A Branch-and-Bound Algorithm for Quadratic 0-1 Optimization

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Extended Abstract

It is well known that several combinatorial optimization problems can be easily and nicely transformed into optimizing a quadratic $0-1$ function (under, or without, linear constraints). This is the case for the the max-flow problem in a directed graph [1]. This is also the case for a number of NP-complete problems such as Max 2-SAT, the maximum stable set/clique problem (see, e.g., [2]), Max-cut, and frequency assignment in a cellular phone network.

For the max-flow problem, given a directed graph $G = (V, E)$ with $|V| = n$ and 2 special nodes $v_1 = s$ (the source) and $v_n = t$ (the sink), and given a capacity c_{ij} for every edge (i, j), find a set of flow values f_{ij} for which the total amount of flow out of s (and into t) is maximum and for which flow conservation is achieved at every node v, different from s and t. For each node i, define $0-1$ variable x_i. For each arc (i, j), define the quadratic term $c_{ij}x_i(1 - x_j)$, and let $x_s = 1$ and $x_t = 0$. Finding a maximum flow from s to t (and a minimum cut separating s from t) is equivalent to minimizing $f = \sum c_{ij}x_i(1 - x_j)$.

For the frequency assignment problem, two interference matrices $A = (a_{ij})$ and $B = (b_{ij})$ are given. Suppose there are n frequencies to be allocated to m cells in such a way as to minimize the total interference, measured as the sum of all values a_{ij} (and b_{ij}) for which cells i and j are using a same frequency f (or two adjacent frequencies, f and $f \pm 1$). Suppose further that cell i must be allocated exactly D_i frequencies. Define variables x_{if} such that $x_{if} = 1$ if and only if frequency f is allocated to cell i. The quadratic objective function is given by the sum of all $a_{ij}x_{if}x_{jf}$ plus the sum of all $b_{ij}x_{if}x_{jg}$, where $g = f \pm 1$. As for the constraints, (i) allocating exactly D_i frequencies to cell i can be expressed as $\sum_f x_{if} = D_i$ for every cell i; (ii) forbidding the allocation of f to i and of g to j when f and g are "too close" can be achieved by adding to the objective function the term $p_{ij}x_{if}x_{jg}$, where p_{ij} is a very high penalty coefficient, etc.

It follows that quadratic $0-1$ optimization is an important problem that has numerous practical applications. Furthermore, it is possible to disregard the set of linear constraints, if any, since these can be replaced by the addition to the objective function of a set of quadratic terms with penalty coefficients.

The purpose of this paper is to describe a branch-and-bound algorithm for the unconstrained quadratic $0-1$ optimization problem (see [2] for an application to the maximum stable set/clique problem). We will focus on the algorithm implementation and the results obtained. The computer code will be made available to interested researchers once a number of problems related to the format of input and output files will be solved.

References

[1] J.-M. Bourjolly, "An efficient pseudo-Boolean method for solving the maximum flow problem", Faculty of Commerce and Administration Working Paper Series #88-015, Concordia University, April 1988.

[2] J.-M. Bourjolly, P. Gill, G. Laporte, H. Mercure, "An exact quadratic 0-1 algorithm for the stable set problem", DIMACS Series in Discrete Mathematics and Theoretical Computer Science, volume 26, 1996, 53-73 (Second DIMACS Challenge on Clique, Coloring and Satisfiability).