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Causal properties of doubly warped spacetimes

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Abstract. In this talk we will describe the characterization of the chronological relation on doubly warped product spacetimes of the form $(a, b) \times M_1 \times M_2$ with Lorentzian metric $g = -dt^2 + \alpha_1 g_1 + \alpha_2 g_2$, and we will discuss when these spacetimes are causally continuos and causally simple. This talk is based on the first section of [1], available at arXiv:1709.00234.

1. Introduction

In the classical book on Lorentzian geometry [2] the causality of warped spacetimes products of the form $((a, b) \times M, -dt^2 + \alpha g_M)$, where $((a, b), -dt^2)$ is a Lorentzian manifold, (M, g_M) is a Riemannian manifold and $\alpha : V \to (0, \infty)$ is a smooth function, is studied and results concerning to the stably causal, strongly causal and global hyperbolicity of these spacetimes are obtained, see [2, Lemma 3.55, Prop. 3.62 and Thm. 3.66]. However, the study of the causally continuous and causally simple stages of the causal hierarchy are not included. Our aim in this talk is to give a characterization of the chronological and causal relations in doubly warped spacetimes and then we will give conditions to obtain the causally continuity and causal simplicity of these spacetimes. The study of the characterization of the chronological relation on these spacetimes will be useful in order to study the causal completion of these spacetimes, see [1].

2. Preliminaries

A doubly warped spacetime can be written as:

$$V := (a, b) \times M_1 \times M_2 \quad \text{and} \quad g = -dt^2 + \alpha_1 g_1 + \alpha_2 g_2. \tag{1}$$

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Where $\alpha_i : (a,b) \to (0,\infty)$ are smooth functions for all $i = 1,2$ and (M_i, g_i) are Riemannian manifolds for all $i = 1,2$.

The chronological relation in these spacetimes is characterized in the following proposition (see [1]):

Proposition 2.1. *Let (V,g) be a doubly warped spacetime as in (1), and $(t^o, x^o), (t^e, x^e) \in V$ with $x^o \neq x^e$. The following conditions are equivalent:*

(i) $(t^o, x^o) \ll (t^e, x^e)$;

(ii) there exist strictly positive constants $\mu_1', \mu_2' > 0$, with $\mu_1' + \mu_2' = 1$, such that

$$\int_{t^o}^{t^e} \frac{\sqrt{\mu_i'}}{\alpha_i(s)} \left(\sum_{k=1}^{2} \frac{\mu_k'}{\alpha_k(s)} \right)^{-1/2} ds > d_i(x_i^o, x_i^e) \qquad \textit{for } i = 1,2. \tag{2}$$

In order to give a proper characterization of the causal relation in doubly warped spacetimes we need the following definition:

Definition 2.1. *A Riemannian manifold (N,h) is L-*convex *if any pair of points $p,q \in N$ with $d_h(p,q) < L$ can be joined by a minimizing geodesic.*

Now, we can establish the announced characterization about the causal relation (see [3, Thm. 2(2)])

Proposition 2.2. *Let (V,g) be a doubly warped spacetime as in (1) whose fibers (M_i, g_i) are L_i-convex for $i = 1,2$. Consider two points $(t^o, x_1^o, x_2^o), (t^e, x_1^e, x_2^e) \in V$, with $t^o \leq t^e$, satisfying $d(x_i^o, x_i^e) < L_i$, $i = 1,2$. Then, the following conditions are equivalent:*

(i) the points are causally related, $(t^o, x_1^o, x_2^o) \leq (t^e, x_1^e, x_2^e)$;

(ii) there exists a causal geodesic joining (t^o, x_1^o, x_2^o) with (t^e, x_1^e, x_2^e);

(iii) there exist constants $\mu_1', \mu_2' \geq 0$, $\mu_1' + \mu_2' = 1$, such that

$$\int_{t^o}^{t^e} \frac{\sqrt{\mu_i'}}{\alpha_i(s)} \left(\sum_{k=1}^{2} \frac{\mu_k'}{\alpha_k(s)} \right)^{-1/2} ds \geq d_i(x_i^o, x_i^e) \qquad \textit{for } i = 1,2. \tag{3}$$

Moreover, if the equalities hold in (3), then there is a lightlike and no timelike geodesic joining the points.

With the characterization of the chronological relation and causality relation, the last under $L-$weakly convex condition, we are ready to study the causally continuity and causally simplicity of doubly warped spacetimes.

Causal hierarchy on doubly warped spacetimes

Next we are going to prove that double warped spacetimes are causally continuous, and, causally simple if and only if each (M_i, g_i) satisfy a notion of $L-$convexity. First we recall the following definitions on the causal hierarchy of spacetimes, see [5] for a complete list:

Definition 2.2. *A spacetime (V, g) is*

- Causal *if it does not contain closed causal curves.*
- Distinguishing *if whenever $I^+(p) = I^+(q)$ and $I^-(p) = I^-(q)$, necessarily $p = q$.*
- Causally continuous *if it is distinguishing and the set valued functions $I^+(\cdot)$ and $I^-(\cdot)$ are outer continuous (say, $I^+(\cdot)$ is* outer continuous at *some $p \in V$ if, for any compact subset $K \subset I^+(p)$ there exists an open neighborhood $U \ni p$ such that $K \subset I^+(q)$ for all $q \in U$). This is equivalent to being distinguishing and* reflecting, *i.e. for any pair of events $p, q \in V$, $I^+(q) \subset I^+(p)$ if and only if $I^-(p) \subset I^-(q)$.*
- Causally simple *if it is causal and $J^{\pm}(p)$ are closed sets for any $p \in V$.*

Prop. 2.1 allow us to prove that all doubly warped spacetimes are causally continuous:

Theorem 2.1. *Any doubly warped spacetime (V, g) as in (1) is causally continuous.*

Idea of the proof: Since (V, g) is stably causal, it is also distinguishing. So, it suffices to show that (V, g) is reflecting. Let $(t^o, x_1^o, x_2^o), (t^e, x_1^e, x_2^e) \in V$ be such that $I^+((t^e, x_1^e, x_2^e)) \subset I^+((t^o, x_1^o, x_2^o))$, and let us prove that $I^-((t^o, x_1^o, x_2^o)) \subset I^-((t^e, x_1^e, x_2^e))$ (the converse is analogous). Consider the sequence $\{(t^e + 1/n, x_1^e, x_2^e)\}_n \subset I^+((t^e, x_1^e, x_2^e))$ and note that, by the hypothesis, this sequence also belongs to $I^+((t^o, x_1^o, x_2^o))$. Therefore, from Prop. 2.1, there exist constants $\mu_1^n, \mu_2^n > 0$, with $\mu_1^n + \mu_2^n = 1$, satisfying the following inequalities:

$$\int_{t^o}^{t^e+1/n} \frac{\sqrt{\mu_i^n}}{\alpha_i(s)} \left(\sum_{k=1}^{2} \frac{\mu_k^n}{\alpha_k(s)} \right)^{-1/2} ds > d_i(x_i^o, x_i^e) \qquad \text{for } i = 1, 2. \tag{4}$$

Up to a subsequence, we can assume that $\{\mu_i^n\}_n$ converges to μ_i, for all i, with $0 \leq \mu_1, \mu_2 \leq 1$ and $\mu_1 + \mu_2 = 1$. Using arguments related to convergence of functions we deduce that:

$$\int_{t^o}^{t^e} \frac{\sqrt{\mu_i}}{\alpha_i(s)} \left(\sum_{k=1}^{2} \frac{\mu_k}{\alpha_k(s)} \right)^{-1/2} ds \geq d_i(x_i^o, x_i^e), \qquad \text{for } i = 1, 2.$$

If we consider $(t^o - 1/n, x_1^o, x_2^o)$, and modify slightly (μ_1, μ_2), by continuity we obtain new coefficients (μ_1', μ_2'), with $\mu_1', \mu_2' > 0$ and $\mu_1' + \mu_2' = 1$ and previous inequalities are strict. Again from Prop. 2.1, we have $(t^o - 1/n, x_1^o, x_2^o) \ll (t^e, x_1^e, x_2^e)$ for all n. So, taking into account that $I^-((t^o, x_1^o, x_2^o)) = \cup_{n\in\mathbb{N}} I^-((t^o - 1/n, x_1^o, x_2^o))$, we deduce the inclusion $I^-((t^o, x_1^o, x_2^o)) \subset I^-((t^e, x_1^e, x_2^e))$, as required.

□

Also, the characterization of the causal relation given in Prop. 2.2 allow us to give a characterization of the causally simple doubly warped products:

Theorem 2.2. *A doubly warped spacetime (V, g) as in (1) is causally simple if and only if (M_i, g_i) is L_i-convex for $L_i = \int_a^b \frac{1}{\sqrt{\alpha_i(s)}} ds$, for all $i = 1, 2$.*

Main idea: For the implication to the right. Let x_1^o and x_1^e with

$$0 < d_1(x_1^o, x_1^e) < L_1 = \int_a^b \frac{ds}{\sqrt{\alpha_1(s)}},$$

then, there exists some c_1 and c_2 with $a < c_1 < c_2 < b$ and $d_1(x_1^o, x_1^e) < \int_{c_1}^{c_2} \frac{ds}{\sqrt{\alpha_1(s)}}$ take some point $x_2 = x_2^0 = x_2^e$ in M_2 and consider the following points (c_1, x_1^o, x_2) and (c_2, x_1^e, x_2), then, by Prop. 2.1 we have that $(c_2, x_1^e, x_2) \in I^+((c_1, x_1^o, x_2))$. Since $(c_1, x_1^e, x_2) \notin I^+((c_1, x_1^o, x_2))$, there exists a point

$$(t^e, x_1^e, x_2)) \in \partial I^+((c_1, x_1^o, x_2)) = J^+((c_1, x_1^o, x_2)) \setminus I^+((c_1, x_1^o, x_2)),$$

(because (V, g) is causally simple), so, there exists a null geodesic between these points. The projection of this geodesic on M_1 will be a minimizing geodesic between x_1^o and x_1^e, therefore (M_1, g_1) is L_1-convex, the same reasoning gives the result for (M_2, g_2).

For the implication to the left, we have to show that

$$\overline{J^\pm((t^o, x_1^o, x_2^o))} = J^\pm((t^o, x_1^o, x_2^o)),$$

for all $(t^o, x_1^o, x_2^o) \in V$. Reasoning for the future case, let

$$(t^e, x_1^e, x_2^e) \in \overline{J^+((t^o, x_1^o, x_2^o))},$$

and consider a sequence $\{(t^e+1/n, x_1^e, x_2^e)\}_n \subset I^+((t^o, x_1^o, x_2^o))$, using Prop. 2.1 and convergence of functions we show that

$$d_i(x_i^o, x_i^e) < \int_a^b \frac{\sqrt{\mu_i}}{\alpha_i(s)} \left(\sum_{k=1}^{2} \frac{\mu_k}{\alpha_k(s)}\right)^{-1/2} ds \leq \int_a^b \frac{1}{\sqrt{\alpha_i(s)}} ds = L_i,$$

and, since (M_i, g_i) is L_i-convex we have that Prop. 2.2 implies that $(t^e, x_1^e, x_2^e) \in J^+((t^o, x_1^o, x_2^o))$. The past case is analogous.

□

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