Extending Attribute Exploration by Means of Boolean Derivatives

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Abstract. We present a translation of problems of Formal Context Analysis into ideals problems in $\mathbb{F}_2[\mathbf{x}]$ through the Boolean derivatives. The Boolean derivatives are introduced as a kind of operators on propositional formulas which provide a complete calculus. They are useful to refine stem basis as well as for extending attribute exploration.

1 Introduction

Attribute exploration (cf. [3]) is a family of interactive procedures for Knowledge Acquisition (KA) in Formal Concept Analysis (FCA), whose goal is to build a knowledge base of the attributes we are working with. The procedures used in FCA have nice computer implementations, existing even generalizations for the management of the background information. Sometimes attribute exploration is hard or tedious to apply. Thus, it may advisable to use automated tools. Many Computer Algebra Systems (CAS) provide tools for working with discrete data, for example, Gröbner basis. Since it is possible to translate entailment problems into ideal problems in finite fields, Gröbner basis is a powerful tool for reasoning in propositional logic [8, 9, 2].

Our aim is to extend the framework of attribute exploration through the introduction of *Boolean derivatives* and the assistance of a CAS. The CAS that we will use CoCoA (`http://cocoa.dima.unige.it/`), very well suited for our pourposes because of its easy management of Gröbner basis and related tools. The paper is organized as follows. The next section reviews the relationship between propositional logic and the ring $\mathbb{F}_2[\mathbf{x}]$, as well as the basics of FCA. In the third section the *Boolean derivatives* are introduced, as well as a complete polynomial calculus based on them. An algebraic characterization of sensitivity for implications in FCA is given in forth section. In fifth and sixth sections new versions of attribute exploration are introduced, and in section 7 an application to graph theory is given. We conclude with some remarks about future work.

2 Background

We assume that the reader is familiar with propositional logic and polynomial algebra on positive characteristics. We setup a propositional language $PV =$

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$\{p_1, \ldots, p_n\}$, $PForm$ will denote the set of propositional formulas, and $var(F)$ denotes the set of variables of the propositional formula F.

The ring in which we are working is $\mathbb{F}_2[\mathbf{x}]$ (where $\mathbf{x} = x_1, \ldots, x_n$). A key ideal is $\mathbb{I}_2 := (x_1 + x_1^2, \ldots, x_n + x_n^2)$. To clarify our proposition, let fix an identification $p_i \mapsto x_i$ (or $p \mapsto x_p$) between PV and the set of indeterminates.

Given $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{N}^n$, let us define $|\alpha| := \max\{\alpha_1, \ldots, \alpha_n\}$, and $sg(\alpha) := (\delta_1, \ldots, \delta_n)$, where δ_i is 0 if $\alpha_i = 0$ and 1 otherwise. The *degree of* $a(\mathbf{x}) \in \mathbb{F}_2[\mathbf{x}]$, is $\deg_\infty(a(\mathbf{x}))$:=max$\{|\alpha| \;:\; \mathbf{x}^\alpha$ is a monomial of $a\}$, and $\deg_i(a(\mathbf{x}))$ is the degree w.r.t. x_i. If $\deg_\infty(a(\mathbf{x})) \leq 1$, $a(\mathbf{x})$ is called a *polynomial formula*.

Three maps represent the relationship between propositional logic and $\mathbb{F}_2[\mathbf{x}]$:

- $\Phi : \mathbb{F}_2[\mathbf{x}] \to \mathbb{F}_2[\mathbf{x}]$ is defined by $\Phi(\sum_{\alpha \in I} \mathbf{x}^\alpha) := \sum_{\alpha \in I} \mathbf{x}^{sg(\alpha)}$.
- The map $P : PForm \to \mathbb{F}_2[\mathbf{x}]$ is defined by the following equations
  - $P(\bot) = 0, P(p_i) = x_i, P(\neg F) = 1 + P(F)$
  - $P(F_1 \wedge F_2) = P(F_1) \cdot P(F_2)$ and $P(F_1 \vee F_2) = P(F_1) + P(F_2) + P(F_1)P(F_2)$
  - $P(F_1 \to F_2) = 1 + P(F_1) + P(F_1)P(F_2)$, and
  - $P(F_1 \leftrightarrow F_2) = 1 + P(F_1) + P(F_2)$
- $\Theta : \mathbb{F}_2[\mathbf{x}] \to PForm$ is defined by
  - $\Theta(0) = \bot,\ \Theta(1) = \top,\ \Theta(x_i) = p_i$,
  - $\Theta(a \cdot b) = \Theta(a) \wedge \Theta(b)$, and $\Theta(a + b) = \neg(\Theta(a) \leftrightarrow \Theta(b))$.

We have that $\Theta(P(F)) \equiv F$ and $P(\Theta(a)) = a$. Since we shall frequently be applying $\Phi \circ P$, we define the *polynomial projection* as $\pi := \Phi \circ P$.

Regarding valuations and polynomials, the key fact is that, if $v : PV \to \{0, 1\}$ is a valutation with $v(p_i) = \delta_i$, then for every $F \in PForm$,

$$v(F) = P(F)(\delta_1, \ldots \delta_n)$$

The behaviour of the ideals of $\mathbb{F}_2[\mathbf{x}]$ is well known: If $A \subseteq (\mathbb{F}_2)^n$, then $V(I(A)) = A$, and for every $\mathfrak{J} \in Ideals(\mathbb{F}_2[\mathbf{x}])$, $I(V(\mathfrak{J})) = \mathfrak{J} + \mathbb{I}_2$. Therefore $F \equiv F'$ if and only if $P(F) = P(F') \pmod{\mathbb{I}_2}$ which is also equivalent to $\Phi \circ P(F) = \Phi \circ P(F')$. The following theorem states the main relationship between propositional logic and $\mathbb{F}_2[\mathbf{x}]$:

Theorem 1. *The following conditions are equivalent:*
(1) $\{F_1, \ldots, F_m\} \models G$.
(2) $1 + P(G) \in (1 + P(F_1), \ldots, 1 + P(F_n)) + \mathbb{I}_2$.
(3) $\mathtt{NF}(1 + P(G), \mathtt{GB}\,[(1 + P(F_1), \ldots, 1 + P(F_m)) + \mathbb{I}_2]) = 0$.
(where `GB` *denotes* Gröbner basis*) and* `NF` *denotes* normal form*.*

In the rest of this section we succintly present some elements of FCA we use, although we assume that the reader knows the basic principles of this theory (the fundamental reference is [3]). We represent a *formal context* as $M = (O, A, I)$, which consists of two sets, O (the *objects*) and A (the *attributes*) and a relation $I \subseteq O \times A$. Finite contexts can be represented by a 1-0-table (representing I as a Boolean function on $O \times A$). The main goal in FCA is the computation of the *concept lattice* associated to the context.

Basic logical expressions in FCA are *implication between attributes*, that is, pair of sets of attributes written as $Y_1 \to Y_2$. Truth with respect to $M = (O, A, I)$ is defined as follows. A subset $T \subseteq A$ *respects* $Y_1 \to Y_2$ if $Y_1 \not\subseteq T$ or $Y_2 \subseteq T$. We say that $Y_1 \to Y_2$ *holds* in M ($M \models Y_1 \to Y_2$) if for all $o \in O$, the set $\{o\}'$ respects $Y_1 \to Y_2$. In that case we say that $Y_1 \to Y_2$ is an *implication* of M.

From a propositional logic viewpoint, $Y_1 \to Y_2$ is the formula $\bigwedge Y_1 \to \bigwedge Y_2$, so it is equivalent to a set of Horn clauses (implications with a singleton as right-hand side). On the other hand, the definition of truth can be extended: Given $Y \subseteq A$, define $\neg Y := Y \to \bot$, and it holds in the context if for all $o \in O$, $Y \not\subseteq \{o\}'$. Given a formula written with $\{\to, \bot\}$, $M \models F$ can be defined in the natural way. Since this set of connectives is functionally complete, truth definition can be extended to $PForm$.

Definition 1. *Let $\mathcal{L}$ be a set of implications and L an implication of M.*
- L follows from $\mathcal{L}$ *($\mathcal{L} \models L$) if each subset of A respecting $\mathcal{L}$ also respects L.*
- $\mathcal{L}$ *is* closed *if every implication following from $\mathcal{L}$ is already in $\mathcal{L}$.*
- $\mathcal{L}$ *is* complete *if every implication of the context follows from $\mathcal{L}$.*
- $\mathcal{L}$ *is* non-redundant *if for each $L \in \mathcal{L}$, $\mathcal{L} \setminus \{L\} \not\models L$.*
- $\mathcal{L}$ *is a* stem basis *for M if it is complete and non-redundant.*

For every context we can obtain a stem basis from the pseudo-intents:

Theorem 2. *[7] The set $\mathcal{L} = \{Y \to Y'' \ : \ Y$ is a pseudointent$\}$ is a stem basis.*

Actually one can choose $Y \to Y'' \setminus Y$ instead of $Y \to Y'$, so we will assume, by default, that for every implication $Y_1 \to Y_2$ belonging to a stem basis Y_1 and Y_2 are disjoint. Such a basis for the example of figure 5 (left) is $\mathcal{L} = \{\emptyset \to N, \{N, A\} \to \{Mo\}, \{N, Le\} \to \{Mo\}\}$.

The called *Amstrong rules* facilitates implicational reasoning:

$$R1: \ \frac{}{X \to X} \qquad R2: \frac{X \to Y}{X \cup Z \to Y} \qquad R3: \ \frac{X \to Y, \ Y \cup Z \to W}{X \cup Z \to W}$$

It has that A set of implications $\mathcal{L}$ is closed if and only if the set is closed by Amstrong rules [1]. A consequence of Amstrong result is that, if $\vdash_A$ denotes the proof notion associated to Amstrong rules, stem basis are $\vdash_A$-complete, that is:

Theorem 3. *Let $\mathcal{L}$ be a stem basis for M, and L an implication. Then $M \models L$ if and only if $\mathcal{L} \vdash_A L$*

The computing of stem basis may be expensive if the set of objects is large. Even it is possible we do not have the complete context M, or it has a potentially infinite set of objects. *Attribute exploration* is an interactive procedure designed to obtain a stem basis starting with a set H of *good examples* generating the subcontext

$$M \restriction_H := (H, A, I \cap (H \times A))$$

One expects that a stem basis associated to $M \restriction_H$ is also a stem basis for the complete context. To guarantee it, we proceed as follows. Assume that $\mathcal{L} =$

1. *Compute pseudo-intent*: Find X a pseudo-intent for $M\restriction_H$.
2. *Soundness of the new implication*: Ask to the user $X \overset{?}{\to} X''$ (the operators $'$ are w.r.t. the subcontext). The user must react:
   - Confirming the suggested implication (adding it to $\mathcal{L}$), or
   - giving o (a counterexample) such that $\{o\}'$ does not respect the implication. This is added to H, and the implication is discarded.

Fig. 1. Attribute exploration

$\{L_1, \ldots, L_k\}$ is a partial set of implications accepted as true, built from pseudo-intents of $M\restriction_H$[1]. Attribute exploration consists in a loop of the two steps shown in fig. 1, and it stops when no new pseudointent is found (see [4] for variants).

3 Boolean derivatives and non-clausal theorem proving

We introduce an operator on propositional formulas as a translation of the usual derivation on $\mathbb{F}_2[\mathbf{x}]$. In this section we review its basic properties (from [2]).

Recall that a derivation on a ring R is a map $d : R \to R$ verifying that $d(a+b) = d(a) + d(b)$ and $d(a \cdot b) = d(a) \cdot b + a \cdot d(b)$

Definition 2. *A map* $\partial : PForm \to PForm$ *is a* Boolean derivation *if there exists a derivation* d *on the ring* $\mathbb{F}_2[\mathbf{x}]$ *such that* $\partial = \Theta \circ d \circ \pi$

If the derivation on $\mathbb{F}_2[\mathbf{x}]$ is $d = \frac{\partial}{\partial x_p}$, we denote ∂ as $\frac{\partial}{\partial p}$. It has that:

$$\frac{\partial}{\partial p} F \equiv \neg(F\{p/\neg p\} \leftrightarrow F)$$

Thus, the value of $\frac{\partial}{\partial p}F$ with respect to a valuation does not depend on p. Therefore, we can apply valuations on $PV \setminus \{p\}$ to this formula.

Definition 3. *The* independence rule *(or* ∂*-rule) on polynomial formulas is*

$$\partial_x(a_1, a_2) : \quad \frac{a_1,\ a_2}{1 + \Phi\left[(1 + a_1 \cdot a_2)(1 + a_1 \cdot \frac{\partial}{\partial x}a_2 + a_2 \cdot \frac{\partial}{\partial x}a_1 + \frac{\partial}{\partial x}a_1 \cdot \frac{\partial}{\partial x}a_2)\right]}$$

In order to simplify the notation, if $a_i = b_i + x_p \cdot c_i$, with $\deg_{x_p}(b_i) = \deg_{x_p}(c_i) = 0$ $(i = 1, 2)$,. Then we can rewrite the values as:

$$\partial_{x_p}(a_1, a_2) : \quad \frac{b_1 + x_p \cdot c_1,\ b_2 + x_p \cdot c_2}{\Phi\left[1 + (1 + b_1 \cdot b_2)[1 + (b_1 + c_1)(b_2 + c_2)]\right]}$$

The rule is symmetric and generalizes resolution of non-tautological polynomial clauses (see lemma 1). For formulas the rule is translated as

$$\partial_p(F_1, F_2) := \Theta(\partial_{x_p}(\pi(F_1), \pi(F_2))).$$

[1] Pseudointents are generated in lexicographic order. This way previously computed pseudointents are preserved by augmentations of H. See th. 27 in [3].

It naturally induces a concept of proof, $\vdash_\partial$. A $\vdash_\partial$-refutation is a proof of $\bot$. In [2] the soundness and the refutational completeness of $\vdash_{partial}$ has been proved

Theorem 4. *[2] Let $v : PV \setminus \{p\} \to \{0,1\}$. The following conditions are equivalent:*

1. *$v \models \partial_p(F_1, F_2)$.*
2. *There exists an extension of v to PV is a model of $\{F_1, F_2\}$.*

For example, $\partial_{x_1}(x_1(1 + x_2), x_1(1 + x_2)) = 1 + x_2$. So the valuation v s.t. $v(\neg p_2) = 1$ is the only one that we can extend to a model of $p_1 \wedge \neg p_2$. When $\partial_p(\pi(F_1), \pi(F_2)) = 1$, every partial valuation is extendable to a model of $\{F_1, F_2\}$.

Theorem 5. *[2] If Γ is inconsistent then $\Gamma \vdash_\partial \bot$.*

Let be $\partial_p[\Gamma]$ defined as $\partial_p[\Gamma] := \{\partial_p(F, G) \; : \; F, G \in \Gamma\}$.

Given $Q = \{q_1, \ldots, q_k\} \subseteq PV$ the operator $\partial_Q := \partial_{q_1} \circ \cdots \circ \partial_{q_k}$ is well defined modulo logical equivalence (by corollary 4, for every $p, q \in PV$, $\partial_p \circ \partial_q[\Gamma] \equiv \partial_q \circ \partial_p[\Gamma]$). A consequence of corollary 4 and theorem 5 is that entailment can be located on variables of the goal;

Corollary 1. $\Gamma \models F \quad \Longleftrightarrow \quad \partial_{PV \setminus var(F)}[\Gamma] \models F$

We can define an explicit equivalent expression for ∂_p when it is applied to implications. To simplify, suppose that the right-side of implications is a singleton.

Lemma 1. *Let $C_i \equiv \bigwedge Y_1^i \to \bigwedge Y_2^i$ be a implications ($i = 1, 2$, $Y_1^i \cap Y_2^j = \emptyset$), and Γ be a set of implications. Let $\partial_p^c(C_1, C_2)$ be the symmetric operator*

$$\partial_p^c(C_1, C_2) := \begin{cases} \{C_1, C_2\} & p \notin var(C_1) \cup var(C_2) \\ \{C_2\} & p \in Y_1^1, \; p \notin var(C_2) \\ \{\bigwedge Y_1^1 \to \bigwedge (Y_2^1 \setminus \{p\}), C_2\} & p \in Y_2^1, \; p \notin var(C_2) \\ \{\top\} & p \in (Y_1^1 \cap Y_1^2) \cup (Y_2^1 \cap Y_2^2) \\ \{Resolvent_p(C_1, C_2)\} & p \in Y_1^1 \cap Y_2^2 \end{cases}$$

If $\partial_p^c[\Gamma] := \bigcup \{\partial_p^c(C_1, C_2) \; : \; C_1, C_2 \in \Gamma\}$, then $\partial_Q^c[\Gamma] \equiv \partial_Q[\Gamma]$ ($Q \subseteq PV$).

4 Algebraic characterization of sensitive implications

We shall provide an algebraic treatment for implications on a fixed $M = (O, A, I)$. It is well know that every set $X \subseteq (\mathbb{F}_2)^n$ is an algebraic set; that is, there exists $a_X \in \mathbb{F}_2[\mathbf{x}]$ such that $V(a_X) = X$. If $|A| = n$, M is identified with a subset $X(M)$ of $(\mathbb{F}_2)^n$ (each object identified with the 1-0 expresion of its intent). Let $a_M \in \mathbb{F}_2[\mathbf{x}]$ denote a polynomial formula such that $V(a_M) = X(M)$. Since $IV(a_M) = (a_M) + \mathbb{I}_2$, the coordinate ring of M is

$$\mathbb{F}_2[\mathbf{x}]/_{I(X(M))} \cong (\mathbb{F}_2[\mathbf{x}]/_{(a_M)})/_{\mathbb{I}_2}$$

One might also use an ideal J_X such that $V(J_X) = X$, if it is better to work with them (for example using CoCoA's command `IdealsofPoints`). Thus we can assume that $\mathbb{I}_2 \subseteq J_M$. We choose a_M only to simplify the proofs.

Also, each $o \in O$ defines a valuation v_o defined by: $v_o(p_i) = 1$ iff oIp_i.

Proposition 1. *Let $F \in PForm$ and let $\mathcal{L}$ be a stem basis. The following conditions are equivalent:*
(1) $M \models F$.
(2) $1 + \pi(F) \in (a_M) + \mathbb{I}_2$.
Moreover, if F is an implication, they are also equivalent to
(3) $\{P(L) \ : \ L \in \mathcal{L}\} \cup \{1 + \pi(F)\} \vdash_\partial 0$.
(4) $\partial^c_{PV \setminus var(F)}[\mathcal{L}] \models F$

Proof
(1) $\Longleftrightarrow$ (2): If $M \models F$, then $V(a_M) \subseteq V(1 + \pi(F))$. Thus, $IV(1 + \pi(F)) \subseteq IV(a_M)$ hence $1 + \pi(F) \in (a_M) + \mathbb{I}_2$. The converse is similar.
If F is an implication and $M \models F$, then $\mathcal{L} \models F$. Therefore $\mathcal{L} \cup \{\neg F\}$ is inconsistent so by completeness, $\mathcal{L} \cup \{\neg F\} \vdash_\partial \perp$ hence we have (3). The converse is true by soundness. (4) is equivalent to $\mathcal{L} \models F$ by lemma 1.

We now deal with the problem of redundant arguments in implications. In the worst case, the recognizing of redundancy requires a complete exploration of intents. An argument is redundant if it is not *sensitive*:

Definition 4. *A formula F is* sensitive *in p w.r.t. a formal context M if $M \not\models F\{p/\neg p\} \leftrightarrow F$. We say that F is sensitive w.r.t. M (or simply sensitive, if M is fixed) if F is sensitive in all its variables.*

Thus, F is not sensitive in p iff $M \models \neg \frac{\partial}{\partial p} F$. In this case, there exists G with var(G) =var(F) $\setminus \{p\}$ such that $M \models F \leftrightarrow G$ (e.g. $F\{p/\perp\}$).

Sensitive implications (also called *proper implications*) have several advantages over implications obtained from pseudo-intents (see [10]). In attribute exploration, sensitivity analysis is justified: it is possible that implications are based on a nonrepresentative set of examples, and thus they can be refined, basically giving witnesses of the role of the arguments in the implication, or making them more precise, removing redundant arguments:

Lemma 2. *Let $L = Y_1 \to Y_2$ be an implication. If $M \models L$ and L is not sensitive in $p \in Y_1$, then $M \models Y_1 \setminus \{p\} \to Y_2$. If $p \in Y_2$, then $M \models \neg Y_1$.*

By default, sensitivity analysis for implications will be always restricted to attributes in the left-hand side.

Proposition 2. *Let $p \in$var(F). The following conditions are equivalent:*
(1) F is sensitive in p w.r.t. M.
(2) $\frac{\partial}{\partial x_p}\pi(F) \neq 0$ in the coordinate ring of M.

Proof. (1) $\Longrightarrow$ (2): Assume $v_o \not\models F \leftrightarrow F\{p/\neg p\}$ for some $o \in O$. Then $v_o \models \frac{\partial}{\partial p} F$, so $V(a_M) \not\subseteq V(\pi(\frac{\partial}{\partial p}F)) = V(\frac{\partial}{\partial x_p}\pi(F))$, hence $\frac{\partial}{\partial x_p}\pi(F) \notin (a_M) + \mathbb{I}_2$.
(2) $\Longrightarrow$ (1): If $\frac{\partial}{\partial x_p}\pi(F) \notin (a_M) + \mathbb{I}_2$, then $O = V(a_M) \not\subseteq V(\pi(\frac{\partial}{\partial p}F))$. Therefore there exists $o \in O$ such that $v_o \models \frac{\partial}{\partial p} F$. Thus F is sensitive in p.

(3) *Sensitivity test*: If the implication has not been discarded, test whether the implication is sensitive in all its arguments w. r. t. the actual set H (using lemma 2 if necessary). If it is not sensitive in some of them, the user must to react:
  - Adding a new example o to H, witness of the sensitivity (that is, he/she thinks that it is sensitive), or
  - eliminating the attribute of the implication (it accepts it is not sensitive), changing the accepted implication by the refined one.

Fig. 2. Sensitivity test to add to algorithm of fig. 1

One can recursively apply the above criteria (w.r.t. an order on PV) to obtain sensitive implications. If $\mathcal{L}$ is a stem basis and $\mathcal{L}'$ is the refinement obtained, since Amstrong's rule R2 states $Y_1 \setminus \{Y\} \to Y_2 \models Y_1 \to Y_2$, one has that $\mathcal{L}' \models \mathcal{L}$. Thus $\mathcal{L}'$ is also a complete set of implications. The set $\mathcal{L}'$ has an advantage over other sets of proper implications (e.g. [10]) that it directly works on Duquenne-Guigues basis so it does not need an specific algorithm to build it.

5 Variants of attribute exploration

We shall propose new steps for attribute exploration. All of them are investigated with the translation into polynomials in mind. Although in the exposition we do not explicitely use polynomials -the results and their proofs are more readable in logical form- in practice they will be useful.

The attribute exploration can be extended by adding a sensitivity test w.r.t H (shown in fig. 2). Note that addition of a new object follows the formula

$$a_{H\cup\{(\delta_1,\ldots,\delta_n)\}} = \Phi(a_H \cdot (1 + \Pi_{i=1}^n (x_i + \delta_i + 1)))$$

For the running example, the implication $N \wedge A \to Mo$ is obtained and considered as sound. In this case, $a_M = x_1x_2x_4 + x_1x_2 + x_1x_3 + x_1x_4 + x_1 + 1$. A Gröbner basis for $a_M + \mathbb{I}_2$ is

$$\{x_4^2 + x_4, x_3^2 + x_3, x_2^4 + x_2, x_3x_4 + x_3, x_2x_4 + x_2 + x_3 + x_4, x_2x_3 + x_2, x_1 + 1\}$$

It verifies (with CoCoA) that $\frac{\partial}{\partial x_1}\pi(N \wedge A \to Mo) = x_2(1 + x_3) \in (a_M) + \mathbb{I}_2$

We think that is not really sensitive in N (every live being needs water), so we accept $A \to Mo$, which is now sensitive. Reasoning similarly with the other one, it obtains $\{N,\ A \to Mo, L \to Mo\}$, a stem basis of sensitive implications.

Sensitivity test can be also added when background knowledge exists. In this case, we deal with hard problems as consistency checking or entailment. It starts with H and a background knowledge Γ for $M\restriction_H$, that is, $M\restriction_H \models \Gamma$. Or, in algebraic terms, $V(a_H) \subseteq V(\{1 + \pi(F)\ :\ F \in \Gamma\})$. The step to add is in given figure 3. Condition $(*)$ means

$$1 + \frac{\partial}{\partial x_p}\pi(L) \notin (\{1 + \pi(F)\ :\ F \in \Gamma\}) + \mathbb{I}_2$$

(3) *Sensitivity test*: Test whether the implication is sensitive in all its arguments with the actual set H and the background knowledge Γ, that is, whether for each attribute p in the antecedent of L

$$\Gamma \not\models L \leftrightarrow L\{p/\neg p\} \qquad (*)$$

If it is not sensitive in some of them, the user must to react:
- **Giving** $o \in O$, a witness of the sensitivity (which is added to H), that is

$$v_o \models \{\frac{\partial}{\partial p} L\} \cup \Gamma \qquad (**)$$

- **or else** removing the attribute of the implication (he/she accepts it is not sensitive).

Fig. 3. Sensitivity test with background knowledge

Regarding to the existence of an object for $(**)$, if the user does not know one, but believes that it really exists, a model search program may be used to give an anonymous object. Test $(**)$ can be fairly translated into algebraic terms.

6 Attribute exploration with new attributes

Now we propose how to extend the context by adding new attributes. Formally, one starts with M_0, a subcontext with partial set of attributes,

$$M_0 = (H, A_0, I \cap (H \times A_0)), \qquad \text{with } A_0 \subsetneq A$$

Assume that, at some stage, full extents for a set H' of objects are introduced, with the aim of expanding the new attributes to initial objects of M_0 (see fig. 5). The user only knows -about the new attributes- a background knowledge Γ, relating old and new attributes. Since it seems not advisable to add many arguments at once (to facilitate the answers of tests), Γ will be relatively small.

It is important to observe that $\Delta = \mathcal{L} \cup \Gamma$, where $\mathcal{L}$ is the partial set of implications, may be inconsistent with ontological commitments implicitely or unconsciously accepted for the old attributes; that is, it may be false for M_0, whenever the extents of H were expanded to the full attribute set. Thus one needs an *expandability test* for objects of H (to simplify assume that the new attributes are $\{p_{k+1}, \dots, p_n\}$):

For each $o \equiv (\delta_1, \dots, \delta_k)$ *of* H*, is there* $(\delta_{k+1}, \dots \delta_n) \in \{0,1\}^{n-k}$ *such that* $\{p_j \ : \ \delta_j = 1 \ \wedge \ j \in \{1, \dots, n\}\}$ *respects* Δ*?*

Theorem 6. *Let M be an expansion of M_0 to the complete attribute set, with the same set of objects. If Γ is a stem basis (respectively a background knowledge) for M, then $\partial^c_{\{p_{k+1},\dots,p_n\}}[\Gamma]$ is a complete set of implications (respectively $\partial_{\{p_{k+1},\dots,p_n\}}[\Gamma]$ is a background knowledge) for M_0.*

Proof. Assume that Γ is a stem basis. Let L be an implication in the language $A \setminus \{p_{k+1}, \dots, p_n\}$. If $M \models L$, then $\Gamma \models L$. By corollary 1, $\partial_p[\Gamma] \models L$ so by lemma 1, $\partial^c_p[\Gamma] \models L$ holds. If Γ is a background knowledge, the result is a straightforward consequence of corollary 1.

(3) *Expansion test*: If implication has not been discarded, test whether the set of implications plus background knowledge is extendable to H.
- **If it is extendable**, the user shall proceed:
  - Confirming the suggested implication, or
  - giving $o \in O$ such that $\{o\}'$ does not respect the implication. This is added to H', and the implication is discarded.
- **Else**, it must revise the background knowledge, or to discard the implication

Fig. 4. Aditional step for exploration with new attributes

	Need water	Aquatic	Mobility	Legs
Cat	1	0	1	1
Leech	1	1	1	0
Frog	1	1	1	1
Maize	1	0	0	0
Fish	1	1	1	0

	Need water	Aquatic	Mobility	Legs	Land
Cat	1	0	1	1	?
Leech	1	1	1	0	?
Frog	1	1	1	1	?
Maize	1	0	0	0	?
Fish	1	1	1	0	0
Dog	1	0	1	1	1
Bean	1	0	0	0	1

Fig. 5. Extension of the context on live beings with new attributes

Corollary 2. *Let $\Delta \subseteq PForm$. The following conditions are equivalent:*
(1) Every extension of objects of H can be expanded to the full attribute set, consistently with Δ.
(2) $\{1 + \pi(F) \ : \ F \in \partial_{\{p_{k+1},\ldots,p_n\}}[\Delta]\} \subseteq (a_H) + \mathbb{I}_2$

Assume now that it has previously certified that Δ is expandable to objects of H, and let L be a new implication. If $\Delta \cup \{L\}$ can be consistently extended to H, but the user thinks that it is not true, in a first stage the user is required to give a counterexample for L by completing the extention of some object of H (in this way it bounds the set of new examples), or, if he/she does not know which, a new example. Summarizing, the new step is shown in figure 4.

For example, suppose that we decide to add a new attribute, *to live in land* (La). Some complete extensions are given (figure 5). We only know as background knowledge that aquatic live beings do not live in land, and we consider the implication *every live being with legs and mobility lives in land*, that is

$$\Delta = \{A \to \neg La, \ Le \wedge Mo \to La\}$$

In this case, $\pi[\Delta] = \{1 + x_A x_{La}, \ 1 + x_{Le}x_{Mo} + x_{Le}x_{Mo}x_{La}\}$.

The set H can not be consistently expanded to a model of Δ, because $\partial_{\{x_{La}\}}[\pi[\Delta]] = \{1 + x_A + x_A x_{Le} x_M, 1\}$ and $x_A + x_A x_{Le} x_M \notin (a_H) + \mathbb{I}_2$.

6.1 A final remark: defining the new attributes

We now see how to extend the above procedure *for learning* the new attribute. We suppose we have a stem basis consistent with old information; and, in a second stage, we wish to find a definition of the new attribute w.r.t the old ones (if the user thinks it is possible). The next theorem states a solution, which is an adaptation of predicate completion procedure (sect. 6.2 in [5]). That is, we are considering the stem basis is a complete knowledge base for the attribute.

Theorem 7. *Let M_0 as in section 6 with $A_0 = A \setminus \{p\}$. Assume that $\mathcal{L}$ is a stem basis, built by attribute exploration with expansion test. Let*

$$\Omega = \{Y_1 \subseteq A_0 \ : \ \textit{there exists } Y \subseteq A_0 \textit{ s.t. } \ Y_1 \to Y \cup \{p\} \in \mathcal{L}\}$$

If M_c is the expansion of M_0 to A by defining the intent w.r.t. $\{p\}$ by

$$p \in \{o\}' \quad \Longleftrightarrow \quad v_o(\bigvee_{Y \in \Omega} \bigwedge Y) = 1$$

then $\mathcal{L}$ is a stem basis for M_c.

Since $M_c \models p \leftrightarrow \bigvee_{Y\in\Omega} \bigwedge Y$, M_c is model of completion formula for p. Thus, the intent of each object o is expanded to p_n by the polynomial

$$v_o(p_n) := \pi(\bigvee_{Y \in \Omega} \bigwedge Y)(v_o(p_1), \ldots, v_o(p_{n-1}))$$

7 An application: discovering tree notion in graph theory

We shall investigate the relationship among several properties on graphs (with three or more nodes), comparing stem basis produced by classical attribute exploration with the result of the new methods. The properties are: *acyclic*, *connected*, *2-connected* (if one edge of the graph is removed, the induced subgraph is connected), *geodetic* (for every two nodes there exists only one shortest path), *bipartite* (it can be partionated the set of nodes in two sets such that every edge joins a node of each set), *nonseparable* (connected and, if one removes a node, the resulting graph remains connected), and *planar*.

We begun (classical) attribute exploration with the two first objects of figure 6 (left). For this we used ConExp, and the result is the formal context of fig. 6 (left. K_5 is the complete graph with five nodes, and K_{33} is the complete bipartite graph with two sets of tree nodes each one as partition). The stem basis is:

$$\begin{cases} L_1 : t \to a, b, c, g, p & L_2 : n \to c, d \quad\quad L_3 : g \to c \quad\quad L_4 : d \to c \\ L_5 : b, c, g, p \to a, t & L_6 : b, c, d, g \to n \quad L_7 : a \to b, p \quad L_8 : a, c, b, p \to g, t \end{cases}$$

One might apply completion procedure on *tree*, obtaining a (messy) definition,

(Bipartite ∧ Connected ∧ Geodetic ∧ Planar) ∨ (Acyclic ∧ Connected ∧ Bipartite ∧ Planar)

Even it is not evident that the first conjuction defines a tree; it is necessary to know the fact that every geodetic and bipartite graph is acyclic. For this context, the ideal generated is

$J_M = (g + b + t + 1, c + d + t, a + d + 1, t^2 + t, pt + t, nt, bt + t, dt, p^2 + p, np + n + p + 1, dp + d + p + 1, n^2 + n, dn + n, b^2 + b, db + d + b + 1, d^2 + d)$.

The first interesting sensitivity analysis is on L_5 ($\pi(L_5) = 1 + bcgp(1 + at)$):

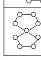
	Acyclic (a)	Connected (c)	2−connected (d)	Geodetic (g)	Bipartite (b)	Nonseparable (n)	Planar (p)	*Tree (t)*	rad.−minimal (r)
	1	1	0	1	1	0	1	1	1
	0	1	1	0	1	1	1	0	0
K_5	0	1	1	1	0	1	0	0	0
	1	0	0	0	1	0	1	0	0
	0	1	1	1	0	1	1	0	0
K_{33}	0	1	1	0	1	1	0	0	0
	0	1	1	0	1	0	1	0	0
	0	1	1	1	0	0	1	0	0

Fig. 6. Formal context on graphs, and the extension obtained for *radius-minimal*

- $\frac{\partial}{\partial b}\pi(L_5) \notin J_M$, thus is sensitive in b, hence preserve the implication.
- $\frac{\partial}{\partial c}\pi(L_5) \in J_M$, hence is not sensitive (in this case, we see that $g \to c$ holds in graphs), hence we redefine $L_5 := b, g \to a, t$.
- $\frac{\partial}{\partial g}\pi(L_5) \notin J_M$, thus it is sensitive in g, so preserve g finishing the analysis.

Other cases (L_6 and L_8) are similarly treated. The resultant is

$$\mathcal{L} = \begin{cases} t \to a, c, g, b, p & n \to c, d \quad\quad g \to c \quad\quad d \to c \\ b, g \to a, t, p & a \to b, p \quad\quad a, c \to g, t \quad\quad d, t \to n \end{cases}$$

The completion of *tree* from this basis is

$$Tree \leftrightarrow (\text{Bipartite } \wedge \text{ Geodetic}) \vee (\text{Acyclic } \wedge \text{ Connected})$$

It easy to see that the first conjunction is equivalent to the second one, the original definition of *tree*.

Our next aim is to expand our set of attributes with a new one, *radius-minimal* (denoted as variable by r). The *distance* of two nodes of a graph is the length of a shortest path between them. The *eccentricity* of a node v is the distance to a node farthest from v. The *radius* of a graph G, $r(G)$, is the minimum eccentricity of the nodes. Finally, a graph is called *radius-minimal* if $r(G - e) > r(G)$ for every edge in G. We used the method shown in section 6; the objects of fig. 6 suffices for it.

The exploration starts with the first two objects of figure 6, knowing that the first one is *radius-minimal* and the second one is not. Also we have the background knowledge $\{\neg c \to \neg r\}$. The procedure gives the basis

$$\mathcal{L} = \begin{cases} L_1 : r \to a, c, g, b, p, t & L_2 : t \to a, c, g, b, p, r & L_3 : n \to c, d \\ L_4 : g \to c & L_5 : d \to c & L_6 : c, g, b \to a, p, t, r \\ L_7 : a \to b, p & L_8 : a, c, b, p \to g, t, r & L_9 : a, c, d, g, b, p, t, r \to n \end{cases}$$

After testing sensitivity, three implications are refined, producing:

$$L_6 : g, b \to a, p, t, r \quad L_8 : a, c \to g, t, r \quad L_9 : d, r \to n$$

and the rest remains. Thus completion for r is

$$\text{Radius-minimal} \leftrightarrow \text{Tree} \vee (\text{Geodetic} \wedge \text{Bipartite}) \vee (\text{Acyclic} \wedge \text{Connected})$$

The last two conjunctions are equivalent to *Tree*, so we conclude that *Radius-minimal* and *Tree* are equivalent. Actually, this result is proved in [6]. Thus we take $v_o(r) := t$ to extend the attribute r for objects.

8 Conclusions and Future work

We present a framework for solving problems of FCA with the assistance of a CAS. We are confident that the tools described here may be useful to facilitate knowledge processing. As mentioned in previous sections, the complexity of some subproblems involved in the improvements of attribute exploration may restrict the method to projects of modest size, if a CAS as CoCoA is not used.

Throughout the paper we remarked some works related with the tools used here. The future work is the extension to many-valued logics and their applications [9].

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