Decompositions of graphs with a given tree

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Abstract

A tree decomposition of a graph G is a family of subtrees whose sets of edges partition the set of edges of G. In this paper we are interested in the structure of the trees involved in a tree decompositions. We show that arbitrary trees may appear in minimal decompositions of maximal planar graphs, maximal planar bipartite graphs and regular graphs.

1 Introduction

Let $G = (V, E)$ be a connected simple graph. An *edge decomposition* of G is a family of subgraphs $G_1, \ldots, G_k$ whose sets of edges partition E. We write

$$G = G_1 \oplus \cdots \oplus G_k.$$

When each subgraph G_i is acyclic we have a *forest decomposition.* The *arboricity* of G is the minimum number of forests in a forest decomposition of G, and it is denoted by $a(G)$. When each forest is connected, we have a *tree decomposition.* The minimum number of trees in a tree decomposition of G is denoted by $\tau(G)$.

Since each forest on m vertices has at most $m-1$ edges, $a_0(G) = \lceil |E|/(|V|-1) \rceil$ is a trivial lower bound for both, the arboricity and $\tau(G)$.

There are several classes of graphs for which $\tau(G)$ attains its minimum value $a_0(G)$. Kampen [4] proved that maximal planar graphs can be decomposed into 3 edge-disjoint trees. Since a maximal planar graph G has $3n-6$ vertices, we have $\tau(G) = a_0(G) = 3$. Ringel [12] proved that maximal planar bipartite graphs have $\tau(G) = a_0(G) = 2$. Chung [2] obtained the nontrivial upper bound, $\tau(G) \leq \lceil |V|/2 \rceil$, for connected graphs with no multiple edges. Thus, for complete graphs, $\tau(K_n) = a_0(K_n) = \lceil n/2 \rceil$. Truszczyński [13] showed that the equality $a_0(G) = \tau(G)$ also holds for complete bipartite graphs and hypercubes. Lladó et al. [6, 8] showed that this is also the case for regular graphs of even degree and maximum edge-connectivity and for graphs of order n and minimum degree $\delta \geq (n-1)/2$. Minyong Shi [9] et al. have shown that a class of graphs with uniform edge-density also satisfy $a_0(G) = \tau(G)$.

In some cases, the structure of the minimal decomposition has also been studied. Ringel pointed out that a decomposition of a maximal planar bipartite graph of order n admits two different kinds of decompositions: a spanning tree and a tree with $n-2$ vertices or two trees each one with $n-1$ vertices. He conjectured that there exist both kinds of tree decompositions in a maximal planar bipartite graph. The conjecture was proved by Ouyang and Liu in [11]. We say that a tree decomposition $G = T_1 \oplus \cdots \oplus T_k$ is of type $(a_1, \ldots, a_k)$ if T_i has $n - a_i$ vertices, $i = 1, \ldots, k$, where n is the order of G. A tree decomposition of a maximal planar graph into three trees can be of type $(1,1,1)$, $(0,1,2)$ or $(0,0,3)$. Minyong Shi et al. [9] proved that all three types of decompositions exist for a general class of graphs $\mathcal{P}_3$ which includes the maximal planar ones. For regular graphs and dense graphs, Lladó et al. [6, 8] obtained tree decompositions of type $(0, \ldots, 0, l)$, where $l = a_0(G)(n-1) - |E(G)|$.

In this paper we are interested in the structure of the trees involved in a tree decomposition. Our main goal consists in showing that arbitrary trees may appear in minimal tree decompositions. More precisely, we prove the following results. For the first one, we denote by $P_3^{r,s}$ the tree obtained from a path P_3 with three vertices by adding $r \geq 1$ leaves to one end vertex and $s \geq 1$ leaves to the other one.

Theorem A. *Let T be an arbitrary tree. There is a maximal bipartite graph G which admits T as a factor in a minimal tree decomposition of type (0,2) or (1,1). Moreover, T can be chosen to be the spanning tree in a decomposition of type $(0,2)$ if and only if T is neither a star S_n, $n \geq 2$, nor a $P_3^{r,s}$-tree, $r, s \geq 1$.*

Theorem B. *Let T be an arbitrary tree. There is a maximal planar graph G which admits T as a factor in a minimal tree decomposition of type (a, b, c) for any choice of $c \geq b \geq a \geq 0$ with $a + b + c = 3$.*

In addition to the theorems A and B, we include proofs of the Ringel conjecture and the existence of the three types of minimal tree decompositions of maximal planar graphs. These proofs are much simpler than the ones in [9] required to prove the result for the general class $\mathcal{P}_k$, $k = 2, 3$.

Theorem C. *Let $r \geq 2$ be an integer and let T be a tree of order $n \geq r + 1$ with nr even and maximum degree $\Delta(T) \leq \lceil \frac{r}{2} \rceil + 1$. Then there is a regular graph G of order n and degree r such that*

$$G = T \oplus T_2 \oplus \cdots \oplus T_k \oplus T_{k+1}$$

is a minimum tree decomposition of type $(0, \ldots, 0, l)$, where $k = \lfloor r/2 \rfloor$, $l = n - k - 1$ if r is even and $l = (n/2) - k - 1$ if r is odd.

2 Maximal planar bipartite graphs

A maximal planar bipartite graph (mpb) is a planar bipartite graph of order $n \geq 4$ in which the addition of any edge results in a graph which is no longer planar or bipartite. All faces in a planar embedding of a mpb graph are 4-cycles. Therefore, a mpb graph G has $2n - 4$ edges and any minimal decomposition of G must be of type $(0, 2)$ or $(1, 1)$. We give below a short proof of Ringel's conjecture about the existence of both types of decompositions in a mpb graph.

Theorem 2.1 *Each maximal planar bipartite graph admits minimal tree decompositions of type $(0, 2)$ and of type $(1, 1)$.*

Proof. Let G be a mpb graph. Let $G = T_1 \oplus T_2$ a minimal tree decomposition of G of type $(0, 2)$. We say that the decomposition is *good* if the two vertices of G which belong to an only tree are in different chromatic classes of the bipartition of G.

The proof is by induction on the order n of the graph. We actually prove the following slightly stronger statement: each maximal planar bipartite graph G admits tree decompositions of type $(1, 1)$ and good tree decompositions of type $(0, 2)$. Such decompositions are shown when $n = 4$ in Figure 1.

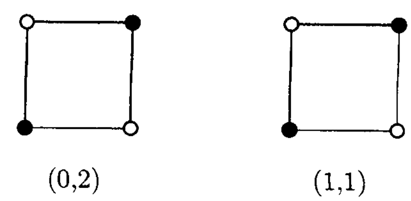

Figure 1: The two types of minimal tree decompositions for $n = 4$.

Let G be a mpb graph of order $n > 4$. Let C be a 4-cycle which is a face in a planar embedding of G. Let v_0, v_1, v_2, v_3 be the vertices of C where v_i is adjacent to $v_{i+1 \pmod 4}$ in the cycle. Since G is planar, one of the two pairs of vertices, $\{v_0, v_2\}$ or $\{v_1, v_3\}$, have no additional common neighbours in G than $\{v_1, v_3\}$ or $\{v_0, v_2\}$ respectively. Assume that the only common neighbours of v_0 and v_2 in G are v_1 and v_3.

Let G' be the graph obtained from G by identifying the vertices v_0 and v_2 in a single vertex v and identifying the pairs of edges v_1v_0, v_1v_2 and v_0v_3, v_2v_3. Then G' is again a mpb graph. By the induction hypothesis, there are decompositions of G' of type $(1, 1)$ and good ones of type $(0, 2)$. Let $G' = T_1' \oplus T_2'$ be a tree decomposition. Color the edges of G' with $i \in \{1, 2\}$ according to the tree they belong. We color the edges of $G - C$ as they are colored in G'. To give a color to the remaining 4 edges of C we consider two cases (see Figure 2.)

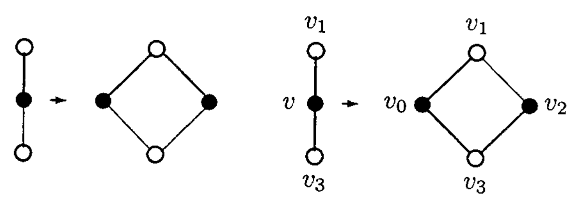

Figure 2: An illustration of cases 1 and 2 in the proof of Theorem 2.1.

Case 1. The edges v_1v and vv_3 have different colors, say 1 and 2 respectively. Then we color the edges v_1v_0, v_1v_2 with 1 and the edges v_3v_0, v_3v_2 with 2.

Case 2. The edges v_1v and vv_3 have the same color, say 1. Since the vertices v_1, v_3 belong to the same chromatic class of the bipartition of G' and the decomposition is assumed to be good if it has type $(0,2)$, these vertices can not both belong to $V(T_1') \setminus V(T_2')$. We may assume that v_1 belongs to $V(T_1') \cap V(T_2')$. If $v \in V(T_1') \cap V(T_2')$ as well, there is an only path in T_2' connecting v_1 with v. Therefore, there is an only path either from v_0 or from v_2 to v_1 in G with all edges colored 2. We may assume that this path connects v_0 with v_1. Then, we color with 1 the edges v_1v_0, v_3v_0, v_3, v_2 and give color 2 to the edge v_1v_2. We make the same assignment if $v \notin V(T_1') \cap V(T_2')$.

Let T_i be the subgraph of G generated by the edges colored i, $i = 1, 2$. We have an edge decomposition $G = T_1 \oplus T_2$. In both cases above, we have increased by one the number of edges and vertices of T_1' and T_2' and the resulting graphs are clearly acyclic. Therefore they are both trees. Hence, $G = T_1 \oplus T_2$ is a minimal tree decomposition of the same type as $G' = T_1' \oplus T_2'$ and, if the type is $(0,2)$, then it is a good decomposition. The proof follows by induction on n. □

We next consider the problem of completing a given tree T to a mpb graph which admits T in a minimal tree decomposition. Note that the star S_n with n edges can not be required to be a spanning tree in a minimal decomposition of a mpb graph. Let $P_3^{r,s}$, $r, s \geq 1$, denote the tree obtained from a path P_3 with three vertices by adding r leaves to an end vertex and s leaves to the other one (see Figure 3.) The following easy Lemma shows that $P_3^{r,s}$ can not be both a factor and a spanning tree of a mpb graph.

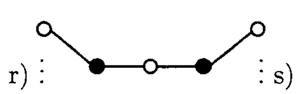

Figure 3: A tree $P_3^{r,s}$.

Lemma 2.2 *Let G be a mpb graph such that $G = P_3^{r,s} \oplus T$ for some tree T. Then $P_3^{r,s}$ is not a spanning tree of G.*

Proof. Suppose that $P_3^{r,s}$ is a spanning tree. There are exactly two vertices u, v in one of the chromatic classes of the bipartition of G, as well as in $P_3^{r,s}$, and they are at distance 2 in the tree. All vertices in $P_3^{r,s}$ adjacent to u must be adjacent to v in G and viceversa, but u and v can not be adjacent in G, so that T is not connected, a contradiction.□

Let us denote by $\mathcal{F}$ the class of stars and $P_3^{r,s}$-trees, $r, s \geq 1$.

Theorem 2.3 *Let T be a given tree of order $n \geq 4$. There is a mpb graph G and a tree T_2 such that $G = T \oplus T_2$ is a minimal tree decomposition of type (a,b) for any choice of $b \geq a \geq 0$ with $a + b = 2$ unless $a = 0$ and $T \in \mathcal{F}$.*

Proof. The proof is by induction on the order n of the given tree T. As in the proof of Theorem 2.1, we prove that the decompositions of type $(0,2)$ may be assumed to be good. For $n = 4$, the corresponding mpb graphs which admit the star S_3 or the path P_4 as factors in minimal decompositions of each type are shown in Figure 4.

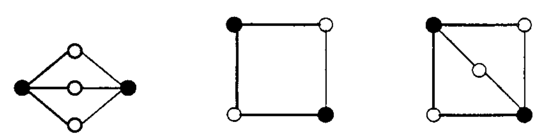

Figure 4: The mpb graphs admiting (a) S_3 and (b) P_4 as factors.

Let T be a given tree of order $n > 4$. Let l be an end vertex in T such that $T' = T - l \notin \mathcal{F}$, or any end vertex if there is no one with this property. Let vl be the edge of T incident to l.

By the induction hypothesis, for any choice of $b \geq a \geq 0$ such that $a + b = 2$, there is a mpb graph G' and a tree T_2' such that $G' = T' \oplus T_2'$ and the decomposition is of type (a,b) unless $a = 0$ and $T' \in \mathcal{F}$. Moreover, if the decomposition is of type $(0,2)$, then the vertices belonging only to the spanning tree are in different chromatic classes of the bipartition of G'.

Let R be the set of faces incident to v in a planar embedding of G', and let U be the set of vertices incident to the faces in R in the same chromatic class as v in the bipartition of G'.

Suppose that there is a vertex $w \in U \cap V(T_2')$. Then the addition of vertex l and edges vl, wl to G'

results in a mpb graph G which admits the decomposition $G = T \oplus T_2$, where $T_2 = T_2' + wl$.

Both T, T_2 are obtained from T', T_2' by adding one vertex and an one edge. Therefore, the decomposition of G is of the same type as the one of G'. Moreover, if the decomposition is of type $(0, 2)$, then the vertices belonging only to the spanning tree are in different chromatic classes of the bipartition of G.

Suppose now that $U \cap V(T_2') = \emptyset$. By the induction hypothesis, we may assume that, if the decomposition is of type $(0, 2)$ then it is good. Therefore, there is an only vertex $u \in U$. This implies that u and v are the only vertices in their chromatic class and they have the same neighbours $w_1, \ldots, w_t$, $t \geq 2$. One of them, say w_1, must be in a path in T' connecting v to u. Therefore, both u and w_1 belong to $V(T') \setminus V(T_2')$ and the decomposition is of type $(0, 2)$. Moreover, $T = P_3^{1,t-1}$.

Let v, w_1, w_2, u be a face in a planar embedding of G'. Let G be the graph obtained from G' by adding two new vertices l and z and edges vl, w_2z, zl, lu. Then G is a mpb graph and $G = T \oplus T_2$, where $T_2 = T_2' + \{u, z, l\} + \{w_2z, zl, lu\}$, is a decomposition of type $(1, 1)$.

To complete the proof we have to show that, if $T \notin \mathcal{F}$ but, for any end vertex l of T, $T - l \in \mathcal{F}$, then there is a mpb graph of order n and a tree T_2 such that $G = T \oplus T_2$ and T is a spanning tree of G. Note that such a tree T must be the path P_6 of six vertices for which a completion to the required mpb graph is shown in Figure 5. □

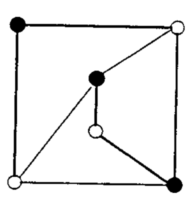

Figure 5: A mpb graph admiting P_6 as factor in a decomposition of type $(0, 2)$.

3 Maximal planar graphs

We recall that a minimal tree decomposition of a maximal planar (mp) graph G of order n must be of type (0,0,3), (0,1,2) or (1,1,1).

If $G = T_1 \oplus T_2 \oplus T_3$, we say that a vertex v is non singular in this tree decomposition if it belongs to the three trees. Otherwise we say that v is *singular.*

We first prove that a mp graph admits the three types of decompositions. The proof below is much simpler than the one by Shi [9] required to show a similar statement for the wider class of graphs $\mathcal{P}_3$.

Theorem 3.1 *Let G be a maximal planar graph of order $n \geq 5$ and x a given vertex in G. There is a minimal tree decomposition G of type (a, b, c) for each choice of $c \geq b \geq a \geq 0$ with $a + b + c = 3$ such that x is a non singular vertex.*

Proof. The proof is by induction on the order n of G.

The result clearly holds for $n = 5$ as shown in Figure 6.

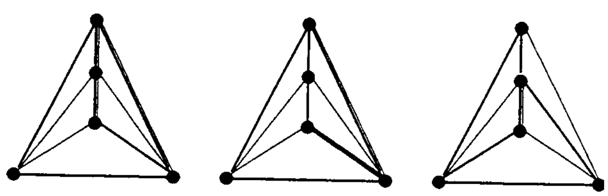

Figure 6: Decompositions of the mp graph of order 5, $K_5 - e$, of each type with a chosen nonsingular vertex.

Let G be a mp graph of order $n > 5$. An edge xy of G is said to be *contractible* if x and y have exactly two common neighbors, say u, w. In this case, the contraction of xy and the identification of the pairs of edges ux, uy and xw, yw gives rise to a mp graph G' of order $n - 1$. Kampen proves in [4] that each vertex of a mp graph of order $n > 3$ is incident to a contractible edge.

Let $c \geq b \geq a \geq 0$ such that $a + b + c = 3$ and let x be an arbitrary vertex in G. Let v_1v_2 be a contractible edge incident to $x = v_1$. Denote by u, w the common neighbours of v_1 and v_2 in G. Let G' be the mp graph obtained from G by contracting v_1v_2 to a vertex v. By the induction hypothesis, there is a minimal decomposition $G' = T_1' \oplus T_2' \oplus T_3'$ of type (a, b, c) such that v is a non singular vertex.

Let H be the subgraph of G induced by the four vertices u, v_1, v_2, w. Color each edge e of $G - H$

with color $c(e) = i \in \{1, 2, 3\}$ if the corresponding edge in G' belongs to the tree T_i'.

We consider two cases.

Case 1. Suppose that both uv and vw belong to the same tree, say T_1'. We may assume that $b \leq 1$. We may also assume that $w \in V(T_2')$ and that the only path in T_2 joining v and w in T_2' corresponds to a path of edges colored 2 in G joining v_1 and w. Then we color the six edges of H as

$$c(uv_1) = c(uv_2) = c(v_1w) = 1,$$
$$c(v_2w) = 2, \quad c(v_1v_2) = 3,$$

see Figure 7.

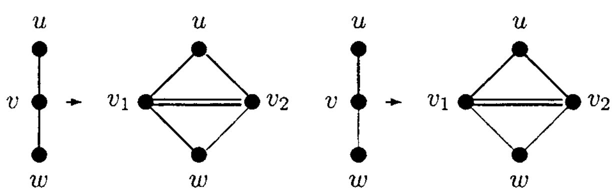

Figure 7: Coloring of H in (i) case 1, and (ii) case 2.

Case 2. Suppose that uv and vw belong to different trees, say $uv \in E(T_1')$ and $vw \in E(T_2')$. Then we color the six edges of H as

$$c(uv_1) = c(uv_2) = 1,$$
$$c(v_1w) = c(v_2w) = 2,$$
$$c(v_1v_2) = 3,$$

see Figure 7.

Let T_i be the graph spanned by edges colored i in G, $i = 1, 2, 3$. Each T_i is obtained from T_i' by the addition of one edge and one vertex without forming any cycle. Therefore each one is a tree and $G = T_1 \oplus T_2 \oplus T_3$ is a decomposition of G of type (a, b, c). Moreover, both v_1, v_2 are non singular in this decomposition. The proof follows by induction. □

We now proceed to prove that every tree is a factor of a mp graph in a decomposition of a chosen type.

Theorem 3.2 *Let T be an arbitrary tree of order $n \geq 5$. There is a maximal planar graph G which admits T as a factor in a minimal tree decomposition of type (a, b, c) for any choice of $c \geq b \geq a \geq 0$ with $a + b + c = 3$.*

Proof. The proof is by induction on the order of T. It can be easily checked that the result holds for trees with $n = 5$ vertices.

Let T be a tree of order $n > 5$ and $c \geq b \geq a \geq 09$ with $a + b + c = 3$. Let l be an end vertex of T and vl the only edge incident to l.

Let $T' = T - l$. By the induction hypothesis there is a mp graph $G' = T' \oplus T_2' \oplus T_3'$ and the decomposition is of type (a, b, c). Let $W = \{w_0, \ldots, w_{k-1}\}$ be the set of neighbours of v in G' numbered in clockwise order in a planar embedding of the graph.

Let us show that there is i such that $w_i \in V(T_2')$ and $w_{i+1 \pmod{k}} \in V(T_3')$ (or viceversa). Set $C = V(G') \setminus V(T_3')$. We can not have $W \subset C$, since otherwise we would have $|W| = 3$ and $v \in C$, contradicting $c \leq 3$. We may assume $w_1 \in W \setminus C$. Since $b \leq c$, we have $b \leq 1$. hence, either $w_0 \in V(T_2')$ or $w_2 \in V(T_2')$.

Let G be the mp graph obtained from G' by adding a new vertex l and the edges vl, w_0l, w_1l. Then $T_2 = T_2' + w_0l$ and $T_3' + w_1l$ are both trees and $G = T \oplus T_2 \oplus T_3$ is a decomposition of type (a, b, c). The proof follows by induction. □

4 Regular graphs

Minimum tree decompositions of regular graphs have been considered in [6, 7]. If G is a regular graph of degree r, then the minimum number of trees in such a decomposition is $\lfloor r/2 \rfloor + 1$, and this bound is achieved for dense graphs $r \geq n/2$, where n is the order of G, and for graphs with good isoperimetric bounds.

Here we consider the opposite problem: given a tree T of order n we ask for regular graphs with minimum degree and order which admit T as a factor in a tree decomposition with $\lfloor r/2 \rfloor + 1$ trees. Then T should be a spanning tree of G, and we consider decompositions of type $(0, \ldots, 0, l)$ for some $l < n$. In this case, the degree r of G must satisfy $r \geq 2\Delta(T) - 3$ and nr must be even. We show that these necessary conditions are also sufficient with two single exceptions. When $\Delta(T) = (n+2)/2$ then $r = 2\Delta(T) - 3 = n - 1$ and G would be the complete graph K_n with n even, which decomposes into $n/2$ spanning trees. Then the maximum degree of each of the trees in such a decomposition is at most $n/2$. The second exception is the double star dS_k consisting of two stars, each one with k end vertices, whose central vertices are joined by an edge. The maximum degree of dS_k is $n/2$ and G would have degree $n-3$. It can be shown that no such graph admits dS_k as a factor in a decomposition of type $(0,\dots,0,l)$.

Theorem 4.1 *Let $r \geq 2$ be an integer and let T be a tree of order $n \geq r+1$ with nr even and maximum degree $\Delta(T) \leq \lceil \frac{r}{2} \rceil + 1$. Then there is a regular graph G of order n and degree r such that*

$$G = T \oplus T_2 \oplus \cdots \oplus T_k \oplus T_{k+1}$$

is a minimum tree decomposition of type $(0,\dots,0,l)$, where $k = \lfloor r/2 \rfloor$, $l = n-k-1$ if r is even and $l = (n/2)-k-1$ if r is odd.

We shall need the following two results. Recall that a sequence $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ is said to be *graphical* if there is a graph G with vertex set $\{x_1,\dots,x_n\}$ such that $d(x_i) = d_i$, $i = 1,2,\dots,n$.

Theorem 4.2 (Kleitman, Wang [5]) *Let $d_1 \geq d_2 \geq \cdots \geq d_n \geq k$ be a graphical sequence such that $\sum_{i=1}^n d_i \geq 2k(n-1)$. Then the sequence is realizable by a graph G that has k edge disjoint spanning trees.*

Graphical sequences are characterized by the following result.

Theorem 4.3 (Erdős, Gallai [3]) *Let $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ be a sequence of integers such that $\sum_{i=1}^n d_i$ is an even number. There is a graph G with vertex set $\{x_1,\dots,x_n\}$ such that $d(x_i) = d_i$ if and only if, for each $l = 1,2,\dots,n$*

$$\sum_{i=1}^{l} d_i \leq l(l-1) + \sum_{i=l+1}^{n} \min\{l, d_i\}. \qquad (1)$$

The strategy of the proof of Theorem 4.1 is to show that the tree T together with an appropriate T_{k+1} can be packed with a graph G_1 which has $k-1$ edge disjoint trees in a regular graph G. The existence of graph G_1 is guaranteed by the above two Theorems. We need two lemmas to proceed with the proof.

Lemma 4.4 *Let T be a tree of order n and maximum degree $\Delta(T)$, and $k \geq 2$ an integer.*

(i) If $n \geq 2k+1$ and $\Delta(T) \leq k+1$ then there is a packing of T with a path P of order $k+1$ such that $\Delta(T \oplus P) \leq k+1$.

(ii) If $n \geq 2k+4$, n is even, $\Delta(T) \leq k+2$ and $T \neq dS_{k+1}$ then there is a packing of T with a path P of order $n/2+k+1$ such that $\Delta(T \oplus P) \leq k+2$.

Proof. Let $1 = d_1 = d_2 \leq \cdots \leq d_n$ be the degree sequence of T.

(i) We have $d_{k+2} \leq 2$, since otherwise $2(n-1) = \sum_{i=1}^n d_i \geq k+1+3(n-k-1)$, which implies $n \leq 2k$.

If $k = 2$, we can permute if necessary x_{k+1} with x_{k+2} such that the subgraph of T induced by x_1, x_2, x_3 has maximum degree 1. Therefore, there is a path P of order 3 which can be packed with $T[x_1,x_2,x_3]$ and then $\Delta(T \oplus P) \leq 3$.

If $k \geq 3$ then the subgraph of T generated by $\{x_1,\dots,x_{k+1}\}$ has maximum degree $2 \leq (k+1)/2$. Hence, its complement is hamiltonian. In particular, there is a path P of order $k+1$ in the complement of $T[x_1,\dots,x_{k+1}]$. Therefore, $\Delta(T \oplus P) \leq k+1$.

(ii) Let us show that $d_j \leq k$ for $j \leq \frac{n}{2}+k-1$ and $d_j \leq k+1$ for $j \leq \frac{n}{2}+k+1$. Suppose that $d_j \geq k+1$ for $j = \frac{n}{2}+k-1$. Then $2(n-1) \geq \frac{n}{2}+k-2+(k+1)(\frac{n}{2}-k+2)$ which implies $n \leq 2k$. Similarly, if $d_j \geq k+2$ for $j = (n/2)+k+1$. Then $2(n-1) = \sum_{i=1}^n d_i \geq \frac{n}{2}+k+(\frac{n}{2}-k)(k+2)$, which implies $(2k+4)(k-1) \geq n(k-1)$. Hence $n = 2k+4$ and the degree sequence is $1,\dots,1,k+2,k+2$, which corresponds to the double Star dS_{k+1}.

Hence, the complement of the subgraph induced by $\{x_1,\dots,x_{n/2+k+1}\}$ has a hamiltonian path P with end vertices $x_{n/2+k}, x_{n/2+k+1}$ and $\Delta(T \oplus P) \leq k+2$. □

Lemma 4.5 *Let $2k-\epsilon \geq d_1 \geq \cdots \geq d_n \geq k-1$, $\epsilon \in \{0,1\}$, $k \geq 2$, be a sequence of integers such that $\sum_{i=1}^n d_i = 2(k-1)(n-1)$. Then the sequence is graphical for $n \geq 2k+4-2\epsilon$.*

Proof. If $k = 2$ then $\sum_{i=1}^n d_i = 2(n-1)$ and there is a tree realizing the degree sequence.

Suppose that $k \geq 3$. Let

$$\varphi(l) = l(l-1) + \sum_{i=l+1}^{n} \min\{l, d_i\} - \sum_{i=1}^{l} d_i.$$

According to Theorem 4.3, we have to show that $\varphi(l) \geq 0$ for $l = 1, \ldots, n$. If $l \geq d_1 + 1$ then $\varphi(l) \geq l(l - d_1 - 1) \geq 0$. Put $d_0 = n$ and $d_{n+1} = 0$.

For each $l = 1, \ldots, d_1$, let $s = s_l$ be the minimum subindex such that $d_{s+1} \leq l$.

Suppose that $s \leq l$. Then

$$\varphi(l) = l(l-1) + \sum_{i=l+1}^{n} d_i - \sum_{i=1}^{l} d_i.$$

If $l = d_1$ then we have $\varphi(l) \geq (n - d_1)d_n - d_1 \geq 3(k-1) - 2k \geq 0$. Suppose that $l < d_1$. If $d_1 \leq 2k - 1$ or $l = 2k - 2$ when $d_1 = 2k$, then

$$\begin{aligned} \varphi(l) &= l(l-1) + 2(k-1)(n-1) - 2\sum_{i=1}^{l} d_i \\ &\geq 2(k-1)(n-1) - l(2d_1 - l + 1) \geq 0. \end{aligned}$$

Finally, if $l = 2k - 1$, we use that $\sum_{i=l+1}^{n} d_i \geq (n - l)(k - 1)$ to prove $\varphi(l) \geq 0$.

Suppose now that $s > l$. Then,

$$\varphi(l) = l(s-1) + \sum_{i=s+1}^{n} d_i - \sum_{i=1}^{l} d_i.$$

If $s \geq d_1 + 1$ then $\varphi(l) \geq l(s - d_1 - 1) \geq 0$. If $s = d_1$ then we have $\varphi(l) \geq (n - d_1)d_n - l \geq 3(k-1) - 2k \geq 0$. Similarly, if $s = d_1 - 1$. Finally, if $s < d_1 - 1$, as $\varphi(l) = l(s-1) + 2(k-1)(n-1) - \sum_{i=1}^{s} d_i - \sum_{i=1}^{l} d_i$, then

$$\begin{aligned} \varphi(l) &\geq l(s-1) + 2(k-1)(n-1) - (s+l)d_1 \\ &\geq (d_1 - 3)^2 + 2(k-1)(n-1) - (2d_1 - 5)d_1 \\ &\geq -d_1^2 - d_1 + 9 + 2(k-1)(n-1) \geq 0. \end{aligned}$$

Therefore, for each $l = 1, \ldots, n$, we have $\varphi(l) \geq 0$ and the sequence is graphical. □

Proof of Theorem 4.1 Let $1 = d_1 = d_2 \leq \cdots \leq d_n$ be the degree sequence of the given tree T.

Suppose first that r is even. We have $d_n \leq k+1$ where $k = r/2$. By Lemma 4.4, there is a path T_{k+1} of order $k+1$ such that $\Delta(T \oplus T_{k+1}) \leq k+1$. Let $1 \leq d_1'' \leq d_2'' \leq \cdots \leq d_n'' \leq k+1$ be the degree sequence of $T \oplus T_{k+1}$. The sequence

$$2k - 1 \geq d_1' \geq d_2' \geq \cdots \geq d_n' \geq k - 1,$$

where $d_i' = r - d_i''$, satisfies $\sum_{i=1}^{n} d_i' = rn - 2(n-1) - 2k = 2(k-1)(n-1)$. If $r = n - 1$, then the sequence corresponds to the complement of $T \oplus T_{k+1}$ and thus it is graphical. If $r < n - 1$ then $n \geq 2k + 2$ and, by lemma 4.5, the sequence is also graphical. By Theorem 4.2 it is realizable by a graph G_1 with $(k-1)$ edge disjoint trees. By construction, the graph $G = G_1 \oplus T \oplus T_{k+1}$ is r-regular and has a minimal decomposition of type $(0, \ldots, 0, n-k-1)$.

The proof is similar if r is odd. Then, $\Delta(T) \leq k+2$ where $r = 2k+1$ and n is even. By Lemma 4.4, there is a path T_{k+1} of order $(n/2) + k + 1$ such that $\Delta(T \oplus T_{k+1}) \leq k+2$. Let $1 \leq d_1'' \leq d_2'' \leq \cdots \leq d_n'' \leq k+1$ be the degree sequence of $T \oplus T_{k+1}$. The sequence

$$2k - 1 \geq d_1' \geq d_2' \geq \cdots \geq d_n' \geq k - 1,$$

where $d_i' = r - d_i''$, satisfies $\sum_{i=1}^{n} d_i' = rn - 2(n-1) - 2((n/2) + k) = 2(k-1)(n-1)$. If $r = n - 1$, then the sequence corresponds to the complement of $T \oplus T_{k+1}$ and thus it is graphical. If $r < n - 1$ then $n \geq 2k + 4$ and, by lemma 4.5, the sequence is also graphical. By Theorem 4.2 it is realizable by a graph G_1 with $(k-1)$ edge disjoint trees. By construction, the graph $G = G_1 \oplus T \oplus T_{k+1}$ is r-regular and has a minimal decomposition of type $(0, \ldots, 0, (n/2) - k - 1)$. □

Note that, as a consequence of Theorem 4.1, we have the following Corollary.

Corollary 4.6 *Every spanning tree with maximum degree $\Delta(T) \leq \lfloor \frac{n}{2} \rfloor + 1$ in a complete graph K_n can be extended to a minimal tree decomposition.*

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