

Small point sets whose graph of triangulations is not connected (abridged¹)

Francisco Santos ²

Abstract

Presentamos la construcción de una configuración de 50 puntos en dimensión 5 cuyo grafo de triangulaciones no es conexo. La versión completa de este artículo contiene otra construcción con 26 puntos. Estos ejemplos mejoran el construido en *J. Amer. Math. Soc.* **13**:3 (2000), 611–637 no solamente en cuanto a su tamaño sino también en los dos aspectos siguientes:

- Pueden perturbarse a posición convexa sin incrementar su dimensión.
- Poseen triangulaciones *unimodulares* en las componentes no regulares del grafo de triangulaciones, lo cual tiene importantes implicaciones algebraicas: los *esquemas de Hilbert tóricos* asociados canónicamente a estas configuraciones de puntos son no conexos.

Introduction

The graph of triangulations of a finite point set $A \subset \mathbf{R}^d$ has as vertices all the triangulations of A and as edges certain local geometric changes to them, analogous to the bistellar flips considered in combinatorial topology [2]. See the precise definition in Section 1.

This graph is interesting in several contexts: In Geometric Combinatorics, its study is a special case of the Generalized Baues problem, which includes as other special cases monotone paths on polytopes, zonotopal tilings, or the extension space of realizable oriented matroids. See [4, 7]. In Computational Geometry, triangulations are a standard tool and flips have been often proposed as a method

to explore the space of all possible triangulations or search for optimal ones (e.g., the Delaunay one). See, for example, [1]. In Algebraic Geometry, polytopes and triangulations of them are closely connected to toric varieties [3, 12]. In particular, every point set has an associated toric variety and a so-called *toric Hilbert scheme*. See Section 1.4.

More details on these interrelations can also be found in [8], where we constructed the first example of a point set whose graph of triangulations is not connected, in dimension 6 and with 324 points. Here we show much smaller examples. The essential new idea is that we construct triangulations with sub-structures which cannot be destroyed by flips, rather than triangulations without flips. This gives more flexibility to the construction, and it also saves us some technicalities in the proofs.

In Section 1 we give the precise definition of the graph of triangulations and the main ingredients needed to prove that the graph is not connected in our examples, summarized in Theorem 5 and its Corollary 6. This section describes also the connection between the graph of triangulations and the toric Hilbert scheme.

Section 2 proves that the graph of triangulations of the following 50 points in dimension 5 is not connected:

$$A_{50} = (A \cup \{O\}) \times \{0, 1\},$$

where A and O are, respectively, the set of vertices and the centroid of the regular 24-cell in \mathbf{R}^4 . The full version of this paper [11] contains also another (smaller, but also more complicated) example A_{26} with only 26 points. Apart of being smaller these new constructions have other two advantages with respect to the one in [8]:

- The graph of triangulations remains non-connected under suitable perturbation of the

¹ Este artículo es una versión resumida de [11]

² Departamento de Matemáticas, Estadística y Computación, Universidad de Cantabria, E-39005, Santander. Email: santos@matemco.unican.es. Parcialmente financiado por el proyecto BFM2001-1153 de la Dirección General de Enseñanza Superior

point set into convex position (Proposition 13). In particular, there are 5-dimensional polytopes with 26 and 50 vertices whose graphs of triangulations are not connected. Point sets in convex position with disconnected graph of triangulations already appear in [8], but the argument used there produces a drastic explosion to dimension 234.

- b) We can prove that the toric Hilbert schemes of these point sets are not connected. The reason is that the graph of *unimodular* triangulations is not connected. Unimodular triangulations (of an integer point set) are those all of whose maximal simplices are bases for the integer lattice. They have a more direct relation to the "toric world" than the non-unimodular ones. See our Corollary 8, based on results of MacLagan and Thomas [5].

Thanks to symmetries, we can be more precise: the graphs of triangulations (and the toric Hilbert schemes) of A_{26} and A_{50} have at least 17 and 13 connected components respectively. In A_{50} , each of these 13 components contains at least 3^{48} unimodular triangulations (Corollary 12).

Even if the present constructions improve considerably those in [8], they still leave open some of the problems mentioned there:

- Can the graphs of triangulations be non-connected for point sets of *dimensions 3 and 4*? Dimension 3 is specially important for the applications in Computational Geometry. In dimension 2 the graph of triangulations is known to be connected.
- Can the graphs of triangulations be non-connected for point sets *in general position*? General position (i.e., no $d + 2$ points lie in an affine hyperplane) would imply that the refinement poset of subdivisions of A is not connected either, a question still open.

1 Triangulations and flips.

1.1 Triangulations and flips of point sets.

A triangulation of a finite point set $A \subset \mathbb{R}^d$ is a geometric simplicial complex with vertex set con-

tained in A and which covers the convex hull of A . Geometric bistellar flips are local operations which transform one triangulation of A into another. Essentially, they correspond to switching between the two triangulations of a minimal affinely dependent subset of A . More precisely: Every minimal dependent subset $Z \subset A$ can be divided in a unique way in two parts Z^+ and Z^- whose convex hulls intersect. We call the ordered pair (Z^+, Z^-) a *circuit* of A . (This deviates slightly from the standard oriented matroid terminology, in which Z is a circuit and (Z^+, Z^-) an oriented circuit). The only two triangulations of Z are:

$$T_+(Z) := \{S \subseteq Z : Z^+ \not\subseteq S\},$$

$$T_-(Z) := \{S \subseteq Z : Z^- \not\subseteq S\}.$$

Definition 1 Let T be a triangulation of A and let $(Z^+, Z^-) \subseteq A$ be a circuit of A . Suppose that the following conditions are satisfied:

- The triangulation $T_+(Z)$ is a sub-complex of T .
- All the maximal simplices of $T_+(Z)$ have the same link L in T . In particular, $T_+(Z) * L$ is a sub-complex of T . Here $A * B := \{S \cup T : S \in A, T \in B\}$.

Then, we can obtain a new triangulation T' of A replacing the sub-complex $T_+(Z) * L$ of T by the complex $T_-(Z) * L$. This operation is called a *geometric bistellar operation* or *geometric bistellar flip* (or a *flip*, for short) supported on the circuit (Z^+, Z^-) .

This definition is literally taken from [8]. It originally comes from [3, Chapter 7], where it is called a *modification*. See also [4, p.287]. The following arguments will help to convince the reader that it is the right concept of minimal or elementary change between triangulations:

- There is a certain subset of all triangulations of A , called *regular* or *coherent*, which are in bijection with the vertex set of a polytope of dimension $|A| - d - 1$ (the *secondary polytope* of A). The edges of this polytope are in bijection with flips between regular triangulations [3].
- Triangulations are the minimal elements in the poset of polyhedral subdivisions of A , with the

partial order given by refinement. Flips are the "next to minimal" elements, in the following well-defined sense: any subdivision whose only proper refinements are triangulations has exactly two proper refinements and they are triangulations related by a flip. Conversely, every flip arises in this way [10, Corollary 4.5 and Proposition 5.3].

1.2 Locally acyclic orientations

Let A be a point set in \mathbf{R}^d . Let $I = [0, 1] \subset \mathbf{R}$. If T is a triangulation of A , we abbreviate as $T \times I$ the polyhedral subdivision of $A \times \{0, 1\}$ into prisms $\sigma \times I$, $\sigma \in T$. We are interested in studying the triangulations of $A \times \{0, 1\}$ which refine a given such subdivision.

For the refining process we need to understand triangulations of the prism $\Delta^d \times I$, where Δ^d is a simplex of dimension d and I is a segment. The following description appears in [3]; see also [9, Section 3].

Proposition 2 *Let the vertices of $\Delta^d \times I$ be labeled $\{a_1, \dots, a_{d+1}, b_1, \dots, b_{d+1}\}$ so that the a_i 's are the vertices of the facet $\Delta^d \times \{0\}$ and each b_i is the vertex corresponding to a_i in the opposite facet $\Delta^d \times \{1\}$. Then:*

- a) *There is a bijection between triangulations of $\Delta^d \times I$ and linear orderings (permutations) of the numbers $\{1, \dots, d+1\}$. The triangulation corresponding to the ordering (s_1, \dots, s_{d+1}) has the following $d+1$ maximal simplices: $\{a_{s_1}, \dots, a_{s_i}, b_{s_i}, \dots, b_{s_{d+1}}\}$, for $i = 1, \dots, d+1$.*
- b) *Two triangulations of $\Delta^d \times I$ differ by a bistellar flip if and only if the corresponding orderings differ by a transposition of a pair of consecutive elements.*

In particular, we get the following result, where a *locally acyclic orientation* of the 1-skeleton of a simplicial complex is an orientation of all its edges which is acyclic on every simplex.

Proposition 3 *The triangulations of $A \times \{0, 1\}$ which can be obtained by refining the product $T \times I$ are in bijection to the locally acyclic orientations of the 1-skeleton of T . Flips between such triangulations correspond to reversal of single edges.*

Proof: A locally acyclic orientation induces a linear ordering on every simplex σ of T , which we can use to triangulate $\sigma \times I$. The triangulations so obtained agree on common faces of any two prisms because the linear orderings agree on the intersection of any two simplices. Conversely, a refinement of $T \times I$ triangulates in particular each prism $\sigma \times I$, hence induces a linear ordering on the vertices of every simplex σ of T . ■

The triangulation of $A \times \{0, 1\}$ obtained from a certain locally acyclic orientation of T is characterized by using the diagonal $\{(v, 0), (w, 1)\}$ in the quadrilateral $\{(v, 0), (w, 0), (v, 1), (w, 1)\}$ of $T \times I$ if and only if the edge $\{v, w\}$ of T is directed from v to w .

We will be specially interested in locally acyclic orientations of T without reversible edges, meaning that every single-edge reversal creates a cycle in a simplex. The smallest one we know of has 15 vertices and dimension 3. It is a Schlegel diagram of the example considered in Remark 3.4 of [8]. Unfortunately, Proposition 3 does not imply that locally acyclic orientations of T produce refinements of $T \times I$ without flips. They only produce refinements of $T \times I$ none of whose flip-neighbors refine $T \times I$.

1.3 Freezing sub-complexes in triangulations

All we have mentioned so far is essentially present in [9]. The new ingredient in this paper is that we focus on restrictions of triangulations to sub-complexes of faces. Let F be a face of the polytope $\text{conv}(A)$. It is obvious that every triangulation of A restricts to a triangulation of $F \cap A$. The next statement says that the same happens for flips:

Proposition 4 *If F is a face of $\text{conv}(A)$ and T and T' are triangulations of A differing by a flip, then T and T' restricted to F either coincide or differ by a flip on a circuit contained in F .*

Proof: The only simplices of a triangulation that are removed by a flip are those containing the negative part of the circuit Z in which the flip is supported. For T restricted to F to be affected by the flip it is necessary that $Z^- \subseteq F$ and, being a face, then $Z^+ \subseteq F$ too. Hence, the circuit is contained in F and the flip restricts to a flip in F . ■

More generally, let \mathcal{K} be a simplicial subcomplex of the face complex of $\text{conv}(A)$. With the word *simplicial* we do not only mean that every $F \in \mathcal{K}$ is a simplex, but also that $F \cap A$ is affinely independent. That is, that $F \cap A$ is the vertex set of F . A triangulation of $\mathcal{K} \times I$ is any geometric simplicial complex T with vertex set contained in $A \times \{0, 1\}$ satisfying that: (1) every simplex of T is contained in one of the products $F \times I$, $F \in \mathcal{K}$; and (2) $\cup_{\sigma \in T} \text{conv}(\sigma) = \cup_{F \in \mathcal{K}} F \times I$.

Since every $F \times I$ is a face of the convex hull of $A \times I$, every triangulation T of $\mathcal{K} \times I$ induces a triangulation of $F \times I$ for every $F \in \mathcal{K}$. The following result has essentially the same proof as Proposition 3, taking into account Proposition 4:

Theorem 5 *Triangulations of $\mathcal{K} \times I$ are in bijection with locally acyclic orientations of the 1-skeleton of \mathcal{K} . Flips between triangulations correspond to locally acyclic orientations differing on the reversal of a single edge.* ■

Corollary 6 *In the above conditions, let $A' = A \times \{0, 1\}$. If \mathcal{K} has a locally acyclic orientation without reversible edges and the corresponding triangulation of $\mathcal{K} \times I$ can be extended to a triangulation of A' , then the graph of triangulations of A' is not connected.*

Proof: By Theorem 5, the restrictions to $\mathcal{K} \times I$ of triangulations of A' can be considered elements in the graph of locally acyclic orientations of \mathcal{K} , and flips between triangulations either reverse a single edge or leave the locally acyclic orientation unchanged, by Proposition 4. The statement holds because a triangulation extending (the triangulation of $\mathcal{K} \times I$ corresponding to) a locally acyclic orientation without reversible edges will not be connected by flips to a triangulation extending any other locally acyclic orientation. For example, one extending any globally acyclic orientation of edges of \mathcal{K} , which clearly has reversible edges and can be extended to a lexicographic triangulation of A' . ■

1.4 Unimodular triangulations and the toric Hilbert scheme.

Let $A = \{a_1, \dots, a_n\} \subset \mathbb{Q}^d$ be a rational point set. We transform the *point* configuration A into a *vector* configuration $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{Q}^{d+1}$, by

choosing a positive integer l_i for each $i = 1, \dots, n$ and letting $a_i = (l_i a_i, l_i) \in \mathbb{Z}^{d+1}$. We assume, without loss of generality, that \mathcal{A} has only integer entries. The standard choice of scaling factors is the *homogeneous* one, with $l_i = 1$ for every i .

As detailed in [12, Chapter 10], we use \mathcal{A} to define a $(d-1)$ -dimensional multigrading on the polynomial ring $\mathbf{K}[x_1, \dots, x_n]$, assigning multidegree \mathbf{a}_i to the variable x_i . Ideals $I \subset \mathbf{K}[x_1, \dots, x_n]$ which are homogeneous with respect to this grading have a well-defined Hilbert function where \mathbb{N} is the set of non-negative integers, $\mathbf{N}\mathcal{A}$ is the semigroup of non-negative integer combinations of \mathcal{A} and for each $\mathbf{b} \in \mathbf{N}\mathcal{A}$, $I_{\mathbf{b}}$ is the part of degree \mathbf{b} of I .

The most natural \mathcal{A} -homogeneous ideal is the toric ideal $I_{\mathcal{A}}$ of \mathcal{A} , generated by the binomials

$$x_1^{\lambda_1} \dots x_n^{\lambda_n} - x_1^{\mu_1} \dots x_n^{\mu_n}$$

where $\lambda, \mu \in \mathbb{N}^n$, and $\sum_{i=1}^n (\lambda_i - \mu_i) \mathbf{a}_i = 0$. For every $\mathbf{b} \in \mathbf{N}\mathcal{A}$, $(I_{\mathcal{A}})_{\mathbf{b}}$ has codimension one in $(\mathbf{K}[x_1, \dots, x_n])_{\mathbf{b}}$. This characterizes the Hilbert function of $I_{\mathcal{A}}$.

Definition 7 *An \mathcal{A} -homogeneous ideal $I \subset \mathbf{K}[x_1, \dots, x_n]$ is called \mathcal{A} -graded if it has the same Hilbert function as the toric ideal. The toric Hilbert scheme, introduced by Peeva and Stillman [6], is the set of all \mathcal{A} -graded ideals with a suitable algebraic structure defined by some determinantal equations.*

\mathcal{A} -graded ideals include all the initial ideals (more generally, all the toric deformations) of $I_{\mathcal{A}}$. Sturmfels [12, Theorem 10.10] proved that every \mathcal{A} -graded ideal I has canonically associated a polyhedral subdivision S_I of A . Observe that subdivisions of A , in the affine setting used in this paper, coincide with subdivisions of \mathcal{A} in the linear setting used in [12, Chapter 10].

If I is monomial, then S_I is a triangulation, whose simplices are the standard monomials in $\mathbf{K}[x_1, \dots, x_n]/\text{Rad}(I)$. In other words, S_I is the triangulation whose Stanley-Reisner ideal equals the radical of I .

This produces a map Φ from the toric Hilbert scheme of \mathcal{A} to the poset of all polyhedral subdivisions of A . Maclagan and Thomas [5] go further and construct a graph of monomial \mathcal{A} -graded ideals (*mono- \mathcal{A} -GIs* for short) by defining a concept of flip between mono- \mathcal{A} -GIs in a way which makes the toric Hilbert scheme connected if and only of the

graph of mono- \mathcal{A} -GIs is connected. This does not directly imply that a non-connected graph of triangulations provides a non-connected toric Hilbert scheme, because the map Φ is in general not surjective. However, MacLagan and Thomas also make the observation, based on [12, Theorem 10.14], that the image of Φ contains all the unimodular triangulations of \mathcal{A} . A *unimodular triangulation* of \mathcal{A} is one in which every maximal simplex is a basis for the lattice spanned by \mathcal{A} .

Corollary 8 *The toric Hilbert scheme has at least as many connected components as connected components of the graph of triangulations contain unimodular triangulations.* ■

2 A construction with 50 points, based on the 24-cell

Let $A \subset \mathbb{R}^4$ denote the vertex set of a regular 24-cell, with centroid O . Let \mathcal{K} be the 2-skeleton of $\text{conv}(A)$, consisting of 96 triangles and 96 edges. In this section we show that \mathcal{K} can be given a locally acyclic orientation with no reversible edges, and that this orientation can be extended to a (unimodular) triangulation of

$$A_{50} := (A \cup \{O\}) \times \{0, 1\}.$$

Via Corollary 6, this implies that the graph of triangulations of A_{50} is not connected. Via Corollary 8, that the toric Hilbert scheme of $A_{50} := A_{50} \times \{1\} \subset \mathbb{R}^6$ is not connected either. At the end of the section we prove more precise quantitative results, and that the graph of triangulations of A_{50} remains not connected when its only two non-vertices $(O, 0)$ and $(O, 1)$ are moved into convex position.

Recall that the 24-cell is the convex hull of the following 24 points in \mathbb{R}^4 :

- The 8 points $\pm 2e_i$.
- The 16 points $(\pm 1, \pm 1, \pm 1, \pm 1)$.

It is a regular 4-polytope whose facets are 24 octahedra. One of them is the octahedron with vertices $(2, 0, 0, 0)$, $(0, 2, 0, 0)$, $(1, 1, 1, 1)$, $(1, 1, -1, -1)$, $(1, 1, 1, -1)$ and $(1, 1, -1, 1)$. We denote $F_{1,1,0,0}$ this octahedron. From this we get other three octahedra $F_{-1,1,0,0}$, $F_{-1,-1,0,0}$ and $F_{1,-1,0,0}$ by the rotation of order 4 on the first two coordinates. And

from these four we get the rest of the octahedra by permuting coordinates. The subindices in each octahedron give the coordinates of its centroid. The 24-cell is self-polar, which implies that these 24 centroids are the vertices of another regular (and smaller) 24-cell. We prefer the choice of vertices we have given because it highlights the symmetries of the 24-cell that we are interested in.

We orient the edges in $F_{1,1,0,0}$ with a source at $(2, 0, 0, 0)$, a sink at $(0, 2, 0, 0)$ and a cycle of length 4 on the equatorial square $(1, 1, 1, 1) \rightarrow (1, 1, -1, 1) \rightarrow (1, 1, -1, -1) \rightarrow (1, 1, 1, -1)$. See Figure 1. Observe that among the 12 edges of this octahedron only the equatorial four can be reversed without creating a local cycle. We let the other 84 edges of the 24-cell be oriented by the action of the affine group G of order 32 generated by the exchange of first two and last two coordinates and the rotation of order 4 on the plane of the first two coordinates (or of the last two coordinates).

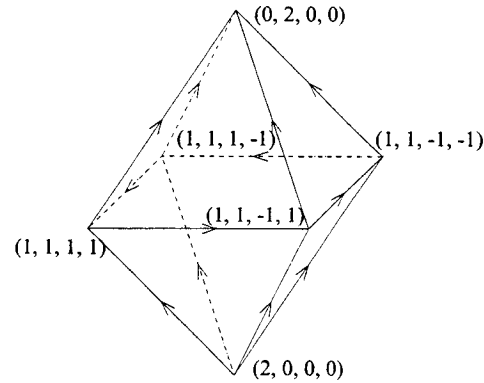


Figure 1: A locally acyclic orientation of the 1-skeleton of the octahedron $F_{1,1,0,0}$, with only 4 reversible edges.

Lemma 9 *This orientation of the 1-skeleton of \mathcal{K} is well-defined, locally acyclic, and has no reversible edges.*

Proof: The orientation is well-defined because edges of $F_{1,1,0,0}$ in the same orbit under G receive orientations compatible with the action. To check this observe that the only symmetries in G sending some edge of $F_{1,1,0,0}$ to another are the rotations on the first two coordinates, and they are compatible with the orientation given.

Under G , there are two orbits of octahedra in the 24-cell, 3 orbits of triangles and edges, and two orbits of vertices. The following are representatives of the three orbits of triangles, with their orientations indicated:

- $(2, 0, 0, 0) \rightarrow (1, 1, 1, -1) \rightarrow (1, 1, 1, 1)$,
- $(1, 1, 1, -1) \rightarrow (1, 1, 1, 1) \rightarrow (0, 2, 0, 0)$, and
- $(1, 1, 1, -1) \rightarrow (0, 0, 2, 0) \rightarrow (1, 1, 1, 1)$.

This proves that the orientation is locally acyclic. The three orbits of edges have as representatives $(2, 0, 0, 0) \rightarrow (1, 1, 1, 1)$, $(1, 1, 1, -1) \rightarrow (0, 2, 0, 0)$, and $(1, 1, 1, -1) \rightarrow (1, 1, 1, 1)$. That is, the non-reversible edges of the three stated representatives of triangles. This proves that the orientation has no reversible edges. ■

Lemma 10 *The triangulation T of $\mathcal{K} \times I$ corresponding (via Theorem 5) to our locally acyclic orientation can be extended to a unimodular triangulation T' of A_{50} .*

In the above statement, a triangulation of a lattice point set $B \subset \mathbb{R}^d$ is called unimodular if every maximal simplex is an affine basis for the lattice spanned by B . Equivalently, if it is unimodular as a triangulation of the vector set \mathcal{B} obtained from B with a homogeneous choice of scaling lengths.

Proof: Let $T_{1,1,0,0}$ be the triangulation of the octahedron $F_{1,1,0,0}$ that uses the axis $\{(2, 0, 0, 0), (0, 2, 0, 0)\}$. This has the property that orienting this new edge from $(2, 0, 0, 0)$ to $(0, 2, 0, 0)$ we extend the given orientation on $F_{1,1,0,0}$ maintaining local acyclicity.

Let T_0 be the triangulation of $A \cup \{O\}$ obtained by first replicating $T_{1,1,0,0}$ to the other octahedra by the action of G and then coning the triangulated boundary of the 24-cell to the centroid O . We also extend the orientation to all the boundary via G , and orient all the edges incident to O with a source at O . This gives a locally acyclic orientation of the 1-skeleton of T_A and, hence, a triangulation T' of A_{50} which refines $T_0 \times I$. Since the orientation of T_0 extends the one we had in \mathcal{K} , T' extends what we had in $\mathcal{K} \times I$.

It remains to check unimodularity. Since a prism over a simplex is totally unimodular (meaning that all its triangulations are unimodular), every refinement of $T_0 \times I$ will be unimodular as long as T_0 itself is unimodular. By symmetry, we only need to check that one of the 96 maximal simplices of T_0 is unimodular. For example, take the 4-simplex with vertex set $(0, 0, 0, 0)$, $(2, 0, 0, 0)$, $(0, 2, 0, 0)$, $(1, 1, 1, 1)$, $(1, 1, 1, -1)$. This is unimodular (in the sub-lattice spanned by $A \cup \{O\}$) because

$$\begin{vmatrix} 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = -8$$

and $A \cup \{O\}$ spans a sub-lattice of index 8 in the integer lattice \mathbb{Z}^4 , given by the conditions “the sum of any two coordinates is even”. ■

Corollary 11 *The 5-dimensional point configuration with 50 elements A_{50} has a non-connected space of triangulations and its homogenized version $A_{50} = A \times \{1\}$ has a non-connected toric Hilbert scheme.* ■

We finish this paper proving the quantitative results mentioned in the introduction, and that the graph of triangulations of A_{50} remains unimodular in a convex-position version of the point set.

Corollary 12 *The graph of triangulations of A_{50} and the toric Hilbert scheme of its homogenized version $A_{50} = A_{50} \times \{1\}$ have at least 13 connected components, each with at least 3^{48} unimodular triangulations/monomial ideals.*

Proof: Observe in Figure 1 that, out of the 48 symmetries of the octahedron $F_{1,1,0,0}$, only the four generated by the rotation around the vertical axis leave our locally acyclic orientation invariant. Hence, there are 12 ways of constructing a locally acyclic orientation with exactly the same properties. Each of these produces a different connected component in the graph of triangulations of A' , since they are isolated elements in the graph of locally acyclic orientations of \mathcal{K} . To these 12 we have to add the regular component of the graph, obtained for example starting with a globally acyclic orientation of \mathcal{K} .

To prove the number 3^{48} , observe the following: let $\sigma \subset A \cup \{O\}$ be (the vertex set of) a 5-simplex in the triangulation T_0 of the proof of Lemma 10. Let v and w be the source and sink of σ . In T' , $\sigma \times \{0\}$ is joined to $w \times \{1\}$ and $\sigma \times \{1\}$ is joined to $v \times \{0\}$.

Now consider one of the 24 octahedra, for example $F_{1,1,0,0}$. Its triangulation has exactly two flips, and the four simplices in it have the same source and have the same sink. The same is true when we consider the join of $F_{1,1,0,0}$ with O : the source becomes O and the sink does not change. This implies that the two flips in $F_{1,1,0,0}$ produce four flips in T' . Two with supporting circuit contained in the octahedron $F_{1,1,0,0} \times \{0\}$ and another two with supporting circuit in $F_{1,1,0,0} \times \{1\}$. Moreover, flips in different octahedra, out of the total of 48, are independent. Hence, from T' we can at least reach the 3^{48} other triangulations obtained by these flips, which are all unimodular. ■

Proposition 13 *Let A'_{50} be the point set obtained from A_{50} by moving the points $(O, 0)$ and $(O, 1)$ to (O, α) and (O, β) , for any $\alpha < 0$ and $\beta > 1$. Then, A'_{50} is in convex position and its graph of triangulations has all the properties claimed in Corollaries 11 and 12 for the graph of A_{50} .*

Proof: A'_{50} is still a point set containing $A \times \{0, 1\}$, and K satisfies the two hypothesis in Corollary 6. The only thing to proof is that our triangulation T of $K \times I$ extends to a triangulation of A'_{50} . Actually, the following is true: the same triangulation T' , considered in A'_{50} with the substitution $(O, 0) \mapsto (O, \alpha)$ and $(O, 1) \mapsto (O, \beta)$ is still a triangulation of A'_{50} . This holds because the facets $A \times \{0\}$ and $A \times \{1\}$ of A_{50} are centrally triangulated in T' and the perturbed points (O, α) and (O, β) lie beyond these facets of A_{50} (with the standard meaning of beyond in polytope theory: a point lies beyond a facet F of the polytope P if only that facet is visible from the point). ■

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