause  $\Sigma$  such that*

- $\Sigma$  is satisfiable.
- $A \vdash_{\delta} \Sigma$ .
- There is not  $\Sigma' \subset \Sigma$  such that  $A \vdash_{\delta} \Sigma'$ .

**Observation 20** *Given a finite set of  $\delta$ -clauses  $A$ ,  $A^{\delta}$  is also finite. The reason is that the set of possible  $\delta$ -clauses formed from a finite set of literals  $U$  (the literals which appear in  $A$ ) is exactly  $\mathcal{P}(U)$  (the power set of  $U$ ). Then,  $A^{\delta}$  is a subset of the former and it should be completed in a finite number of applications of the  $\delta$ -resolution rule, when it is not possible to generate any new  $\delta$ -clause.*

**Corollary 21 (Fundamental property of saturation)** *Let  $A$  be the  $\delta$ -clausal form of  $\alpha$ . Then,  $A^{\delta}$  is the set of minimal models of  $\alpha$ .*

**Proof:** Theorem 15 states that every  $\Sigma$  which is a minimal model of  $\alpha$  is provable by  $\delta$ -resolution from  $A$ . So, by definition 19,  $\Sigma$  should belong to  $A^{\delta}$ , because  $\Sigma$  is satisfiable (definition 3) and there is no  $\Sigma' \subset \Sigma$  such that  $A \vdash_{\delta} \Sigma'$  (because then, by theorem 13,  $\Sigma' \models A$ , and it contradicts that  $\Sigma$  is a minimal model of  $\alpha$ ). So, every minimal model of  $\alpha$  belongs to  $A^{\delta}$ . On the other hand, let us prove that every  $\delta$ -clause  $\Sigma$  of  $A^{\delta}$  is a minimal model of  $\alpha$ . As, by definition 19,  $\Sigma$  is satisfiable and  $A \vdash_{\delta} \Sigma$ , so  $\Sigma \models A$  (theorem 13) and  $\Sigma \models \alpha$  (definition 7), the only way of not being  $\Sigma$  a minimal model of  $\alpha$  is (observation 4) that  $\Sigma' \subset \Sigma$  is a minimal model of  $\alpha$ . But then, by theorem 15,  $A \vdash_{\delta} \Sigma'$ , what contradicts what definition 19 states about  $\Sigma$ . So,  $\Sigma$  is a minimal model of  $\alpha$ . ■

**Observation 22** *Given  $A$ , a way for constructing  $A^{\delta}$  is to obtain every possible  $\delta$ -resolvent from the original  $\delta$ -clauses and those which appear, and then eliminate all the contradictory and subsumed ones. But this process may be hard (anyway, its complexity is exponential), so it will be good to use (specially in an implementation) some kind of searching strategies, like those followed by resolution theorem provers. As, thought semantically dual,  $\delta$ -resolution calculus is syntactically identical to de classical resolution one, it is possible to use in the former the same searching strategies of the latter. For example,*

corollaries 17 and 18 allow us to eliminate the subsumed and contradictory  $\delta$ -clauses in any moment, without waiting until the end of the process.

**Theorem 23 (Fundamental theorem of  $\delta$ -resolution)** *For a given abductive problem  $\langle \{\theta_1, \dots, \theta_n\}, \phi \rangle$ , if  $N_\Theta$  and  $O$  are respectively the  $\delta$ -clausal forms of  $\neg(\theta_1 \wedge \dots \wedge \theta_n)$  and  $\phi$ , then*

$$Abd(\Theta, \phi) = (N_\Theta^\delta \cup O^\delta)^\delta - (N_\Theta^\delta \cup O^\delta)$$

**Proof:** First, consider that  $\Sigma \in Abd(\Theta, \phi)$ . As explained in observation 5, this means that:

- $\Sigma$  is a minimal model of  $\neg(\theta_1 \wedge \dots \wedge \theta_n) \vee \phi$ . But this formula has  $N_\Theta \cup O$  as its  $\delta$ -clausal form, so by corollary 21,  $\Sigma \in (N_\Theta \cup O)^\delta$ . But  $(N_\Theta^\delta \cup O^\delta)^\delta = (N_\Theta \cup O)^\delta$  (the order in which the  $\delta$ -resolution rule is applied to the  $\delta$ -clauses of a set  $A$  does not change the resulting set  $A^\delta$ ), so  $\Sigma \in (N_\Theta^\delta \cup O^\delta)^\delta$ .
- $\Sigma \not\models \neg(\theta_1 \wedge \dots \wedge \theta_n)$ , and  $\Sigma \not\models \phi$ . So, by definition 3,  $\Sigma$  is not a minimal model of  $\neg(\theta_1 \wedge \dots \wedge \theta_n)$  nor of  $\phi$ , so by corollary 21,  $\Sigma \notin N_\Theta^\delta$  and  $\Sigma \notin O^\delta$ , respectively.

Hence,  $\Sigma \in (N_\Theta^\delta \cup O^\delta)^\delta - (N_\Theta^\delta \cup O^\delta)$ . Now, suppose that  $\Sigma \in (N_\Theta^\delta \cup O^\delta)^\delta - (N_\Theta^\delta \cup O^\delta)$  to prove  $\Sigma \in Abd(\Theta, \phi)$ . We have:

- $\Sigma \in (N_\Theta^\delta \cup O^\delta)^\delta$ . As observed above, this is equivalent to  $\Sigma \in (N_\Theta \cup O)^\delta$ , and for being  $N_\Theta \cup O$  the  $\delta$ -clausal form of  $\neg(\theta_1 \wedge \dots \wedge \theta_n) \vee \phi$ , then by corollary 21,  $\Sigma$  is a minimal model of  $\neg(\theta_1 \wedge \dots \wedge \theta_n) \vee \phi$ .
- $\Sigma \notin N_\Theta^\delta$ . Then, given that  $N_\Theta$  is the  $\delta$ -clausal form of  $\neg(\theta_1 \wedge \dots \wedge \theta_n)$ , by corollary 21 we have that  $\Sigma$  is not a minimal model of  $\neg(\theta_1 \wedge \dots \wedge \theta_n)$ . But suppose that  $\Sigma \models \neg(\theta_1 \wedge \dots \wedge \theta_n)$ ; then, by observation 4, there is a  $\Sigma' \subset \Sigma$  such that  $\Sigma'$  is a minimal model of  $\neg(\theta_1 \wedge \dots \wedge \theta_n)$ . But then  $\Sigma' \in N_\Theta^\delta$  (corollary 21), and this contradicts that  $\Sigma \in (N_\Theta^\delta \cup O^\delta)^\delta$ , because  $\Sigma$  is subsumed by  $\Sigma'$ . So,  $\Sigma \not\models \neg(\theta_1 \wedge \dots \wedge \theta_n)$ .
- $\Sigma \notin O^\delta$ . Then, as  $O$  is the  $\delta$ -clausal form of  $\phi$ , by corollary 21 we have that  $\Sigma$  is not a minimal model of  $\phi$ . Suppose that  $\Sigma \models \phi$ . Then, by observation 4, there is a  $\Sigma' \subset \Sigma$  such that  $\Sigma'$  is a minimal model of  $\phi$ , and so  $\Sigma' \in O^\delta$  (corollary 21), and this contradicts that  $\Sigma \in (N_\Theta^\delta \cup O^\delta)^\delta$ , because  $\Sigma$  is subsumed by  $\Sigma'$ . So,  $\Sigma \not\models \phi$ .

Hence, by observation 5,  $\Sigma \in Abd(\Theta, \phi)$ . ■

**Corollary 24** *Given  $\alpha, \beta \in \mathcal{L}_p$  such that their  $\delta$ -clausal forms are, respectively,  $A$  and  $B$ , then  $\alpha \models \beta$  iff for every  $\Sigma \in A^\delta$  there is a  $\Sigma' \subseteq \Sigma$  such that  $\Sigma' \in B^\delta$ .*

**Proof:** Suppose that  $\alpha \models \beta$  and  $\Sigma \in A^\delta$ . Then, by corollary 21,  $\Sigma$  is a minimal model of  $\alpha$ , and by definition 3 this means that  $\Sigma \models \alpha$ . But  $\alpha \models \beta$ , so  $\Sigma \models \beta$ . As commented in observation 4, there is a  $\Sigma' \subseteq \Sigma$  such that  $\Sigma'$  is a minimal model of  $\beta$ . So, by corollary 21,  $\Sigma' \in B^\delta$ .

Now suppose that for every  $\Sigma \in A^\delta$  there is a  $\Sigma' \subseteq \Sigma$  such that  $\Sigma' \in B^\delta$ , in order to prove that  $\alpha \models \beta$ . Let  $v$  be any boolean valuation such that  $v \models \alpha$ . Then, let  $\Sigma^*$  be the  $\delta$ -clause composed by all the literals satisfied by  $v$  which are either propositional variables occurring in  $\alpha$  or their negations. It is obvious that  $\Sigma^* \models \alpha$ . Then (observation 4) there exists a  $\delta$ -clause  $\Sigma \subseteq \Sigma^*$  such that  $\Sigma$  is a minimal model of  $\alpha$ , and by corollary 21,  $\Sigma \in A^\delta$ . Then, there is a  $\Sigma' \subseteq \Sigma$  such that  $\Sigma' \in B^\delta$ . So,  $\Sigma'$  is a minimal model of  $\beta$ , and so  $\Sigma' \models \beta$ , and as  $\Sigma' \subseteq \Sigma^*$ ,  $v \models \Sigma'$  and then  $v \models \beta$ . So,  $\alpha \models \beta$ . ■

## 4 An abductive process by $\delta$ -resolution

Theorem 23 establishes that for a given abductive problem  $\langle \Theta, \phi \rangle$  it is possible to find the set  $\mathcal{Abd}(\Theta, \phi)$  by only  $\delta$ -resolution operations. In this section we present a sketch of an abductive process that, for a given pair  $\langle \Theta, \phi \rangle$  first determines whether it is an abductive problem (by studying semantically  $\Theta$ ,  $\phi$  and their relation) and only in the positive case constructs  $\mathcal{Abd}(\Theta, \phi)$ . This is all done by  $\delta$ -resolution.

**Algorithm 25 (Abductive process)** *The input is  $\langle \Theta, \phi \rangle$ ,  $\Theta = \{\theta_1, \dots, \theta_n\}$ , where  $\Theta \subset \mathcal{L}_p$  and  $\phi \in \mathcal{L}_p$ . Then:*

**Step 1: Theory Analysis.** *Let  $N_\Theta$  be the  $\delta$ -clausal form of  $\neg(\theta_1 \wedge \dots \wedge \theta_n)$ . Then:*

- *If  $N_\Theta$  does not contain any satisfiable  $\delta$ -clause, then  $\Theta$  is universally valid, and the process stops<sup>1</sup>.*
- *Else,  $N_\Theta^\delta$  is obtained, and:*
  - *If  $\diamond \in N_\Theta^\delta$ , then  $\Theta$  is not satisfiable, and the process stops<sup>2</sup>.*
  - *Else,*

**Step 2: Observation Analysis.** *Let  $O$  be the  $\delta$ -clausal form of  $\phi$ . Then:*

- *If  $O$  does not contain any satisfiable  $\delta$ -clause, then  $\phi$  is not satisfiable, and the process stops<sup>3</sup>.*
- *Else,  $O^\delta$  is obtained, and:*

<sup>1</sup>When the theory  $\Theta$  is universally valid,  $\langle \Theta, \phi \rangle$  cannot have abductive solutions in the sense of definition 2, because every  $\delta$ -clause  $\Sigma$  which satisfies (3) does not satisfy (5).

<sup>2</sup>Now,  $\langle \Theta, \phi \rangle$  cannot be an abductive problem, because  $\Theta \models \phi$  (and also  $\Theta \models \neg\phi$ ).

<sup>3</sup>In this case, every  $\Theta$  verifies  $\Theta \models \neg\phi$ , so  $\langle \Theta, \phi \rangle$  is not an abductive problem.



- If  $\diamond \in O^\delta$ , then  $\phi$  is universally valid, and the process stops<sup>4</sup>.
- Else,

**Step 3: Refutation search.** If for every  $\delta$ -clause  $\Sigma \in O^\delta$  there is a  $\Sigma' \subseteq \Sigma$  such that  $\Sigma' \in N_\Theta^\delta$ , then  $\Theta \models \neg\phi$ , and the process stops. Else,

**Step 4: Explanations search.** From  $N_\Theta^\delta$  and  $O^\delta$ ,  $(N_\Theta^\delta \cup O^\delta)$  and then  $(N_\Theta^\delta \cup O^\delta)^\delta$  are obtained. Then,

- If  $\diamond \in (N_\Theta^\delta \cup O^\delta)^\delta$ , then  $\Theta \models \phi$  and the process stops.
- Else,  $\langle \Theta, \phi \rangle$  is an abductive problem. The output is,

$$Abd(\Theta, \phi) = (N_\Theta^\delta \cup O^\delta)^\delta - (N_\Theta^\delta \cup O^\delta)$$

**Corollary 26 (Soundness of the abductive process)** For every  $\Theta \subset \mathcal{L}_p$  and  $\phi \in \mathcal{L}_p$ , the abductive process described in algorithm 25 for  $\langle \Theta, \phi \rangle$  is sound.

**Proof:** For a given formula  $\alpha \in \mathcal{L}_p$  with  $A$  as its  $\delta$ -clausal form,  $\diamond \in A^\delta$  iff  $\models \alpha$ . Also,  $\alpha$  is not satisfiable iff  $A$  is not satisfiable, what means, given definitions 6 and 7, that there is no satisfiable  $\delta$ -clause in  $A$ . With this observation, the proof of the soundness of steps 1 and 2 of algorithm 25 is direct. Step 3 is a consequence of corollary 24. For step 4, if  $\diamond \in (N_\Theta^\delta \cup O^\delta)^\delta$ , then  $N_\Theta^\delta \cup O^\delta$  is universally valid (theorem 13). But this is the  $\delta$ -clausal form of  $\neg(\theta_1 \wedge \dots \wedge \theta_n) \vee \phi$ , so  $\models \neg(\theta_1 \wedge \dots \wedge \theta_n) \vee \phi$ , what is equivalent to  $\Theta \models \phi$ . Else, the way that step 4 follows to construct the set  $Abd(\Theta, \phi)$  is correct by corollary 23. ■

## 5 Comments

Let us finish with some remarks about the  $\delta$ -resolution calculus. We have defined it as an *abductive calculus*, due to its duality with respect to the standard resolution calculus. The duality between deduction and abduction is commonly pointed out in the abductive literature, sometimes related to the duality between forward and backward reasoning (abduction is usually characterized as *backward deduction*). We find suggestive that the  $\delta$ -resolution calculus catches this duality.

It is possible to extend the  $\delta$ -resolution calculus to first order logic, by using *Herbrandization* [2] (dual to Skolemization) and *reversed skolemization* [4]. The undecidability problem appears, but it is possible to restrict the abductive search to finite domains, in a similar way to [5].

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<sup>4</sup>Now,  $\Theta \models \phi$

## References

- [1] Atocha Aliseda. *Abductive Reasoning: Logical Investigations into Discovery and Explanation*, volume 330 of *Synthese Library*. Springer, 2005.
- [2] Samuel R. Buss. On Herbrand's theorem. *Lecture Notes in Computer Science*, 960:195–209, 1995.
- [3] Marta Cialdea Mayer and Fiora Pirri. First order abduction via tableau and sequent calculi. *Bulletin of the IGPL*, 1:99–117, 1993.
- [4] Philip T. Cox and Tomasz Pietrzykowski. A complete, nonredundant algorithm for reversed skolemization. *Theoretical Computer Science*, 28:239–261, 1984.
- [5] Ángel Nepomuceno. Scientific explanation and modified semantic tableaux. In *Logical and Computational Aspects of Model-Based Reasoning*, Applied Logic Series, pages 181–198. Kluwer Academic Publishers, 2002.
- [6] John Alan Robinson. A machine-oriented logic based on the resolution principle. *Journal of the ACM*, 12:23–41, 1965.