The polyhedron of non-crossing graphs on a planar point set[1]

David Orden and Francisco Santos[2]

Abstract

Adding marks to some vertices, we introduce the notion of marked graphs and pseudo-triangulations. We extend to them some properties of usual pseudo-triangulations, among them the notion of flips of interior edges between pointed vertices of a pseudo-triangulation to flips of any interior edge or mark of a marked pseudo-triangulation.

These flips produce a graph whose vertices are all the pseudo-triangulations of the given point set, containing in particular all triangulations and pointed pseudo-triangulations. The graph is regular of degree $3i + b - 3$, where i and b are the numbers of interior and boundary points in the point set

We construct the polyhedron of marked non-crossing graphs of a point set in the plane, defined as a convenient perturbation of a polyhedral cone. Its 1-skeleton is the regular graph mentioned above. The face poset of the polyhedron is opposite to the poset of marked non-crossing graphs, so in particular we obtain a (sub)polyhedron whose face poset is the poset of non-crossing graphs.

1 Introduction

Let P be a fixed point set of n points in the plane, in general position. The main result of this paper is that the poset of non-crossing graphs drawn on P can be embedded as part of the face poset of a simple polyhedron of dimension $3n - 3 - n_b$, where n_b is the number of points in the boundary of the convex hull. Observe that there is a trivial way of embedding this poset in the face poset of the $\binom{n}{2}$-cube, simply using one dimension for each of the possible edges in the graph. Hence, our crucial

point is the dimension $3n - 3 - n_b$, which is the minimum possible one since it equals the number of edges in any triangulation of the point set.

Our techniques are based on those of [4], who construct a polyhedron of dimension $2n - 3$ whose face poset is that of "pointed" non-crossing graphs drawn on P. Our main new ingredient is that we consider "marked" non-crossing graphs, meaning non-crossing graphs with a subset of their pointed vertices marked.

Pseudo-triangulations have arisen on many Computational Geometry applications, among them visibility [2, 3, 5], ray shooting and kinetic data structures [1]. Streinu [6] introduced the *minimum* or *pointed pseudo-triangulations*, used to prove the Carpenter's Rule Theorem and having applications to non-colliding motion planning of planar robot arms. They also have very nice combinatorial and rigidity theoretic properties, and the polyhedron constructed in [4] encodes their combinatorial structure.

We show that considering "marked pseudo-triangulations", we can generalize pointed pseudo-triangulations, extending their combinatorial properties as well as the existence of such a polyhedron.

In Section 2 we give some preliminaries and results about pseudo-triangulations and pointed graphs, and introduce the notions of *marked pseudo-triangulation* and *marked graph*, leading to results which extend the former. Section 3 is devoted to the construction of the polyhedron of non-crossing marked graphs.

Throughout this paper, we will consider planar point sets and will assume they are in general position, i.e. no three points are collinear.

[2]Departamento de Matemáticas, Estadística y Computación. Universidad de Cantabria. E-mail: {ordend,santos}@matesco.unican.es

2   Marked graphs on a planar point set

Pseudo-triangulations. A *pseudo-triangle* is a simple polygon with only three convex vertices (called *corners*) joined by three inward convex polygonal chains (called *pseudo-edges* of the pseudo-triangle), see Figure 1(a).

A *pseudo-triangulation of P* is a partitioning, such that every point of P is used, of the convex hull of P into pseudo-triangles. They are then *graphs embedded on P*; drawn in the plane, having vertex set P and edges the straight-line segments. Below we will work with other graphs embedded in the plane.

Planar and pointed graphs. Pointed pseudo-triangulations. Given a graph G embedded in the plane, it is said to be a *non-crossing* or *planar graph* if its edges intersect only at their endpoints. Observe that this is the case for pseudo-triangulations. A vertex v of the graph is called *pointed* if all its incident edges are strictly contained in a half-plane based on v (i.e. all contained in an angle smaller than π with vertex v).

A graph is *pointed* if it is pointed at every vertex. If, in addition, it is a pseudo-triangulation, then it is called a *pointed pseudo-triangulation*, which is usually abbreviated as *p.p.t.* We present now some of the good combinatorial properties of pointed pseudo-triangulations and pseudo-triangulations:

Proposition 1 (Streinu [6]) *With the previous definitions:*

1. *Every pointed and planar graph on n points has at most $2n - 3$ edges, and can be completed to a pointed pseudo-triangulation.*

2. *Every pseudo-triangulation on n points, n_γ non-pointed and n_ϵ pointed, has:*

   $2n - 3 + n_\gamma = 3n - 3 - n_\epsilon$ *edges, and*

   $n - 2 + n_\gamma = 2n - 2 - n_\epsilon$ *pseudo-triangles.*

3. **(Definition of Flips)** *In a pseudo-triangulation, every interior edge (not on the convex hull) between pointed vertices can be flipped; once removed, there is a unique way to put back another edge to obtain a different pseudo-triangulation.* ■

Corollary 2 *In the situation above,*

1. *Every pointed pseudo-triangulation on n points has exactly $2n - 3$ edges.*

2. *Pointed pseudo-triangulations are exactly the maximal planar pointed graphs.*

3. *The graph of flips between pointed pseudo-triangulations of a given point set is regular of degree $2n - 3 - n_b$, for n_b the number of vertices of the convex hull.* ■
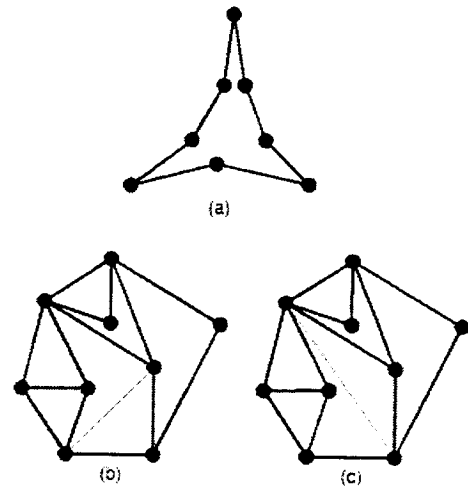

Figure 1:   (a) A pseudo-triangle.   (b) A pointed pseudo-triangulation. (c) The thin edge in (b) is flipped, giving another pointed pseudo-triangulation.

Marked pseudo-triangulations. We introduce here a new object, the *marked pseudo-triangulation*, and explain how it extends the previous properties of pseudo-triangulations.

Definition 3 Given a planar point set $P = \{p_1, \ldots, p_n\}$, a *marked pseudo-triangulation* is a partitioning of the convex hull of P into pseudo-triangles, along with a subset of the pointed vertices in this partitioning, which we define to be the *marked vertices*.

Observe that, in particular, a "pseudo-triangulation" is a "marked pseudo-triangulation with marked vertices the empty set".

Fully-marked graphs. Fully-marked pseudo-triangulations. We define now the property of being marked in a more general context, as was done above for pointedness:

Definition 4 A *marked graph* is a graph G embedded in the plane together with a subset of its pointed vertices. We define a graph to be *fully-marked* if all its pointed vertices are marked. If, in addition, it is a pseudo-triangulation, then it is called a *fully-marked pseudo-triangulation*, which will be abbreviated as *f.m.p.t.*

An interpretation of what marked vertices mean, can be the following; given a graph embedded in the plane, we consider this plane contained in a three dimensional space and a point at the infinity. Then each mark on a pointed vertex represents an edge joining it with the point at the infinity.

Let us show that the analogous to the properties stated above for "pointed pseudo-triangulations" and "pseudo-triangulations" hold when rewritten in terms of "fully-marked pseudo-triangulations" and "marked pseudo-triangulations", respectively:

Proposition 5 *With the previous definitions:*

1. *Every marked and planar graph on n points has at most $3n - 3$ edges plus marks, and can be completed to a fully-marked pseudo-triangulation.*

2. *Every marked pseudo-triangulation on n points, n_γ non- pointed, n_ϵ pointed and n_m marked, has:*

    $2n - 3 + n_\gamma + n_m = 3n - 3 - n_\epsilon + n_m$ *edges plus marks, and*

    $n - 2 + n_\gamma = 2n - 2 - n_\epsilon$ *pseudo-triangles.*

3. **(Definition of Flips)** *In a marked pseudo-triangulation, every interior edge or mark (not on the convex hull) can be flipped; once removed, there is a unique way to put back another edge or mark to obtain a different marked pseudo-triangulation.*

Proof: The second statement is trivial provided Lemma 1, so let us prove the others: For the first

one, observe that every *pseudo-n-gon*, $n \geq 4$ has diagonals between non-adjacent corners. A *diagonal* between two points of a pseudo-n-gon is the shortest path contained in the pseudo-n-gon and joining those points. Thus, the marked graph can be completed to a marked pseudo-triangulation which, in order to be fully-marked, only needs to have marks added, in such a case, at the non-marked vertices of the original graph.

To prove the third property we start with the edge case; since the edge e is not on the convex hull, there are two pseudo-triangles T_1, T_2 containing e as an edge. The union $T_1 \cup T_2$ can only be either a pseudo-quadrangle or another pseudo-triangle; the first arises when both edge endpoints are pointed in $T_1 \cup T_2$ and then it is enough to perform a flip of the type defined in Lemma 1 in the pseudo-quadrangle $T_1 \cup T_2$. On the other hand, when one of the two endpoints p, is non-pointed, $T_1 \cup T_2$ is another pseudo-triangle. We define the flip of this configuration removing the edge e and marking p (which became pointed).

It remains to define flips on an interior marked vertex p'. In order to do that, we focus on the two extreme adjacent edges of p'; there is a unique pseudo-triangle T' having them as part of one of its pseudo-edges. The flip of the marked vertex p' is defined then removing the mark and adding the diagonal between p' and the opposite vertex of its pseudo-edge (so p' becomes non-pointed). ■
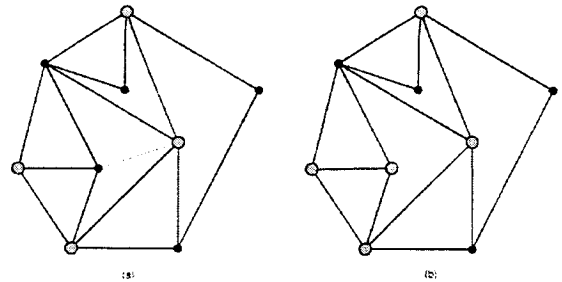

Figure 2: (a) A marked pseudo-triangulation. (b) The flip between the thin edge in (a) and the new marked vertex in (b).

As above, from these three properties we can easily obtain the following important facts about fully-marked pseudo-triangulations:

Corollary 6 *In the previous situation,*

1. *Every fully-marked pseudo-triangulation on n points has exactly $3n - 3$ edges plus marks.*

2. *Fully-marked pseudo-triangulations are exactly the maximal planar marked graphs.*

3. *The graph of flips between fully-marked pseudo-triangulations of a given point set is regular of degree $3n - 3 - 2n_b$, for n_b the number of vertices of the convex hull.* ∎

3   The polyhedron of non-crossing graphs.

In this section we prove our main result; we construct a polyhedron of dimension $3n - 3$ whose face poset is opposite to the poset of marked non-crossing graphs. This extends the construction given in [4] of a polyhedron of dimension $2n - 3$ whose face poset is opposite to the poset of pointed non-crossing graphs.

Construction of the polyhedron.   We consider a $(3n - 3)$-dimensional space in which we first define a collection of hyperplanes passing through the origin. The positive region of this hyperplane arrangement gives a polyhedron. This is not the one we are looking for, so we perturbe the hyperplanes in a convenient way to obtain a new polyhedron having the combinatorial structure we want.

Definition 7 *Given a set of n points $P = \{p_1, \ldots, p_n\}$ in $\mathbf{R}^2$ such that some of them are marked, we consider the following $(3n - 3)$-dimensional space;*

$$M := \{(v_1, \ldots, v_n, t_1, \ldots, t_n) \in \mathbf{R}^{2n} \times \mathbf{R}^n :$$
$$v_1^1 = v_1^2 = v_2^1 = 0\} \subset \mathbf{R}^{3n} \qquad (1)$$

where we define the following hyperplanes; for every pair of points $p_i, p_j \in P$,

$$H_{ij} := \{(v_1, \ldots, v_n, t_1, \ldots, t_n) \in M :$$
$$\langle p_i - p_j, v_i - v_j \rangle - |p_i - p_j|(t_i + t_j) = 0\}$$

and for every point p_j,

$$H_{0j} := \{(v, t) \in M : t_j = 0\}$$

We orient these hyperplanes as

$$H_{ij}^+ := \{(v_1, \ldots, v_n, t_1, \ldots, t_n) \in M :$$
$$\langle p_i - p_j, v_i - v_j \rangle - |p_i - p_j|(t_i + t_j) \geq 0\} \qquad (2)$$

and

$$H_{0j}^+ := \{(v, t) \in M : t_j \geq 0\} \qquad (3)$$

Definition 8 *We define the polyhedron $\bar{X}_0(P)$ to be the positive region of the hyperplane arrangement given by those of the previous definition:*

$$\bar{X}_0(P) := \bigcap_{i,j} H_{ij}^+$$

Lemma 9 *The polyhedron $\bar{X}_0(P)$ is a pointed polyhedral cone of full dimension $3n - 3$ in the subspace $M \subset \mathbf{R}^{3n}$. (Note that in this context "pointed" means that the origin is a vertex of the cone).*

Proof: The vector $v_i := p_i, t_i := \min_{k,l}\{|p_k - p_l|\}/4$ satisfies all inequalities $(2),(3)$ strictly. Equations (1) can be satisfied too, without changing the status of the inequalities, by adding a suitable rigid motion, giving a relative interior point in the subspace M. On the other hand, if the cone were not pointed, it would contain two opposite vectors (v, t) and $-(v, t)$. From this we would conclude that $\langle v_j - v_i, p_j - p_i \rangle - |p_i - p_j|(t_i + t_j) = 0$ for all i, j and $t_j = 0$ for all j, and hence by the equations (1), (v, t) must be $(0, 0)$. ∎

When permitted by the clearness of the context, we will omit the point set P and denote $\bar{X}_0$. Some of the inequalities in $(2),(3)$ may be satisfied with equality by a solution (v, t); the corresponding edges and vertices $E(v, t)$ are said to be *tight* for that solution. In the same way, given a face K of $\bar{X}_0$, we call *tight edges and vertices* of K and denote $E(K)$ the edges and vertices whose equations are satisfied with equality over K.

Lemma 10 *Consider the set $E(v, t)$ of tight edges and vertices for any feasible point $(v, t) \in \bar{X}_0$. If $E(v, t)$ contains*

(i) *two crossing edges,*

(ii) *a set of edges incident to a common vertex with no angle larger than π (witnessing that $E(v, t)$ is not pointed at this vertex), or*

(iii) *a convex subpolygon,*

then $E(v,t)$ *must contain the complete graph between the endpoints of all involved edges. In case* (iii), *this complete graph also includes all points inside the convex subpolygon.*

Proof: The same proof given in [4] works here noting that $E(v,t)$ can be rewritten as $E(v,t) = E(v) \cap \{i : t_i = 0\}$ for $E(v)$ the set of edges for which $\langle p_i - p_j, v_i - v_j \rangle = 0$. ∎

In order to get the wanted combinatorial structure for our polyhedron, we have to give up homogeneity and perturb the constraints (2),(3), translating faces of the cone:

Definition 11 *We define the following hyperplanes; for every pair of points $p_i, p_j \in P$,*

$$\tilde{H}_{ij} := \{(v_1, \ldots, v_n, t_1, \ldots, t_n) \in M :$$
$$\langle p_i - p_j, v_i - v_j \rangle - |p_i - p_j|(t_i + t_j) = f_{ij}\}$$

and for every point p_j,

$$\tilde{H}_{0j} := \{(v, t) \in M : t_j = f_{0j}\}$$

which, as above, we orient as

$$\tilde{H}_{ij}^+ := \{(v_1, \ldots, v_n, t_1, \ldots, t_n) \in M :$$
$$\langle p_i - p_j, v_i - v_j \rangle - |p_i - p_j|(t_i + t_j) \geq f_{ij}\} \quad (4)$$

and

$$\tilde{H}_{0j}^+ := \{(v, t) \in M : t_j \geq f_{0j}\} \quad (5)$$

Definition 12 *We define the polyhedron $\bar{X}_f(P)$ to be the positive region of the hyperplane arrangement given by those of the previous definition:*

$$\bar{X}_f(P) := \bigcap_{i,j} \tilde{H}_{ij}^+$$

And we define $E(v,t)$ and $E(K)$ for $\bar{X}_f$ in the same way we did for $\bar{X}_0$.

Corollary 13 *From Lemma 9, we conclude that $\bar{X}_f(P)$ is a $(3n-3)$-dimensional unbounded polyhedron with at least one vertex, for any choice of parameters f.* ∎

Combinatorial structure of the polyhedron.
In this paragraph we introduce some definitions and results leading to the main one; the construction of a polyhedron whose face poset is (opposite to) the poset of non-crossing and marked graphs. In particular, this gives a polyhedron with face poset opposite to the poset of non-crossing graphs in $\mathbf{R}^2$.

Definition 14 In our context, a *stress* on a graph $G = (P, E)$ embedded on P is an assignment of scalars w_{ij} to edges and α_j to vertices, such that for every $(v,t) \in \mathbf{R}^{3n}$:

$$\sum_{ij \in E} w_{ij}(\langle p_i - p_j, v_i - v_j \rangle -$$
$$|p_i - p_j|(t_i + t_j)) + \sum_{j=1}^{n} \alpha_j t_j = 0 \quad (6)$$

Lemma 15 *Let $\sum_{i=1}^{n} \lambda_i p_i = 0$, $\sum \lambda_i = 0$, be an affine dependence on a point set $P = \{p_1, \ldots, p_n\}$. Then,*

$$w_{ij} := \lambda_i \lambda_j \text{ for every } i, j$$

and

$$\alpha_j := \sum_{i: ij \in E} \lambda_i \lambda_j |p_i - p_j| \text{ for every } j$$

defines a stress of the complete graph G on P.

Proof: Considering (6) on variables v gives

$$\sum_{ij \in E} w_{ij} \langle p_i - p_j, v_i - v_j \rangle = 0, \quad \forall (v,t) \in \mathbf{R}^{3n} \quad (7)$$

what is equivalent to

$$\forall j \in P, \quad \sum_{ij \in E} w_{ij}(p_i - p_j) = 0 \quad (8)$$

This is fulfilled by our w_{ij}'s, using that the λ_i's are an affine dependence.

Then, from (6) and (7) we obtain

$$\alpha_j := \sum_{i: ij \in E} w_{ij} |p_i - p_j|.$$

∎

In particular, four points in $\mathbf{R}^2$ have a unique (up to constants) affine dependence, what determines uniquely the w_{ij}'s and α_j's of the previous Lemma. The coefficients of this dependence are:

$$\lambda_i = (-1)^i \det([p_1, \ldots, p_4] \setminus \{p_i\}).$$

(Recall that $\det(q_1, q_2, q_3)$ is two times the signed area of the triangle spanned by q_1, q_2, q_3).

Then, dividing the w_{ij}'s and α_j's of the previous lemma by the constant

$$-\prod_{i=3,4} \det(p_1, p_2, p_i) \prod_{j=1,2} \det(p_j, p_3, p_4)$$

we obtain next expressions for the case of four points:

$$w_{ij} = \frac{1}{\det(p_i, p_j, p_k) \det(p_i, p_j, p_l)}$$

and

$$\alpha_j = \sum_{i:ij \in E} w_{ij} |p_i - p_j| \qquad (9)$$

where k and l are the two indices other than i and j.

Lemma 16 *The previous expressions give positive w_{ij} and α_j on boundary edges and points and negative w_{ij} and α_j on interior edges and point, for both possible cases on four points; "one point inside the triangle formed by the other three" and "four points in convex position".*

Proof: The part concerning w_{ij}'s can be easily checked. For the α_j's, we use that (8) gives a triangle around j, and so a triangular inequality; for $w_{ij} |p_i - p_j|$ with i s.t. $ij \in E$, the sum of any two of them is greater than the other. Considering the signs of the edges around each vertex, it is easy to check that every $\alpha_j = \sum_{i:ij \in E} w_{ij} |p_i - p_j|$ has the claimed sign. ∎

Note that fully-marked pseudo-triangulations on four points are, in both cases, the graphs obtained deleting from the complete graph any single interior edge and marking every non-interior point.

Definition 17 We define a choice of the constants f to be *valid* if for every four points $\{p_1, p_2, p_3, p_4\}$ of P and w_{ij}'s, α_j's those of (9), they satisfy:

$$\sum_{1 \le i < j \le 4} w_{ij} f_{ij} + \sum_{j=1}^{4} \alpha_j f_{0j} > 0 \qquad (10)$$

Lemma 18 *The choice of constants $f_{ij} := \det(a, p_i, p_j) \det(b, p_i, p_j)$, $f_{0j} := 0$ for a and b any two points in the plane, is valid.*

Proof: Let us consider the four points p_i as fixed and regard $R := \sum w_{ij} f_{ij} + \sum \alpha_i f_{0j} = \sum w_{ij} f_{ij}$ as a function of a and b.

$$R(a, b) = \sum_{1 \le i < j \le 4} \det(a, p_i, p_j) \det(b, p_i, p_j) w_{ij}.$$

For fixed b, $R(a, b)$ is clearly an affine function of a. We claim that $R(p_i, b) = 1$ for each of the four points $p_1, \ldots, p_4$, what implies that $R(a, b)$ is constantly equal to 1.

To prove the claim; fixed one of the four points p_i, $R(p_i, b)$ is an affine function of b, so it is enough to prove that it equals 1 for b any of the other three points. But $R(p_i, p_j)$ trivially equals 1. ∎

We state now our main result; the polyhedron of planar marked graphs:

Theorem 19 *For every set $P = \{p_1, \ldots, p_n\}$ of planar points in general position, any valid choice of f produces a $\bar{X}_f$ simple, of dimension $3n - 3$ and with the following properties:*

1. *The face poset of the polyhedron equals the opposite of the poset of marked and non-crossing graphs on P, by the map sending each face K to the marked graph $E(K)$ having as edges and marked vertices those which are tight over K. In particular:*

   (a) *Vertices of the polyhedron are in 1-to-1 correspondence with fully-marked pseudo-triangulations of P.*

   (b) *Bounded edges correspond to flips of interior edges or marks in fully-marked pseudo-triangulations, i.e., to fully-marked pseudo-triangulations with one interior edge or mark removed.*

   (c) *Extreme rays correspond to fully-marked pseudo-triangulations with one convex hull edge or mark removed.*

2. *The face $X_f(P)$ obtained turning into equalities those inequalities from (4),(5) which correspond to convex hull edges or marks of P is bounded (hence a polytope) and contains all vertices. In other words, it is the unique maximal bounded face, and its 1-skeleton is the graph of flips among fully-marked pseudo-triangulations.*

Proof: Next four Lemmas, whose proof will appear in the full version, together with the existence of valid f's shown in Lemma 18, prove part 1.

Lemma 20 *For any choice of f's, $X_f(P)$ is a bounded set.*

Lemma 21 *The following three statements are equivalent:*

(i) *A choice of f's makes $\bar{X}_f$ have the combinatorial structure of the Theorem.*

(ii) *The graph $E(v,t)$ of tight edges and vertices corresponding to any feasible point $(v,t) \in \bar{X}_f(P)$ is non-crossing and marked.*

(iii) *The graph $E(v,t)$ of tight edges and vertices corresponding to any vertex $(v,t) \in \bar{X}_f(P)$ has exactly $3n - 3$ incident faces and is a fully-marked pseudo-triangulation.*

Lemma 22 *A choice of f's makes $\bar{X}_f(P)$ have the combinatorial structure of the Theorem if, and only if, it makes $\bar{X}_{f'}(\{p_1, p_2, p_3, p_4\})$ have that structure for every four points of P (where f' is f restricted to $\{p_1, p_2, p_3, p_4\}$).*

Lemma 23 *A choice of constants f makes $\bar{X}_{f'}(\{p_1, p_2, p_3, p_4\})$ have the combinatorial structure of the Theorem for every four points of P if, and only if, it is a valid choice.*

Finally, part 2 can be easily derived from part 1: For every vertex or bounded edge of $\bar{X}_f(P)$, the set $E(v,t)$ contains all convex hull edges and marks of P. On the contrary, for any unbounded edge (ray) of $\bar{X}_f(P)$, the set $E(v,t)$ misses some convex hull edge or mark of P. Thus, turning into equalities those inequalities corresponding to convex hull edges and marks gives a face $X_f(P)$ of $\bar{X}_f(P)$ which contains all vertices and bounded edges of $\bar{X}_f(P)$, but no unbounded edge. ■

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