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# Contractions of certain Lie algebras

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**Abstract.** This communication is focused on the study of contractions of certain types of Lie algebras of low dimension, in order to address the implementation of the results to certain physical concepts, such as the boundary process by which quantum mechanics contracts to classical mechanics. To do this, we consider in the first place the contractions of filiform Lie algebras, which were introduced by M. Vergné in 1966, by using the psi and phi invariant functions, introduced in 2008 by Hrivnák and Novotny. These functions are also dealt with other types of algebra, as Heisenberg algebras among others, and we study the existing contractions between these algebras.

## 1. Introduction

At present, the study of certain physical concepts has significantly increased due to its significance in *limit processes* which allow us to relate Lie algebras between themselves. These processes were first investigated by Segal [5] in 1951 and two are the better known examples of them. The first of them involves the connection between classical mechanic and relativistic mechanic, with their respective Poincaré symmetry group and Galilean symmetry group. The second one is the limit process by which quantum mechanic is contracted to classical mechanic, when $\hbar \to 0$, which actually corresponds to a contraction of the Heisenberg algebra to the abelian algebra of the same dimension.

For these reasons physical or mathematical contractions are of great interest nowadays, not only for their applications but for the proper study of their algebraic properties. So, continuing with this research, the main goal of this paper is to study the proper contractions of certain types of Lie algebras, mainly filiform, although other types of Lie algebras of lower dimensions are also considered. To do this, we use the invariant functions $\psi$ and $\varphi$, introduced in 2007 by Hrivnák and Novotný [4] as a tool.

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## 2. Preliminaries

We show in this section some preliminaries on filiform Lie algebras, invariant functions in Lie algebras and proper contractions of Lie algebras. A complete review on Lie algebras can be consulted in [3].

The *lower central series* of a Lie algebra $\mathfrak{g}$ is defined as

$$\mathfrak{g}^1 = \mathfrak{g}, \ \mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}], \ \ldots, \ \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \ \ldots \tag{1}$$

If there exists $m \in \mathbb{N}$ such that $\mathfrak{g}^m \equiv 0$, then $\mathfrak{g}$ is called *nilpotent.*

The *nilpotency index* of $\mathfrak{g}$ if the smallest integer $c$ such that $\mathfrak{g}^c \equiv 0$.

An $n$-dimensional nilpotent Lie algebra $\mathfrak{g}$ is said to be *filiform* if it is verified that

$$\dim \mathfrak{g}^k = n - k, \ \text{for all } k \in \{2, \ldots, n\}. \tag{2}$$

The only $n$-dimensional filiform Lie algebra for $n < 3$ is the abelian.

For $n \geq 3$, it is always possible to find a so-called *adapted basis* $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$ such that $[e_1, e_2] = 0$, $[e_1, e_j] = e_{j-1}$, for all $j \in \{3, \ldots, n\}$, $[e_2, e_j] = [e_3, e_j] = 0$, for all $j \in \{3, \ldots, n\}$. All these brackets are called *brackets due to filiformity.*

If all the structure constants of a filiform Lie algebra, with the exception of those corresponding to the brackets due to the filiformity of the algebra, are zeros, then that filiform Lie algebra $\mathfrak{g}$ is called *model.*

Now, we recall the definitions and main properties of invariant functions $\psi$ and $\varphi$, obtained by Hrivnák and Novotný [4] in 2007.

**Definition 2.1.** *Let $\mathfrak{g}$ be a Lie algebra. An isomorphism $d$ of $\mathfrak{g}$ is called a* $(\alpha, \beta, \gamma)$-derivation *of $\mathfrak{g}$ if there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that*

$$\alpha\, d[X, Y] = \beta\, [dX, Y] + \gamma\, [X, dY], \quad \forall X, Y \in \mathfrak{g}$$

*The set of* $(\alpha, \beta, \gamma)$-derivations *of $\mathfrak{g}$ is denoted by $Der_{(\alpha,\beta,\gamma)}\mathfrak{g}$.*

**Definition 2.2.** *The function $\psi_{\mathfrak{g}} : \mathbb{C} \mapsto \{0, 1, 2, ..., (\dim \mathfrak{g})^2\}$ defined as*

$$(\psi_{\mathfrak{g}})(\alpha) = \dim Der_{(\alpha,1,1)}\mathfrak{g}$$

*is called the $\psi_{\mathfrak{g}}$* invariant function *corresponding to the $(\alpha, \beta, \gamma)$-derivations of $\mathfrak{g}$.*

A $\kappa$-twisted cocycle (simply $\kappa$-cocycle ) is any $c \in C^q(\mathfrak{g}, V)$, with $q \in \mathbb{N}$, verifying $0 = \sum_{i=1}^{q+1} (-1)^{i+1} \kappa_{ii} f(x_i) c(x_1, \ldots, \hat{x_i}, \ldots, x_{q+1}) + \sum_{i,j=1,i<j}^{q+1} (-1)^{i+j}$ $\kappa_{ij} c([x_i, x_j], x_1, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_{q+1})$.

**Definition 2.3.** *The $\varphi$ invariant function corresponding to an $n$-dimensional Lie algebra $\mathfrak{g}$ is the mapping*

$$\varphi : \mathbb{C} \mapsto \{0, 1, \ldots, \frac{n^2(n-1)}{2}\} \text{ defined by } (\varphi\mathfrak{g})(\alpha) = \dim coc_{(1,1,1,\alpha,\alpha,\alpha)}\mathfrak{g}.$$

**Definition 2.4.** *Let $\mathfrak{g} = (V, [,])$ be an $n$-dimensional* Lie algebra *and $U : (0,1] \mapsto \mathfrak{gl}(V)$ be an one-parameter mapping. If the limit*

$$[X,Y]_0 = \lim_{\varepsilon \to 0^+} U^{-1}(\varepsilon)\,[U(\varepsilon)X, U(\varepsilon)Y]$$

*exists for all $X, Y \in \mathfrak{g}$, we say that $\mathfrak{g}_0 = (V, [,]_0)$ is an one-parameter contraction of the algebra $\mathfrak{g}$ and we write $\mathfrak{g} \mapsto \mathfrak{g}_0$.*

Finally, with respect to contractions of Lie algebras, let us recall

**Definition 2.5.** *Let $\mathfrak{g} = (V, [,])$ be an $n$-dimensional* Lie algebra *and $U : (0,1] \mapsto \mathfrak{gl}(V)$ be an one-parameter mapping. If the limit*

$$[X,Y]_0 = \lim_{\varepsilon \to 0^+} U^{-1}(\varepsilon)\,[U(\varepsilon)X, U(\varepsilon)Y]$$

*exists for all $X, Y \in \mathfrak{g}$, we say that $\mathfrak{g}_0 = (V, [,]_0)$ is an one-parameter contraction of the algebra $\mathfrak{g}$ and we write $\mathfrak{g} \mapsto \mathfrak{g}_0$.*

The contraction $\mathfrak{g} \mapsto \mathfrak{g}_0$ is said to be *proper* if $\mathfrak{g}$ is not isomorphic to $\mathfrak{g}_0$. The following results were shown in [3].

If $\mathfrak{g}_0$ is a proper contraction of the complex Lie algebra $\mathfrak{g}$, then

1. $\dim Der(\mathfrak{g}) < \dim Der(\mathfrak{g}_0)$.
2. $\psi\mathfrak{g} \le \psi\mathfrak{g}_0$ and $\psi\mathfrak{g}(1) < \psi\mathfrak{g}_0(1)$.
3. $\varphi\mathfrak{g} \le \varphi\mathfrak{g}_0$ and $\varphi^0\mathfrak{g} \le \varphi^0\mathfrak{g}_0$.

Moreover, it is satisfied that, in dimension 3, Condition 2 is a characterization of proper contractions of $\mathfrak{g}$.

# 3. Extending the definitions of the previously defined invariant functions to the case of filiform Lie algebras. New results obtained.

With the objective of extending the definitions of the invariant functions $\psi$ and $\varphi$ introduced by Hrivnák and Novotný [4] to the case of filiform Lie algebras of lower dimensions, authors have obtained in a previous paper the following values of these invariant functions for the case of filiform Lie algebras of dimension $3, 4$ and 5 (see [2])

- The $\psi$ invariant function for 3-dimensional filiform Lie algebras. According to the notation used in [4], we have obtained that $\psi_{\mathfrak{f}_3(\alpha)} = 6$, for all $\alpha \in \mathbb{C}$.

- The $\varphi$ invariant function for 3-dimensional filiform Lie algebras. We have obtained that $\varphi_{\mathfrak{f}_3}(\lambda) = 9$, if $\lambda = 0$, and $\varphi_{\mathfrak{f}_3}(\lambda) = 8$, if $\lambda \in \mathbb{C} \setminus \{0\}$.

- The invariant functions $\psi$ for the 4-dimensional filiform Lie algebra $\mathfrak{f}_4$. We have obtained $\psi_{\mathfrak{f}_4}(\alpha) = 7$, for all $\alpha \in \mathbb{C}$.

- The $\varphi$ invariant functions for 4-dimensional filiform Lie algebra. We have that $\varphi_{\mathfrak{f}_4}(\alpha) = 16$, if $\alpha = 0$, and $\varphi_{\mathfrak{f}_4}(\alpha) = 18$, if $\lambda \in \mathbb{C} \setminus \{0\}$.

- Finally, in the case of the $\psi$ invariant function for filiform Lie algebras of dimension 5, we have found that $\psi_{\mathfrak{f}_5}(\alpha) = 9$, for all $\alpha \in \mathbb{C}$.

## 4. Some examples of proper contractions between different types of algebras

In this section we study proper contractions from filiform Lie algebras of lower dimensions to different types of algebras.

### 4.1. Proper contractions of 3-dimensional filiform Lie algebras

Now, by using some results got by Novotný and Hrivnák in [3], we have obtained the invariant function $\psi$ for some particular Lie algebras. Indeed,

$\mathfrak{g}_1$: Abelian Lie algebra

| $\alpha$ | $\forall\alpha \in \mathbb{C}$ |
|---|---|
| $\psi_{3\mathfrak{g}_1}(\alpha)$ | 9 |

$\mathfrak{g}_{3,1}$: $[e_2, e_3] = e_1$

| $\alpha$ | $\forall\alpha \in \mathbb{C}$ |
|---|---|
| $\psi_{\mathfrak{g}_{3,1}}(\alpha)$ | 6 |

$\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1 : [e_1, e_2] = e_2$

| $\alpha$ | $\forall\alpha \in \mathbb{C} \setminus \{0\}$ | 0 |
|---|---|---|
| $\psi_{\mathfrak{g}_{2,1}\oplus\mathfrak{g}_1}(\alpha)$ | 4 | 6 |

$\mathfrak{g}_{3,2} : [e_1, e_3] = e_1, [e_1, e_3] = e_1 + e_2$

| $\alpha$ | 1 | $\forall\alpha \in \mathbb{C} \setminus \{1\}$ |
|---|---|---|
| $\psi_{\mathfrak{g}_{3,2}}(\alpha)$ | 4 | 3 |

$\mathfrak{g}_{3,3} : [e_1, e_3] = e_1, [e_2, e_3] = e_2$

| $\alpha$ | 1 | $\forall\alpha \in \mathbb{C} \setminus \{1\}$ |
|---|---|---|
| $\psi_{\mathfrak{g}_{3,3}}(\alpha)$ | 6 | 3 |

### 4.2. The invariant function $\psi_{\mathbb{H}_5}$

The computation of the invariant functions of filiform Lie algebras of dimension odd allows us to compare these algebras with Heisenberg algebras (which are only defined for these dimensions).

With respect to the invariant function $\psi_{\mathbb{H}_5}$ we have obtained that $\psi_{\mathbb{H}_5}(\alpha) = 15$, for all $\alpha \in \mathbb{C}$.

## 5. Conclusions

- As $\psi_{\mathfrak{f}_3} \leq \psi_{\mathfrak{g}_1}$ and $\psi_{\mathfrak{f}_3}(1) < \psi_{\mathfrak{g}_1}(1)$, a previous result assures the existence of a proper contraction from $\mathfrak{f}_3$ to $\mathfrak{g}_1$. Analogously, the same occurs between $\mathfrak{g}_{3,2}$ and $\mathfrak{f}_3$ since $\psi_{\mathfrak{g}_{3,2}} \leq \psi_{\mathfrak{f}_3}$ and $\psi_{\mathfrak{g}_{3,2}}(1) < \psi_{\mathfrak{f}_3}(1)$.
- Note that $\psi_{\mathfrak{f}_3}(1) = 6$ and $\psi_{\mathfrak{g}_{3,1}}(1) = 6$. Therefore, the same result assures that there is not any proper contraction between $\mathfrak{g}_{3,1}$ and $\mathfrak{f}_3$. Similarly, the same occurs between $\mathfrak{g}_{3,3}$ and $\mathfrak{f}_3$ due to that $\psi_{\mathfrak{f}_3}(1) = 6$ and $\psi_{\mathfrak{g}_{3,3}}(1) = 6$.
- We have also computed the invariant function $\psi$ for the 5-dimensional Heisenberg algebra and have proved that there is not a contraction from the 5-dimensional filiform Lie algebra to the Heisenberg algebra of this dimension.

  Therefore, we can conclude that the 5-dimensional classical-mechanical model built upon a 5-dimensional filiform Lie algebra cannot be obtained as a limit process of a quantum-mechanical model based on a 5-dimensional Heisenberg algebra.

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