A Brief Survey on Graph Labelings[1]

Francesc Antoni Muntaner Batle [2]

Abstract

The goal of this paper is to introduce the reader to the topic of graph labelings. In particular we will focus our attention to edge magic and super edge magic labelings.

1 Introduction

Through this document we will understand a graph to be finite and simple, that is to say without loops or multiple edges. Also by a pseudograph we will understand a finite simple graph which may contain loops, multiple edges or both.

The area of graph theory has experimented a fast development during the last 60 years. Among all the different kinds of problems that appear while studying graph theory, one that has been growing strong during the last three decades is the area that studies labelings of graphs. This is not only due to its mathematical importance but also because of the wide range of applications arising from this area. For instance we can find labelings of graphs showing up in x-rays, crystallography, coding theory, radar, astronomy, circuit design, and communication design. A good account on specific applications of graph labelings can be found in [1] and [2]. The goal is the study of some kinds of graph labelings, namely edge magic and super edge magic labelings, which were first introduced by Kotzig and Rosa in 1970 [17] and by Enomoto, Lladó, Nakamigawa and Ringel in 1998 respectively [4]. However some other kinds of labelings will appear, as for instance graceful labelings, harmonious labelings, cordial labelings, etc. In this document we will give a precise situation of the problem to be attacked, as well as the tools used in order to accomplish our objectives. We will do this in the way presented next.

Section 3 (*Problem Status*) In this section we give a general introduction to the subject of graph labelings, introducing the most popular types of labelings as well some of the most challenging open problems. This will allow us to introduce our problem in its historical contest. Then we will introduce the basic definitions and main results obtained about magic and super magic labelings up to this date as well as the main open problems in this area. We will also introduce some related topics, as well as their current status.

Section 4 (*Summary of the results obtained by the authors*) We will structure the results obtained by the authors into different classes, providing a brief summary of the results found in each class.

2 Problem Status

The subject of graph labelings has developed enormously during the last three decades, and is nowadays a very active area of research in combinatorics. With more than 400 papers in the literature, and a very complete dynamic survey by Joseph Gallian [14], this new branch of mathematics has caught the attention of many authors, and many new results appear day after day.

It is important to distinguish between two major classes of labelings, vertex labelings and total labelings (see [14]).

A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces a label for each edge uv of G, depending on the labels $f(u)$ and $f(v)$. The oldest and more popular vertex labeling is the one introduced by Rosa [24] in 1967, called graceful labeling. Graceful labelings were introduced in order to provide an alternative way to attack the conjecture of Ringel [22] which states that the complete graph K_{2n+1} is decomposable into $2n+1$ subgraphs that are all isomorphic

[1]Trabajo parcialmente subvencionado por Universidad Politécnica de Cataluña

[2]Dpto. de Matemática Aplicada IV. Universidad Politécnica de Cataluña. E-mail: muntaner@mat.upc.es

to a given tree of size n.

A function f is called a graceful labeling of a graph G with size q if f is an injective function from $V(G)$ to the set $\{0,1,2,...,q\}$ with the property that the function $\overline{f}$ with domain $E(G)$ and range in $\mathbb{Z}$, defined by the rule $\overline{f}(uv) = |f(u) - f(v)|$, assigns different labels to the edges of G. If a graph G admits a graceful labeling, then we say that G is a graceful graph.

Although Erdös proved in an unpublished paper that almost no graphs are graceful, many particular families of graphs have been proven to admit graceful labelings and some other families of graphs have been conjectured to admit such labelings, for more information see [14]. In particular, the most important conjecture in this direction (and probably in the whole area of graph labelings) is the one known as the Ringel-Kotzig conjecture that all trees are graceful. This conjecture has been the focus of many papers, see [14], but, in spite of all efforts, no major progress has been made towards the final solution. It is worth mentioning that one of the main reasons this conjecture has become so popular, is because a positive answer to it implies the truth of Ringel's conjecture, mentioned before. Another well-known conjecture concerning graceful labelings is the one that states that all unicyclic graphs except C_n where $n \equiv 1$ or $2 (\mathrm{mod}\, 4)$ are graceful. This one is known as Truszczyński's conjecture [25]. Truszczyński proved some particular families of unicyclic graphs to be graceful, and Doma [3], studied the gracefulness of several unicyclic graphs of order less than or equal to q. In spite of all this work, general results about Truszczyński's conjecture are non-existing up to the present time.

Another important vertex labeling, that has also been the focus of study of many papers, is the harmonious labeling. Harmonious labelings were first introduced in 1980 by Graham and Sloane [15] as a possible way to study additive basses. A labeling f of the vertices of a graph G of size q is called harmonious if f is an injective function from $V(G)$ to the additive group Z_q such that the function $\overline{f}$ from the set $E(G)$ to Z_q defined by the rule $\overline{f}(uv) = f(u) + f(v) (\mathrm{mod\, q})$ assigns different labels to the edges of G. If G is a tree, then the condition that f is injective is relaxed and exactly two vertex labels are allowed to be equal. If a graph G admits a harmonious labeling, then it is said to be a harmonious graph. As in the case of graceful labelings, in the original paper, Graham and Sloane proved that almost no graphs are harmonious and also conjectured that all trees admit harmonious labelings. Since then many different families of graphs have been proven to be harmonious, see [14], however general results are rare, and it seems that at least in the near future the conjecture that all trees are harmonious is out of reach.

Motivated by these two types of vertex labelings, many authors have defined a large amount of different vertex labelings that Gallian [14] divides into two main groups. The first group is called "variations of graceful labelings" and the second one is called "variations of harmonious labelings".

Among the most important labelings in the first group, we find α -labelings, odd graceful labelings, graceful like labelings, cordial labelings, k-equitable labelings and hamming-graceful labelings. Among the most important labelings of the second group we find sequential and strongly C -harmonious labelings, elegant labelings and felicitious labelings. For more information about these labelings, the reader is directed to Gallian's survey [14].

The other important major class of labelings, as we said before, is the class of total labelings. A total labeling is a function from the set of vertices union the set of edges of a given graph G to a set of labels. The most important labelings of this type are k-sequential labelings, sequentially additive labelings, edge magic labelings and super edge magic labelings. The main focus of our work will be on edge magic and super edge magic labelings, to which we will pay special attention in the next lines.

Edge Magic Labelings

The concept of edge magic labeling was first introduced by Kotzig and Rosa [17] in 1970. They defined an edge magic labeling of a graph G with order p and size q to be a bijective function f from the set $V(G) \cup E(G)$ to the set $\{1,2,...,p+q\}$ with the property that for every edge uv of $E(G)$ the sum $f(u) + f(uv) + f(v) = k$, called the valence of the labeling f, is constant. If a graph G admits an edge magic labeling then we call such graph an edge magic graph. In their first paper, Kotzig and Rosa proved that the complete bipartite graph $K_{m,n}$ and the cycle C_n are edge magic. They also proved that the graph mK_2 is edge magic if and only if m is odd. Also, in this first paper, they studied several classes of forests proving that they admit edge magic labelings. Another interesting definition that appears in this paper is the concept of magic deficiency. Kotzig and Rosa defined the magic deficiency of a graph G to be the minimum number of isolated vertices that we have to union G with, in order to obtain an edge magic graph. In relation to this matter, they proved that the magic deficiency of a graph G with n vertices is upper bounded by the quantity $F_{n+2} - 2 - \binom{n+1}{2}$ where F_n denotes the n^{th} Fibonacci number. They concluded the paper with two questions that nowadays are still open and will be stated next.

Problem 1: Does there exist an edge magic labeling for any tree?

Problem 2: What is the necessary and sufficient condition for a regular graph of degree two (three and four respectively) in order to have an edge magic labeling?

Later in 1972 Kotzig and Rosa [18] showed by using tools that today we would include in the area of additive number theory, that the complete graph K_n is edge magic if and only if $n = 1, 2, 3, 5$. At this point the research about edge magic labelings stopped, and it was not until 1996 when Ringel and Lladó [18] unaware of the work done by Kotzig and Rosa introduced the same concept, and in their paper, they reproved some of the Kotzig and Rosa results. But in addition, they proved that a graph G of order p and size q is not edge magic if q is even, $p + q \equiv 2 (\bmod 4)$ and the degree of every vertex is odd. They also conjectured that all trees admit magic labelings. This paper was enormously helpful in order to popularize the subject.

Super Edge Magic Labelings

Motivated by the concept of edge magic labelings, Enomoto, Lladó, Nakamigawa, and Ringel defined in 1998 [4] the concept of super edge magic labelings. Let G be a graph of order p and size q, a function f is a super edge magic labeling of G, if f in addition of being an edge magic labeling of G has the extra property that $f(V(G)) = \{1, 2, 3, ..., p\}$. If a graph G admits a super edge magic labeling then G is called a super edge magic graph. In their first paper about this subject, they proved that the cycle C_n is super edge magic if and only if n is odd and that if a graph G of order p and size q is super edge magic then $q \leq 2p - 3$. Also, it was conjectured that all trees are super edge magic. Later on, Enomoto, Masuda and Nakamigawa [5], using results from additive number theory proved that any graph H is an induced subgraph of some super edge magic connected graph G.

Product Magic Labelings

Another type of labelings that have been studied are the product magic and antimagic labelings. Next we present the necessary definitions, not without first specifying that the only paper that appears, up to this point in the literature, is the one introduced by Figueroa, Ichishima and Muntaner in 2000 [7]. A graph G of size q is product magic if there exists a labeling from $E(G)$ onto $\{1, 2, ..., q\}$ such that at each vertex v, the product of the labels on the edges incident with v is constant, while it is called product antimagic if all products are different. Also, a (p, q) graph G is product edge magic if there exists a labeling f from $V(G) \cup E(G)$ onto $\{1, 2, ..., p + q\}$ with the property that $f(u) \cdot f(uv) \cdot f(v) = k$ is constant, where k is called the valence of the labeling, for every edge uv of G. On the other hand it is said to be the product edge antimagic if all the valences are distinct.

3 Summary of the work done by the authors

3.1 Examples and basic results

The first few papers were basically devoted to exhibit examples of infinite families of graphs which admit edge magic and super edge magic labelings. We also introduced some basic results about edge magic and super edge magic labelings and graphs. For instance we provided necessary conditions for graphs to admit edge magic and super edge magic labelings, as well as a formula that counts the number of super edge magic labeled graphs in the set of graphs with order p and size q. This work can be found in [7] and [10].

3.2 The place of super edge magic labelings among other labelings

The goal of this work was to exhibit the relations existing between edge magic and super edge magic labelings with other well studied types of labelings. For instance harmonious labelings, sequential labelings, cordial labelings, and some times even graceful labelings. This work can be found in [7].

3.3 Results involving graph operations

In this section we can include the following results. If a bipartite or a tripartite graph G is edge magic or super edge magic, then the graph $H = (2k+1)G$ where $k \in N \cup \{0\}$ is also edge magic or super edge magic respectively. This result can be found in [11]. We also proved that if a graph $G = \bigcup_{i=1}^{K(G)} C_{n_i}$, where $K(G)$ denotes the number of components of G, is an edge magic or super edge magic graph and m is any odd interger, then the graph $H = \bigcup_{i=1}^{K(G)} (m, n_i)\ C_{[m,n_i]}$, where (m, n_i) and $[m, n_i]$ denotes the greatest common divisor and the least common multiple respectively, is also edge magic or super edge magic respectively, as a first approach to the question of Kotzig and Rosa [17] of characterizing the set of magic 2-regular graphs. This last result was attained very recently and the paper is still under preparation. We also suspect that the following generalization of the previous result may be even true. If $G_1 = \bigcup_{i=1}^{K(G_1)} C_{n_i}$ and $G_2 = \bigcup_{i=1}^{K(G_2)} C_{m_i}$ are two 2-regular edge magic or super edge magic graph, then the graph $\bigcup_{i=1}^{K(G_1)} \bigcup_{j=1}^{K(G_2)} (n_i, m_j) C_{[n_i,m_j]}$ is also edge magic or super edge magic respectively.

We have also studied different ways of combining super edge magic graphs using the corona product in order to generate new families of super edge magic graphs. This study has lead us to obtain some nice general results as well as to the solution of Yegnayaranan conjecture [26]. See [8] for further information.

In Order to conclude this section we will mention that new results involving amalgamation of trees with other trees and unicyclic graphs have been descovered recently. This work can be found at [12] and [13].

3.4 Matrices and labelings

The concept of super edge magic labeling, can be naturally generalized to pseudographs as follows.

A *pseudograph* P of order p and size q is super edge magic if there exists a bijective function $f : V(P) \cup E(P) \to \{1, 2, ..., p+q\}$ satisfying that $f(V(P)) = \{1, 2, ..., p\}$ and $f(u)+f(uv)+f(v) = k$ is constant for any edge uv of $E(P)$. If such a labeling f exists, then f is called a super edge magic labeling of P and k is called the valence of f. It is easy to see that the only pseudographs that may admit super edge magic labelings are those without multiple edges nor multiple loops.

With the aid of matrices, we studied how starting with a super edge magic labeling of either a graph or a pseudograph P, we can obtain a super edge magic labeling of a bipartite graph G with degree sequence depending on the degree sequence of P. This result has been already presented in [21].

We also studied how we can use the Kronecker product of matrices in order to obtain super edge magic labelings of new graphs starting with super edge magic labelings of other graphs. This work is still on preparation.

3.5 Super edge magic deficiency and isolated vertices

We introduced a possible way of studying "how close" is a graph G to be super edge magic, by means of the minimum number of isolated vertices that we have to union G with, in order to obtain a super edge magic graph. This concept is called the super edge magic deficiency, and it was motivated by the concept of magic deficiency introduced in the original paper by Kotzig and Rosa.

The super edge magic deficiency of a graph G was defined as follows:

Let G be any graph and define the set $M(G)$ to be
$M(G)=\{n \in N \cup \{0\} : G \cup nK_1 \text{ is super edge magic}\}$.

Then, the super edge magic deficiency of G, written as $\mu_s(G)$, is defined to be

$$\mu_s = \begin{cases} \min M(G) & \text{if M(G)} \neq \emptyset, \\ +\infty & \text{if M(G)} = +\infty. \end{cases}$$

We gave sufficient conditions for a graph G, to have super edge magic deficiency $+\infty$. Also, using ideas of additive number theory, we proved that

$$\mu_s(K_n) = \begin{cases} 0 & \text{if n} \leq 3 \\ 1 & \text{if n} = 4 \\ +\infty & \text{if n} \geq 5 \end{cases}$$

The super edge magic deficiency of the complete bipartite graphs and acyclic graphs has been studied, and exact formulas for the super edge magic deficiency of some infinite classes of two regular graphs and forests have been computed. This work can be found in [9].

3.6 Super edge magic labelings and additive number theory

We already mentioned in this document, that we have attained some results about super edge magic labelings using tools from additive number theory. The main objective here was to study "how powerful" may additive number theory be in order to study edge magic and super edge magic labelings. In paticular we have been able to obtain reults involving the size of the largest clique in a super edge magic conected graph improving older results obtained by Enomoto et al. (see [5]). Also other extremal results involving forests imbeded in super edge magic trees have been obtained. This work can be found in [19] and [20].

3.7 Magical and antimagical product labelings

We borrowed some well known tools from classical number theory, as for instance, Bertrand's postulate and Ingham's theorem, in order to completely characterize product magic and product edge magic graphs without isolated vertices. Also several classes of graphs have been shown to admit product antimagic labelings and results involving joins and corona products of graphs have also been discussed. Finally, a characterization of product edge antimagic graphs was provided. These results are available in [6].

3.8 Probability

The main goal of this work has been to study the number of super edge magic graphs from an asymptotic point of view. This work can be found in [16].

References

[1] G. S. Bloom and S. W. Golomb. *Applications of Numbered Undirected Graphs.* Proc. IEEE. vol.5. pp.562-570. 1977

[2] G. S. Bloom and S. W. Golomb. *Numbered Complete Graphs, Unusual Rulers, and Assorted Applications.* Theory and Applications of Graphs. Lecture notes in math. Springer- Verlag, 642 New York pp.53-65. 1978.

[3] J. Doma. *Unicyclic Graceful Graphs.* M. S. Thesis, Macaster univ. 1991.

[4] H. Enomoto and A. S. Lladó and T. Nakamigawa and G. Ringel. *Super Edge-Magic Graphs.* SUT J. Math. vol.34. pp.105-109. 1998.

[5] H. Enomoto and K. Masuda, and T. Nakamigawa. *Induced Graph Theorem on Magic Valuations.* Ars Combin. vol.56. pp.25-32. 2000.

[6] R. M. Figueroa-Centeno and R. Ichishima and F. A. Muntaner-Batle. *Bertrand's Postulate and Magical Product Labelings.* Bulletin of the ICA. vol.30. pp.53-65. 2000.

[7] R. M. Figueroa-Centeno and R. Ichishima and F. A. Muntaner-Batle. *The Place of Super Edge-Magic Labelings among other Classes of Labelings.* Discrete Math. vol.231. pp.153-168. 2001.

[8] R. M. Figueroa-Centeno and R. Ichishima and F. A. Muntaner-Batle. *Magical Coronations of Graphs.* Australasian Journal of Combinatorics. vol.26. pp.199-208. 2002.

[9] R. M. Figueroa-Centeno and R. Ichishima and F. A. Muntaner-Batle. *On the Super Edge-Magic Deficiency of Graphs.* Discrete Math. to appear.

[10] R. M. Figueroa-Centeno and R. Ichishima and F. A. Muntaner-Batle. *On Super Edge-Magic Graphs.* Ars Combin. to appear.

[11] R. M. Figueroa-Centeno and R. Ichishima and F. A. Muntaner-Batle. *On Edge Magic Labelings of Certain Disjoint Union of Graphs.* preprint.

[12] R. M. Figueroa-Centeno and R. Ichishima and F. A. Muntaner-Batle. *A Magical Approach to some Labeling Conjectures.* preprint.

[13] R. M. Figueroa-Centeno and R. Ichishima and F. A. Muntaner-Batle. *Two Construction Methods Concerning the Conjecture of Truszczyński.* preprint.

[14] J. Gallian. *A Dynamic Survey of Graph Labeling.* The electronic Journal of Combinatorics. vol.5. http://www.combinatorics.org. 1998.

[15] R. L. Graham and N. J. A. Sloane. *On Additive Bases and Harmonious Graphs.* SIAM J. Alg. Discrete Math. vol.1. pp.382-404. 1980.

[16] R. Ichishima and A. Lladó and F. A. Muntaner-Batle. *How Many Graphs Are Super Edge Magic?.* preprint.

[17] A. Kotzig and A. Rosa. *Magic Valuations of Finite Graphs.* Canad. Math. Bull. vol.13. pp.451-461. 1970.

[18] A. Kotzig and A. Rosa. *Magic Valuations of Finite Graphs.* Centre de Recherches Mathematiques, Universite de Montreal, CRM-175. 1972.

[19] A. Llaó and F.A. Muntaner-Batle. *Super Magic Graphs and Sidon Sets.* preprint.

[20] A. Llaó and F.A. Muntaner-Batle. *Any Forest is a Subgraph of a Super Magic Tree.* preprint.

[21] F. A. Muntaner-Batle. *Special Super Edge-Magic Labelings of Bipartite Graphs.* JCMCC. vol.39. pp.107-120. 2001.

[22] G. Ringel. *Problem 25, in Theory of Graphs and its Applications.* Proc. Symposium Smolenice 1963. Prague 162. 1964.

[23] G. Ringel and A. Lladó. *Another Tree Conjecture.* Bull. Inst. Combin. Appl. vol.18. pp.83-85. 1996.

[24] A. Rosa. *On Certain Valuations of the Vertices of a Graph.* Theory of Graphs (Internat. Symposium, Rome, July 1996). Gordon and Breach, N. Y. and Dunod Paris pp.349-355. 1967

[25] M. Truszyński. *Graceful Unicyclic Graphs.* Demonstatio Mathematica. vol.17. pp.377-387. 1984.

[26] V. Yegnanarayanan. *On some Additive Analogues of Graceful Theme: Cycle Related Graphs.* Southeast Asian Bulletin of Math. vol.23. pp.317-333. 1999.