# PhD Thesis

## Model-based estimation and control for systems over communication networks

Departamento de Ingeniería de Sistemas y Automática
Universidad de Sevilla


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| Author: | D. Diego Luis Orihuela Espina |
| | PhD Research Student |
| Supervisors: | Prof. Francisco Rodríguez Rubio |
| | Dr. Fabio Gómez-Estern Aguilar |
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Photography:

http://www.ima.umn.edu/2008-2009/PUB1.22.09/Network_Image_Robert_Ghrist_PL_Card.jpg

# Model-based estimation and control for systems over communication networks

by

DIEGO LUIS ORIHUELA ESPINA

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# Abstract

In the last years we have witnessed the introduction in the control loop of different telecommunication technologies, such as data networks, smart sensors, modern mobile telephony and internet. The control of systems over communication networks constitutes a recent branch of automatic control. The adoption of these new communication capabilities incur in additional issues that ought to be accounted for. In particular, communications delays, packet-based communication, possible data losses, quantization effects, bandwidth limitations and energy consumption, to name a few, are relevant features to be faced in this new paradigm. These issues become critical in real-time applications.

This thesis proposes new solutions for the control and estimation of systems over communication networks. Although the thesis is mainly focused in problems where bandwidth and energy consumption constraints apply, other effects, such as delays and packet dropouts, will also be considered where appropriate.

First of all, the thesis studies the stability of time-delay systems (TDS) and networked control systems (NCS) affected by delays and packet dropouts. A new stability criterion is proposed, achieving less conservative results than existing methods in the literature.

Secondly, the thesis presents a novel method to design $H_2/H_\infty$ controllers applicable to TDS and NCS. The method is demonstrated to synthesize controllers that achieve an upper bound of the cost index lower than other approaches.

Furthermore, the reduction of the traffic over the network is explored by introducing a model of the plant at the controller end of the communication. A periodic and a self-triggered sampling policy are proposed.

Concerning decentralized large-scale systems, the sensor scheduling problem is of great interest when the available bandwidth is severely limited. The thesis proposes two novel solutions in this line: a scheduling based on a predefined periodic pattern and a Kalman-based aperiodic filter. Although the former is a mathematically simpler solution and more energy-efficient, the latter yields better performance. Then, it is shown that, under some assumptions, *a priori* aperiodic solutions eventually produce periodic patterns, providing with the benefits of both approaches.

Finally, the thesis tackles a problem that, despite its importance, has received little attention in the literature: the joint problem of estimation and control for distributed systems. The objective is to propose a design method that ensures the system stability, providing a cost-guaranteed solution with respect to a given quadratic index. The reduction of the bandwidth usage is attained exploiting an event-based communication policy between agents.

Most of the contributions of this thesis are of theoretical nature. Notwithstanding, experimental applications have not been forgotten. Two experimental testbeds have been considered, namely a networked control of a direct drive robot manipulator and an educational four-tank level control system.

*to family and friendship*

# Acknowledgements

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*Luis Orihuela Espina*

Almonte, Diciembre de 2012

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# List of Abbreviations

| | |
|---|---|
| AUV | Autonomous Underwater Vehicle |
| DLF | Discretized Lyapunov Functional |
| dof | degrees of freedom |
| FDE | Functional Differential Equation |
| GUUB | Globally Uniformly Ultimately Bounded |
| iid | identically independently distributed |
| IP | Internet Protocol |
| IQC | Integral Quadratic Constraint |
| LKF | Lyapunov-Krasovskii Functional |
| LMI | Linear Matrix Inequality |
| LQG | Linear Quadratic Gaussian control |
| LQR | Linear Quadratic Regulator |
| LTI | Linear Time-Invariant |
| mpQP | multi parametric Quadratic Problem |
| NCS | Networked Control System |
| PID | Proportional Integral Derivative |
| ODE | Ordinary Differential Equation |
| QoS | Quality of Service |
| QP | Quadratic Problem |
| TCP | Transmission Control Protocol |
| TDMA | Time Division Multiple Access |
| TDS | Time-Delay System |
| UAV | Unmanned Aerial Vehicle |
| UDP | User Datagram Protocol |
| WSN | Wireless Sensor Network |
| ZOH | Zero Order Holder |

# Notation

| | |
|---|---|
| $A, B, C, D$ | State-space system representation |
| $C^1$ | Set of functions whose derivative is continuous |
| $diag\{X_1, \dots, X_n\}$ | Block-diagonal matrix with diagonal elements $X_1, \dots, X_n$ |
| $\delta$ | Model error |
| $e$ | Observation error |
| $\mathcal{E}$ | Set of links of a graph |
| $f^n(\cdot)$ | $f \circ \overbrace{\cdots}^{n \text{ times}} \circ f(\cdot)$ |
| $\mathcal{G}$ | Graph |
| $I_n$ | Identity $n \times n$ matrix |
| $K$ | State-feedback controller matrix |
| $L_2$ | Set of signals with bounded $L_2$ norm |
| $\mathcal{N}_i$ | Set of neighbours of agent $i$ |
| $\mathbb{N}$ ($\mathbb{N}^+$) | Set of (positive) natural numbers |
| $P$ | Lyapunov matrix |
| $\mathbb{R}$ ($\mathbb{R}^+$) | Set of (positive) real numbers |
| $\mathbb{R}^{n \times m}$ | Set of matrices of dimension $n \times m$ |
| $\mathcal{V}$ | Set of vertices of a graph |
| $x$ | Plant state |
| $\hat{x}$ | Estimated plant state |
| $y$ | Plant output |
| $z$ | Controlled output |
| $\langle \cdot, \cdot \rangle$ | Inner product |
| $\lVert \cdot \rVert_p$ | Vector or induced matrix $p$-norm |
| $\lVert x(\cdot) \rVert_{L_2}$ | $\int_{t_0}^{\infty} x^T(s)x(s)ds$ or $\sum_{j=k_0}^{\infty} x^T(j)x(j)$ |
| $X > Y$ ($X \geq Y$) | $X - Y$ is a positive definite (semidefinite) matrix |
| $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ | Symmetric matrix. Entries $*$ implied by symmetry |

# Chapter 1

# Introduction

## 1.1 Historical introduction

**Automatic control** is defined as "the science that aims to substitute the human operator in such a way that the systems or processes operate autonomously". Automatic control arises, then, to facilitate and tackle work in a more comfortable way, to augment the reliability and precision in the process, and to achieve a higher level of productivity and quality of product.

Since its inception, automatic control has incorporated ideas and concepts that have arisen in other scientific areas [242].

Until the mid-nineteenth century, control theory did not constitute a real scientific theory, because it was dedicated to solving problems with a simple trial and error method. During the Industrial Revolution and in early years, several mechanisms were introduced to control some processes. Some mechanisms of note include the pressure regulator by Papin (1681), the centrifugal governor by James Watt (1769) (illustrated in Figure 1.1[1]), and the temperature regulators by Bonnemain (1777). However, none of those methods were based in any theory that establishes a common mathematical basis for automatic control.

Beginning in the mid-nineteenth century, mathematics was used for the stability analysis of dynamical systems, and would later become the language of the theory of automatic control. The newly presented theory of differential equations, boosted by the development of infinitesimal calculus by Sir I. Newton (1642-1727) and G. W. Leibniz (1646-1716), as well as the works carried out by the Bernoulli brothers (the late 1600s and early 1700s), J. F. Riccati (1676-1754) and others, was successfully applied to the analysis of dynamical systems by J. L. Lagrange (1736-1813) y W. R.

[1]http://en.wikipedia.org/wiki/File:Boulton_and_Watt_centrifugal_governor-MJ.jpg

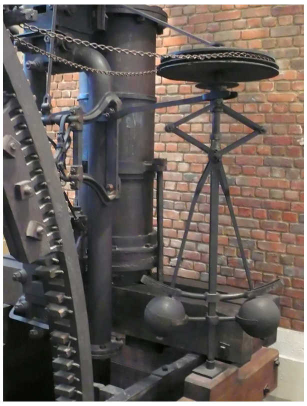


Figure 1.1: Boulton and Watt centrifugal governor

Hamilton (1805-1865). The works of E. J. Routh (1877) and A. M. Lyapunov (1892) in the field of stability were also relevant.

Until the early twentieth century, the analysis of control systems involved differential equations in the time domain. But between 1920 and 1930 at Bell labs, analysis in the frequency domain gained interest. Some figures deserving to be mentioned for their contribution to the control theory are P. S. Laplace (1749-1827), J. B. Fourier (1768-1830) and A. L. Cauchy (1789-1857). In 1938, H. W. Bode studied the frequency response of systems and investigated closed-loop stability using concepts such as gain and phase margin.

In the mid-1940s, stochastic techniques were introduced in the theory of control and communication. For instance, in 1942 N. Wiener analyzed a series of information processing systems employing models of stochastic processes. In the frequency domain, he developed a stochastic optimal filter for stationary signals in continuous time that improved the signal to noise ratio. At the same time, A. N. Kolmogorov (1941) established a theory for stochastic stationary processes in discrete time.

Beginning in the mid-twentieth century, the control community focused on optimality problems. In 1958, L. S. Pontryagin developed his maximum principle, which solved those control optimal problems using the calculus of variations studied by L. Euler (1707-1783). He gave a solution for the minimum time problem and proposed an on/off control law suitable for optimal control. In the United States during 1950, the calculus of variations was applied to solve general optimal control problems at the University of Chicago and other major universities.

In 1960, R. Kalman and his collaborators published three important papers. In one of them, the importance of the work developed by Lyapunov in the field of nonlinear control was pointed out. The second argued about the optimal control of systems, providing a set of equations for the optimal quadratic regulator (LQR). Finally, the third discussed optimal filtering and estimation theory, and provided equations for the discrete Kalman filter. The continuous time Kalman filter was developed later by Kalman himself and Bucy (1961). The work done by Kalman was characterized by the introduction of linear algebra and matrices in such a way that systems with multiple inputs and outputs could be easily handled. He used the concept of the internal state of a system, leading to a characterization based not only on outputs and inputs, but also on the internal dynamics of the systems.

With the arrival of the microprocessor, a new area began. Control systems implemented in computers must previously have been discretized. The studies of C. E. Shannon in the mid-1950s showed the importance of sampled data techniques in signal processing. With the introduction of the PC in 1983, the design of modern control systems was extended to reach every engineers. Hence, the theory of digital control starts, or what is known today as the theory of modern control.

In the last two decades, due to the ubiquity of telecommunication technologies such as mobile phones, internet and data networks, the control community has been challenged by a new and attractive problem: the inclusion of a communication network in the control loop. The control of systems over communication networks constitutes a recent branch of automatic control.

## 1.2 Control of systems over communication networks

The name of Networked Control System (NCS) can encompass a relatively large number of situations and problems. Favoured by the large number of applications and difficulties involved, in the last few years NCS has become a common issue for many control research groups all around the world. A number of efforts have been made towards classifying, organizing and defining what networked systems are [83, 86, 237, 253, 259].

Needless to mention, the feature that distinguishes a NCS from a classical control system is the presence of a communication network affecting or inside the loop. In classical control schemes, like the one presented in Figure 1.2, the controller used to be physically near the plant in such a way that the control commands could be directly applied to the system. Similarly, they assumed that the controller has continuous access to the plant output.

Those assumptions do not hold when a network mediates the connection among the different elements, at least generally speaking. Even when dedicated, standard

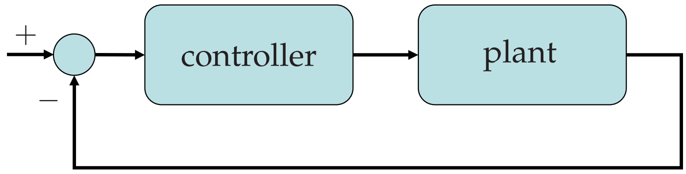



Figure 1.2: Classic control scheme

communication networks are usually designed to preserve data integrity and do not suit the stringent real-time requirements of closed-loop control. These problems become particularly apparent when wireless or non-dedicated networks are used. A large number of systems may be using the communication channel concurrently in such a way that the channel communication capabilities must be shared between them.

Despite emergence of new challenges, a number of benefits are gained. Namely,

**Low cost:** Using a point-to-point communication in large-scale systems or geographically distributed plants may result in higher cost. Wireless or even wired networks, however, reduce the connections and the wire length. Concomitantly, the deployment and maintenance costs are shortened.

**Reliability:** Some network protocols provide reliable communication by means of retransmissions and acknowledgment packets. Additionally, fault detection algorithms can be easily implemented.

**Maintenance:** The reduction of wiring complexity facilitates the diagnosis and maintenance of the system.

**Flexibility:** The inclusion of new elements, such as sensors or actuators, in an operating installation is relatively easy in NCS, whereas in classical systems doing so may incur significant changes in hardware and software.

**Accessibility:** In some plants, some of the elements may occupy locations difficult to reach. The inclusion of the network, especially wireless, could improve reachability.

In addition to these advantages, in other systems the use of networked schemes may be recommended due to space and weight limitations, such as in modern avionics and automotion. In large-scale systems, such as chemical and solar plants, the use of the network will help to cover huge distances. Furthermore, some applications impose the use of a network by the very nature of the problem they are

Figure 1.3: Fleets of submarines, planes and teleoperated robot

addressing. Think, for instance, of a fleet of underwater or flying vehicles, or of teleoperated systems, in which the master and the slave could be separated by hundreds of kilometers (see Figure 1.3[2]).

For these and other reasons, the control community now considers the so-called co-design between control and communication. Not only must the controller be aware of the network, but the communication protocols should also be suitably designed to guarantee and improve the control performance. For instance, see the expert panel report on *Future Directions in Control, Dynamics, and Systems* [165].

## Introducing a communication network in the control loop

Compared to the classical scheme depicted in Figure 1.2, in NCS the various elements are connected by means of a communication network as depicted in Figure 1.4. Communication through such a medium is imperfect and may be affected by some of the following problems (see Figure 1.5).

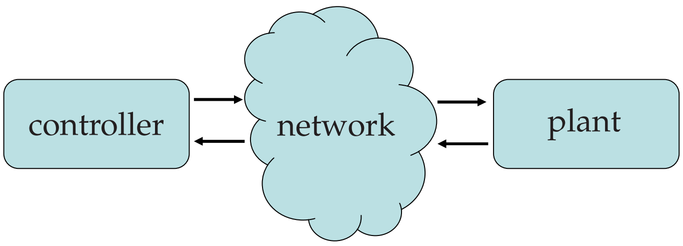



Figure 1.4: Networked control scheme

[2]http://raweb.inria.fr/rapportsactivite/RA2010/necs/uid58.html
http://www.armedforces-int.com/news/usaf-researches-uav-anti-collision-systems.html
http://www.space.mech.tohoku.ac.jp/research/parm/parm-e.html

**Sampling:** If a certain medium has a maximum bandwidth, then the sampling appears as a problem of the channel. In some cases it may have to be taken into account. The information transmitted through the network is included in most cases wrapped inside packets. These packets are sent at a rate that, obviously, is not infinite. Therefore, continuous models must be discretized with an adequate sampling time. In some network protocols, such as WiFi or Ethernet, this sampling time is not constant, as it strongly depends on the network traffic and congestion.

**Delay:** The delay is mainly consequence of the congestion of the medium. Packets travelling through the network are received belatedly and this delay is time-varying and often unpredictable. It is certainly common to receive one packet before another released earlier. Some protocols, such as TCP/IP, implement mechanisms accounting for this, but at the cost of increasing the delay.

**Loss of information:** Some packets may also be lost, mainly because of the capacity of the reception buffer. If an element is receiving packets at a higher rate than it can process them, the buffer could overflow at any instant. The problem is circumvented by retransmitting the packet, but again, the delay is increased.

**Quantization:** Quantization arises when information must be coded using a finite number of bits. Some communication channels, such as underwater mediums, have very low bit rates, so that the packet length must be short to have an adequate transmission rate.

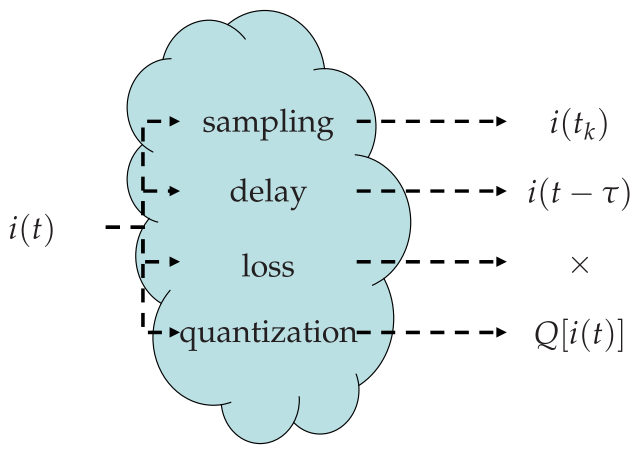



Figure 1.5: The various problems affecting information $i(t)$ transmitted through a network

These effects may affect the information separately but, more often that not, they tend to appear together. For instance, in a non-dedicated congested network, there are two options: either decrease the sampling rate aiming to reduce the congestion and consequently the delay as well as packet losses; or increase the sampling rate sending shorter packages aiming to maximize the use of the available bandwidth, but risking longer delays and dropouts. Both situations are undesirable for a real-time application, and might lead to instability and lower performance.

From the point of view of a control engineer, three approaches could be implemented to cope with the network problems [259]:

**Control of networks:** The objective is to provide a certain level of performance to network data flow, while simultaneously achieving efficient and fair utilization of the network resources. The different elements connected to the network are directed to keep the congestion, delays and dropouts under a prescribed bound.

**Control over networks:** The design of feedback strategies is adapted to control systems in which control data is exchanged through unreliable communication links. The controllers must take into account the possible delays, losses and quantization effects in order to preserve stability, while guaranteeing some performance indexes.

**Co-design:** The ideal situation is the mixed solution. Both the traffic network and the system are controlled, leading to the best accomplishment of the objectives.

Finally, it is worth mentioning another issue that is gaining interest: energy efficiency. In all the fields of engineering, energy-saving is becoming not only an added value, but an imposition in the design of products. In networked control systems, reducing the consumption is beneficial in two ways. Firstly, energy-saving gives rise to lower cost and greater environmental care. Secondly, the life of the batteries is increased, reducing the number of times that they must be replaced and, indirectly, reducing costs. In some networks, the devices could be located in dangerous or unreachable locations. In these situations, replacing the batteries may be expensive or hazardous. Note that most of the energy is consumed when the radio is on, so having an adequate policy in regards to reducing transmissions over the network leads to an energy and cost efficient protocol.

## 1.3 Motivation and objectives

In recent years, the control of systems over communication networks has attracted great attention. Previous sections detail on one hand, the benefits that can be obtained with the inclusion of such a network and, on the other hand, the new problems that arise. Although the literature concerning NCS is huge, there is still much room for further developments that justify and motivate this thesis.

Ensuring the stability of the proposed solutions is of capital importance in the field of automatic control, specially when the controllers are to be applied in real systems. In NCS, the very nature of the systems under consideration advises to apply stability results that take into account the sampling effects, delays and dropouts. Several methods are available in the literature to deal with these problems, such as the Lyapunov-Krasovskii and Lyapunov-Razumikhin approaches. However, both methods suffer from excessive conservatism [62, 81, 99, 114, 162, 200, 216, 247, 251]. The first objective of this thesis is to develop a new and less conservative stability criterion, valid for time-delay systems and networked control systems.

The design of state-feedback controllers constitutes a common topic in the related literature. Several problems have been tackled: network-induced drawbacks [10, 39, 93, 174, 250, 254, 255], disturbance rejection [60, 63, 98, 101, 144, 174], cost-guaranteed solutions [34, 53, 64, 120, 121, 246] and others [90, 170, 244, 268]. But, in most cases, these issues have been studied separately. In this line, $H_2/H_\infty$ controllers for systems affected by communication problems seem to be an adequate solution [33, 65, 113, 258]. However, they have not received sufficient attention. What is the trade-off between performance and delays? And between disturbance rejection and reduction of a given cost index? These and others questions have not yet been answered in NCS.

Additionally, some authors hace proposed an extension which consists of outfitting the controllers with a model of the plant [160, 161, 166]. With this extension, they pursue a reduction of the communication between plant and controller, as the model provides an open-loop estimation of the system state between two consecutive measurements. The inclusion of the network in both paths of the communication (sensor to controller and controller to actuator) and asynchronous sampling policies are two open problems that motivate this research. Both solutions, if possible, could reduce the bandwidth usage and, indirectly, the energy consumption.

In the field of estimation and control of large-scale systems, the available literature is minor. One problem that has received considerable attention is sensor scheduling, in which a set of sensors must share a common medium to transmit their measurements of the plant [43, 44, 57, 76, 88, 89, 130, 198, 206, 238, 262, 263, 264]. The objective is twofold: manage the transmissions, and use the information to reconstruct the state of the plant. Periodic and aperiodic solutions have been proposed. Notwithstanding, problems such as finding optimal patterns, the co-design of both the communication protocol and the observer gains, and the periodic phenomenon appearing in *a priori* aperiodic schemes, remain unsolved.

The problems of distributed control or distributed estimation have also been profusely studied [26, 42, 49, 91, 92, 134, 135, 168, 209, 210, 225, 241]. These schemes are applicable when the plant is estimated or controlled from different, possibly spatially separated, locations. However, in the author's opinion, the joint problem of estimation and control in distributed systems has not been suitably studied in the literature. Stability, sampling policies, network-induced problems and others, are issues that motivate this thesis.

Therefore, the main goal of this research is to afford innovative solutions to some of the new problems that arise when controlling a system through a communication network.

The thesis has been mainly developed in the Automation, Control and Robotic Research Group at the University of Seville (Spain), within the framework of several national and international research projects related to NCS:

- COYAR (Spanish acronym corresponding with 'Control and Analysis of Systems through Communication Networks'). National project.
- FeedNetBack ('Feedback Design for Wireless Network Systems'). European project.
- CONRED (Spanish acronym corresponding with 'Feedback Control of Systems Embedded in Wireless Networks'). National project.

Therefore, the thesis finds an adequate framework in the present research paradigm.

## 1.4 Thesis overview and contributions

The thesis is structured in seven main chapters, the contents of which are summarized below.

**Chapter 2. Preliminaries on NCS and observation techniques.**

This chapter intends to serve as a background to the rest of the thesis. Regarding networked control systems, it gives a particular classification of NCS with respect to the networking scheme. In this way, it is possible to differentiate between small-size systems where the controller is located at the other end of the network, large-scale systems controlled using a network of agents, and multi-agent systems. The second part of the chapter surveys observation techniques, some of which will be employed throughout the thesis.

**Chapter 3. Stability of time-delay systems.**

Doubtlessly, a thesis in the field of control must be characterized by a thorough analysis of the stability of the proposed solutions. The results presented in this chapter are crucial for the rest of the thesis. They will serve as a mathematical foundation to prove the stability of the controllers proposed later. The main contribution of this chapter is a new delay-dependent stability criterion for systems affected by bounded delays. In order to prove the stability, a Lyapunov-Krasovskii approach is followed (Appendix A). A reduced conservatism is achieved by dividing the delay range into a number of intervals.

**Chapter 4. Control of delayed and networked systems.**

This chapter deals with the design of $H_2/H_\infty$ controllers for time delay systems and networked control systems. It presents a general design method that synthesizes optimal controllers that can be applicable to different sorts of time-delay systems (TDS) and to different choices of the Lyapunov-Krasovskii functional. The main contribution here is the demonstration that the presented method produces controllers that achieve an upper bound of the cost index lower than other approaches.

Moreover, by using appropriate mathematical transformations, it is shown that this design method can also be applied to NCS. An experimental application is given at the end of the chapter. The objective is to control a two degree-of-freedom robot at the surroundings of its unstable equilibrium point.

**Chapter 5. Model-based networked control systems.**

This chapter explores the benefits that can be achieved by introducing a model of the plant at the controller end of the communication. In particular the reduction of the necessary transmissions between plant and controller is studied. Two sampling schemes are proposed: periodic and asynchronous. In both cases, the stability of the system is preserved in spite of reducing the bandwidth usage.

**Chapter 6. Scheduled communication for state estimation and control.**

The sensor scheduling problem is applicable in large-scale systems, or when the available bandwidth is severely limited. The plant's outputs are sensed by a set of sensors that share a communication network. With these measurements, a centralized unit estimates the plant state and generates the control commands. Therefore, the problem is twofold. First, the observer and controller gains must be designed. Second, a suitable communication protocol must be defined.

The chapter proposes two solutions. First, a scheduling based on a predefined periodic pattern is proposed. Issues such as pole-placement and the optimal choice of the pattern are studied. Second, a Kalman-based aperiodic filter, in which the sensor with access to the communication channel is the one that minimizes the variance of the observation error. Although the former is a mathematically simpler solution and more energy-efficient, the latter yields enhanced performance.

One of the main contributions is that it is shown that, under some mild assumptions, *a priori* aperiodic solutions eventually produce periodic patterns. Therefore, the benefits of both approaches could be obtained.

**Chapter 7. Distributed estimation and control.**

When the plant is being controlled from different, spatially separated locations, distributed schemes are of interest. The thesis contributes a novel method that allows us to design both the controllers and the observers at once. The objective is to synthesize stabilizing suboptimal controllers, in the sense that the upper bound of a given cost function is minimized. The reduction of the bandwidth usage is attained exploiting an event-based communication policy between agents.

The results have been applied to an experimental plant consisting of a four coupled tank system. The efficiency of the proposed method, in terms of reduction of the traffic and tuning capabilities, is shown.

**Chapter 8. Conclusions.**

The last chapter summarizes the main achievements, and points out the potencial weakness and limitations of the results as well. Furthermore, the impact of the research is evaluated. Finally, different lines of future research are listed.

To summarize, the thesis proposes a number of solutions for both small-size and large-scale plants controlled over communication networks. The persistent goal is to reduce network traffic, as well as the energy consumption. Together with the thesis of Pablo Millán [147], they serve as a comprehensive solution package for NCS.

## 1.5 List of publications supporting this thesis

**Journal papers**

1. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Robust stability of nonlinear time-delay systems with interval time-varying delays*. International Journal of Robust and Nonlinear Control. 21(7):709-724, 2011. [188]

2. P. Millán, L. Orihuela, G. Bejarano, C. Vivas, T. Álamo, F. R. Rubio. *Design and application of suboptimal mixed $H_2/H_\infty$ controllers for networked control systems*. IEEE Transactions on Control Systems Technology. 20(4):1057-1065, 2012. [148]

3. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio. *Control óptimo $L_2$ basado en red mediante funcionales de Lyapunov-Krasovskii*. Revista Iberoamericana de Informática y Automática Industrial. 09(1):14-23, 2012. [157]

4. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio. *Distributed consensus-based estimation considering network induced delays and dropouts*. Automatica. 48(10):2726-2729, 2012. [155]

5. P. Millán, L. Orihuela, I. Jurado, C. Vivas, F. R. Rubio. *Distributed estimation in networked systems under periodic and event-based communication policies*. International Journal of Systems Science. Accepted. [149]

6. L. Orihuela, F. Gómez-Estern, F. R. Rubio. *Scheduled communication in sensor networks*. IEEE Transactions on Control Systems Technology. Under review. [184]

7. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *$H_2/H_\infty$ control for discrete TDS with application to networked control systems: periodic and asynchronous communication*. Optimal Control Applications and Methods. Under review. [192]

8. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Computationally efficient distributed $H_\infty$ observer for sensor networks*. International Journal of Control. Under review. [191]

9. P. Millán, L. Orihuela, I. Jurado, C. Vivas, F. R. Rubio. *Control of autonomous underwater vehicle formations subject to inter-vehicle communication problems*. IEEE Transactions on Control Systems Technology. Under review. [150]

10. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio, D. V. Dimarogonas, K. H. Johansson. *Sensor-network-based robust distributed control and estimation*. Control Engineering Practice. Submitted. [156]

11. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Suboptimal distributed control and observation: application to a four coupled tanks system*. Journal of Process Control. Submitted. [190]

12. L. Orihuela, A. Barreiro, F. Gómez-Estern, F. R. Rubio. *Periodicity of the optimally scheduled distributed Kalman filter*. Automatica. Submitted. [182]

**International conferences**

1. L. Orihuela, F. Gómez-Estern, F. R. Rubio. *Model-based networked control systems under parametric uncertainties*. 18th IEEE International Conference on Control Applications. Saint Petersburg, Russia. pp:7-12. 2009. [194]

2. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio. *Improved delay-dependent stability criterion for uncertain networked control systems with induced time-varying delays*. 1st IFAC Workshop on Estimation and Control of Networked Systems. Venice, Italy. pp:346-351, 2009. [153]

3. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Delay-dependent robust stability analysis for systems with interval delays*. American Control Conference. Baltimore, Maryland, USA. pp:4993-4998, 2010. [186]

4. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio. *An optimal control $L_2$-gain disturbance rejection design for networked control systems*. American Control Conference. Baltimore, MD, USA. pp:1344-1349, 2010. [154]

5. L. Orihuela, P. Millán, G. Bejarano, C. Vivas, F. R. Rubio. *Optimal networked control of a 2 degree-of-freedom direct drive robot manipulator*. IEEE Conference on Emerging Technologies and Factory Automation. Bilbao, Spain. pp:1-8, 2010. [185]

6. P. Millán, L. Orihuela, D. Muñoz de la Peña, C. Vivas, F. R. Rubio. *Self-triggered sampling selection based on quadratic programming*. 18th IFAC World Congress. Milano, Italy. pp:8896-8901, 2011. [151]

7. L. Orihuela, F. Gómez-Estern, F. R. Rubio. *Stability and performance of networked control systems with time-multiplexed sensors and oversampled observer*. 18th IFAC World Congress. Milano, Italy. pp:9200-9205, 2011. [183]

8. L. Orihuela, X. Yan, S. K. Spurgeon, F. R. Rubio. *Variable structure observer for discrete-time multi-output systems*. 12th International Workshop on Variable Structure Systems. Mumbai, India. pp:34-39, 2012. [196]

9. L. Orihuela, S. K. Spurgeon, X. Yan, F. R. Rubio. *A variable structure observer for unknown input estimation in sampled systems*. UKACC International Conference on Control. Cardiff, UK. pp:601-606, 2012. [195]

10. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Improved Performance $H_2/H_\infty$ Controller Design for Time Delay Systems*. European Control Conference. Zürich, Switzerland. Submitted, 2013. [189]

**National conferences**

1. L. Orihuela, F. R. Rubio. *Control adaptativo basado en la estimación de los parámetros de calidad de una red inalámbrica*. XXIX Jornadas de Automática. Tarragona, Spain. 2008. [193]

2. L. Orihuela, T. Álamo, D. Muñoz de la Peña, F. R. Rubio. *Algoritmo de minimización para control predictivo con restricciones*. XXIX Jornadas de Automática. Tarragona, Spain. 2008. [180]

3. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio. *Control óptimo de sistemas de control a través de redes mediante funcionales de Lyapunov-Krasovskii*. XXX Jornadas de Automática. Valladolid, Spain. 2009. [152]

4. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Control y observación distribuida en sistemas de control a través de redes*. XXXII Jornadas de Automática. Sevilla, Spain. 2011. [187]

5. L. Orihuela, A. Barreiro, F. Gómez-Estern, F. R. Rubio. *¿La gestión óptima del canal de comunicaciones para la estimación implica un patrón de muestreo periódico?*. XXXIII Jornadas de Automática. Vigo, Spain. 2012. [181]

# Chapter 2

# Preliminaries on networked control systems and observation techniques

## 2.1 Introduction

This chapter summarizes the background necessary for the rest of the thesis. The first part of this chapter gives the reader a complete view of networked control systems and the state of the art in the field. It presents a classification by the type of plant, that is, small-size systems controlled through a network, large-scale systems and multi-agent systems. The chapter reviews the state of the art regarding each one of these.

The second part of the chapter is devoted to observation techniques. Over the years, several solutions in the context of estimation and prediction have been presented: from the early works of Luenberger [132], that intended to observe the state of a linear system, to modern particle filters [70], which deal with nonlinear time-varying system. Throughout the thesis, some of these solutions will be used and adapted to networked structure. In order to facilitate the understanding, it has been considered of interest to include some remarks related to those estimation techniques that are to be used in subsequent chapters.

## 2.2 Networked schemes

There exist a wide range of possible configurations for a networked control system. It is possible to use a network to connect some or all the elements in the control loop -plant, sensors, controller, actuators- so the possibilities are vast. This section organizes the schemes under three main approaches. A classification is suggested based on the characteristics of the plant: small plants where the controller sends the control commands through a network; large-scale plants, where the sensors and actuators are connected by means of a shared medium; and multi-agent systems where a group of agents are coordinately controlled pursuing a global objective.

Previously, other authors have reviewed networked control systems. Different configurations and models applicable in NCS are presented in [86, 237]. The study of the degradation of the performance and stability due to the network-induced problems is surveyed in [83, 102, 253]. The work [122] reviews the different industrial control networks and their applicability to NCS. Finally, it is worth mentioning the survey [259], in which future trends are presented.

### 2.2.1 Small plants: systems controlled through a network

The simplest configuration of a NCS is perhaps a plant and controller linked through a network. Figure 2.1 depicts this configuration.

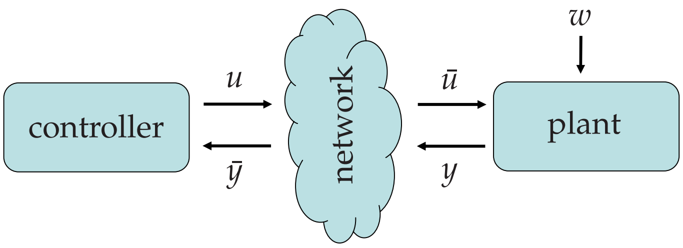



Figure 2.1: The communication network links the system and the controller

The plant output is wrapped within a packet that is sent the network to the controller side. With this information, the controller builds a control signal that is in return wrapped in another packet. It is only when this control packet reaches the other side of the network, that the control input is effectively applied to the plant. It is fairly standard to assume that the different signals -plant output and control input- remain constant between two consecutive measurements. The idea is equivalent to that of a Zero Order Holder (ZOH). Under these assumptions, the plant dynamics can be modelled in continuous,

$$\dot{x}(t) = f_c\left(x(t), \bar{u}(t), w(t)\right), \quad (2.1)$$

$$y(t) = g_c\left(x(t), \bar{u}(t), v(t)\right), \quad (2.2)$$

or in discrete time,

$$x(k+1) = f_d\left(x(k), \bar{u}(k), w(k)\right), \tag{2.3}$$
$$y(k) = g_d\left(x(k), \bar{u}(k), v(k)\right), \tag{2.4}$$

where $x \in \mathbb{R}^n$ is the state of the plant, $\bar{u} \in \mathbb{R}^r$ is the control signal applied, $y \in \mathbb{R}^m$ is the output and $w \in \mathbb{R}^s$, $v \in \mathbb{R}^q$ represent external disturbances and noises. Functions $f_c, f_d : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ and $g_c, g_d : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^q \to \mathbb{R}^m$ model the dynamics of the state and the output, respectively.

The control signal is given by

$$u(t) = h_c\left(\bar{y}(t)\right), \tag{2.5}$$

or, in discrete time,

$$u(k) = h_d\left(\bar{y}(k)\right), \tag{2.6}$$

where $u \in \mathbb{R}^r$ is the generated control signal, and $\bar{y} \in \mathbb{R}^m$ is the output received. Functions $h_c, h_d : \mathbb{R}^m \to \mathbb{R}^r$ model the dynamics of the controller. It could be from a simple state feedback linear controller to a more sophisticated predictive controller.

Because of the mediating network, the control signal generated by the controller $u$ and the one actually applied to the plant $\bar{u}$ may differ. And the same occurs with the system output:

$$\bar{u} = n_u(u), \tag{2.7}$$
$$\bar{y} = n_y(y), \tag{2.8}$$

where maps $n_u, n_y$ represent the effect of sampling, delay and other perjudicial effects.

There exists a vast literature dedicated to this class of NCS. Many authors have studied the conditions that ought to be verified to ensure the stability of the whole system using different controllers and under specific conditions of the network. To mention some of them, it is possible to find find works studying the stability of NCSs with delays [39, 112, 124, 166, 254, 255, 265]; packet losses [10, 93, 174, 236, 250, 254, 255]; or with limited bandwidth [84, 128, 235].

In this scheme, some remarkable results have been obtained based on both the Lyapunov-Krasovskii and Lyapunov-Razumikhin approaches [138]. Using an appropriate functional, it is possible to account for delays and packet dropouts affecting the communication. Both techniques have been widely used to study the stability of time-delay systems, see for instance [252, 260] and references therein. It is from

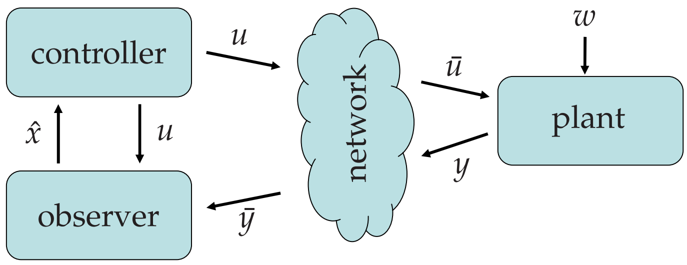



Figure 2.2: Incorporation of an observer at the controller end of the network

the work in [146] and the input delay approach to sampled-data system, when the contributions of time-delay systems were widely applied to this networked control system scheme [30, 61, 60, 99, 101, 112, 162, 233, 257].

In the design of controllers assuring either deterministic or stochastic stability, the main issue has been the conservativeness of the Lyapunov-based methods since its inception. For this reason, given a characterization of the communication channel, a number of techniques have been proposed that have progressively reduced this drawback [58, 62, 80, 99, 162, 200, 216, 247, 269].

In addition to stability considerations, some authors have proposed the application of optimal and robust controllers to NCS. $H_2$ or cost guaranteed controllers have been studied in [93, 115, 123, 203, 223]. And $H_\infty$ controllers for NCS have been proposed in [63, 60, 101, 144, 257]. To the best of the author's knowledge, none addresses the joint problem of optimality and disturbance rejection. One classical approach to this is the $H_2/H_\infty$ control problem, where a certain performance index is minimized ($H_2$), concomitantly with a $L_2$-gain disturbance rejection constraint ($H_\infty$). Chapters 3 and 4 deal with the stability conditions and design of $H_2/H_\infty$ controllers for TDS and NCS.

Alternatively to the scheme depicted in Figure 2.1, it is possible to conceive a slightly different configuration presented in Figure 2.2. Some authors have proposed the inclusion of state observers and/or model-based controllers at the controller end of the communication network [159, 166, 226, 227]. The objective is twofold. Firstly, the performance of the closed loop can be enhanced. And, secondly, the observer/model could reduce the number of transmissions made, overall in the plant-controller path.

Now, the pair observer & controller is defined by

$$\dot{\hat{x}}(t) = \hat{f}_c\left(\hat{x}(t), \bar{y}(t), u(t)\right), \tag{2.9}$$

$$u(t) = \hat{h}_c\left(\hat{x}(t)\right), \tag{2.10}$$

or in discrete time,

$$\hat{x}(k+1) = \hat{f}_d\left(\hat{x}(k), \bar{y}(k), u(k)\right), \tag{2.11}$$
$$u(k) = \hat{h}_d\left(\hat{x}(k)\right), \tag{2.12}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimated state. Functions $\hat{f}_c, \hat{f}_d : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^n$ and $\hat{h}_c, \hat{h}_d : \mathbb{R}^n \to \mathbb{R}^r$ model the dynamics of the observer and of the controller, respectively.

Under this paradigm, two main approaches have been pursued. On one hand, some authors have explored model-based solutions for NCS, in which a model of the plant is running in open loop between two consecutive samplings [40, 124, 125, 126, 160, 161, 164]. The claimed benefits are twofold: (i) reduction of the bandwidth usage by separating the samples using either a periodic or an aperiodic sampling scheme; and (ii) increased performance through a continuous estimation of the state of the plant.

The second approach modifies either the Luenberger or Kalman filter to propose a state observer robust against the conditions of the network [36, 110, 166, 212, 213, 214, 226, 222].

Model-based controllers are the subject of Chapter 5.

### 2.2.2 Large-scale systems

Large-scale systems are informally plants whose elements occupy vast areas or are spatially located in remote positions. Typical examples are buildings, solar plants, or big industrial processes. In those systems, there are a large number of sensors and actuators monitoring and acting over the plant. In order to link and manage them, the use of a network seems perfectly justified.

For these systems it is possible to differentiate two networked schemes, based on the sort of controller. In the first scheme, there is a single centralized controller, which gathers all the information measured by sensors and creates the many control signals to be delivered to the different actuators. In contrast, in the second scheme the controllers could be distributed or decentralized, in the sense that they generate control inputs based on partial and often neighboring information.

#### 2.2.2.1 Centralized controller

A general diagram is depicted in Figure 2.3. A set of sensors measure some aspects of the plant state and send it to the centralized unit located at the other end of the

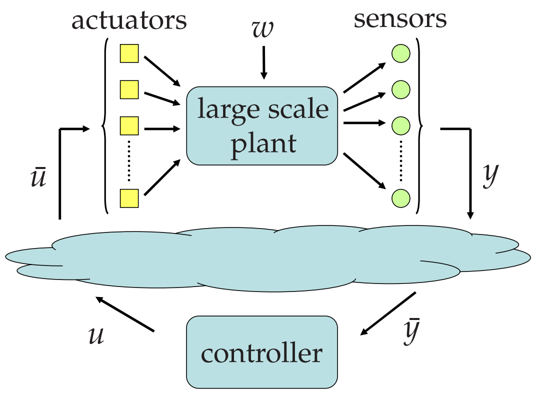



Figure 2.3: Centralized schemes for the control (and estimation) of a large-scale plant.

network. The observer, if exists, in turn estimates the overall plant state. The controller, either using the estimated state or the received outputs, builds the control signals that are sent to the different actuators.

The plant, the controller and the observer are defined as above in equations (2.1)-(2.4) and (2.9)-(2.12).

As the network is, in general, shared, the way in which the different elements divide the available bandwidth must be adequately defined. This is subject of a very interesting line of research in co-design called sensor scheduling [43, 44, 57, 76, 88, 89, 130, 198, 206, 238, 262, 263, 264]. Chapter 6 proposes several solutions for controlling and estimating large-scale systems under a centralized scheme.

#### 2.2.2.2 Decentralized controller

In this case a global controller (observer & controller) is lacking, and instead a set of them are distributed. However, they collaborate pursuing the same goal, namely, the control of the same plant. This demands that consequently the network now supports traffic between controllers (observers & controllers). They share some information with fellow controllers (observers & controllers) through the network. Figure 2.4 describes this situation.

The agents in the network can play the role of observers, estimating the state of the plant, the role of controllers, providing a control signal to a subset of the plant's control inputs, or both. The interconnected nature of the approach allows agents to enrich their estimates not only with the information that they collect directly from the plant, but also with the information exchanged with their neighbors.

The plant is described by equations (2.1)-(2.4). As Figure 2.4 illustrates, in agent-based control is usually assumed that only the transmissions between agents are

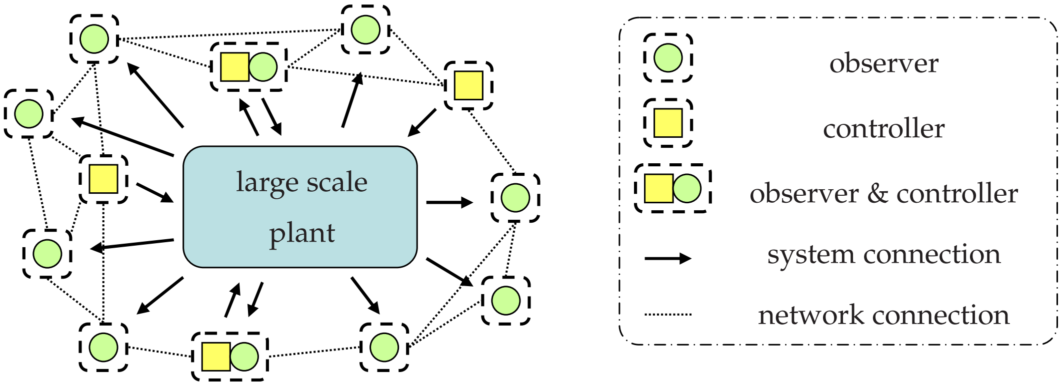



Figure 2.4: Decentralized scheme for the control of a large-scale plant

sent through the network. Control inputs and plant outputs are directly applied and measured, respectively, i.e. $u = \bar{u}, y = \bar{y}$. The complete control signal $\bar{\mathcal{U}}$ applied to the plant is comprised of the partial inputs generated by every controller agent:

$$\bar{\mathcal{U}} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_p \end{bmatrix}, \tag{2.13}$$

where $\bar{u}_i \in \mathbb{R}^{r_i}$ $(i = 1, \ldots, p)$ is the control signal that agent $i$ applies to the system and $p$ is the number of agents in the network. Assuming that $\sum_{i=1}^{p} r_i \geq r$, overlapping is considered. Also notice that if agent $i$ is not a controller agent, then $\bar{u}_i \equiv 0$.

The network in Figure 2.4 is topologically defined by its graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $l$ links between $p$ agents. The graph $\mathcal{G}$ is, in general, directed, with agents $\mathcal{V} = \{1, 2, \ldots, p\}$ and links $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The set of agents which are connected to agent $i$ is named the *neighborhood* of $i$ and is denoted by $\mathcal{N}_i \equiv \{j : (i,j) \in \mathcal{E}\}$. Link $(i,j)$ implies that agent $i$ receives information from agent $j$.

A generic agent $i$ may receive information from the plant $\bar{y}_i \in \mathbb{R}^{r_i}$ and may deliver some control input $u_i \in \mathbb{R}^{m_i}$. Additionally, each agent may be running an observer of the plant. Then, the dynamics of agent $i$ are given by the following equations:

$$\dot{\hat{x}}_i(t) = \hat{f}_{ci}\left(\hat{x}_i(t), \bar{y}_i(t), u_i(t), \hat{x}_j(t)(j \in \mathcal{N}_i)\right), \tag{2.14}$$

$$u_i(t) = \hat{h}_{ci}\left(\hat{x}_i(t)\right), \tag{2.15}$$

or in discrete time,

$$\hat{x}_i(k+1) = \hat{f}_{di}\left(\hat{x}_i(k), \bar{y}_i(k), u_i(k), \hat{x}_j(k)(j \in \mathcal{N}_i)\right), \tag{2.16}$$

$$u_i(k) = \hat{h}_{di}\left(\hat{x}_i(k)\right), \tag{2.17}$$

where $\hat{x}_i \in \mathbb{R}^{n_i}$ is the estimated state. Functions $\hat{f}_{ci}, \hat{f}_{di} : \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \times \mathbb{R}^{r_i} \times \mathbb{R}^{r_j} \rightarrow \mathbb{R}^{n_i}$ and $\hat{h}_{ci}, \hat{h}_{di} : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{r_i}$ model the dynamics of the observer and of the controller, respectively. The output measured from the plant is given by

$$\bar{y}_i(t) = g_{ci}\left(x(t), \bar{u}(t), v(t)\right), \tag{2.18}$$

or

$$\bar{y}_i(k) = g_{di}\left(x(k), \bar{u}(k), v(k)\right), \tag{2.19}$$

where $g_{ci}, g_{di} : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^q \rightarrow \mathbb{R}^{m_i}$ model the dynamics of the output $i$.

Decentralized and distributed control of large-scale plants have been subject of intense research in the last years [3, 24, 28, 42, 72, 135, 136, 137, 175, 176, 177, 209, 218, 219, 220, 230, 256, 270]. Chapter 7 studies both the estimation and control of large-scale systems under a decentralized scheme.

### 2.2.3 Fleets, groups or multi-agent systems

The last scheme is structurally radically different from previous ones. There is not necessarily a unique plant to be controlled, but a set of different agents. Although each agent may have its own inner objective, there exists a common goal for all the group of agents. In order to adequately accomplish their task, the agents share some information with their neighbours. Think, for instance, in a group of Unmanned Aerial Vehicles (UAV) that are flying together. To preserve the formation of the fleet, the UAVs benefits from the knowledge of positions and velocities of other vehicles. Therefore, a communication network between agents seems important. Figure 2.5 shows this particular situation.

Each agent is defined by its internal state $x_i$, whose dynamics is given by

$$\begin{aligned} \dot{x}_i(t) &= f_{ci}\left(x_i(t), u_i(t), w_i(t), \bar{y}_k(t)(k \in \mathcal{N}_i)\right), & (2.20)\\ u_i(t) &= h_{ci}\left(x_i(t)\right), & (2.21)\\ y_{ij}(t) &= g_{cij}\left(x_i(t), u_i(t), w_i(t)\right),\ i \in \mathcal{N}_j, & (2.22) \end{aligned}$$

or in discrete time,

$$\begin{aligned} x_i(k+1) &= f_{di}\left(x_i(k), u_i(k), w_i(k), \bar{y}_k(k)(k \in \mathcal{N}_i)\right), & (2.23)\\ u_i(k) &= h_{di}\left(x_i(k)\right), & (2.24)\\ y_{ij}(k) &= g_{dij}\left(x_i(k), u_i(k), w_i(k)\right),\ i \in \mathcal{N}_j, & (2.25) \end{aligned}$$

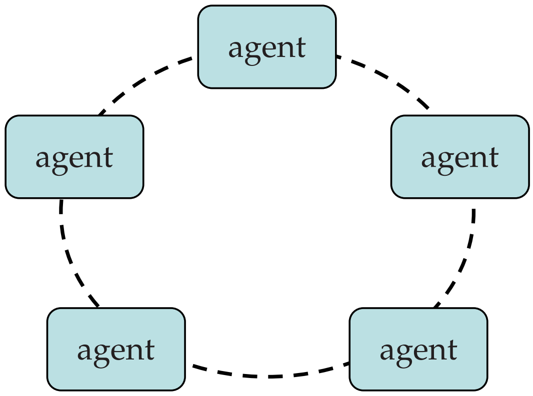



Figure 2.5: Multi-agent system

where $x_i \in \mathbb{R}^{n_i}$ is the state of agent $i$, $u_i \in \mathbb{R}^{r_i}$ is the local control signal, and $w_i \in \mathbb{R}^{s_i}$ represents external disturbances. Note that the control signal is locally computed in each agent, unlike previous schemes.

The output $y_{ij} \in \mathbb{R}^{m_{ij}}$ is the information that agent $i$ shares with agent $j$. Note that the state of each agent may be influenced by the information received from every neighboring agent.

Consensus problems are a very interesting line of research for multi-agent systems, where all the agents intend to get a common estimation of some dynamic or static variable [12, 87, 178, 179, 231]. Control of multi-agent systems and, in particular, fleet of vehicles have received much attention for the control community [20, 22, 32, 35, 48, 68, 118, 119, 204, 207]. This kind of systems is not covered in this thesis, and the reader is directed to more specialized literature [18, 55, 96, 116, 178, 201, 217].

## 2.3 Observation techniques

This second part of the chapter is dedicated to review some well-known observation techniques existing in the literature. Following [52], the principle of an observer is that by combining a measured feedback signal with knowledge of the control-system components (primarily the plant and feedback system itself), the behavior of the plant can be estimated with greater precision than shall the feedback signal be used in isolation.

In some cases, the observer can be used to enhance system performance. It can be more accurate than sensors and/or can reduce the phase lag inherent in the sensor. Observers can also provide observed disturbance signals, which can be used in turn to improve the disturbance response. In other cases, observers can reduce the system cost by augmenting the performance of a low-cost sensor so that the two together can afford a performance equivalent or superior to a higher cost sensor. In the extreme case, observers can make a sensor redundant altogether, eliminating the sensor cost and the associated wiring.

Observer technology is not without hinders. Observers add complexity to the system and require computational resources. They may be less robust than physical sensors, especially when plant parameters vary substantially during operation. Still, an observer developed with skill can bring substantial performance benefits and do so, in many cases, whilst reducing cost and/or increasing reliability.

From the early works of Kalman [107] and Luenberger [131, 132], a multitude of different adaptive filters and estimators have been proposed. A comprehensive review of these is beyond the scope of this thesis. Nevertheless, in order to facilitate the understanding of the following chapters, it is considered of interest to give some notions about two classical solutions: the Luenberger observer and the Kalman filter. Both estimators can be used to observe the state of a linear system. There exist continuous and discrete versions of both approaches, but only the discrete one will be presented.

### 2.3.1 Luenberger observer

Consider the following unperturbed discrete linear time-invariant (LTI) system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned}$$

where $A, B, C$ are known matrices of appropriate dimensions. The Luenberger observer[1] is defined by

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k)),$$

where $L$ is the observer gain. The schematic diagram of the observer is depicted in Figure 2.6.

The observation error is given by $e(k) = x(k) - \hat{x}(k)$. The dynamics of the error depends directly on the observer gain $L$,

$$e(k+1) = (A - LC)e(k).$$

[1]This is the simplest state-space representation of the observer. More detailed forms including uncertainties or disturbances can be found in the literature, see [52] and references therein.

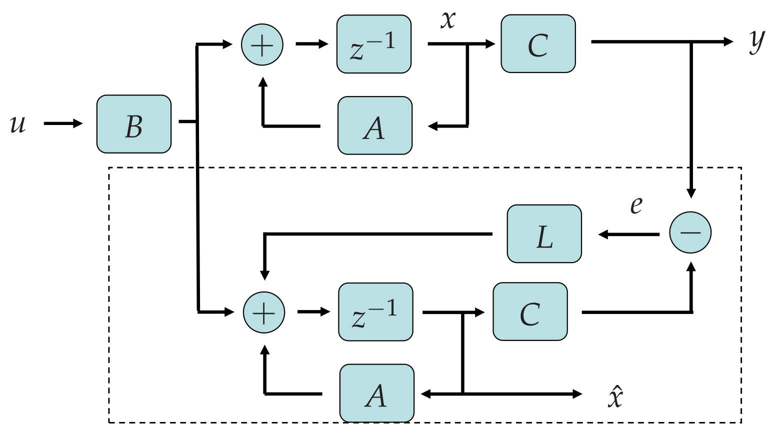



Figure 2.6: Schematic block-diagram of the Luenberger observer

If the pair $(A, C)$ is observable, the poles of the closed-loop can be freely placed [52]. A necessary condition for the stability of the observation error is the detectability of the pair $(A, C)$. The design of the observer gain $L$ can be made with different techniques: optimality, pole placement, robustness, etc. [52, 71].

### 2.3.2 Kalman filter

The Kalman filter [107] is, arguably, the most famous observer proposed. The 'filter' name is related with the fact that it tries to estimate the actual state of a system. Smoothers or predictors intend, on the other hand, to yield an estimation of past instants of the state, or predict the future evolution of it, respectively [11]. The Kalman filter is an algorithm which capitalises on a series of measurements observed over time, containing noise and other inaccuracies, and produces estimates of unknown variables that tend to be more precise than those that would be based on a single measurement alone.

The Kalman filter has found application in a vast number of fields, not only in control. From a theoretical standpoint, the main assumption is that the underlying system is a linear dynamical system and that all error terms and measurements follow a Gaussian distribution. Extensions and generalizations to the method have been developed, such as the Extended Kalman Filter [127] and the Unscented Kalman filter which work on nonlinear systems [105, 106].

In the following, the simplest form of the Kalman filter is shown. Figure 2.7 illustrates the block diagram.

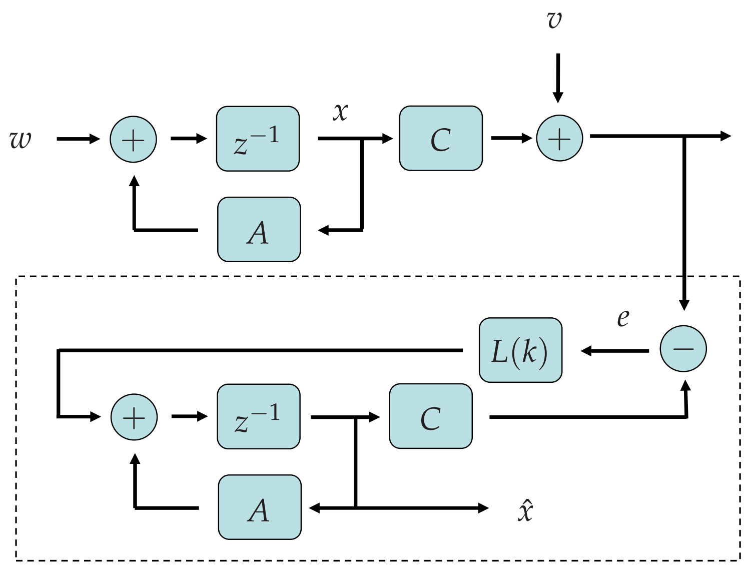



Figure 2.7: Schematic block-diagram of the Kalman filter

Consider a discrete-time linear perturbed system, whose dynamics is given by[2]

$$\begin{aligned} x(k+1) &= Ax(k) + w(k), && (2.26)\\ y(k) &= Cx(k) + v(k), && (2.27)\end{aligned}$$

where the processes $w(k), v(k)$ are Gaussian i.i.d., with

$$\begin{aligned} E[w(k)] &= 0,\\ E[w(k)w(k)^T] &= Q,\\ E[v(k)] &= 0,\\ E[v(k)v(k)^T] &= R,\end{aligned}$$

and represents uncertainties of the model, noises and disturbances. The dynamics of the filter is determined by

$$\hat{x}(k+1) = A\hat{x}(k) + L(k)\left[y(k) - C\hat{x}(k)\right].$$

The error between the state of the system and the state of the observer is again $e(k) = x(k) - \hat{x}(k)$. The evolution of the error is given by

$$e(k+1) = (A - L(k)C)e(k) + w(k) - L(k)v(k).$$

[2]Note that the control input has been removed from (2.26), because the measurement under estimation does not necessarily represent the output of a linear control system.

The Kalman filter aims at minimizing the expected value of the observation error, as well as its variance. To do that, it tries to minimize the trace of the error covariance matrix, defined as

$$P(k) = E\left[e(k)e(k)^T\right].$$

The adaptation of the covariance matrix can be calculated as

$$\begin{aligned} P(k+1) &= AP(k)A^T + Q - AP(k)C^T L(k)^T - L(k)CP(k)A^T \\ &\quad + L(k)CP(k)C^T L(k)^T + L(k)RL(k)^T. \end{aligned} \tag{2.28}$$

The Kalman gain is the one that minimizes the trace of $P(k+1)$ and can be obtained as

$$L(k)^* = AP(k)C^T \left(CP(k)C^T + R\right)^{-1}.$$

Using this value for the observer gain in equation (2.28) the matrix covariance at instant $k+1$ is

$$P(k+1) = AP(k)A^T + Q - AP(k)C^T \left(CP(k)C^T + R\right) CP(k)A^T,$$

which is a Riccati recursion with initial condition $P(0) = P_0$. Note that $P(k)$ can be computed before any observations are made. Thus, the estimation error covariance can be calculated before getting any observed data.

As in the Linear Quadratic Regulator (LQR), the Ricatti recursion for $P(k)$ converges to steady-state value $\hat{P}$ provided that $(A, C)$ is detectable and $(A, Q)$ stabilizable. Note that $\hat{P}$ satisfies the following algebraic Riccati equation:

$$\hat{P} = A\hat{P}A^T + Q - A\hat{P}C^T \left(C\hat{P}C^T + R\right) C\hat{P}A^T,$$

which can be directly solved [78].

The steady-state filter is a time-invariant observer, as in the previous section,

$$\hat{x}(k+1) = A\hat{x}(k) + \hat{L}\left[y(k) - C\hat{x}(k)\right],$$

where $\hat{L} = A\hat{P}C^T \left(C\hat{P}C^T + R\right)^{-1}$.

Therefore, the estimation error propagates according to a linear system, with closed-loop dynamics $(A - \hat{L}C)$, driven by the process $w(k) - \hat{L}v(k)$, which is i.i.d. zero mean and covariance $Q + \hat{L}R\hat{L}^T$. Provided that $(A, Q)$ is stabilizable and $(A, C)$ is detectable, the closed-loop dynamics is stable [117].

## 2.4 Chapter summary

This chapter has offered, in its first part, an overview of systems controlled over a communication network. It has suggested a classification based on the sort of plant or plants being controlled. The different schemes reviewed will serve as a background for the rest of the thesis as was hinted through this chapter.

The second part reviews two classical observation schemes used in control theory, namely, the Luenberger observer and the Kalman filter. Throughout the thesis, a number of modifications of these two will be proposed to adapt them to NCS.

# Chapter 3

# Stability of time-delay systems

## 3.1 Introduction

This chapter is mainly dedicated to the study of stability of time-delay systems, sometimes called by the initials TDS. As the name suggests, these are systems whose internal dynamics are affected by non negligible delays. It is well known that time-delays are a major source of instability in the control loop and naturally arise in a number of practical control problems such as networked control systems, chemical processing systems, transportation systems, and power systems [208].

The reader may find surprising the inclusion of such chapter in a thesis about networked systems. However, there exists a compelling reason. As it will be shown in the following chapter, a system controlled through a network can be modelled as a TDS under some assumptions. Therefore, most of the ideas and developments in this field can be extrapolated to NCS.

During the past two decades, considerable attention has been devoted to the problem of stability analysis and control design for TDS. Several reasons justify this. First, time-delay is an applied problem, since many real-world applications include delays in their inner dynamics. To name a few, examples can be found in biology, chemistry, economics, mechanics, physics, population dynamics, and engineering (see [171] for detailed examples). Remarkably, TDS are involved in feedback loops in challenging areas of communication and information technologies: stability of networked controlled systems [23] or Internet video transmission [145]. A second reason is that TDS entail important theoretical challenges from the point of view of stability analysis and controller design. They belong to the class of functional differential equations (FDEs) which are infinite dimensional, as opposed to ordinary differential equations (ODEs). The first implication of this, is that TDS cannot be adequately analyzed resorting to classical tools. Approaches to the problem in terms of finite dimensional approximations often lead to conservative results or even unstable behaviors when dealing with time-varying delays [208].

Two main Lyapunov-based approaches are usually invoked to study the stability of time-delay systems: Lyapunov-Razumikhin and Lyapunov-Krasovskii theorems, see Appendix A. Both approaches can handle time-varying delays, but results using Lyapunov-Krasovskii functionals are usually less conservative, since they allow to incorporate additional information on the derivative of the time-varying delay. A completely different approach was proposed using the ideas of Integral Quadratic Constraints (IQC) in [108, 143]. These are standard methods that allow to provide a way of representing relationships between processes evolving in a complex dynamical system in a convenient form for analysis. With regards to Lyapunov-Krasovskii methods, there is no theoretical result that suggests that one method outperforms the other in any way. The rest of the chapter, and some of the stability results on this thesis, are based in the Lyapunov-Krasovskii theorem.

In this framework, many recent works have addressed the problem of finding delay-dependent sufficient conditions to ensure the stability of linear TDS. Delay-dependent conditions introduce information about the characteristics of the delay (lower and upper bounds, time derivative, etc), thus obtaining better results, in general, than delay-independent approaches. As delay-independent conditions do not use information on the characteristics of the delay, they can only be applied to systems containing instantaneous negative stabilizing feedback terms: for instance, linear systems in the state-space realization which contain $Ax(t)$, where $A$ is a Hurwitz matrix, see [114] and the references therein.

First works on delay-dependent stability analysis assumed constant but unknown delays [114, 162, 200, 251]. However, there are a number of practical applications in which the delay is inherently time-varying. For instance, in networked control systems applications. In such cases, some authors have derived delay-dependent conditions that assume a known upper bound for the delay [62, 81]. Moreover, in many practical situations, a lower delay bound (not necessarily zero) can be assumed. This fact has been recently exploited by a number of works [99, 216], showing that it is possible to improve results if the information about both lower and upper delay bounds are taken into account.

Nonetheless, the criteria to guarantee asymptotic stability of time-delay systems suffered from important conservatism since its inception. The standard methodology in this context typically consists in proposing a Lyapunov-Krasovskii functional whose derivative along the trajectories of the system is proven negative definite. In this process, specific terms are usually required to be bounded in order to cast conditions as Linear Matrix Inequalities (LMI), and this is usually the major source of conservatism.

Over the recent years, the control community has witnessed a continuous race aiming at reducing the conservatism. For instance, in [162, 200] the authors introduced novel bounds for the inner product of two vectors that typically arise in the derivative of proposed Lyapunov-Krasovskii functionals. In [62], a *descriptor representation* for time-delay systems was introduced bringing a reduction in the conservatism. Nowadays, a fairly standard technique, that was introduced in [247, 251], is the use of free weighting matrices (also called *slack matrices*). The mathematical argument consists in adding null terms to the derivative of the Lyapunov-Krasovskii functional, by using the Leibniz-Newton formula. These null terms include free matrices to provide additional degrees of freedom. Recently, some authors have worked to improve the bounds of some integral terms appearing frequently in this context, see [80]. The use of polytopes to describe the delay was introduced in [216] with interesting results.

After giving some remarks about the different formulations of time-delay systems, the purpose of this chapter is to derive an improved stability criterion for a particular type of time-delay systems based on the Lyapunov-Krasovskii approach. The system is affected by norm-bounded time-varying nonlinear uncertainties. The lower and upper bounds of the delay interval are assumed to be known. With respect to its variation, a finite upper bound on the time derivative of the delay is given. Additionally, the proposed results incorporate the analysis of the $L_2$-gain of the system.

In order to reduce conservatism in the derivation of the stability conditions, this work resorts to the idea of splitting the known bounds of the delay interval in multiple regions or subintervals. This allows to reduce the conservatism due to the fact that less restrictive bounds for specific terms in the Lyapunov-Krasovskii functional are derived separately in each subinterval. The idea has points of similarity with *discretized Lyapunov functionals* (DLF) (see [59, 73, 74]), but there are some relevant differences: to the best of our knowledge, available publications on DLF deal only with systems with constant delay. Moreover, DLFs use piecewise linear matrices in the functional which depend on each subinterval and impose constraints to guarantee that the Lyapunov functional is, indeed, positive definite. On the contrary, the approach proposed in this chapter deals with time-varying delays. Furthermore, an unique Lyapunov-Krasovskii functional is defined for all subintervals. Each subinterval covers a delay range which imposes an LMI constraint on the functional. Reduced conservatism is achieved by introducing appropriate slack matrices in each region, providing additional degrees of freedom.

### Related publications

1. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Delay-dependent robust stability analysis for systems with interval delays*. American Control Conference. Baltimore, Maryland, USA. pp:4993-4998, 2010. [186]

2. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Robust stability of nonlinear time-delay systems with interval time-varying delays*. International Journal of Robust and Nonlinear Control. 21(7):709-724, 2011. [188]

## 3.2 The family of time-delay systems

Time-delay systems can be described in several ways: transfer functions, state-space realizations, functional differential equations or more complex models. See [208] for a complete list of models. In this thesis, a state-space description is used. This section reviews the different kinds of linear TDS considered in the literature, and proposes a common notation to encompass all these descriptions under an unified approach.

Let us first recall the structure and characteristics of the different TDS considered:

1. *Standard time-delay systems*

   The first TDS model considered, (3.1), represents a functional differential equation (FDE) that accounts for time-varying delayed state:

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + A_h \int_{t-\eta(t)}^{t} x(s)ds. \tag{3.1}$$

2. *Descriptor systems with delay*

   The descriptor system represents a generalization of the previous one in which a singular matrix $E$ is introduced in the dynamics as follows

$$E\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + A_h \int_{t-\eta(t)}^{t} x(s)ds. \tag{3.2}$$

**Remark 3.1.** The descriptor form introduced in [58] corresponds to a 2-D model that resorts to the Leibniz-Newton formula by posing the system into the singular system structure given by (3.2). It is then a particular case of this descriptor linear system with delay.

3. *Neutral systems with delays*

   Neutral systems are also delayed systems, but involve the same highest derivation order for some components of $x(t)$ at both time $t$ and past time $t' < t$, which implies an increased mathematical complexity. The general model can be expressed as

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + A_h \int_{t-\eta(t)}^{t} x(s)ds + A_2 \dot{x}(t - v(t)). \tag{3.3}$$

All these sorts of TDS can be nonetheless expressed under a unified description as it is shown next. Let us introduce the following linear state-space equation:

$$E\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + A_h \int_{t-\eta(t)}^{t} x(s)ds + A_2 \dot{x}(t - hv(t)). \tag{3.4}$$

It is fairly straightforward to verify that model (3.4) can describe the dynamics of the three forms of TDS considered.

Unlike non delayed system, the initial condition must be given for the whole time interval in which the delays are defined. Concretely, for all these descriptions the initial condition is defined as $x_{t_0}(\theta) = \phi(t + \theta)$, for all $\theta \in [-r, 0]$, being $r = \max\{\max d(t), \max \eta(t), \max v(t)\}$, that is, the maximum delay of all delayed components.

## 3.3 Problem statement

In this chapter, a simplified sort of the general TDS given in (3.4) is considered, but including nonlinear uncertainties and external disturbances:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t - d(t)) + h(t, x(t)) + B_w w(t), && (3.5) \\ z(t) &= Cx(t) + C_d x(t - d(t)), && (3.6) \end{aligned}$$

where $h(t, x(t)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ represents nonlinear uncertainties of the plant. A diagram of the system is depicted in Figure 3.1.

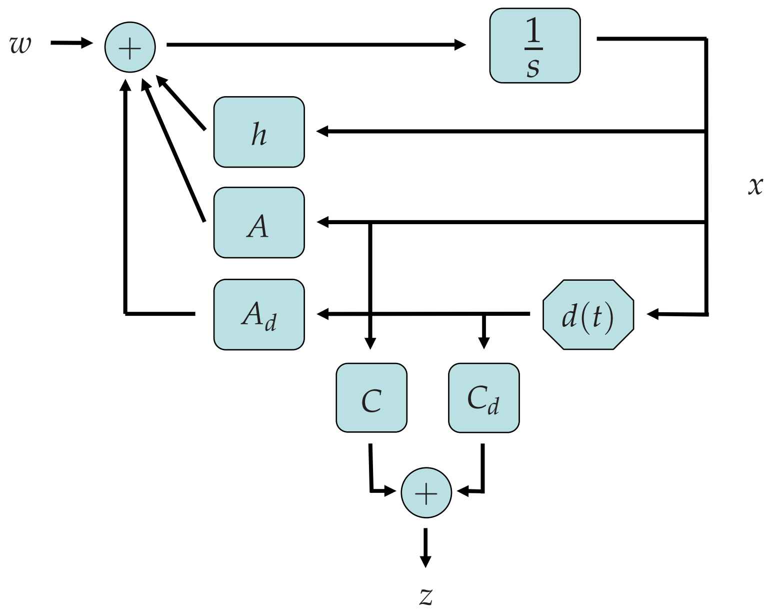



Figure 3.1: Schematic diagram of the perturbed nonlinear time-delay system

It is assumed that $h(t, x(t))$ is a piecewise-continuous nonlinear function in $t$ and $x$, that satisfies the following quadratic constraint condition:

$$h^T(t, x(t))h(t, x(t)) \leq \alpha^2 x^T(t)H^T Hx(t), \quad \forall t \geq 0, \tag{3.7}$$

where $\alpha$ is the bounding parameter and $H$ is a constant matrix.

Observe that, for any given $H$, inequality (3.7) defines a class of piecewise-continuous functions such that

$$H_\alpha = \left\{ h(t, x(t)) : h^T(t, x(t))h(t, x(t)) \leq \alpha^2 x^T(t)H^T Hx(t) \text{ for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \right\}.$$

The set $H_\alpha$ is comprised of functions satisfying $h(t, 0) = 0$ in their domains of continuity. It is assumed that $x(t) = 0$ is an equilibrium point of system (3.5).

The time delay $d(t)$ is a time-varying continuous function that satisfies:

$$d_m \leq d(t) < d_M, \tag{3.8}$$

$$\dot{d}(t) \leq \mu. \tag{3.9}$$

Note that no hard constraint on the derivative of the time delay is imposed, as $\mu$ can take any positive finite value.

The initial condition for the system is $x_{t_0}(\theta) = \phi(t + \theta)$, where $\phi(t + \theta)$ is a continuous vector-valued function of $\theta \in [-d_M, 0]$.

The objective of the chapter is to present a stability criterion for system (3.5)-(3.6) in such a way that:

- System (3.5) is robustly asymptotically stable with degree $\alpha$, that is, the equilibrium point $x(t) = 0$ is globally asymptotically stable for $w \equiv 0$ and for all $h(t, x(t)) \in H_\alpha$ [224].
- Under the assumption of zero initial condition, output $z(t)$ satisfies $\|z(t)\|_{L_2} \leq \gamma \|w(t)\|_{L_2}$ for any nonzero disturbance $w(t) \in L_2[0, \infty)$.

In order to derive a less restrictive stability criterion, the complete delay range is divided into multiple disjoint subintervals of the same size,

$$[d_m, d_M) \equiv [d_1, d_2) \cup [d_2, d_3) ... \cup [d_i, d_{i+1}) ... \cup [d_N, d_{N+1}) \tag{3.10}$$

where $d_1 \triangleq d_m$, $d_{N+1} \triangleq d_M$ and $d_{i+1} - d_i \triangleq \Delta d$.

The parameter $N$ provides the method with an additional degree of freedom. As it will be shown, less conservative results can be obtained when substituting the delay bounds in (3.8)-(3.9) in each subinterval. This idea will be used in the next section to derive the stability criterion proposed in the chapter.

## 3.4 Robust stability and $L_2$-gain analysis

This section presents the main result of this chapter. The aim is to develop a novel stability criterion for the nonlinear system (3.5)-(3.6) under $L_2$-bounded disturbances. Recall from equation (3.8) that $d_m \leq d(t) < d_M$, and from (3.10) that constants $d_i$ are defined in such a way that $d_m = d_1 < d_2 < ... < d_{N+1} = d_M$.

The following theorem presents a delay-dependent criterion in terms of LMIs.

**Theorem 3.1.** *Given scalars $0 < d_m < d_M$, $\mu$, $\alpha$ and $\gamma$, system (3.5)-(3.6) is robustly asymptotically stable with degree $\alpha$ for all admisible uncertainties $h(t, x(t)) \in H_\alpha$ and presents an $L_2$-gain lower than $\gamma$, if there exist any matrices $P, Z_1, Z_2, Q_1, \ldots, Q_{N+2}, > 0$, any matrices $N_{ji}, R_{ji}$, $(j = 1, 2;\ i = 1, \ldots, N)$, and a scalar $\epsilon > 0$ such that the following $N$ LMIs for $i = 1, \ldots, N$, are satisfied:*

$$\begin{bmatrix} \Gamma_i & \bar{P} & \bar{P_w} & \Delta d \bar{N}_i & \Delta d \bar{R}_i & \bar{A} U & \epsilon \bar{H} & \bar{C} \\ * & -\epsilon I & 0 & 0 & 0 & U & 0 & 0 \\ * & * & -\gamma^2 I & 0 & 0 & B_w^T U & 0 & 0 \\ * & * & * & -\Delta d(Z_1 + Z_2) & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Delta d(Z_1 + Z_2) & 0 & 0 & 0 \\ * & * & * & * & * & -U & 0 & 0 \\ * & * & * & * & * & * & -\frac{\epsilon}{\alpha^2} I & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{3.11}$$

*where*

$$\Gamma_1 = \begin{bmatrix} \theta_{11} & \theta_{1,12} & R_{11} + \frac{Z_1}{d_m} & -N_{11} & 0 & \cdots & 0 & 0 \\ * & \theta_{1,22} & R_{21} & -N_{21} & 0 & \cdots & 0 & 0 \\ * & * & -Q_1 - \frac{Z_1}{d_m} & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & -Q_2 - \frac{Z_1+Z_2}{d_M-d_2} & 0 & \cdots & 0 & \frac{Z_1+Z_2}{d_M-d_2} \\ * & * & * & * & -Q_3 & \cdots & 0 & 0 \\ * & * & * & * & * & \ddots & \vdots & \vdots \\ * & * & * & * & * & * & -Q_N & 0 \\ * & * & * & * & * & * & * & -Q_{N+1} - \frac{Z_1+Z_2}{d_M-d_2} \end{bmatrix},$$

$$\Gamma_N = \begin{bmatrix} \theta_{11} & \theta_{N,12} & \frac{Z_1}{d_m} & 0 & \cdots & 0 & R_{1N} & -N_{1N} \\ * & \theta_{N,22} & 0 & 0 & \cdots & 0 & R_{2N} & -N_{2N} \\ * & * & -Q_1 - \frac{Z_1}{d_m} - \frac{Z_1+Z_2}{d_N-d_m} & 0 & \cdots & 0 & \frac{Z_1+Z_2}{d_N-d_m} & 0 \\ * & * & * & -Q_2 & \cdots & 0 & 0 & 0 \\ * & * & * & * & \ddots & \vdots & \vdots & \vdots \\ * & * & * & * & * & -Q_{N-1} & 0 & 0 \\ * & * & * & * & * & * & -Q_N - \frac{Z_1+Z_2}{d_N-d_m} & 0 \\ * & * & * & * & * & * & * & -Q_{N+1} \end{bmatrix},$$

*and for* $i = 2, \ldots, N-1$,

$$\Gamma_i = \begin{bmatrix} \theta_{11} & \theta_{i,12} & \frac{Z_1}{d_m} & 0 & \cdots & 0 & R_{1i} & -N_{1i} & 0 & \cdots & 0 & 0 \\ * & \theta_{i,22} & 0 & 0 & \cdots & 0 & R_{2i} & -N_{2i} & 0 & \cdots & 0 & 0 \\ * & * & -Q_1 - \frac{Z_1}{d_m} - \frac{Z_1+Z_2}{d_i-d_m} & 0 & \cdots & 0 & \frac{Z_1+Z_2}{d_i-d_m} & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & -Q_2 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & * & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & * & -Q_{i-1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & * & * & * & -Q_i - \frac{Z_1+Z_2}{d_i-d_m} & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & * & * & * & * & -Q_{i+1} - \frac{Z_1+Z_2}{d_M-d_{i+1}} & 0 & \cdots & 0 & \frac{Z_1+Z_2}{d_M-d_{i+1}} \\ * & * & * & * & * & * & * & * & -Q_{i+2} & \cdots & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \ddots & \vdots & \vdots \\ * & * & * & * & * & * & * & * & * & * & -Q_N & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -Q_{N+1} - \frac{Z_1+Z_2}{d_M-d_{i+1}} \end{bmatrix}.$$

$$\begin{aligned} \theta_{11} &= PA + A^T P + \sum_{i=1}^{N+2} Q_i - \frac{Z_1}{d_m} \\ \theta_{i,12} &= PA_d + N_{1i} - R_{1i}, \quad i = 1, \ldots, N \\ \theta_{i,22} &= N_{2i} + N_{2i}^T - R_{2i} - R_{2i}^T - (1-\mu) Q_{N+2}, \quad i = 1, \ldots, N \\ U &= d_M Z_1 + (d_M - d_m) Z_2 \\ \bar{P} &= [P \quad 0 \quad \ldots \quad 0]^T \\ \bar{P_w} &= [B_w^T P \quad 0 \quad \ldots \quad 0]^T, \\ \bar{N_i} &= \left[ N_{1i}^T \quad N_{2i}^T \quad 0 \quad \ldots \quad 0 \right]^T, \quad i = 1, \ldots, N \\ \bar{R_i} &= \left[ R_{1i}^T \quad R_{2i}^T \quad 0 \quad \ldots \quad 0 \right]^T, \quad i = 1, \ldots, N \end{aligned}$$

$$
\begin{aligned}
\bar{A} &= [A \quad A_d \quad 0 \quad \dots \quad 0]^T \\
\bar{H} &= [H \quad 0 \quad \dots \quad 0]^T \\
\bar{C} &= [C \quad C_d \quad 0 \quad \dots \quad 0]^T
\end{aligned}
$$

**Proof.** In order to prove Theorem 3.1, the delay range is divided in $N$ subintervals $[d_i, d_{i+1})$ $(i = 1, \dots, N)$ by taking into account: $[d_m, d_M) \equiv [d_1, d_2) \cup \dots \cup [d_N, d_{N+1})$.

The following Lyapunov-Krasovskii functional candidate is chosen:

$$
\begin{aligned}
V(t) &= x^T(t)Px(t) + \sum_{i=1}^{N+1} \int_{t-d_i}^{t} x^T(s)Q_i x(s)ds + \int_{t-d(t)}^{t} x^T(s)Q_{N+2}x(s)ds \\
&+ \int_{-d_M}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta + \int_{-d_M}^{-d_m} \int_{t+\theta}^{t} \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta, \qquad (3.12)
\end{aligned}
$$

where $P > 0, Q_i > 0$, $(i = 1, \dots, N+2)$ and $Z_i > 0$, $(i = 1, 2)$. From the Leibniz-Newton formula, the derivative of (3.12) takes the form

$$
\begin{aligned}
\dot{V}(t) &= 2x^T(t)P\dot{x}(t) + x^T(t)\left[\sum_{i=1}^{N+2} Q_i\right]x(t) - (1-\dot{d}(t))x^T(t-d(t))Q_{N+2}x(t-d(t)) \\
&- \sum_{i=1}^{N+1} x^T(t-d_i)Q_i x(t-d_i) + \dot{x}^T(t)(d_M Z_1 + (d_M - d_m)Z_2)\dot{x}(t) \\
&- \int_{t-d_M}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-d_M}^{t-d_m} \dot{x}^T(s)Z_2\dot{x}(s)ds. \qquad (3.13)
\end{aligned}
$$

To prove the theorem, it is sufficient to prove that if condition (3.11) holds, the derivative of the functional (3.13) is negative along the solutions of (3.5)-(3.6).

Consequently, the derivative of the functional will be proved to be negative for each subinterval. The first and last subintervals are special cases, but they can be carefully subsumed in the generic case, if we consider that in the first subinterval, the states $x(t-d_m)$, and $x(t-d_i)$ are the same. In the last subinterval, both states $x(t-d_M)$ and $x(t-d_{i+1})$ are identical.

In the following, the derivative for a generic subinterval is analyzed. The required particularizations for the first and last subintervals will be made when applicable.

Thus, consider interval $i$, $d_i \le d(t) < d_{i+1}$. Integral terms on the right-hand side in (3.13) can be rewritten as follows:

$$\int_{t-d_M}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds + \int_{t-d_M}^{t-d_m} \dot{x}^T(s)Z_2\dot{x}(s)ds =$$
$$= \int_{t-d_M}^{t-d_{i+1}} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds + \int_{t-d_{i+1}}^{t-d(t)} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds$$
$$+ \int_{t-d(t)}^{t-d_i} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds + \int_{t-d_i}^{t-d_m} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds + \int_{t-d_m}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds.$$

It is worth mentioning that in the first (respectively, last) subinterval the fourth (first) integral is removed due to the fact that $d_m = d_1$ ($d_M = d_{N+1}$).

Consider now the addition of the following null terms to $\dot{V}(t)$:

$$\begin{aligned}
0 &= 2[x^T(t)N_{1i} + x^T(t-d(t))N_{2i}]\left[x(t-d(t)) - x(t-d_{i+1}) - \int_{t-d_{i+1}}^{t-d(t)} \dot{x}(s)ds\right],\\
0 &= 2[x^T(t)R_{1i} + x^T(t-d(t))R_{2i}]\left[x(t-d_i) - x(t-d(t)) - \int_{t-d(t)}^{t-d_i} \dot{x}(s)ds\right],\\
0 &= [Cx(t) + C_d x(t-d(t))]^T[Cx(t) + C_d x(t-d(t))] - z^T(t)z(t),\\
0 &= \gamma^2 w^T(t)w(t) - \gamma^2 w^T(t)w(t),\\
0 &= \epsilon h^T(t,x(t))h(t,x(t)) - \epsilon h^T(t,x(t))h(t,x(t)),
\end{aligned}$$

where $\epsilon$ is a positive scalar. Defining the augmented state

$$\xi^T(t) = \left[x^T(t)\;\; x^T(t-d(t))\;\; x^T(t-d_1)\;\ldots\; x^T(t-d_{N+1})\;\; h^T(t,x(t))\;\; w^T(t)\right],$$

and adding the previous null terms, equation (3.13) can be rewritten as:

$$\begin{aligned}
\dot{V}(t) \leq\; & \xi^T(t)\left(\begin{bmatrix} \tilde{\Gamma}_i & \bar{P} & \bar{P}_w \\ * & -\epsilon I & 0 \\ * & * & -\gamma^2 I \end{bmatrix} + \alpha^2\epsilon\bar{H}\bar{H}^T + \bar{C}\bar{C}^T\right)\xi(t)\\
+\; & \dot{x}^T(t)(d_M Z_1 + (d_M - d_m)Z_2)\dot{x}(t)\\
-\; & \int_{t-d_{i+1}}^{t-d(t)} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds - 2\xi^T(t)\bar{N}_i\int_{t-d_{i+1}}^{t-d(t)} \dot{x}(s)ds\\
-\; & \int_{t-d(t)}^{t-d_i} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds - 2\xi^T(t)\bar{R}_i\int_{t-d(t)}^{t-d_i} \dot{x}(s)ds\\
-\; & \int_{t-d_M}^{t-d_{i+1}} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds - \int_{t-d_i}^{t-d_m} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds\\
-\; & \int_{t-d_m}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds - z^T(t)z(t) + \gamma^2 w^T(t)w(t),
\end{aligned} \tag{3.14}$$

where

$$\tilde{\Gamma}_i = \begin{bmatrix} \theta_{11} + \frac{Z_1}{d_m} & \theta_{i,12} & 0 & \cdots & R_{1i} & -N_{1i} & \cdots & 0 \\ * & \theta_{i,22} & 0 & \cdots & R_{2i} & -N_{2i} & \cdots & 0 \\ * & * & -Q_1 & \cdots & 0 & 0 & \cdots & 0 \\ * & * & * & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & -Q_i & 0 & \cdots & 0 \\ * & * & * & * & * & -Q_{i+1} & \cdots & 0 \\ * & * & * & * & * & * & \ddots & \vdots \\ * & * & * & * & * & * & * & -Q_{N+1} \end{bmatrix}.$$

Please note that in the first and last subintervals, the terms in the columns corresponding to $Q_i$ and $Q_{i+1}$ appear in the columns corresponding to $Q_1$ and $Q_{N+1}$, respectively. Notice also the similarities between the first addend of (3.14) and the corresponding block of the LMI (3.11).

Up to this point, no conservatism has been introduced (except for the choice of a particular structure for the functional $V(t)$ and for the substitution of $\dot{d}(t)$ by its upper bound $\mu$), as no bounding terms have been required for $\dot{V}(t)$. Nevertheless, in order to treat the integrals of (3.14), it is necessary to bound these terms appropriately. This is the main source of conservatism of almost all approaches available in the literature. The key idea in this work is to divide the delay range in $N$ disjoint subintervals, in such a ways that if one bounds the contribution of each individual term, the overall conservatism can be reduced.

Let us recall now the well-known upper bound for the inner product of two vectors

$$-2b^T a - a^T X a \leq b^T X^{-1} b, \quad X > 0.$$

Choosing vectors $a$ and $b$ appropriately, the resulting inequalities can be integrated in $s$, yielding

$$\begin{aligned} -2\underbrace{\xi^T(t)\bar{N}_i}_{b^T}\int_{t-d_{i+1}}^{t-d(t)}\underbrace{\dot{x}(s)}_{a}\,ds \quad &- \quad \int_{t-d_{i+1}}^{t-d(t)}\underbrace{\dot{x}^T(s)}_{a^T}\underbrace{(Z_1+Z_2)}_{X}\underbrace{\dot{x}(s)}_{a}\,ds \\ &\leq \quad \int_{t-d_{i+1}}^{t-d(t)}\underbrace{\xi^T(t)\bar{N}_i}_{b^T}\underbrace{(Z_1+Z_2)^{-1}}_{X^{-1}}\underbrace{\bar{N}_i^T\xi(t)}_{b}\,ds, \\ -2\xi^T(t)\bar{R}_i\int_{t-d(t)}^{t-d_i}\dot{x}(s)ds \quad &- \quad \int_{t-d(t)}^{t-d_i}\dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds \\ &\leq \quad \int_{t-d_{i+1}}^{t-d(t)}\xi^T(t)\bar{R}_i(Z_1+Z_2)^{-1}\bar{R}_i^T\xi(t)ds. \end{aligned}$$

Observe that the terms that finally bound the integral terms do not depend on $s$, and their integrations result in the presence of $d(t)$ in the final bounds. The time-varying delay must be substituted for the worst cases, which are $d_i$ and $d_{i+1}$ instead of $d_m$ and $d_M$. Therefore, less restrictive bounds are being used with respect to the case where no partition of the delay interval is considered. This is the main advantage of our approach compared to previous works as [98, 257], where partitions of the delay interval are introduced for operational convenience, but no reduction of conservatism is obtained for this reason, since no specific information of every subinterval is exploited.

The final bound of the integral terms is given by

$$\begin{aligned} -2\xi^T(t)\bar{N}_i \int_{t-d_{i+1}}^{t-d(t)} \dot{x}(s)ds &- \int_{t-d_{i+1}}^{t-d(t)} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds \\ &\leq (d_{i+1}-d_i)\xi^T(t)\bar{N}_i(Z_1+Z_2)^{-1}\bar{N}_i^T\xi(t), \\ -2\xi^T(t)\bar{R}_i \int_{t-d(t)}^{t-d_i} \dot{x}(s)ds &- \int_{t-d(t)}^{t-d_i} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds \\ &\leq (d_{i+1}-d_i)\xi^T(t)\bar{R}_i(Z_1+Z_2)^{-1}\bar{R}_i^T\xi(t). \end{aligned} \tag{3.15}$$

The rest of the integral terms in (3.14) are bounded using the Jensen's inequality, which can be stated as:

$$-\int_a^b z^T(s)Xz(s)ds \leq -\left[\int_a^b z(s)ds\right]^T \frac{X}{b-a}\left[\int_a^b z(s)ds\right], \quad X>0.$$

Therefore, it yields

$$\begin{aligned} -\int_{t-d_M}^{t-d_{i+1}} \underbrace{\dot{x}^T(s)}_{z^T(s)}\underbrace{(Z_1+Z_2)}_{X}\underbrace{\dot{x}(s)}_{z(s)}ds &\leq -\left[\int_{t-d_M}^{t-d_{i+1}} \dot{x}(s)ds\right]^T \frac{Z_1+Z_2}{\underbrace{d_M-d_{i+1}}_{b-a}}\left[\int_{t-d_M}^{t-d_{i+1}} \dot{x}(s)ds\right] \\ -\int_{t-d_i}^{t-d_m} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds &\leq -\left[\int_{t-d_i}^{t-d_m} \dot{x}(s)ds\right]^T \frac{Z_1+Z_2}{d_i-d_m}\left[\int_{t-d_i}^{t-d_m} \dot{x}(s)ds\right] \\ -\int_{t-d_m}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds &\leq -\left[\int_{t-d_m}^{t} \dot{x}(s)ds\right]^T \frac{Z_1}{d_m}\left[\int_{t-d_m}^{t} \dot{x}(s)ds\right]. \end{aligned} \tag{3.16}$$

Then, combining (3.14) with (3.15)-(3.16), it can be shown that for the generic interval $d_i \leq d(t) < d_{i+1}$, it holds

$$\dot{V}(t) \leq \xi^T(t)\left(\begin{bmatrix} \Gamma_i & \bar{P} & \bar{P_w} \\ * & -\epsilon I & 0 \\ * & * & -\gamma^2 I \end{bmatrix} + \Delta d\bar{N}_i(Z_1+Z_2)^{-1}\bar{N}_i^T + \Delta d\bar{R}_i(Z_1+Z_2)^{-1}\bar{R}_i^T + \right.$$
$$\left. + \bar{A}U\bar{A}^T + \alpha^2\epsilon\bar{H}\bar{H}^T + \bar{C}\bar{C}^T\right)\xi(t) - z^T(t)z(t) + \gamma^2 w^T(t)w(t), \quad (3.17)$$

where the term $\dot{x}^T(t)(d_M Z_1 + (d_M - d_m)Z_2)\dot{x}(t)$ in (3.14) has been written in the form $\xi^T(t)\bar{A}U\bar{A}^T\xi(t)$.

Let $\Xi$ be defined by

$$\Xi = \begin{bmatrix} \Gamma_i & \bar{P} & \bar{P_w} \\ * & -\epsilon I & 0 \\ * & * & -\gamma^2 I \end{bmatrix} + \Delta d\bar{N}_i(Z_1+Z_2)^{-1}\bar{N}_i^T + \Delta d\bar{R}_i(Z_1+Z_2)^{-1}\bar{R}_i^T +$$
$$+ \bar{A}U\bar{A}^T + \alpha^2\epsilon\bar{H}\bar{H}^T + \bar{C}\bar{C}^T.$$

By Schur complement (see Property B.4), it can be seen that if (3.11) holds, then matrix $\Xi$ is negative definite $\forall d(t) \in [d_i, d_{i+1})$. Then, since the term $\xi^T(t)\Xi\xi(t)$ is negative (see Property B.2), it yields

$$\dot{V}(t) \leq -z^T(t)z(t) + \gamma^2 w^T(t)w(t). \quad (3.18)$$

Obviously, if LMIs (3.11) are satisfied for all subintervals, then condition (3.18) will hold for the complete delay range.

Suppose that external disturbances are null, $w(t) \equiv 0$. If (3.11) holds, from (3.18) one can obtain that $V(t)$ decreases for all $t$. Then $\dot{V}(t) < -\varrho\|\xi(t)\|^2$, $\forall t$, for a sufficiently small $\varrho > 0$. Therefore, the asymptotic stability of system (3.5)-(3.6) is ensured [77].

Next, the fact that the $L_2$-gain of the system is bounded by $\gamma$ will be proved. In this case external perturbations are assumed to be nonzero. Both sides of (3.18) are integrated from $t_0$ to $t$,

$$V(t) - V(t_0) \leq -\int_{t_0}^{t} z^T(s)z(s)ds + \int_{t_0}^{t} \gamma^2 w^T(s)w(s)ds.$$

Then, by letting $t \to \infty$ and under zero initial condition ($V(t_0) = 0$) yields,

$$\int_{t_0}^{\infty} z^T(s)z(s)ds \leq \int_{t_0}^{\infty} \gamma^2 w^T(s)w(s)ds,$$

thus $\|z(t)\|_{L_2} \leq \gamma\|w(t)\|_{L_2}$. Therefore $\gamma$ is an upper bound for the $L_2$-gain of the mapping $w(t) \mapsto z(t)$. $\square$

It is interesting to compare this approach to that in [99]. Notice that in that work the authors introduce one slightly different functional for each subinterval, while here a unique functional for all the subintervals is proposed. Moreover, this method allows to choose the number of subintervals as an additional degree of freedom. The intuition and the numerical examples say that the more intervals, the less conservatism is introduced. However, as $N$ grows, the computational burden increases. Therefore, conservatism and computational effort can be traded off, as will be shown in the numerical examples.

**Remark 3.2.** Without loss of generality, an evenly spaced partition of the delay interval has been considered. Nonetheless, it is straightforward to apply the results in Theorem 3.1 to general, and possibly variable, partitions of the delay interval.

By definition, it has been supposed that $d_m > 0$. In the case that the lower bound on the time delay was unknown or exactly zero, it would be necessary to consider the case $d_m = 0$. In such circumstances, LMIs in Theorem 3.1 cannot be solved, because some infinity terms arise in the diagonal of the matrices. In this particular case, the following corollary is stated to establish the stability of system (3.5)-(3.6).

**Corollary 3.1.** *Given scalars $0 = d_m < d_M$, $\mu$, $\alpha$ and $\gamma$, system (3.5)-(3.6) is robustly asymptotically stable with degree $\alpha$ for all admisible uncertainties $h(t, x(t)) \in H_\alpha$ and presents an $L_2$-gain lower than $\gamma$, if there exist any matrices $P, Q_2, Q_3, \ldots, Q_{N+2}, Z_1, Z_2 > 0$, any matrices $N_{ji}, R_{ji}$, $(j = 1, 2,\ i = 1, \ldots, N)$, and a scalar $\epsilon > 0$ such that the following $N$ LMIs for $i = 1, \ldots, N$, are satisfied:*

$$\begin{bmatrix} \Lambda_i & \bar{P} & \bar{P}_w & \Delta d\bar{N}_i & \Delta d\bar{R}_i & \bar{A}U & \epsilon\bar{H} & \bar{C} \\ * & -\epsilon I & 0 & 0 & 0 & U & 0 & 0 \\ * & * & -\gamma^2 I & 0 & 0 & B_w^T U & 0 & 0 \\ * & * & * & -\Delta d(Z_1+Z_2) & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Delta d(Z_1+Z_2) & 0 & 0 & 0 \\ * & * & * & * & * & -U & 0 & 0 \\ * & * & * & * & * & * & -\frac{\epsilon}{\alpha^2}I & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \quad (3.19)$$

*where*

$$\Lambda_1 = \begin{bmatrix} \lambda_{1,11} & \lambda_{1,12} & -N_{11} & 0 & \cdots & 0 & 0 \\ * & \lambda_{1,22} & -N_{21} & 0 & \cdots & 0 & 0 \\ * & * & -Q_2 - \frac{Z_1+Z_2}{d_M-d_2} & 0 & \cdots & 0 & \frac{Z_1+Z_2}{d_M-d_2} \\ * & * & * & -Q_3 & \cdots & 0 & 0 \\ * & * & * & * & \ddots & \vdots & \vdots \\ * & * & * & * & * & -Q_N & 0 \\ * & * & * & * & * & * & -Q_{N+1} - \frac{Z_1+Z_2}{d_M-d_2} \end{bmatrix},$$

$$\Lambda_N = \begin{bmatrix} \lambda_{N,11} & \lambda_{N,12} & 0 & \cdots & 0 & R_{1N} + \frac{Z_1+Z_2}{d_N} & -N_{1N} \\ * & \lambda_{N,22} & 0 & \cdots & 0 & R_{2N} & -N_{2N} \\ * & * & -Q_2 & \cdots & 0 & 0 & 0 \\ * & * & * & \ddots & \vdots & \vdots & \vdots \\ * & * & * & * & -Q_{N-1} & 0 & 0 \\ * & * & * & * & * & -Q_N - \frac{Z_1+Z_2}{d_N} & 0 \\ * & * & * & * & * & * & -Q_{N+1} \end{bmatrix},$$

*and for* $i = 2, \ldots, N-1$,

$$\Lambda_i = \begin{bmatrix} \lambda_{i,11} & \lambda_{i,12} & 0 & \cdots & 0 & R_{1i} + \frac{Z_1+Z_2}{d_i} & -N_{1i} & 0 & \cdots & 0 & 0 \\ * & \lambda_{i,22} & 0 & \cdots & 0 & R_{2i} & -N_{2i} & 0 & \cdots & 0 & 0 \\ * & * & -Q_2 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & -Q_{i-1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & * & * & -Q_i - \frac{Z_1+Z_2}{d_i} & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & * & * & * & -Q_{i+1} - \frac{Z_1+Z_2}{d_M-d_{i+1}} & 0 & \cdots & 0 & \frac{Z_1+Z_2}{d_M-d_{i+1}} \\ * & * & * & * & * & * & * & -Q_{i+2} & \cdots & 0 & 0 \\ * & * & * & * & * & * & * & * & \ddots & \vdots & \vdots \\ * & * & * & * & * & * & * & * & * & -Q_N & 0 \\ * & * & * & * & * & * & * & * & * & * & -Q_{N+1} - \frac{Z_1+Z_2}{d_M-d_{i+1}} \end{bmatrix}.$$

$$\begin{aligned} \lambda_{1,11} &= PA + A^T P + \sum_{i=2}^{N+2} Q_i + R_{11} \\ \lambda_{i,11} &= PA + A^T P + \sum_{i=2}^{N+2} Q_i - \frac{Z_1 + Z_2}{d_i}, \quad i = 2, \ldots, N \\ \lambda_{1,12} &= PA_d + N_{11} - R_{11} + R_{21}^T \\ \lambda_{i,12} &= PA_d + N_{1i} - R_{1i}, \quad i = 2, \ldots, N \\ \lambda_{i,22} &= N_{2i} + N_{2i}^T - R_{2i} - R_{2i}^T - (1-\mu)Q_{N+2}, \quad i = 1, \ldots, N \\ U &= d_M(Z_1 + Z_2) \end{aligned}$$

**Proof.** The proof follows the same steps than that of Theorem 3.1. Taking the time derivative of the LKF and adding the same null terms, equation (3.14) is obtained.

Substituting $d_m = 0$ the integral $-\int_{t-d_m}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds$ disappears from (3.14). Therefore, if the terms

$$-\int_{t-d_m}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds \leq -\left[\int_{t-d_m}^{t} \dot{x}(s)ds\right]^T \frac{Z_1}{d_m}\left[\int_{t-d_m}^{t} \dot{x}(s)ds\right]$$

are suppressed in each interval, the proof can be made in a similar way to that of Theorem 3.1. Obviously, the state $x(t-d_m)$ has to be suppressed from the augmented state vector, as in the first subinterval $x(t-d_1) = x(t-0) = x(t)$. $\square$

Corollary 3.1 can be used in order to establish the stability of the system (3.5)-(3.6) when the minimum time delay $d_m$ is equal to zero. However, Theorem 3.1 can solve the same problem by choosing $d_m \to 0^+$, as well. With an appropriate computational software, it can be checked that the results of Theorem 3.1 (for instance, in terms of the maximum allowed time delay $d_M$) converge to those of Corollary 3.1 as the minimum delay approaches zero.

**Remark 3.3.** In case that $d_m = 0$, matrices $Z_1$ and $Z_2$ appears together in the functional (3.12) and in the LMIs of Corollary 3.1 inside the term $Z_1 + Z_2$. Therefore, it would be possible to define $Z \triangleq Z_1 + Z_2$ and solve the LMIs in the corollary using $Z$ without loss of generality. However, in order to use a consistent notation throughout the section, these two matrices have been retained.

**Remark 3.4.** Assuming that the bound on the derivative of the delay is unknown, all results still hold if the integral $\int_{t-d(t)}^{t} x^T(s)Q_{N+2}x(s)ds$ is suppressed from the Lyapunov-Krasovskii functional and all the terms with the variable $Q_{N+2}$ are removed from the LMIs.

Scalars $\gamma$ and $\alpha$ are given constants in the LMIs. In some situations, it could be interesting to minimize any of them. In this case, there are two possibilities:

- Let $\delta \triangleq \gamma^2$. Choosing $\delta$ as a new decision variable, the LMIs can be solved aiming to minimize $\delta$ with a given constant $\alpha$:

  $$\begin{aligned}&\min \delta \\ &\text{subject to: (3.11)}\end{aligned}$$

  To do so, appropriate computational software (as `mincx` in MATLAB is available.

- Let $\delta \triangleq \gamma^2$. Choose $\alpha$ and $\delta$ as decision variables. The problem becomes:

$$\begin{aligned}&\min \alpha\\ &\text{subject to: (3.11)}\end{aligned}$$

  Note that conditions (3.11) are not LMIs if $\alpha$ is chosen as a decision variable. Therefore, to solve this optimization problem a bisection algorithm could be used.

Example 3.3 discusses the correlation of $\gamma$ and $\alpha$.

## 3.5 Numerical examples

In this section the presented methods are tested over a number of examples which have become a standard to compare stability criteria from different authors.

**Example 3.1.** Consider the following system:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix} x(t-d(t)) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w(t), \\ z(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -0.375 & -1.15 \end{bmatrix} x(t-d(t)).\end{aligned}$$

Table 3.1 provides the maximum delay $d_M$ for which the stability of system is ensured using the most remarkable (to the best of the author knowledge) methods described in the literature. To obtain comparable results with our method, the maximum delay is computed by using a bisection search with $d_m = 0$ seconds.

It can be observed that the proposed method slightly outperforms all previous results in terms of $d_M$, even for the case of taking just two subintervals. If the number of subintervals is increased, the maximum delay can be further improved reaching a maximum around $N = 15$.

Figure 3.2 shows the influence of the number of subintervals $N$ in the delay $d_M$ and in the relative computation time (RCT). The relative computation time is defined here as the ratio between the computation time required for N subintervals and that required for $N = 2$. This variable is useful to obtain results independently of the characteristics of the computer platform. As a reference, for $N = 2$ the computer used[1] requires 0.20 s of computation time.

[1] Pentium IV microprocessor, 2GHz with 1 GB RAM memory.

| Method | $d_M(s)$ |
|---|---|
| Zhang et al. [266] | $4.5 \times 10^{-4}$ |
| Park et al. [199] | 0.0538 |
| Kim et al. [112] | 0.7805 |
| Naghshtabrizi et al. [167] | $< 0.8871$ |
| Yue et al. [257] | 0.8871 |
| Peng et al. [202] | 0.9410 |
| Jiang et al. [100] | 1.0081 |
| Corollary 3.1 (N = 2) | 1.0240 |
| Corollary 3.1 (N = 15) | 1.0402 |

Table 3.1: Maximum delay with different methods

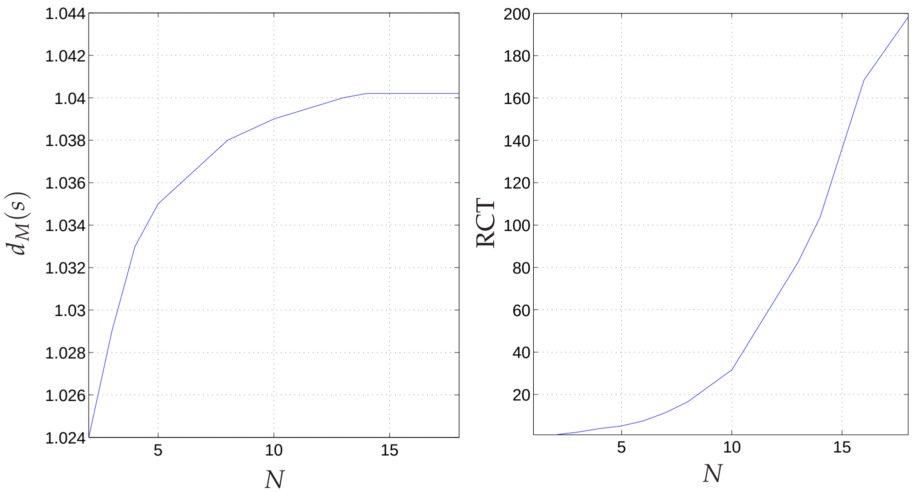



Figure 3.2: Maximum delay and relative computation time vs. Number of subintervals

Next, we proceed with the analysis of the $L_2$-gain of the system. For the case with $d_m = 0$ and $d_M = 0.8695$ seconds, Table 3.2 provides the value of $\gamma_{\min}$ for a number of different methods. It can be observed that the proposed methodology also outperforms the results in previous works in terms of the $L_2$-gain estimated for the system.

**Example 3.2.** Consider the following system in which the lower bound of the delay is greater than zero,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t - d(t)).$$

Different bounds for the maximum delay can be obtained in this case depending

| Method | $\gamma_{\min}$ |
|---|---|
| Yue et al. [257] | 6.82 |
| Jiang et al. [100] | 1.0005 |
| Corollary 3.1 (N = 2) | 0.9035 |

Table 3.2: $L_2$-gain estimation obtained based on different methods

| Method | $d_m(s)$ | 0.3 | 0.5 | 0.8 | 1 | 2 |
|---|---|---|---|---|---|---|
| Jiang et al. [99] | $d_M(s)$ | 0.91 | 1.07 | 1.33 | 1.50 | 2.39 |
| He et al. [80] | $d_M(s)$ | 0.943 | 1.099 | 1.348 | 1.519 | 2.400 |
| Shao et al. [216] | $d_M(s)$ | 1.072 | 1.219 | 1.454 | 1.617 | 2.480 |
| Theorem 3.1 (N = 25) | $d_M(s)$ | 1.223 | 1.360 | 1.582 | 1.738 | 2.572 |

Table 3.3: Maximum delay for various $d_m$ and unknown $\mu$

on whether the upper bound of time-delay derivative $\mu$ is known or not (see equation (3.9) and Remark 3.4). Thus Table 3.3 shows the obtained $d_M$ for a variety of methods with different lower bounds $d_m$, and for unknown $\mu$. Similar results are shown in Table 3.4 for the case of known $\mu = 0.3$.

From both Tables 3.3 and 3.4, it can be observed the goodness of the presented method, which achieves in some cases a reduction of the conservatism up to 15% in terms of the maximum delay.

To conclude the example, Figure 3.3 shows the influence of the number of subintervals $N$ in $d_M$ and also in the relative computation time. The lower bound of the delay is $d_m = 1$ and $\mu$ is unknown. As expected, the maximum $d_M$ is improved as the number of subintervals increases, approaching an asymptotic value. Though this plot has been obtained for this particular case, similar results have been observed in other cases.

| Method | $d_m(s)$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| He et al. [80] | $d_M(s)$ | 2.213 | 2.409 | 3.334 | 4.280 | 5.239 |
| Shao et al. [216] | $d_M(s)$ | 2.247 | 2.480 | 3.389 | 4.330 | 5.277 |
| Theorem 3.1 (N = 25) | $d_M(s)$ | 2.400 | 2.700 | 3.462 | 4.384 | 5.327 |

Table 3.4: Maximum delay for various $d_m$ and $\mu = 0.3$.

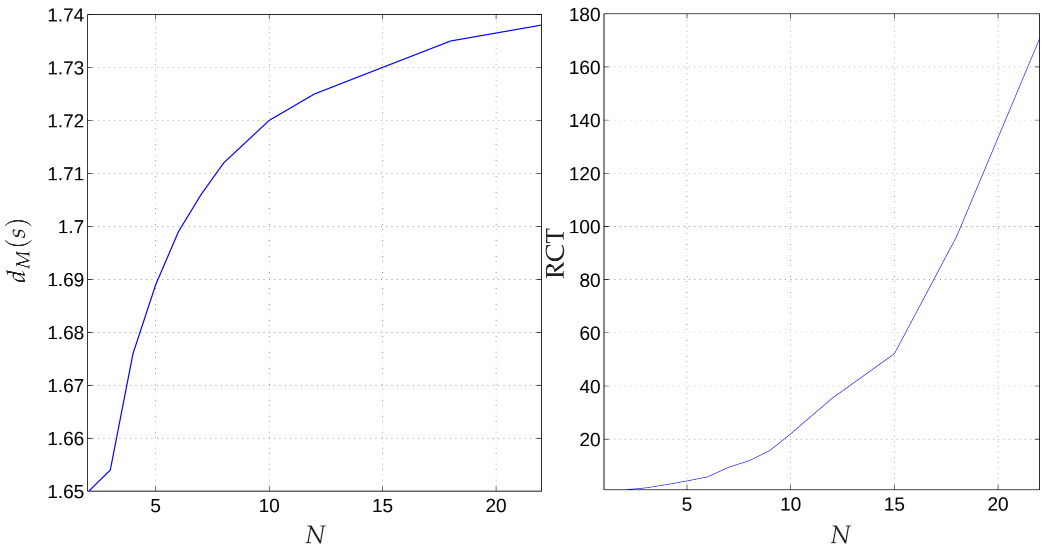



Figure 3.3: Maximum delay and relative computation time vs. Number of subintervals

| Method | N | $\alpha_{max}$ |
|---|---|---|
| Peng et al. [202] | 1 | 0.1636 |
| Theorem 3.1 | 2 | 0.2760 |
| | 4 | 0.2760 |
| | 6 | 0.2760 |

Table 3.5: Maximum stability degree $\alpha$ vs. Number of subintervals

**Example 3.3.** Consider the following system with nonlinear uncertainties,

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0.99 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -3.715 & -3.514 \end{bmatrix} x(t-d(t)) + h(t, x(t)),$$

with $h(t, x(t))$ defined according to equation (3.7) with $H = [1 \ \ 0]$.

Assume $d_m = 0$ and $d_M = 0.2509$ seconds. Table 3.5 lists the maximum stability degree $\alpha$ achieved for different methods.

Once again, the previous table provides the improvement of the proposed method with respect to previous results. Note that the number of subintervals is not an influential parameter when trying to estimate the stability degree $\alpha$, so in the following we will restrict our analysis to $N = 2$.

When the effects of the external perturbation are considered, the previous system

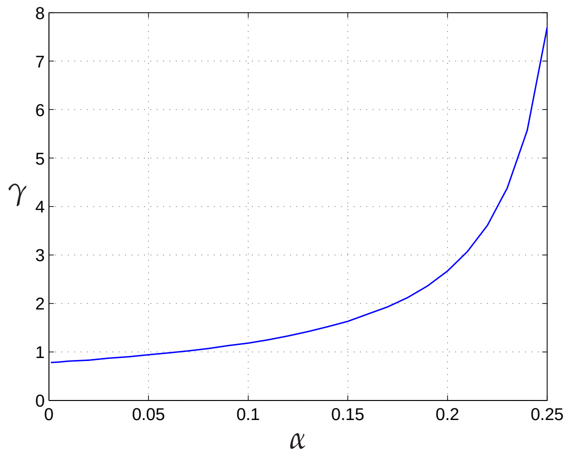



Figure 3.4: Tradeoff between $\alpha$ and $\gamma$

can be described by

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 1 & 1 \\ 0 & 0.99 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -3.715 & -3.514 \end{bmatrix} x(t-d(t)) \\
&\quad + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w(t) + h(t, x(t)), \\
z(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -0.03715 & -0.03514 \end{bmatrix} x(t-d(t)).
\end{aligned}$$

Here, variables $\alpha$ and $\gamma$ can be optimized. Suppose that we want to get the best $L_2$-gain disturbance rejection estimation. In such case, and following the discussion after Remark 3.4, a variable $\delta \triangleq \gamma^2$ is defined and with appropriate software (function `mincx` in Matlab for example), values of $\gamma = 0.7832$ and $\alpha = 0.001$ can be obtained. On the other hand, if the objective is to achieve the best stability degree $\alpha$, by using Theorem 3.1 with free $\gamma$, it can be obtained $\gamma = 191.5$ and $\alpha = 0.275$.

In Figure 3.4, the tradeoff between the minimum $L_2$-gain estimation $\gamma$, given the uncertainty bound $\alpha$, is illustrated for $N = 2$. As expected, the ability to reject disturbances ($L_2$-gain $\gamma$) decreases as the uncertainty in the knowledge of the plant increases ($\alpha$). This plot has been obtained for this particular case, though similar results have been consistently observed for a number of other test systems of higher dimension and with uncertainties.

## 3.6 Chapter summary

The importance of this chapter is crucial for the rest of the thesis and, overall, for the following chapter. Most of the stability results for delayed and networked systems are based in the Lyapunov-Krasovskii theory, so this chapter establishes the mathematical foundations that will be employed hereinafter.

Concretely, this chapter investigates the robust stability and the $L_2$-gain analysis of a class of nonlinear uncertain time-delay system. Sufficient conditions are given in Theorem 3.1 in terms of delay-dependent LMIs which can be efficiently solved with available computational software.

Most results using a Lyapunov-Krasovskii functional to ensure the stability of the system suffer from excessive conservatism. In this respect, it has been shown that by dividing the time-varying delay range into multiple subintervals, the uncertainty in each subinterval decreases and a less conservative stability criterion can be obtained.

Section 3.5 has presented some examples to illustrate the reduced conservatism of this approach compared to previous results in the literature. With the proposed method, the stability of the systems can be ensured for higher bounds on the delay.

# Chapter 4

# Control of delayed and networked systems

## 4.1 Introduction

Chapters 3 and 4 are completely connected. Whereas the stability of time-delay systems was profusely studied previously, this chapter is focused on the control of those systems.

As it has been argued before, the study of TDS has great interest in the control community, as many real applications include delays in their inner dynamics. Furthermore, it involves a theoretical challenge from the point of view of stability analysis, as previous chapter clearly illustrates. Lyapunov-Krasovskii theory has demonstrated to be a suitable tool when studying the stability of delayed systems, and hence, it is also employed in this chapter to prove the stability of the closed-loop system.

Optimal control techniques have been widely studied in many areas, yielding controllers that guarantee good performance when the real system does not deviate too much from the nominal system. In particular, the application to TDS and NCS has been prolific yielding cost-guaranteed controllers for a variety of time-delay conditions [34, 53, 64, 120, 121, 246]. In addition, when the system is affected by disturbances or uncertainties it becomes necessary a robustification of the control method. In this line, $H_\infty$ control has been successfully applied to this framework [60, 63, 98, 101, 144, 174].

Notwithstanding, in some problems it is of interest the obtain both benefits with the same controller, that is, disturbance rejection and optimality. One possibility to deal with those drawbacks is the so called *mixed $H_2/H_\infty$ control* [14, 111]. The problem consists in finding an internally stabilizing controller that minimizes an $H_2$ performance measure subject to a constraint on the $H_\infty$ norm in any output. The $H_2/H_\infty$ control has demonstrated its properties in different control frameworks, achieving good performance as well as disturbance rejection capabilities [228].

The design of $H_2/H_\infty$ controllers for TDSs has been solved specifically for each kind of TDS studied in Section 3.2. Thus, for standard time-delay systems, the synthesis of $H_2/H_\infty$ controllers is addressed in [113]. Yue et al. [258] applied this control technique for descriptor systems with delays. Lastly, a reference paper when dealing with neutral systems with delays using this approach is [33].

This chapter proposes a new method for the design of suboptimal $H_2/H_\infty$ controllers for TDS. The proposed framework is characterized by its generality, as it can be applied to different sorts of TDSs and a variety of different constraints on the delay. Moreover, unlike other works, the contribution does not lie in the use of novel Lyapunov-Krasovskii functionals nor in the mathematical manipulations, but in the optimization method that is general in the sense that it can be used for different functionals. Once the design method is introduced, the chapter focuses on demonstrating the benefits of using this method with respect to other classical approaches, namely:

- The existence of a controller satisfying the constraints involved in the classical designs guarantees the existence of a controller satisfying the constraints imposed in the new design proposed in this chapter.
- The converse implication does not hold.
- For the same constraint in the disturbance attenuation, a controller designed according to the method proposed in this chapter always outperforms controllers designed by means of the classical method in terms of the upper bound of the cost index.

An additional remarkable feature of the proposed methodology is that control design does not need information concerning the initial state of the system.

The second part of the chapter presents an extended model for networked control systems in continuous time in which bounded delays and packet dropouts affect the communication between sensor, controller and actuator. Using the *input delay approach* introduced in [61, 146], it is shown that the NCS can be modeled as a particular TDS with some limitations. Therefore, the general aforementioned method can be applied to these networked systems, leading to an $H_2/H_\infty$ controller synthesis for NCS. This method has been tested in an experimental platform, in which a robotic arm is controlled at the surroundings of its upright unstable equilibrium point.

## Related publications

1. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio. *Control óptimo de sistemas de control a través de redes mediante funcionales de Lyapunov-Krasovskii*. XXX Jornadas de Automática. Valladolid, Spain. 2009. [152]

2. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio. *Improved delay-dependent stability criterion for uncertain networked control systems with induced time-varying delays*. 1st IFAC Workshop on Estimation and Control of Networked Systems. Venice, Italy. pp:346-351, 2009. [153]

3. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio. *An optimal control $L_2$-gain disturbance rejection design for networked control systems*. American Control Conference. Baltimore, MD, USA. pp:1344-1349, 2010. [154]

4. L. Orihuela, P. Millán, G. Bejarano, C. Vivas, F. R. Rubio. *Optimal networked control of a 2 degree-of-freedom direct drive robot manipulator*. IEEE Conference on Emerging Technologies and Factory Automation. Bilbao, Spain. pp:1-8, 2010. [185]

5. P. Millán, L. Orihuela, G. Bejarano, C. Vivas, T. Álamo, F. R. Rubio. *Design and application of suboptimal mixed $H_2/H_\infty$ controllers for networked control systems*. IEEE Transactions on Control Systems Technology. 20(4):1057-1065, 2012. [148]

6. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio. *Control óptimo $L_2$ basado en red mediante funcionales de Lyapunov-Krasovskii*. Revista Iberoamericana de Informática y Automática Industrial. 09(1):14-23, 2012. [157]

7. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *$H_2/H_\infty$ control for discrete TDS with application to networked control systems: periodic and asynchronous communication*. Optimal Control Applications and Methods. Under review. [192]

8. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Improved Performance $H_2/H_\infty$ Controller Design for Time Delay Systems*. European Control Conference. Zürich, Switzerland. Submitted, 2013. [189]

## 4.2 Problem statement

Consider the general formulation of a time-delay system given in (3.4). Now, it is assumed that this system is being controlled, so it can be written as

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-d(t)) + A_h \int_{t-\eta(t)}^{t} x(s)ds + A_2\dot{x}(t-v(t)) \\ &+ Bu(t) + B_d u(t-\tau(t)) + B_w w(t), \end{aligned} \tag{4.1}$$

where the initial condition is defined as $x_{t_0}(\theta) = \phi(t+\theta)$, for all $\theta \in [-r, 0]$, being $r = \max\{\max d(t), \max \eta(t), \max v(t), \max \tau(t)\}$.

The control action applied to the system is divided in two parts: one part corresponds to the signal directly applied to the plant; and another signal which is a delayed plant input. The controller to be designed is a linear state-feedback with the structure:

$$u(t) = Kx(t). \tag{4.2}$$

Let $z_2$ and $z_\infty$ denote two outputs of the system defined as:

$$z_2(t) = C_2 x(t) + D_2 u(t), \tag{4.3}$$

$$z_\infty(t) = C_\infty x(t) + D_\infty u(t), \tag{4.4}$$

where matrices $C_2, D_2, C_\infty, D_\infty$ are known. These outputs will serve to test the optimality and the disturbance rejection of the system, respectively. It is possible to use the same output for both performance measurements, as in SISO systems, i.e. $z_2 \equiv z_\infty$.

Based on the first output, a cost functional is defined to evaluate the control system performance:

$$J_2 = \begin{cases} \int_{t_0}^{\infty} z_2^T(s) z_2(s) ds \\ \text{or} \\ \int_{t_0}^{\infty} x^T(s)Qx(s) + u^T(s)Ru(s)ds \end{cases} \tag{4.5}$$

where $Q, R > 0$.

In the following, the mixed $H_2/H_\infty$ control problem is formally stated. A general solution for this problem is given in the next section.

**Definition 4.1. The suboptimal mixed $H_2/H_\infty$ control problem.** Consider the system described by (4.1). Given:

- A desired level of disturbance attenuation $\gamma$, and

- A quadratic cost function $J_2$ in the form (4.5),

the suboptimal mixed $H_2/H_\infty$ control problem consists in finding a linear controller $K$ such that:

1. The closed-loop system is asymptotically stable for $w(t) \equiv 0$,
2. The controller minimizes the upper bound of the cost function $J_2$ for $w(t) \equiv 0$,
3. Under the assumption of zero initial conditions, the output $z_\infty(t)$ satisfies $\|z_\infty(t)\|_{L_2} \leq \gamma \|w(t)\|_{L_2}$ for any nonzero disturbance $w(t) \in L_2[0, \infty)$.

## 4.3 General solution for the suboptimal mixed $H_2/H_\infty$ control problem

The solution proposed in this chapter for the mixed $H_2/H_\infty$ control problem is based on the Lyapunov-Krasovskii theory, because the system involves delayed loops. The design method is intended to be general, applicable to different kinds of systems and different sorts of functionals. However, some assumptions need to be imposed in order to get a proper solution.

**Assumption 4.1.** Let $V(t)$ be a a continuous quadratic Lyapunov-Krasovskii functional. The time derivative $\dot{V}(t)$ can be bounded by

$$\dot{V}(t) \leq \begin{cases} \xi^T(t)\Xi(K)\xi(t), & w \equiv 0 \\ \begin{bmatrix} \xi(t) \\ w(t) \end{bmatrix}^T \Theta(K,\gamma) \begin{bmatrix} \xi(t) \\ w(t) \end{bmatrix} - z_\infty^T(t) z_\infty(t) + \gamma^2 w^T(t) w(t), & w \neq 0 \end{cases} \tag{4.6}$$

where $\xi(t) \in \mathbb{R}^{n_\xi}$ is an augmented state vector which depends, among others, on the state of the system, and

$$\Theta(K,\gamma) = \begin{bmatrix} \Xi(K) + \bar{C}_z(K) & \bar{B}_w(K) \\ * & -\gamma^2 I + \bar{D}_w(K) \end{bmatrix}. \tag{4.7}$$

The symmetric matrices $\Xi(K), \bar{C}_z(K) \in \mathbb{R}^{n_\xi \times n_\xi}, \bar{D}_w(K) \in \mathbb{R}^{s \times s}$ and the matrix $\bar{B}_w(K) \in \mathbb{R}^{n_\xi \times s}$ might depend, among others, on the controller matrix $K$.

**Remark 4.1.** When dealing with TDS through the Lyapunov-Krasovskii approach, it is usual that the time derivative of the functional can be posed as (4.6) [101, 144, 257]. Therefore, Assumption 4.1 makes a mild hypothesis on the problem structure.

**Assumption 4.2.** The cost functional $J_2$ can be written in the following way:

$$J_2 = \int_{t_0}^{\infty} [\xi^T(s)\Phi(K)\xi(s)]ds, \tag{4.8}$$

where $\Phi(K)$ is a positive semidefinite matrix that might depend on the controller $K$.

**Remark 4.2.** Assumption 4.2 imposes also a mild restriction. For instance, if the augmented state vector is defined as $\xi(t) = x(t)$ and $J_2$ is chosen as the second option in (4.5), it is easy to check that $\Phi(K) = Q + K^T RK$. Similar compositions can be obtained for augmented states including a variety of delayed states and disturbances of the system.

Once the assumptions have been stated, the general design method is presented in the following lemma.

**Lemma 4.1.** *Suppose that Assumptions 4.1 and 4.2 are satisfied. Then, the suboptimal mixed $H_2/H_\infty$ control problem stated in Definition 4.1 can be solved by finding a controller $K$ such that:*

$$\begin{aligned} \min_{K} \quad & \alpha, && (4.9)\\ \textit{subject to} \quad \alpha &> 0, && (4.10)\\ \alpha\Xi(K) &< -\Phi(K), && (4.11)\\ \Theta(K,\gamma) &< 0. && (4.12) \end{aligned}$$

**Proof.** To prove Lemma 4.1 it will be shown that a controller that solves the optimization problem (4.9) with conditions (4.10)-(4.12) also satisfies the statements in Definition 4.1. Thus, let us proceed with each statement individually.

1. For $w(t) \equiv 0$, considering (4.6), $\dot{V}(t) \leq \xi^T(t)\Xi(K)\xi(t)$ holds. From (4.10)-(4.11) and according to Assumption 4.2, one can easily see that $\Xi$ is negative definite, and therefore $V(t)$ decreases, which ensures the asymptotic stability of system [77].

2. For $w(t) \equiv 0$, taking into account equation (4.11) and Assumption 4.2:

$$\dot{V}(t) \leq \xi^T \Xi(K)\xi(t) < -\frac{1}{\alpha}\xi^T(t)\Phi(K)\xi(t). \tag{4.13}$$

   Integrating both sides of (4.13) from $t_0$ to $t$, it yields

$$V(t) - V(t_0) < -\frac{1}{\alpha}\int_{t_0}^{t}[\xi^T(s)\Phi(K)\xi(s)]ds.$$

When $t \to \infty$, the asymptotic stability of the system implies that $V(t) \to 0$, therefore,

$$\begin{aligned} -V(t_0) &< -\frac{1}{\alpha}\int_{t_0}^{\infty}[\xi^T(s)\Phi(K)\xi(s)]ds \\ \Rightarrow J_2 &< \alpha V(t_0). \end{aligned}$$

The value of $V(t_0)$ depends on the initial condition $x_{t_0}$. Nevertheless, by minimizing $\alpha$ the upper bound of the cost function $J_2$ is minimized regardless of the initial conditions. This is a relevant difference between this and other works, such as [33, 109, 113, 258].

3. From condition (4.12), the term

$$\begin{bmatrix} \xi^T(t) & w^T(t) \end{bmatrix} \Theta(K,\gamma) \begin{bmatrix} \xi(t) \\ w(t) \end{bmatrix}$$

is negative definite. Thus, for $w \neq 0$ and under zero initial conditions, it holds

$$\dot{V}(t) \leq -z_\infty^T(t)z_\infty(t) + \gamma^2 w^T(t)w(t). \tag{4.14}$$

The computation of the integral of both sides of (4.14) yields

$$V(t) - V(t_0) \leq -\int_{t_0}^{t} z_\infty^T(s)z_\infty(s)ds + \int_{t_0}^{t}\gamma^2 w^T(s)w(s)ds.$$

Then, by letting $t \to \infty$, taking into account that under zero initial condition $V(t_0) = 0$ and the positive definitiveness of the functional, it can be shown that

$$0 \leq -\int_{t_0}^{\infty} z_\infty^T(s)z_\infty(s)ds + \int_{t_0}^{\infty}\gamma^2 w^T(s)w(s)ds.$$

Thus $\|z_\infty(t)\|_{L_2} \leq \gamma \|w(t)\|_{L_2}$.

The three statements have been proved. □

Lemma 4.1 proposes a general solution to the suboptimal mixed $H_2/H_\infty$ control problem. It can be used for different LKFs and for different kinds of time-delay systems, as it will be shown next.

### 4.3.1 Application to the family of time-delay systems

This section shows that the design tools provided in Lemma 4.1 can be particularized to encompass all three kinds of time-delay descriptions introduced in Section 3.2 of the previous chapter.

The objective does not consist in providing less conservative results by an adequate choice of the functional or by means of advanced mathematical manipulation, but to show the applicability and optimality of the method.

To this purpose, previous works are analyzed in order to compare the results with the method proposed in this research. Thus, the work [113] is selected as a well referenced work for $H_2/H_\infty$ design for *standard time-delay systems*; [258] is chosen for results resorting to the form of *descriptor systems with delays*; and [33] is selected as a reference paper for *neutral systems with delays*.

All these works have been carefully picked according to their relevance in the field and proximity to ours in terms of problem description and assumptions imposed to the problem.

#### 4.3.1.1 Standard time-delay systems

The dynamics of the system presented in [113] is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d(t)) \\ &+ Bu(t) + B_d u(t-\tau(t)) + B_w w(t), \end{aligned} \tag{4.15}$$

with initial condition $x_{t_0}(\theta) = \phi(t+\theta)$ for all $\theta \in [-r, 0]$, being $r = \max\{d(t), \tau(t)\}$. Upper bounds on the derivative of the delay are imposed so the following inequalities are satisfied

$$\dot{d}(t) \leq d_D, \quad \dot{\tau}(t) \leq \tau_D.$$

The author in [113] proposes the following Lyapunov-Krasovskii functional:

$$V(t) = x^T(t)Px(t) + \int_{t-d(t)}^{t} x^T(s)Z_1 x(s)ds + \int_{t-\tau(t)}^{t} x^T(s)K^T Z_2 K x(s)ds. \tag{4.16}$$

The cost function is $J_2 = \int_{t_0}^{\infty} z_2^T(s) z_2(s) ds$. This problem can be studied as a particular case of the results provided in this work in Lemma 4.1, as the following theorem states.

**Theorem 4.1.** *Given $h_D, \tau_D, \gamma > 0$, if matrices $X, Y_1, Y_2 > 0$ and matrix $W$ solve the following optimization problem:*

$$\min_{X,Y_1,Y_2,W} \quad \alpha \tag{4.17}$$

*subject to*

$$\begin{bmatrix} \Phi_1 & X & W^T & XC_2^T + W^T D_2^T \\ * & -Y_1 & 0 & 0 \\ * & * & -Y_2 & 0 \\ * & * & * & -\alpha I \end{bmatrix} < 0, \tag{4.18}$$

$$\begin{bmatrix} \Phi_1 + \frac{1}{\gamma^2} B_w B_w^T & X & W^T & XC_\infty^T + W^T D_\infty^T \\ * & -Y_1 & 0 & 0 \\ * & * & -Y_2 & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{4.19}$$

*where*

$$\Phi_1 = AX + XA^T + BW + W^T B^T + \frac{1}{1-d_D} A_d Y_1 A_d^T + \frac{1}{1-\tau_D} B_d Y_2 B_d^T,$$

*then the mixed $H_2/H_\infty$ controller for the system (4.15) is given by $K = WX^{-1}$.*

**Proof.** To prove Theorem 4.1 it suffices to show that the time derivative of the LKF (4.16) can be written as Assumption 4.1 requires in (4.6), and also that the optimization problem (4.17) is equivalent to the one in Lemma 4.1.

First suppose $w(t) \equiv 0$. Taking the time derivative of $V(t)$ one can obtain that

$$\begin{aligned} \dot{V}(t) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) + x^T(t)Z_1x(t) + x^T(t)K^T Z_2 Kx(t) \\ &- (1-\dot{d}(t))x^T(t-d(t))Z_1 x(t-d(t)) \\ &- (1-\dot{\tau}(t))x^T(t-\tau(t))K^T Z_2 Kx(t-\tau(t)). \end{aligned}$$

The time derivative of the functional can be bounded as follows:

$$\begin{aligned} \dot{V}(t) &\le \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) + x^T(t)Z_1x(t) + x^T(t)K^T Z_2 Kx(t) \\ &- (1-d_D)x^T(t-d(t))Z_1 x(t-d(t)) \\ &- (1-\tau_D)x^T(t-\tau(t))K^T Z_2 Kx(t-\tau(t)), \end{aligned} \tag{4.20}$$

obtaining the same result given in equation (11) in [113]. The augmented state vector is defined as: $\xi^T(t) = \begin{bmatrix} x^T(t) & x^T(t-d(t)) & x^T(t-\tau(t))K^T \end{bmatrix}$. Thus, the time derivative (4.20) can be written as $\dot{V}(t) \le \xi^T(t)\Xi(K)\xi(t)$, where

$$\Xi(K) = \begin{bmatrix} A_K^T P + PA_K + Z_1 + K^T Z_2 K & PA_d & PB_d \\ * & -(1-d_D)Z_1 & 0 \\ * & * & -(1-\tau_D)Z_2 \end{bmatrix},$$

with $A_K = A + BK$. Therefore equation (4.6) for $w \equiv 0$ holds.

Consider now the presence of disturbances. Then, the following null terms are added to the functional:

$$\begin{array}{rcl} \dot{V}(t) & = & \dot{V}(t) - \gamma^2 w^T(t)w(t) + \gamma^2 w^T(t)w(t) \\ & - & z_\infty^T(t)z_\infty(t) + z_\infty^T(t)z_\infty(t). \end{array}$$

It is easy to show that $z_\infty^T(t)z_\infty(t) = x^T(t)C_{\infty K}^T C_{\infty K}x(t)$, where $C_{\infty K} = C_\infty + D_\infty K$. Thus, it yields

$$\begin{array}{rcl} \dot{V}(t) & \leq & \left[\begin{array}{cc} \xi^T(t) & w^T(t) \end{array}\right] \left[\begin{array}{cc} \Xi(K) + \bar{C}_z(K) & \bar{B}_w \\ * & -\gamma^2 I \end{array}\right] \left[\begin{array}{c} \xi(t) \\ w(t) \end{array}\right] \\ & - & z_\infty^T(t)z_\infty(t) + \gamma^2 w^T(t)w(t), \end{array}$$

where $\bar{C}_z(K) = diag\{C_{\infty K}^T C_{\infty K}, 0, 0\}$ and $\bar{B}_w = [B_w^T P \;\; 0 \;\; 0]^T$. Therefore, the derivative of the LKF (4.16) can be written as in equation (4.6). That way, Assumption 4.1 holds.

The cost function (4.5) can be rewritten in the following way:

$$J_2 = \int_{t_0}^{\infty} \xi^T(s)\bar{C}_2^T \bar{C}_2 \xi(s)ds,$$

with $\bar{C}_2 = [C_2 + D_2 K \;\; 0 \;\; 0 \;\; 0]$, satisfying also Assumption 4.2 (4.8) if $\Phi(K)$ is defined as $\Phi(K) = \bar{C}_2^T \bar{C}_2$. Note that $\Phi(K)$ is positive semidefinite.

It remains to prove that the optimization problems in Lemma 4.1 and Theorem 4.1 are equivalent. Consider now equation (4.11) in Lemma 4.1.

$$\alpha \Xi(K) < -\Phi(K) \Leftrightarrow \Xi(K) - \left( -\frac{\bar{C}_2^T \bar{C}_2}{\alpha} \right) < 0 \Leftrightarrow \left[\begin{array}{cc} \Xi(K) & \bar{C}_2^T \\ * & -\alpha I \end{array}\right] < 0, \qquad (4.21)$$

Note that this condition also appears in [113] if $\alpha = 1$. From equation (4.21), after some mathematical manipulations and applying Schur complements, a matrix inequality with the same structure of (4.18) is obtained.

In a similar manner, from equation (4.12), which is identical to that in [113], it can be obtained an inequality with the structure of (4.19). To obtain (4.18)-(4.19) the following definitions are introduced: $X = P^{-1}$, $W = KP^{-1}$, $Y_i = Z_i^{-1}$, $i = 1, 2$. Then, pre- and post- multiplying the matrix inequality by $diag\{X, I, I, I\}$ and its transpose, (4.18)-(4.19) are finally obtained. □

#### 4.3.1.2 Descriptor systems with delays

In this case, the problem in [258] is taken. In that work the authors propose a TDS with the structure

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-d) + A_h \int_{t-\eta}^{t} x(s)ds \\ &+ Bu(t) + B_w w(t), \end{aligned} \tag{4.22}$$

with $x_{t_0}(\theta) = \phi(t+\theta)$ for all $\theta \in [-r, 0]$, where $r = \max\{h, \eta\}$. The delays are assumed to be constant.

The solution proposed in [258] is based on the selection of a Lyapunov-Krasovskii functional as

$$V(t) = x^T(t)PEx(t) + a\int_{t-d}^{t} x^T(s)Zx(s)ds + \frac{1-a}{\eta}\int_{t-\eta}^{t}\int_{s}^{t} x^T(u)Zx(u)duds, \tag{4.23}$$

with the scalar $a$ verifying $0 < a < 1$. The cost function takes the form $J_2 = \int_{t_0}^{\infty} x^T(s)Qx(s) + u^T(s)Ru(s)ds$.

The following theorem gives a solution for the suboptimal mixed $H_2/H_\infty$ control problem for system (4.22) through Lemma 4.1.

**Theorem 4.2.** *Given scalars $\eta, a, \gamma > 0$, if matrix $Y > 0$ and matrices $W, X$, solve the following optimization problem,*

$$\min_{Y,W,X} \quad \alpha$$

*subject to*

$$\begin{bmatrix} \Phi_1 & X & X & W^T \\ * & -Y & 0 & 0 \\ * & * & -\alpha Q^{-1} & 0 \\ * & * & 0 & -\alpha R^{-1} \end{bmatrix} < 0, \tag{4.24}$$

$$\begin{bmatrix} \Phi_1 + \frac{1}{\gamma^2}B_w B_w^T & X & XC_\infty^T + W^T D_\infty^T \\ * & -Y & 0 \\ * & * & -I \end{bmatrix} < 0, \tag{4.25}$$

*constrained by*

$$EX^T = XE^T \geq 0,$$

*where*

$$\Phi_1 = AX^T + XA^T + BW + W^T B^T + \frac{1}{a}A_d Y A_d^T + \frac{\eta^2}{1-a}A_h Y A_h^T,$$

*then the $H_2/H_\infty$ controller for system (4.22) is given by $K = WX^{-T}$.*

The proof of this theorem follows the same steps that the one for Theorem 4.1 so it is omitted.

#### 4.3.1.3 Neutral systems with delays

Lastly, the results in [33] are taken as comparison for the case of neutral systems with delays. In this work the author analyzes a system with the structure

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d(t)) + A_h \int_{t-\eta(t)}^{t} x(s)ds + A_2\dot{x}(t-d(t)) \\ &+ Bu(t) + B_d u(t-\tau(t)) + B_w w(t), \end{aligned} \tag{4.26}$$

with $x_{t_0}(\theta) = \phi(t+\theta)$ for all $\theta \in [-r, 0]$, where $r = \max\{d(t), \tau(t), \eta(t)\}$, and time-varying delays satisfying

$$\begin{aligned} &0 \le d(t) < d_M, \quad 0 \le \tau(t) < \tau_M, \quad 0 \le \eta(t) < \eta_M, \\ &\dot{d}(t) \le d_D < 1, \quad \dot{\tau}(t) \le \tau_D < 1, \quad \dot{\eta}(t) \le \eta_D < 1. \end{aligned} \tag{4.27}$$

The Lyapunov-Krasovskii functional is chosen as

$$\begin{aligned} V(t) &= x^T(t)Px(t) + \int_{t-d(t)}^{t} x^T(s)Z_1x(s)ds + \int_{t-d(t)}^{t} \dot{x}^T(s)Z_2\dot{x}(s)ds \\ &+ \int_{t-\eta(t)}^{t} [s-(t-\eta(t))]x^T(s)Z_3x(s)ds \\ &+ \int_{t-\tau(t)}^{t} x^T(s)K^TZ_4Kx(s)ds \\ &+ \int_{t-d(t)}^{t} [s-(t-d(t))]\dot{x}^T(s)Z_5\dot{x}(s)ds \\ &+ \int_{t-\tau(t)}^{t} [s-(t-\tau(t))]\dot{x}^T(s)K^TZ_6K\dot{x}(s)ds, \end{aligned} \tag{4.28}$$

and the cost function is given by $J_2 = \int_{t_0}^{\infty} x^T(s)Qx(s) + u^T(s)Ru(s)ds$.

The following theorem gives a solution to the suboptimal mixed $H_2/H_\infty$ control problem by using Lemma 4.1.

**Theorem 4.3.** *Given scalars $d_D, d_M, \tau_D, \tau_M, \eta_D, \eta_M$ and $\gamma > 0$, if matrices $X, T, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 > 0$ and matrix $W$, solve the following optimization problem:*

$$\min_{X,T,Y_1\ldots Y_6,W} \quad \alpha$$
$$\textit{subject to} \quad (4.29)-(4.30),$$

*then the $H_2/H_\infty$ controller for system (4.26) is given by $K = WX^{-1}$.*

The result is not difficult to prove following the procedure in the proof of Theorem 4.1, so that it has been omitted.

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & XA^T+W^TB^T & XA^T+W^TB^T & XA^T+W^TB^T & \Phi_{16} & X & W^T \\ * & \Phi_{22} & \Phi_{23} & \Phi_{23} & \Phi_{23} & \bar{0} & \bar{0} & \bar{0} \\ * & * & Y_2 & 0 & 0 & \bar{0} & 0 & 0 \\ * & * & * & -\frac{1}{d_M}Y_5 & 0 & \bar{0} & 0 & 0 \\ * & * & * & * & -\frac{1}{\tau_M}T & \bar{0} & 0 & 0 \\ * & * & * & * & * & \Phi_{66} & 0 & 0 \\ * & * & * & * & * & * & -\alpha Q^{-1} & 0 \\ * & * & * & * & * & * & * & -\alpha R^{-1} \end{bmatrix} < 0, \tag{4.29}$$

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & B_w & XA^T+W^TB^T & XA^T+W^TB^T & XA^T+W^TB^T & \Phi_{16} & XC_\infty^T+W^TD_\infty^T \\ * & \Phi_{22} & 0 & \Phi_{23} & \Phi_{23} & \Phi_{23} & \bar{0} & \bar{0} \\ * & * & -\gamma^2 I & B_w^T & B_w^T & B_w^T & \bar{0} & 0 \\ * & * & * & Y_2 & 0 & 0 & \bar{0} & 0 \\ * & * & * & * & -\frac{1}{d_M}Y_5 & 0 & \bar{0} & 0 \\ * & * & * & * & * & -\frac{1}{\tau_M}Q & \bar{0} & 0 \\ * & * & * & * & * & * & \Phi_{66} & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{4.30}$$

constrained by

$$\begin{bmatrix} -2X+T & W^T \\ * & -Y_6 \end{bmatrix} < 0,$$

where

$$\begin{aligned}
\Phi_{11} &= (A+A_d)X + X(A+A_d)^T + (B+B_d)W + W^T(B+B_d)^T \\
\Phi_{12} &= \begin{bmatrix} 0 & A_2Y_2 & A_hY_3 & 0 & -A_dY_5 & -B_dY_6 \end{bmatrix} \\
\Phi_{22} &= diag\{-(1-d_D)Y_1, -(1-d_D)Y_2, -\frac{(1-\eta_D)}{\eta_M}Y_3, -(1-\tau_D)Y_4, -\frac{(1-\tau_D)}{\tau_M}Y_5, -\frac{(1-\tau_D)}{\tau_M}Y_6\} \\
\Phi_{23} &= \begin{bmatrix} A_dY_1 & A_2Y_2 & A_hY_3 & B_dY_4 & 0 & 0 \end{bmatrix}^T \\
\Phi_{16} &= \begin{bmatrix} X & \eta_M X & W^T \end{bmatrix} \\
\Phi_{66} &= diag\{-Y_1, -\eta_M Y_3, -Y_4\}
\end{aligned}$$

### 4.3.2 Optimality of the method

In this section the main result of this chapter is introduced. It will be shown that the unified method proposed in Section 4.3 is less conservative than the design method classically employed in the literature.

To demonstrate this assertion, it is necessary to describe the basis of the design methods used in the literature for the different TDSs. Although there is a wide variety of designs which solve the control problem through different functionals and bounding techniques, the main guidelines of the design methods can be summarized as follows.

First of all, the authors propose a certain Lyapunov-Krasovskii functional. Then, after some mathematical manipulation, they obtain an optimization problem with a pair of matrix inequalities to carry out the controller design:

- One inequality accounting for the $H_\infty$ disturbance attenuation constraint $\gamma$.
- One inequality to guarantee an upper bound of cost $J_2$.

Therefore, given the restriction on the $H_\infty$ norm, the different methods design a cost guaranteed controller, that is, a controller with a bound on the $H_2$ norm.

The main difference between the standard design methods in the literature and the one proposed here is the minimization of the $H_2$ cost function, as the $H_\infty$ constraint is directly embedded in the optimization problem in the same way. The standard procedure bounds each of the quadratic terms of the initial condition of the functional $V(t_0)$ by the trace of certain matrices $M_i$ and then, the controller is chosen to minimize the sum of the traces of these matrices $M_i$. This implies the minimization of the upper bound of $V(t_0)$, which is an upper bound of the cost $J_2$. Then $V(t_0) = \sum_i V_i(t_0)$, where

$$V_i(t_0) < tr(M_i),\ \forall i. \tag{4.31}$$

Therefore, the standard design methods are based on solving the following optimization problem [33, 113, 258].

**Definition 4.2. Optimization Problem 1 (OP1).** The suboptimal mixed $H_2/H_\infty$ controller $K$ can be obtained by solving the following optimization problem:

$$\min \quad \sum_i tr(M_i), \tag{4.32}$$

$$\begin{aligned} \text{subject to} \quad \Xi(K) &< -\Phi(K) & (4.33)\\ \Theta(K,\gamma) &< 0 & (4.34)\\ V_i(t_0) &< tr(M_i),\ \forall i, & (4.35) \end{aligned}$$

where matrices $\Xi(K), \Phi(K)$ and $\Theta(K,\gamma)$ were defined in Section 4.3.

The comparison between Optimization Problem 1 and Lemma 4.1 leads to three important differences:

1. The function to be minimized.

2. The upper bound of $J_2$, directly related to conditions (4.11) and (4.33).

3. OP1 requires additional constraints, (4.35), which are no needed in Lemma 4.1.

Next, the following theorem demonstrates the main result of this chapter, showing that the proposed design method can always achieve better controllers in terms of optimality, guaranteeing lower bounds for the cost function.

**Theorem 4.4.** *For any system of the TDS family described in Section 3.2, and given an $H_\infty$ norm bound $\gamma$, if there exists a controller $K_2$ obtained through OP1, there also exists a controller $K_1$ that can be obtained by solving the optimization problem in Lemma 4.1. Furthermore, the controller $K_1$ obtained through Lemma 4.1 outperforms (reduces) the upper bound of the $J_2$ cost with respect to the upper bound that can be obtained with $K_2$.*

**Proof.** Let $\bar{J}_{K_1}$ denote the upper bound of the cost obtained through controller $K_1$. Similarly, $\bar{J}_{K_2}$ will denote the upper bound of the cost obtained through controller $K_2$.

Observe that, if a feasible controller $K_2$ solves the optimization problem OP1, all the constraints of that problem are satisfied and, specifically the constraints

$$\begin{aligned} \Xi(K) &< -\Phi(K), \\ \Theta(K,\gamma) &< 0. \end{aligned}$$

Thus, there will exist a controller $K_1$ that satisfies Lemma 4.1 given that, by choosing the same LKF $V(t)$, one obtains the same constraints for $\alpha = 1$. Therefore, it has been proved that the existence of $K_2$ implies the existence of $K_1$.

To complete the proof, it remains to prove that $\bar{J}_{K_1} < \bar{J}_{K_2}$. This is straightforward to see. For $\alpha = 1$ and the same LKF, it holds that $\bar{J}_{K_1} = V(t_0)$, whereas $\bar{J}_{K_2} = \sum_i tr(M_i)$. From constraint (4.35), $V(t_0) = \sum_i V_i(t_0) < \sum_i tr(M_i)$, thus $\bar{J}_{K_1} < \bar{J}_{K_2}$. □

Theorem 4.4 ensures that, given a controller $K_2$, the existence of a controller $K_1$ is guaranteed for $\alpha = 1$ and the same LKF. Additionally, it might exist another controller that solves the optimization problem in Lemma 4.1 for $\alpha < 1$. In this case, the upper bound of the cost $J_2$ is further reduced.

**Corollary 4.1.** *Given a controller for a time-delay system obtained through Lemma 4.1, the existence of a controller designed through OP1 is not guaranteed.*

**Proof.** The proof is straightforward, for the extra constraint (4.35) makes it impossible to ensure that the existence of a controller $K_1$ (that verifies all the constraints in Lemma 4.1) implies the existence of a solution for OP1. □

**Corollary 4.2.** *Given a system of the TDS family and an $H_\infty$ bound $\gamma$, if the controller $K_1$ found through Lemma 4.1 is obtained with $\alpha > 1$, then it is not possible to find any controller $K_2$ that verifies the constraints in OP1.*

The proof follows immediate by Theorem 4.2. A direct implication of the results given in this section is that, for all the systems of the TDS family verifying Assumptions 4.1-4.2, it is always more convenient to use Lemma 4.1 to carry out the controller design than any of the theorems based in the classical procedure generalized under OP1.

### 4.3.3 Numerical examples

Previous section mathematically demonstrates that the proposed design method outperforms the classical approaches. In this section a number of different examples are studied in order to quantify this improvement numerically. The unified design method is particularized and applied to two types of TDSs. After that, the obtained results are compared with well-referenced designs. Please note that the application of the proposed design method is performed by using the LKFs employed in the referenced works in order to show that the improvements come from the new design method rather than from the use of more sophisticated functionals or bounds.

**Example 4.1.** Consider the standard TDS borrowed from [113]:

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -0.3 & 0.1 \\ -0.1 & 0.1 \end{bmatrix} x(t-d(t)) \\
&\quad + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} u(t-\tau(t)), \\
z_2 &= \begin{bmatrix} 2 & 1 \end{bmatrix} x(t) + u(t), \\
z_\infty &= \begin{bmatrix} 1 & 3 \end{bmatrix} x(t) + u(t).
\end{aligned}
$$

The bounds for the time delay derivatives are $d_D = 0.2$, $\tau_D = 0.4$, and the initial condition $x_{t_0}(\theta) = \begin{bmatrix} e^{\theta+1} \\ 0 \end{bmatrix}$ for all $\theta \in [-0.4, 0]$.

For an $H_\infty$ norm $\gamma = 1$, an $H_2/H_\infty$ controller is obtained in [113] with a bound of the $H_2$ cost $\bar{J}_{K_2} = 5.1897$. With the proposed result, the upper bound of the $H_2$ cost is given by $\bar{J}_{K_1} = \alpha V(t_0)$, where

$$\begin{aligned} V(t_0) &= \phi^T(t_0)P\phi(t_0) + \int_{t_0-d(t)}^{t_0} \phi^T(s)Z_1\phi(s)ds \\ &+ \int_{t_0-\tau(t)}^{t_0} \phi^T(s)K^T Z_2 K\phi(s)ds. \end{aligned}$$

Assuming with lost of generality that $t_0 = 0$, then $V(t_0)$ can be easily obtained for the worst case, i.e. $\max d(t) = \max \tau(t) = \infty$. If Theorem 4.1 is used with $\gamma = 0.5$, it can be obtained $\bar{J}_{K_1} = 1.1119$, improving the previous results. Therefore, the proposed design method reduces the guaranteed cost in more than a 75%, while reducing at the same time the disturbance attenuation level to 50%. The designed controller is $K_1 = [-1.7824 \quad -1.4632]$.

**Example 4.2.** Consider the following uncertain descriptor system with delays introduced in [258]:

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) &= \left( \begin{bmatrix} -2 & 0.2 \\ 0 & -1 \end{bmatrix} + \Delta A(t) \right) x(t) \\ &+ \left( \begin{bmatrix} -0.5 & 0 \\ 0.5 & -0.4 \end{bmatrix} + \Delta A_d(t) \right) x(t-1) \\ &+ \int_{t-0.6}^{t} \left( \begin{bmatrix} -0.3 & 0.1 \\ 0.2 & -0.25 \end{bmatrix} + \Delta A_h(s) \right) x(s)ds \\ &+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t), \\ z_\infty(t) &= \begin{bmatrix} 2 & 1 \end{bmatrix} x(t) + u(t), \end{aligned}$$

where $\|\Delta A(t)\|_2 \le 0.05$, $\|\Delta A_d(t)\|_2 \le 0.05$, $\|\Delta A_h(t)\|_2 \le 0.05$. The uncertainties are assumed to verify

$$\begin{bmatrix} \Delta A(t) & \Delta A_d(t) & \Delta A_h(t) \end{bmatrix} = DF(t) \begin{bmatrix} E_A & E_{Ad} & E_{Ah} \end{bmatrix},$$

where $F(t)$ is an time-varying unknown matrix such that $\|F(t)\|_2 \le 1$ and

$$D = I; \quad E_A = E_{Ad} = E_{Ah} = diag\{0.05, 0.05\}.$$

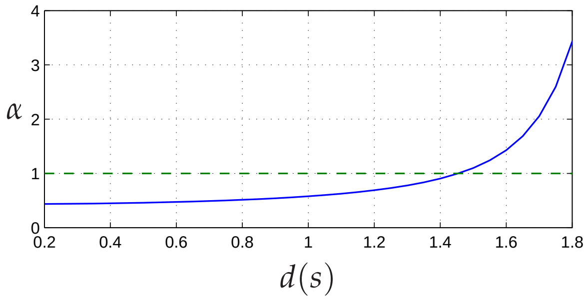



Figure 4.1: $\alpha$ vs. Delay $d$

The cost function $J_2$ is chosen with $Q = I$ y $R = 1$. The initial condition for the system is $x_{t_0}(\theta) = \begin{bmatrix} e^{\theta+1} \\ 0 \end{bmatrix}$ for all $\theta \in [-1, 0]$.

Choosing $a = 0.5$ and an $H_\infty$ norm $\gamma = 1$, the authors obtain in [258] an $H_2/H_\infty$ controller with an upper bound of the cost $\bar{J}_{K_2} = 10.8512$.

With the method of this chapter, the upper bound of the cost is $\bar{J}_{K_1} = \alpha V(t_0)$. The value of $V(t_0)$ can be easily obtained numerically. If Theorem 4.2 and the extension to uncertain systems given in Appendix D.2 are used with $\gamma = 1$, the obtained upper bound is $\bar{J}_{K_1} = 8.0943$, improving the previous results. The designed controller for this system is $K_1 = [-1.6609 \quad -3.4059]$.

With respect to the upper bound for the time delay, the authors in [258] solve the aforementioned problem with a maximum constant delay of $d = 0.6074$ seconds. With the method proposed in this chapter, the maximum constant delay can be extended to $d = 2.1704$ seconds.

In Figure 4.1, the dependence between $\alpha$ and the maximum constant delay $d$ is illustrated for this example. It is worth mentioning that, for the particular case of $\alpha = 1$, the proposed method also achieves better results than the obtained in [258]. This clearly indicates that the constraints $V_i(t_0) < tr(M_i),\ \forall i$, which are necessary in the previous papers, introduce extra conservatism.

## 4.4 Control of systems over networks

The previous section has presented a general method to design robust and optimal controllers for time-delay system through the Lyapunov-Krasovskii theory. Now, the chapter moves on to the framework of networked systems, which is indeed the main focus of the thesis. The inclusion of both areas in the same chapter is well justified, as the explained theory for TDS can be applied to networked systems.

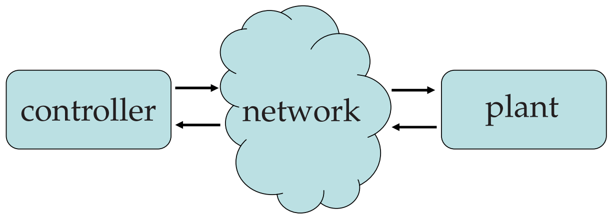



Figure 4.2: System controlled through a network

The results presented in this section are motivated by the following key idea: under some assumptions, a system controlled over a network can be seen as a time-delay system. Then, the ideas and techniques proposed to study the stability and to control a time-delay system can be directly inherited and used for networked systems.

Therefore, after presenting the mathematical formulation, the design method will be applied to an experimental plant consisting of a two-degree-of-freedom robotic arm.

### 4.4.1 Formulation of a networked system as a particular time-delay system

Consider the scheme depicted in Figure 4.2, where the controller is physically located at the other end of a communication network. The plant is described by the following LTI model:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t). \tag{4.36}$$

The communication between system and controller needs to be performed by transmitting packets at discrete time instants. The sampling period is $h$ seconds, that is, the system sends its state periodically. When the information is received at the other end of the network, the controller computes the control signal and sends it immediately to the system. The control action is applied straight away as it reaches the plant.

These packets, when crossing the imperfect channel, may be affected by delays and/or dropouts in both paths, system to controller and controller to system. Let $t_s^k \triangleq kh$ ($k \in \mathbb{N}$) denote the sampling instant in which packet $k$ is sent. However, only a subset of packets arrive to the plant, as some of them may be lost. Let $\{k_1\}$ denote the sequence of the packets received, as Figure 4.3 illustrates. There exists a relation between the discrete time instants $k$ and each element $k_1$, that is, $k = j(k_1)$.

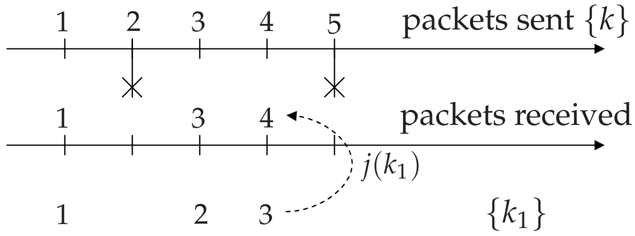



Figure 4.3: Packets sent, packets received and sequence $\{k_1\}$

Then, define $t_s^{k_1} \triangleq j(k_1)h$ ($j(k_1) \in \mathbb{N}$) as the sampling times when received packets were sent. It is worth recalling the following ideas:

- Sequence $\{k\}$ represents the set of packets sent.
- Sequence $\{k_1\}$ represents the sequence of packets received.
- Sequence $\{j(k_1)\}$ represents a subset of $\{k\}$. Missing numbers represent dropouts. Obviously, $j(k_1) < j(k_1+1)$.

Finally, let $d(k_1)$ be the complete delay introduced by the channel in both links to the packet transmitted at $t_s^{k_1}$.

When taking into account the packet-based communication, continuous control inputs in the form of $u(t) = Kx(t)$ cannot be applied, as the system states are not available for the controller at every time. Instead, a typical solution consists in using a piecewise constant signal that is updated whenever a new measurement is received from the controller[1]. In the following, the input delay approach introduced in [146] is used.

The idea consists in unifying the effects of sampling, delays, and dropouts under an unique artificial delay. Define $t \in [t_{k_1}, t_{k_1+1})$ as the time interval between two consecutive measurements received from the controller, where $t_{k_1}$ is the time instant when the control action, calculated with the information of the system at $t_s^{k_1}$, reaches the plant (see Figure 4.4). Note that

$$\begin{aligned} t_{k_1} &= t_s^{k_1} + d(k_1), \\ &= j(k_1)h + d(k_1). \end{aligned}$$

The artificial delay $\tau(t)$ represents the time difference between the current time instant $t$ and the instant when the last packet received by the plant was sent. The

[1]This typical solution uses a zero-order holder to build the continuous signal from discrete samples. Other first-order holders (or higher) can also be used.

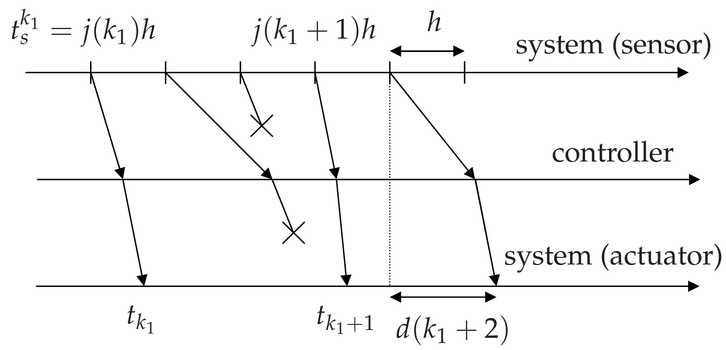



Figure 4.4: Schematic diagram of the packets send through the network

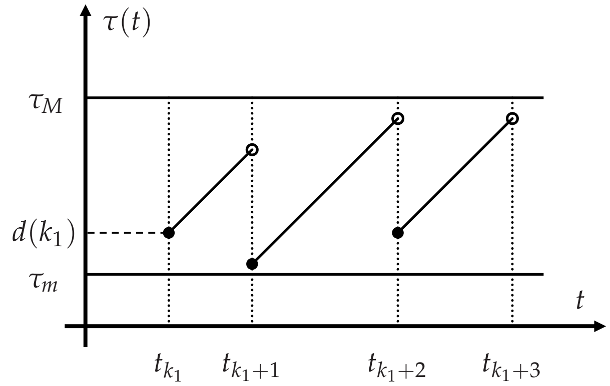



Figure 4.5: Qualitative evolution of $\tau(t)$

artificial delay is then defined as

$$\begin{aligned} \tau(t) &= t - t_s^{k_1}, \\ &= t - t_{k_1} + d(k_1), \quad t \in [t_{k_1}, t_{k_1+1}). \end{aligned}$$

Note that sampling and both undesired effects, delays and packet dropouts, have been merged into a common delay $\tau(t)$. Figure 4.5 illustrates a possible evolution of $\tau(t)$.

The state feedback control law can now be written as

$$u(k) = Kx(t_{k_1} - d(k_1)), \quad t \in [t_{k_1}, t_{k_1+1}). \tag{4.37}$$

Thus, the system (4.36) under the control law (4.37) can be rewritten as

$$\dot{x}(t) = Ax(t) + BKx(t - \tau(t)) + B_w w(t), \quad t \in [t_{k_1}, t_{k_1+1}).$$

Comparing with the dynamics of the generic time-delay system (4.1), it is easy to see the similarities, as previous equation is just a particular case. The main difference is the domain of definition, since this one is only defined inside an interval, due to the switching nature of the artificial delay.

The following assumption characterizes the network conditions with respect to induced delays and packet dropouts. It imposes fairly standard and realistic constraints in the NCS framework.

**Assumption 4.3.** Three constants $\underline{d}, \overline{d}, n_p \geq 0$ exist such that:

- The network-induced delay from sensor to actuator $d(k_1)$ satisfies $\underline{d} \leq d(k_1) \leq \overline{d}, \forall k$.
- The maximum number of consecutive packet dropouts from sensor to actuator is bounded by $n_p$. That is, $j(k_1+1) - j(k_1) \leq n_p$.

Given the network conditions detailed in Assumption 4.3, it turns out that $\tau(t)$ verifies the bounding assumptions typically required for TDS, although the nature of them is different. The following proposition, whose proof is not needed, gives the numerical bounds of the artificial delay $\tau(t)$ in function of the network-induced delays and the packet dropouts.

**Proposition 4.1.** *Suppose that Assumption 4.3 holds. Then, two constants $\tau_M > \tau_m \geq 0$ exist such that*

$$\begin{aligned} \tau(t) &\geq \underline{d} = \tau_m, & (4.38)\\ \tau(t) &\leq (1+n_p)h + \overline{d} = \tau_M. & (4.39) \end{aligned}$$

### 4.4.2 Controller design for networked control systems

In this section the problem of designing a mixed $H_2/H_\infty$ controller to stabilize the network controlled system (4.36) is addressed. The result is a direct application of Lemma 4.1. Controlled outputs $z_2$ and $z_\infty$ are defined in (4.3)-(4.4), and the cost function is given by

$$J_2 = \int_{t_0}^{\infty} x^T(s)Qx(s) + u^T(s)Ru(s)ds$$

Consider the following Lyapunov-Krasovskii functional:

$$\begin{aligned} V(t) &= x^T(t)Px(t) + \int_{t-\tau_m}^{t} x^T(s)Z_1x(s)ds + \int_{t-\tau_M}^{t} x^T(s)Z_2x(s)ds \\ &+ \int_{-\tau_M}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_3\dot{x}(s)dsd\theta + \int_{-\tau_M}^{-\tau_m}\int_{t+\theta}^{t} \dot{x}^T(s)Z_4\dot{x}(s)dsd\theta, \quad (4.40) \end{aligned}$$

where $\tau_m, \tau_M$ are the known bounds of the artificial delay (4.38)-(4.39), and matrices $P, Z_1, \ldots, Z_4$ are positive definite. Note the this LKF is inspired on those proposed for TDS in equations (3.12)-(4.16)-(4.23)-(4.28).

The following theorem provides a solution for the mixed $H_2/H_\infty$ problem in the NCS framework.

**Theorem 4.5.** *Given scalars $\tau_m, \tau_M, \gamma, \varepsilon > 0$ and the weighting matrices $Q, R > 0$, if matrices $X, \tilde{Z}_1, \ldots, \tilde{Z}_4 > 0$ and any matrices $Y, \tilde{N}_i, \tilde{M}_i, \tilde{S}_i$ $(i = 1, 2)$ solve the following optimization problem for the two vertices of the polytope $\tau(t) \in [\tau_m, \tau_M]$:*

$$\begin{array}{c} \min\limits_{X,Y,\tilde{Z}_i,\tilde{N}_i,\tilde{M}_i,\tilde{S}_i} \alpha \\ \textit{subject to } (4.42)-(4.43) \end{array} \tag{4.41}$$

*then, the $H_2/H_\infty$ controller for the NCS* (4.36) *is given by $K = YX^{-1}$.*

**Proof.** The proof follows the same steps that the one of Theorem 4.1. It will be shown that the time derivative of the LKF (4.40) can be written as is required by Assumption 4.1. Furthermore, the optimization problem (4.41) will be proved to be equivalent to that in Lemma 4.1.

First, suppose $w(t) \equiv 0$. The time derivative of $V(t)$ is given by

$$\begin{aligned} \dot{V}(t) &= 2x^T(t)P\dot{x}(t) + x^T(t)(Z_1 + Z_2)x(t) - x^T(t-\tau_m)Z_1x(t-\tau_m) \\ &- x^T(t-\tau_M)Z_2x(t-\tau_M) + \dot{x}^T(t)(\tau_M Z_3 + \Delta\tau Z_4)\dot{x}(t) \\ &- \int_{t-\tau_M}^{t} \dot{x}^T(s)Z_3\dot{x}(s)ds - \int_{t-\tau_M}^{t-\tau_m} \dot{x}^T(s)Z_4\dot{x}(s)ds, \end{aligned} \tag{4.44}$$

with $\Delta\tau = \tau_M - \tau_m$.

Similar to Theorem 3.1, the integral terms in the previous equation can be split as follows:

$$\begin{aligned} \int_{t-\tau_M}^{t} \dot{x}^T(s)Z_3\dot{x}(s)ds &= \int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)Z_3\dot{x}(s)ds + \int_{t-\tau(t)}^{t} \dot{x}^T(s)Z_3\dot{x}(s)ds, \\ \int_{t-\tau_M}^{t-\tau_m} \dot{x}^T(s)Z_4\dot{x}(s)ds &= \int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)Z_4\dot{x}(s)ds + \int_{t-\tau(t)}^{t-\tau_m} \dot{x}^T(s)Z_4\dot{x}(s)ds. \end{aligned}$$

Moreover, the following null terms are added to the right-hand side of (4.44):

$$\begin{aligned} 0 &= 2[x^T(t)N_1 + x^T(t-\tau(t))N_2]\left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s)ds\right], \\ 0 &= 2[x^T(t)S_1 + x^T(t-\tau(t))S_2]\left[x(t-\tau(t)) - x(t-\tau_M) - \int_{t-\tau_M}^{t-\tau(t)} \dot{x}(s)ds\right], \\ 0 &= 2[x^T(t)M_1 + x^T(t-\tau(t))M_2]\left[x(t-\tau_m) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t-\tau_m} \dot{x}(s)ds\right]. \end{aligned}$$

$$\begin{bmatrix} \tilde{\Xi}_{11} & \tilde{\Xi}_{12} & \tau_M \tilde{A} & \Delta\tau \tilde{A} & \tilde{Q} & \tilde{R} \\ * & \tilde{\Xi}_{22} & 0 & 0 & 0 & 0 \\ * & * & -\tau_M X \tilde{Z}_3^{-1} X & 0 & 0 & 0 \\ * & * & * & -\Delta\tau X \tilde{Z}_4^{-1} X & 0 & 0 \\ * & * & * & * & -\alpha Q^{-1} & 0 \\ * & * & * & * & * & -\alpha R^{-1} \end{bmatrix} < 0, \tag{4.42}$$

$$\begin{bmatrix} \tilde{\Xi}_{11} & \tilde{\Xi}_{12} & \tilde{\Xi}_{13} & \tau_M \tilde{A} & \Delta\tau \tilde{A} & \tilde{C} \\ * & \tilde{\Xi}_{22} & 0 & 0 & 0 & 0 \\ * & * & -\gamma^2 I & -\tau_M B_w^T & -\Delta\tau B_w^T & 0 \\ * & * & * & -\tau_M X \tilde{Z}_3^{-1} X & 0 & 0 \\ * & * & * & * & -\Delta\tau X \tilde{Z}_4^{-1} X & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{4.43}$$

where

$$\tilde{\Xi}_{11} = \begin{bmatrix} \tilde{\Gamma}_{11} & \tilde{\Gamma}_{12} & \tilde{M}_1 & -\tilde{S}_1 \\ * & \tilde{\Gamma}_{22} & \tilde{M}_2 & -\tilde{S}_2 \\ * & * & -\tilde{Z}_1 & 0 \\ * & * & * & -\tilde{Z}_2 \end{bmatrix}, \quad \begin{cases} \tilde{\Gamma}_{11} = AX + XA^T + \tilde{Z}_1 + \tilde{Z}_2 + \tilde{N}_1 + \tilde{N}_1^T \\ \tilde{\Gamma}_{12} = BY - \tilde{N}_1 + \tilde{S}_1 - \tilde{M}_1 + \tilde{N}_2^T \\ \tilde{\Gamma}_{22} = -\tilde{N}_2 - \tilde{N}_2^T + \tilde{S}_2 + \tilde{S}_2^T - \tilde{M}_2 - \tilde{M}_2^T \end{cases}$$

$$\tilde{\Xi}_{12} = \begin{bmatrix} (\tau(t)+\varepsilon)\tilde{N} & (\tau(t)-\tau_m+\varepsilon)\tilde{M} & (\tau_M-\tau(t)+\varepsilon)\tilde{S} \end{bmatrix},$$

$$\tilde{\Xi}_{22} = \begin{bmatrix} -(\tau(t)+\varepsilon)\tilde{Z}_3 & 0 & 0 \\ * & -(\tau(t)-\tau_m+\varepsilon)\tilde{Z}_4 & 0 \\ * & * & -(\tau_M-\tau(t)+\varepsilon)(\tilde{Z}_3+\tilde{Z}_4) \end{bmatrix},$$

$$\tilde{\Xi}_{13}^T = \begin{bmatrix} B_w^T & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{N}^T = \begin{bmatrix} \tilde{N}_1^T & \tilde{N}_2^T & 0 & 0 \end{bmatrix},$$

$$\tilde{M}^T = \begin{bmatrix} \tilde{M}_1^T & \tilde{M}_2^T & 0 & 0 \end{bmatrix},$$

$$\tilde{S}^T = \begin{bmatrix} \tilde{S}_1^T & \tilde{S}_2^T & 0 & 0 \end{bmatrix},$$

$$\tilde{A}^T = \begin{bmatrix} AX & BY & 0 & 0 \end{bmatrix},$$

$$\tilde{C}^T = \begin{bmatrix} CX & DY & 0 & 0 \end{bmatrix},$$

$$\tilde{Q}^T = \begin{bmatrix} X & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{R}^T = \begin{bmatrix} 0 & Y^T & 0 & 0 \end{bmatrix}.$$

Let $\xi^T(t) = \left[x^T(t) \;\; x^T(t-\tau(t)) \;\; x^T(t-\tau_m) \;\; x^T(t-\tau_M)\right]$. Therefore, equation (4.44) can be rewritten as:

$$\begin{aligned}
\dot{V}(t) &= \xi^T(t)\Gamma\xi(t) + \dot{x}^T(t)(\tau_M Z_3 + \Delta\tau Z_4)\dot{x}(t) \\
&- \int_{t-\tau(t)}^{t} \dot{x}^T(s)Z_3\dot{x}(s)ds - 2\xi^T(t)\bar{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds \\
&- \int_{t-\tau(t)}^{t-\tau_m} \dot{x}^T(s)Z_4\dot{x}(s)ds - 2\xi^T(t)\bar{M}\int_{t-\tau(t)}^{t-\tau_m} \dot{x}(s)ds \\
&- \int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)(Z_3+Z_4)\dot{x}(s)ds - 2\xi^T(t)\bar{S}\int_{t-\tau_M}^{t-\tau(t)} \dot{x}(s)ds,
\end{aligned} \tag{4.45}$$

where:

$$\begin{aligned}
\Gamma &= \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & M_1 & -S_1 \\ * & \Gamma_{22} & M_2 & -S_2 \\ * & * & -Z_1 & 0 \\ * & * & * & -Z_2 \end{bmatrix}, \\
\Gamma_{11} &= PA + A^TP + Z_1 + Z_2 + N_1 + N_1^T, \\
\Gamma_{12} &= PBK - N_1 + S_1 - M_1 + N_2^T, \\
\Gamma_{22} &= -N_2 - N_2^T + S_2 + S_2^T - M_2 - M_2^T.
\end{aligned}$$

Please notice the equivalence between this equation and equation (3.14) in the proof of Theorem 3.1. Similar integral terms appears. Therefore, the same bounding technique is used, that is, the upper bound for the inner product of two vectors:

$$-2b^Ta - a^TXa \leq b^TX^{-1}b, \quad X > 0.$$

Hence, the integral terms in (4.45) can be bounded by

$$\begin{aligned}
-2\xi^T(t)\bar{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds - \int_{t-\tau(t)}^{t} \dot{x}^T(s)Z_3\dot{x}(s)ds &\leq (\tau(t)+\varepsilon)\xi^T(t)\bar{N}Z_3^{-1}\bar{N}^T\xi(t), \\
-2\xi^T(t)\bar{M}\int_{t-\tau(t)}^{t-\tau_m} \dot{x}(s)ds - \int_{t-\tau(t)}^{t-\tau_m} \dot{x}^T(s)Z_4\dot{x}(s)ds &\leq (\tau(t)-\tau_m+\varepsilon)\xi^T(t)\bar{M}Z_4^{-1}\bar{M}^T\xi(t), \\
-2\xi^T(t)\bar{S}\int_{t-\tau_M}^{t-\tau(t)} \dot{x}(s)ds - \int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)(Z_3+Z_4)\dot{x}(s)ds &\leq \\
(\tau_M-\tau(t)+\varepsilon)\xi^T(t)\bar{S}(Z_3+Z_4)^{-1}\bar{S}^T\xi(t),
\end{aligned}$$

where $\varepsilon > 0$ is introduced in order to avoid null elements in the diagonal terms of the resulting matrix inequalities.

Combining these bounds with (4.45), it can be shown that for $t \in [t_{k_1}, t_{k_1+1})$,

$$\dot{V}(t) \leq \xi^T(t)\Xi(K)\xi(t), \tag{4.46}$$

where

$$\Xi(K) = \Big(\Gamma + (\tau(t)+\varepsilon)\bar{N}Z_3^{-1}\bar{N}^T + (\tau(t)-\tau_m+\varepsilon)\bar{M}Z_4^{-1}\bar{M}^T \\ + (\tau_M-\tau(t)+\varepsilon)\bar{S}(Z_3+Z_4)^{-1}\bar{S}^T + \bar{A}\tau_M Z_3\bar{A}^T + \bar{A}\Delta\tau Z_4\bar{A}^T\Big).$$

Therefore, equation (4.6) holds for $w \equiv 0$. Matrices $\bar{N}, \bar{M}, \bar{S}, \bar{A}$ are given by

$$\bar{N} = \begin{bmatrix} N_1 \\ N_2 \\ 0 \\ 0 \end{bmatrix};\ \bar{M} = \begin{bmatrix} M_1 \\ M_2 \\ 0 \\ 0 \end{bmatrix};\ \bar{S} = \begin{bmatrix} S_1 \\ S_2 \\ 0 \\ 0 \end{bmatrix};\ \bar{A} = \begin{bmatrix} A^T \\ K^TB^T \\ 0 \\ 0 \end{bmatrix}.$$

Consider now the presence of disturbances. Then, the following null terms are added to the functional:

$$\dot{V}(t) = \dot{V}(t) - \gamma^2 w^T(t)w(t) + \gamma^2 w^T(t)w(t) - z_\infty^T(t)z_\infty(t) + z_\infty^T(t)z_\infty(t).$$

It is easy to show that $z_\infty^T(t)z_\infty(t) = \xi^T(t)\bar{C}_{\infty K}^T\bar{C}_{\infty K}\xi(t)$, where:

$$\bar{C}_{\infty K} = \begin{bmatrix} C_\infty & D_\infty K & 0 & 0 \end{bmatrix}.$$

Thus, it yields

$$\begin{aligned} \dot{V}(t) &\leq \begin{bmatrix} \xi^T(t) & w^T(t) \end{bmatrix} \begin{bmatrix} \Xi(K)+\bar{C}_z(K) & \bar{B}_w(K) \\ * & -\gamma^2 I + \bar{D}_w \end{bmatrix} \begin{bmatrix} \xi(t) \\ w(t) \end{bmatrix} \\ &- z_\infty^T(t)z_\infty(t) + \gamma^2 w^T(t)w(t), \end{aligned}$$

where $\bar{C}_z(K) = \bar{C}_{\infty K}^T\bar{C}_{\infty K}$, $\bar{D}_w = B_w^T P B_w$ and

$$\bar{B}_w(K) = \begin{bmatrix} A^T Z_3 B_w + PB_w \\ K^T B^T Z_3 B_w \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, the derivative of the LKF (4.16) can be written as in equation (4.6) by defining $\Theta(K,\gamma) = \begin{bmatrix} \Xi(K)+\bar{C}_z(K) & \bar{B}_w(K) \\ * & -\gamma^2 I + \bar{D}_w \end{bmatrix}$. This way, Assumption 4.1 holds.

The cost function $J_2$ can be rewritten in the following way:

$$J_2 = \int_{t_0}^{\infty} \xi^T(s)\Phi(K)\xi(s)ds,$$

with

$$\Phi(K) = \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & K^T R K & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

satisfying also Assumption 4.2 (4.8).

It remains to prove that the optimization problems in Lemma 4.1 and Theorem 4.5 are equivalent. Consider now equation (4.11) in Lemma 4.1.

$$\alpha \Xi(K) < -\Phi(K). \tag{4.47}$$

From equation (4.47), after some mathematical manipulations and by applying Schur complements, a matrix inequality with the same structure of (4.42) is obtained. To finally obtain (4.42), it is sufficient to introduce the definitions $X = P^{-1}, \tilde{Z}_i = XZ_iX, \tilde{N}_i = XN_iX, \tilde{M}_i = XM_iX, \tilde{S}_i = XS_iX$, and then pre- and post-multiply the matrix inequality by $diag\{X, X, X, X, X, X, X, X, X, I, I\}$ and its transpose.

In a similar way, consider condition (4.12):

$$\Theta(K, \gamma) < 0. \tag{4.48}$$

From this equation, it can be obtained an inequality with the structure of (4.19) by using the same definitions and pre- and post- multiplying the matrix inequality by $diag\{X, X, X, X, X, X, X, I, X, X, I\}$ and its transpose. □

**Remark 4.3.** The scalar parameter $\varepsilon > 0$ needs to be introduced in order to make the problem feasible. It is worth mentioning that this modification does not introduce any conservatism, since $\varepsilon$ can be chosen as small as necessary, i.e., $\varepsilon \to 0^+$.

Notice that (4.42)-(4.43) are not linear matrix inequalities due to the presence of the terms $X\tilde{Z}_3^{-1}X, X\tilde{Z}_4^{-1}X$, so the optimization problem cannot be solved as it is posed. However, two standard solutions can be employed in order to deal with those nonlinearities. The first one introduces an additional constraint which lets us address the problem by means of a set of linear matrix inequalities. The second solution uses the cone complementary algorithm to transform the nonlinear inequality into an iterative optimization problem with linear constraints. Comparing both solutions, the former could be more conservative, but it is computationally more efficient, as the number of constraints and variables is lower. Appendix C gives details of both methods.

In the following section, that describes some experiments on a robotic manipulator, all controllers are designed using the *cone complementary algorithm*.

### 4.4.3 Experimental application

This section describes the platform in which the experiments are performed, then presents the plant and finally, shows the results of the experiments.

#### 4.4.3.1 Platform description

This part describes the test bed built to prove different networked control systems. Figure 4.6 shows a diagram of the connections. The selected control software is the xPC Target environment [141, 163] with MATLAB/Simulink.

There are two Target PCs and a Host PC. The *local* Target PC, which is connected to the system to be controlled, employs a shared Ethernet network to connect with the *remote* Target PC which implements the control tasks. Both Target PCs are connected to the Host PC through the previously mentioned network. For communication tasks, the unreliable UDP/IP protocol has been selected as, compared with reliable TCP/IP, it is commonly recommended for real-time application [172, 197]. The reason is clear, as the transmission speed is slower for TCP due to the error-checking algorithms. Furthermore, there are not retransmissions in UDP.

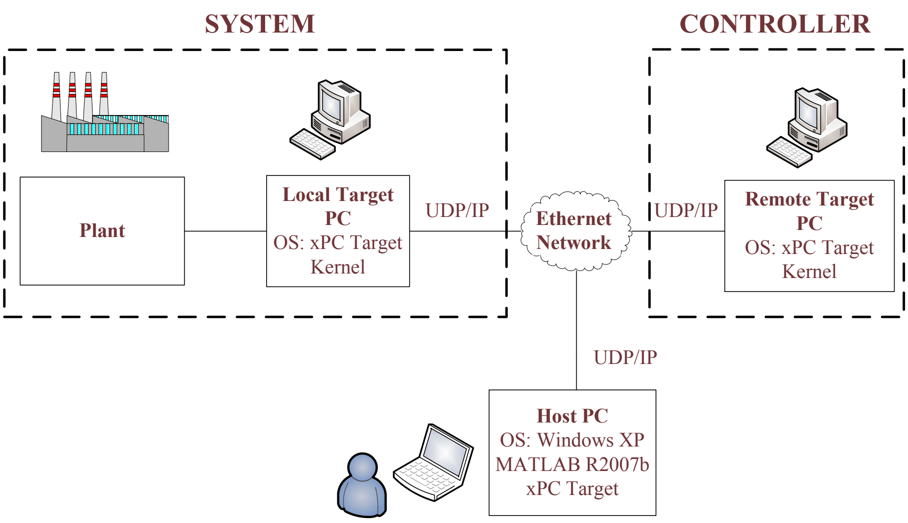



Figure 4.6: Diagram of equipment connections

The Host PC is dedicated to the creation of the Simulink models which are run in real time in the Target PCs. Also, it supervises all the test bed: starts and stops both Target PCs; controls the communication between the PCs; and receives all the data when experiments end.

The *local* Target PC has the following functions:

- Read output signals from the plant.
- Send data packets to the *remote* Target PC.
- Receive control data packets from the *remote* Target PC.
- Apply control signals to the plant.

On the other hand, the *remote* Target PC functions are the following:

- Receive position data packets from the *local* Target PC.
- Calculate control signals from the received information.
- Send control data packets to the *local* Target PC.

The communication network has been identified by means of performing several experiments. In these experiments, from a certain time reference, the maximum and minimum delay of a control cycle through the network have been identified, that is, the total time passed since a position data packet is sent from the *local* Target PC until the control data packet calculated using that information is received in the *local* Target PC. Those values are the following:

- Minimum round-trip delay: $\underline{d} = 9$ ms.
- Maximum round-trip delay: $\overline{d} = 15$ ms.

However, it would be interesting to manipulate the Quality of Service (QoS) for this kind of experiments, in such a way that the controllers can be tested for worse network conditions. This way, the *remote* Target PC includes the possibility of degrading the QoS. More precisely, extra delays (upper and lower bounded) and extra packet dropouts (percentage of losses and maximum number of consecutive dropouts) can be introduced.

#### 4.4.3.2 Plant description. Modeling and control.

The experimental platform described above has been used to control a two-degree-of-freedom direct drive robot, designed and developed by the Department of Systems Engineering and Automation at University of Seville. Figure 4.7 shows a photography of the robot.

The objective of the experiments will be to maintain the robot at its upright equilibrium point, similarly as an inverted pendulum. Maybe, this is an uncommon

Figure 4.7: Photography of the two-degree-of-freedom robot

application of a networked control, but the choice of this plant lets us face interesting control problems, such as uncertainties, nonlinearities (friction), saturation, etc. Furthermore, note that this is a system with fast dynamics, so the presence of delays and dropouts may deeply affect the performance.

The robot configuration is schematically shown in Figure 4.8. The robot has two aluminum joints in an open-chain arrangement in the vertical plane. The first link (which is between both motors) will be termed as *shoulder* whereas the second link (which is between the smaller motor and the edge of the robot) is the *elbow*. Both links are actuated and driven by *Kollmorgen* motors.

This robotic manipulator can be modelled by the following dynamic equation:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + F_f(\dot{q}) = \tau, \tag{4.49}$$

where $\tau(t) \in \mathbb{R}^2$ is the vector of control torque, $M(q) \in \mathbb{R}^{2\times 2}$ is the inertia matrix, $C(q,\dot{q})\dot{q} \in \mathbb{R}^2$ is the vector of Coriolis and centripetal torques, $G(q) \in \mathbb{R}^2$ is the gravitational term and $F_f(\dot{q}) \in \mathbb{R}^2$ is the friction term. In addition, $q = (q_1\ q_2)^T \in \mathbb{R}^2$ is the vector of joint variables. The control torque is applied by means of a voltage signal between -10 and 10 V.

It is well known that the dynamics of a robotic manipulator is extremely nonlinear. In order to apply the results of this chapter, the robot will be operated around the unstable upright equilibrium defined by $q_e = [\pi\ 0], \dot{q}_e = [0\ 0]$. To linearize the system and obtain matrices $A$ and $B$ of equation (4.36), a mean square iterative

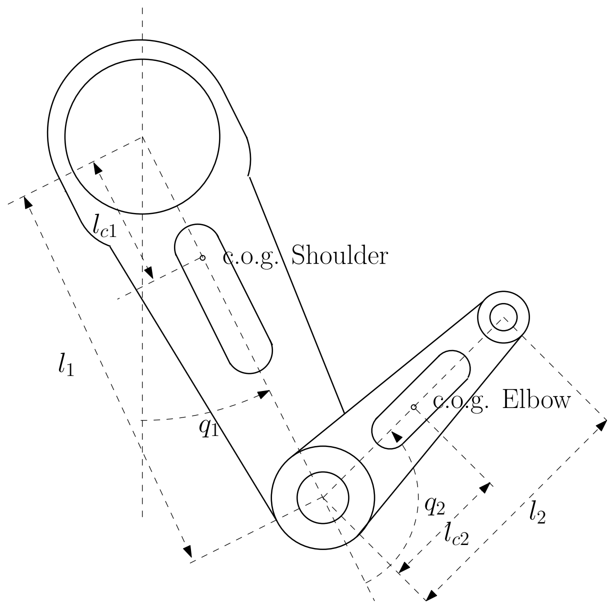



Figure 4.8: Two-degree-of-freedom robot diagram

identification procedure has been followed.

An $H_\infty$ controller is synthesized by means of Theorem 4.5. Using a robust controller, the robot will be stabilized despite the nonlinearities and uncertainties. Moreover, the state has been augmented to include the integral of the position errors. Therefore, the complete state of the system is

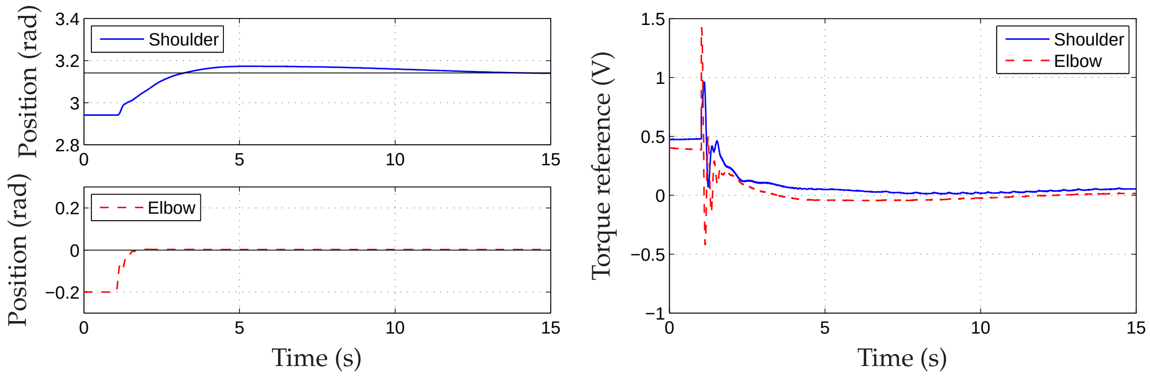
$$x(t) = \begin{bmatrix} \int_0^t (q^T(s) - q_e^T) ds \\ q^T(t) - q_e^T \\ \dot{q}^T(t) - \dot{q}_e^T \end{bmatrix}.$$

The interested reader may find the application of an $H_2/H_\infty$ controller for this manipulator in [147].

In order to perform the tests an initial control based on feedback linearization is applied, which steers the robot from its stable downward position (both links stopped in their lower positions) to the surroundings of its unstable upright equilibrium. This controller is applied on the *local* Target PC, in other words, without using the network. Once that position is reached, the linear networked controller is switched on.

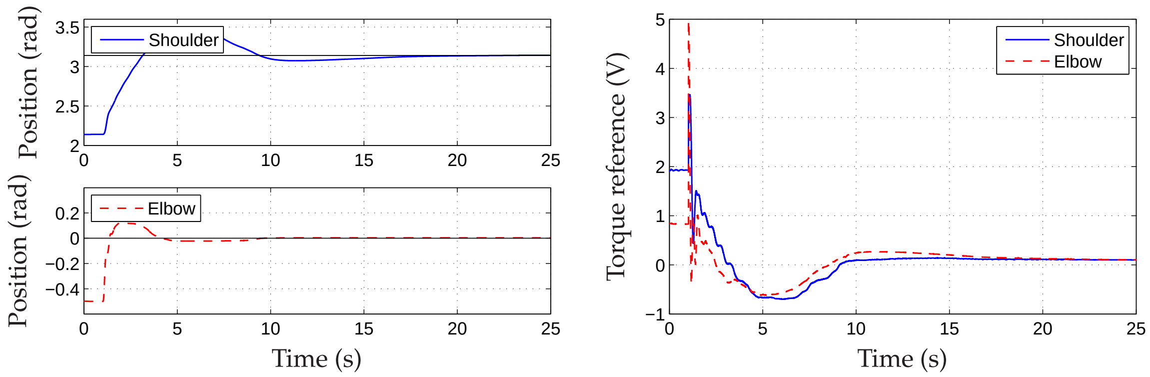
#### 4.4.3.3 Experiments

The first set of experiments is performed without extra delays or dropouts. Only the natural delays of the Ethernet network affect the communication.




(a) Position (b) Torque reference

Figure 4.9: Experiment without extra delays



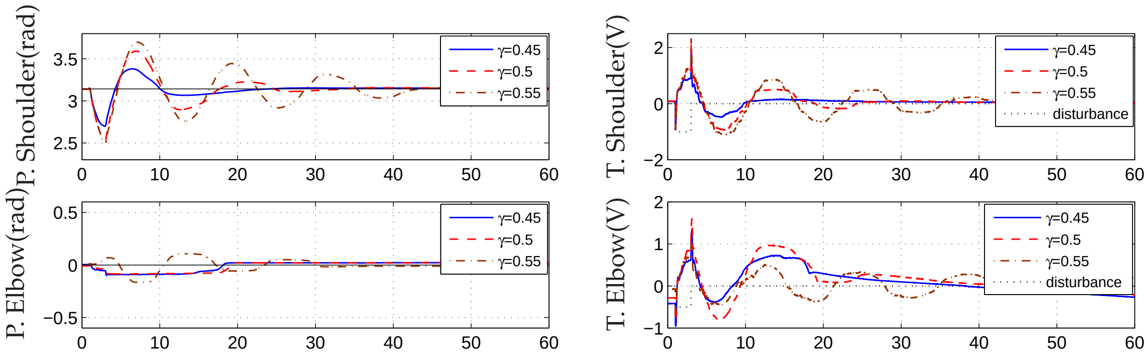

(a) Position (b) Torque reference

Figure 4.10: Reaching the upright equilibrium from a further position

Figure 4.9(a) depicts the evolution of the position when the robot tries to reach the upright equilibrium from an initial state close to that point. Both links attain the final position in spite of the friction and delays. The control signal applied is plotted in Figure 4.9(b). It is worth mentioning that, when the equilibrium state is reached, the control signal is different than zero. This effect is due to the static friction. A small torque is needed to cancel it.

The linearization of the robot dynamics is less reliable when the robot is moving far from the equilibrium point. In the next experiment, the robot is requested to reach the upright position starting from a further position. However, Figure 4.10(a) shows that the same controller still stabilizes the system, at a cost of a more aggressive torque (see Figure 4.10(b)). Again, the effect of static friction appears when the robot is close to the steady state.

The following experiments are quite different form previous ones. Once we know that the robust controller stabilizes the system, we want to explore its charac-



(a) Position (b) Torque reference

Figure 4.11: Disturbance rejection capabilities

teristics and performance. More precisely, the effect of the disturbance attenuation level is studied. The controlled output is chosen as

$$z_\infty(t) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} x(t),$$

in such a way that only the positions of the two links are considered. Three different controllers have been designed for three different values of $\gamma$, keeping the same conditions of the network. In order to test the controllers, the following experiment is executed. Once the robot reaches the upright equilibrium point, a disturbance is added to the torque. The disturbance remains constant for two seconds, and then disappears.

Figure 4.11(a) depicts the evolution of the positions for the three configurations. Great differences can be seen between them. The disturbance is rejected faster, and with a softer response, as the value of $\gamma$ decreases. The experiments for $\gamma \geq 0.6$ show that the controller fails to stabilize the perturbed system.

Notice the presence of the steady-state error, which is more important in the elbow, as this joint is more affected by the static friction. Although it has not yet been cancelled, the integral effect is growing as the increasing torque shows at the end of Figure 4.11(b). At some time instant in the future, the torque will eventually grow to overcome the friction. This effect is negligible for the shoulder.

Finally, the controller is tested for worse conditions of the network. The Ethernet network introduces a round-trip delay between 9 and 15 milliseconds. Hence, the previous controller has been synthesized for $\tau_m = 0.009$ and $\tau_M = 0.015$ seconds. However, as it has been shown throughout the thesis, Lyapunov-Krasovskii controllers suffer from excessive conservatism and, probably, they are able to stabilize the system with bigger delays.

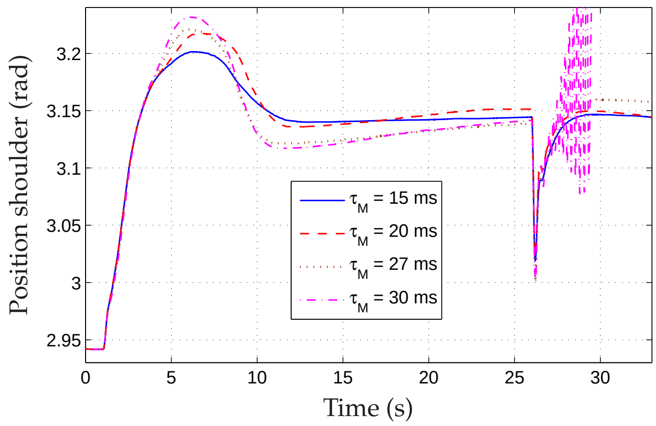



Figure 4.12: Effects of the delay in the position of the shoulder

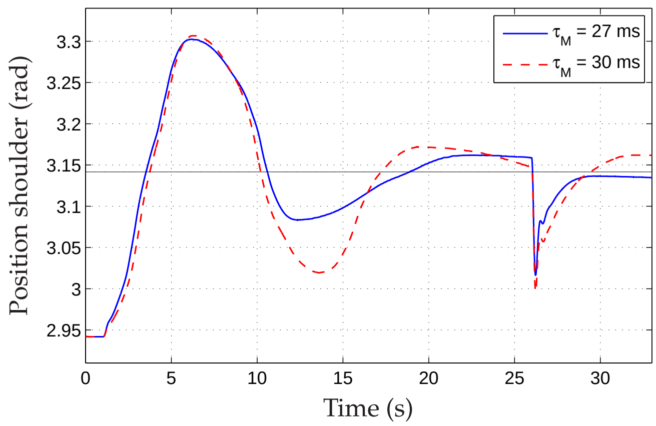



Figure 4.13: Position of the shoulder when additional delay is introduced

The experiment planning is similar to previous ones. First, the robot reaches a position close to the equilibrium (by means of a nonlinear local controller). After that, the linear controller is switched on. Finally, when the arm is at the equilibrium, a small step disturbance is applied to the torque. Figure 4.12 depicts the results of the whole experiment. It turns out that the controller stabilizes the system for poorer QoS than the ones for which it has been designed for. For $\tau_M \geq 0.03$ seconds the controller is unable to control the perturbed system.

Another controller has been designed for $\tau_m = 0.009$ and $\tau_M = 0.030$ seconds. The response for the same experiment is depicted in Figure 4.13. Although the transient seems worse (slower and with more oscillations), it achieves the stabilization of the system around the unstable equilibrium point for higher bounds of the delay.

## 4.5 Chapter summary

This chapter can be seen as the natural extension of the previous one. After studying its stability, the next step consists in controlling a time-delay system. Using the Lyapunov-Krasovsii theory presented before, this chapter proposes a solution for the so called *Mixed* $H_2/H_\infty$ *control problem* in Lemma 4.1. The objective is to synthesize a controller that reduces the upper bound of the $H_2$ cost index, given a fixed bound on the $H_\infty$ part.

The importance of this lemma is twofold. First, it proposes a general solution, applicable to different time-delay systems and different choices of the funcional, that does not require any information about the initial condition, as happens in other works in the literature. And, what is more important and impressive, its solution is always closer to the optimum with respect to those obtained by other methods. This fact has been theoretically proved in Theorem 4.4.

The second part of the chapter is dedicated to networked control systems. The connexion between TDS and NCS is clear, as Section 4.4.1 shows. Using the *input delays approach*, a NCS can be described as a particular TDS, where the effects of delays, packet dropouts and sampling are merged into a common artificial delay. The assumptions and conditions that must be imposed for this delay are similar to the ones for TDS.

Therefore, the general result of Lemma 4.1 can be inherited for NCS, as it has been done in Section 4.4.2. Finally, the performance and tuning capabilities of the controller has been tested in an experimental platform consisting of a two-degree-of-freedom robotic arm.

# Chapter 5

# Model-based networked control systems

## 5.1 Introduction

State-feedback control for NCS has been studied in the previous section. This chapter takes a step forward and proposes the use of the so-called model-based controllers in networked control systems. In general, model-based controllers obtain better performance compared to classical designs, as they introduce information about the plant to compute a new control action (see Figure 5.1).

Model-based techniques have been widely used in classical control. For instance, the use of models makes possible the design of auto-tuning PID controllers [9]. Smith predictors [229] or model predictive control [25] are other classical examples. When the model of the plant is accurate enough, all these approaches achieve remarkable results.

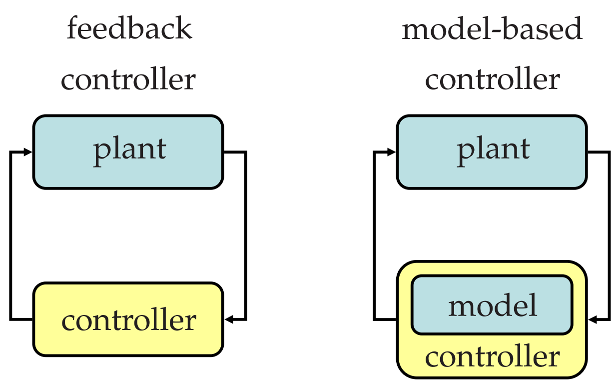



Figure 5.1: Feedback controller vs. Model-based controller

In the context of networked systems, this idea has not been overlooked. Authors have leveraged the model with two different objectives. First, they looked for the foreseeable increase in the control system performance. Second, a reduction of the traffic through the network could be obtained, given that the model allows to make predictions on the actual evolution of the plant. This chapter is mainly focused on the second point, that is, to attain a better exploitation of the available bandwidth.

It is possible to classify the different solutions in the literature according to the way that the samples are sent through the network. Thus, aperiodic approaches emerge in contrast to periodic ones, in which the different elements (plant and controller) deliver information at a given fixed rate.

The theoretical background is obviously vaster for the periodic solutions. For instance, Montestruque et al. [160, 161] have studied the reduction of the network traffic by incorporating a model at the controller end. Nevertheless, their scenario assumes the controller to have direct access to the plant actuators, that is, communication problems are not present in the controller-actuator path. Moreover, Naghshtabrizi et al. [166] consider a more realistic scenario taking into account the effects of delays and dropouts occurring within the communication network. However, they assume an error-free plant model.

In this line, this chapter proposes a novel method for reducing the data being sent through the network, but still ensuring the stability of the closed-loop system. Compared with the aforementioned papers, this work considers the presence of a network in both paths of the communication, sensor to controller and controller to actuator. One of the major issues of most of these approaches is that they do not deal with parametric uncertainties in the model of the plant. In this chapter, techniques based on the stability of the interval matrices [215] are employed aiming at increasing the practical applicability of the method. Moreover, as the identification error grows, the model-based predictions at the controller side could be counterproductive for the system stability. Therefore, this proposal explores the limits of uncertainties that the proposed model-based controller can support before becoming counterproductive.

On the other hand, the aperiodic solution consists in controlling the plant while minimizing the access to the network by using a variable sampling rate in which measurements are only sent when they are indeed necessary. Minimizing the network load is critical in large-scale systems in which the amount of data transmitted may be very large. Instead of using a constant sampling period, network access is scheduled and used only when necessary.

Two different approaches to the problem of scheduling the transmissions can be found in the literature: event-based and self-triggered control. Under the former, the controller execution is triggered according to the state or output of the plant, which requires a continuous monitoring of it [8, 40, 85, 133, 234]. This drawback does not appear in the latter. Self-triggered systems try to emulate the event-based ideas avoiding a continuous sampling of the state and, hence, the implementation problems it carries [5, 6, 40, 142]. Model-based predictions are essential to achieve a notable reduction of the traffic in self-triggered solutions.

It is worth mentioning the difference existing between these approaches and other control schemes in the context of robust stability of NCS subject to time-varying sampling instants in which, although the intervals between sampling times are also time-varying, there is no freedom of choice for these [66, 67, 232].

In this chapter, the problem of reducing the use of a bandwidth-limited channel is tackled in a different way. A scenario with a communication network in the sensor-to-controller path is considered. The system is a linear time-invariant plant, subject to bounded additive disturbances. Starting from the knowledge of a stabilizing feedback controller and an associated Lyapunov function, a model-based controller predicts the system state in open loop between two consecutive samples. The sampling times are chosen by the controller, in such a way that practical stability is guaranteed while reducing the access by maximizing the time between successive samples. In order to decide the sampling times, the controller solves on-line quadratic optimization problems (QP). It will be shown that this self-triggered strategy allows to stabilize the system with low data rates.

### Related publications

1. L. Orihuela, F. Gómez-Estern, F. R. Rubio. *Model-based networked control systems under parametric uncertainties*. 18th IEEE International Conference on Control Applications. Saint Petersburg, Russia. pp:7-12. 2009. [194]

2. P. Millán, L. Orihuela, D. Muñoz de la Peña, C. Vivas, F. R. Rubio. *Self-triggered sampling selection based on quadratic programming*. 18th IFAC World Congress. Milano, Italy. pp:8896-8901, 2011. [151]

## 5.2 Problem statement

Chapter 4 dealt with the problem of designing stabilizing controllers for systems controlled over communication networks. The presence of such networks introduce

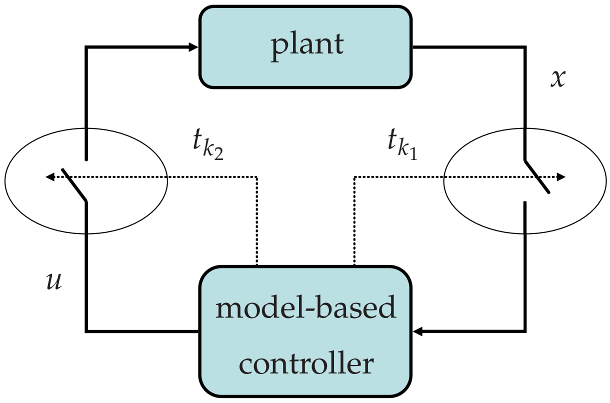



Figure 5.2: The model-based controller decides, in general, both sampling times

undesired effects, as delays and packet dropouts, that were taken into account in the controller synthesis. That problem can be catalogued in the field of control over networks, as it was defined in Chapter 2. This chapter, however, tackles a different problem. Given a pre-designed controller, the question to answer is: is it possible to reduce the network traffic preserving system stability? It is of undeniable interest to reduce the load in a shared medium, since the congestion is a source of problems such as delays and dropouts.

Consider the scheme depicted in Figure 5.2, where the plant is assumed to be a LTI system possibly affected by external disturbances:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t). \tag{5.1}$$

The sensor sends samples of the plant state to the controller at discrete time instants $t_{k_1}$. When this information is received at the model-based controller, the state of the model is updated. The controller is defined by:

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c u(t), \quad \forall t \in [t_{k_1}, t_{k_1+1}) && (5.2) \\ x_c(t_{k_1}) &= x(t_{k_1}), \quad k_1 \in \mathbb{N}, && (5.3) \end{aligned}$$

where the control signal is a piecewise continuous signal depending on the state of the model that is updated at discrete instants $t_{k_2}$:

$$u(t) = Kx_c(t_{k_2}), \quad \forall t \in [t_{k_2}, t_{k_2+1}). \tag{5.4}$$

The objective is to choose adequate sampling times $t_{k_1}$ and $t_{k_2}$ such that the stability of the system is preserved while reducing the traffic over the network. Two solutions are proposed: periodic and aperiodic sampling policies. The next sections study the details of both approaches.

## 5.3 Periodic solutions

Compared with asynchronous sampling policies, periodic implementations are simpler, since continuous systems can be easily described as discrete ones. Since the theory of discrete systems is mature enough, many properties can be directly inherited. Nevertheless, some problems must be carefully tackled, because network-induced problems complicate the theory.

In this section, the following pair plant-model is considered:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5.5}$$

$$\dot{x_c}(t) = A_c x_c(t) + B_c u(t), \tag{5.6}$$

where the model and the control signal are updated following (5.3) and (5.4) respectively. It is assumed that the system is not affected by disturbances, i.e. $w(t) \equiv 0$. However, uncertainties are present since the pairs $(A, B)$ and $(A_c, B_c)$ are different.

The sensor sends the state every $h_1$ seconds, i.e., the update times verify $t_{k_1+1} - t_{k_1} = h_1$, $\forall k_1$. Let $\delta(t)$[1] denote the error between the state of the plant and the state of the model, that is, $\delta(t) \triangleq x(t) - x_c(t)$. At update times $t_{k_1}$, it turns out that $\delta(t_{k_1}) = 0$, $\forall k_1 \in \mathbb{N}$.

Because of the network connecting the controller and the actuator, the samples of the control signal shall arrive to the plant every $h_2$ seconds. That is, the update times satisfy $t_{k_2+1} - t_{k_2} = h_2$, $\forall k_2 \in \mathbb{N}$. Both sampling periods are related, as the following assumption states.

**Assumption 5.1.** Sampling period $h_1$ is a multiple of $h_2$, i.e., $h_1/h_2 = N \in \mathbb{N}$.

The actuator function is to apply the control signal arriving at instant $t_{k_2}$ during the whole interval $t \in [t_{k_2}, t_{k_2+1})$.

Let $\xi(t) = \left[x^T(t) \;\; \delta^T(t)\right]^T$ denote the augmented state vector . Without loss of generality, the equilibrium point is defined by $\xi_e \triangleq [\bar{0} \;\; \bar{0}]^T$.

Under this paradigm, the following section studies the dynamics of the overall system and provides sufficient conditions to ensure global exponential stability around the equilibrium point. The evaluation of the eigenvalues of a test matrix will allow the selection of the update times $h_1, h_2$ to maintain the stability of the system.

[1]Note that a different notation has been used in this chapter for the model state $x_c$ and the error $\delta$, in contrast with $\hat{x}$ and $e$ used in Chapter 2. The reason is that the objective of the model-based controller is to make predictions about the plant state, instead of computing a estimation of it.

### 5.3.1 System dynamics and stability conditions

From the starting point detailed above, it can be seen that the dynamics of the overall system for $t \in [t_{k_2+i}, t_{k_2+i+1})$ can be described by

$$\dot{\xi}(t) = \Lambda\xi(t) + \Upsilon\xi(t_{k_2}),\ \forall t \in [t_{k_2+i}, t_{k_2+i+1}), \tag{5.7}$$

where $\Lambda$ is of the form

$$\begin{aligned}\Lambda &= \begin{bmatrix} A & 0 \\ \tilde{A} & A_c \end{bmatrix}, \\ \Upsilon &= \begin{bmatrix} BK & -BK \\ \tilde{B}K & -\tilde{B}K \end{bmatrix},\end{aligned}$$

with $\tilde{A} \triangleq A - A_c$ and $\tilde{B} \triangleq B - B_c$. At update times $t_{k_1}$, the augmented state verifies

$$\begin{aligned}\xi(t_{k_1}) &= \begin{bmatrix} x(t_{k_1}^-) \\ 0 \end{bmatrix}, \\ \xi(t_{k_2}) &= \begin{bmatrix} x(t_{k_2}^-) \\ \delta(t_{k_2}^-) \end{bmatrix} = \xi(t_{k_2}^-),\end{aligned}$$

where the notation $t^-$ indicates the time instant just before $t$.

The following proposition states the evolution of $\xi(t)$ for a generic interval.

**Proposition 5.1.** *Assuming a nonsingular $\Lambda$, the system described by (5.7) with initial condition $\xi_0 \triangleq \xi(t_0) = \begin{bmatrix} x^T(t_0) & 0 \end{bmatrix}^T$ has the following response for $t \in [t_{k_2+i}, t_{k_2+i+1})$:*

$$\xi(t) = \left(e^{\Lambda(t-t_{k_2+i})} - \Lambda^{-1}\left[I - e^{\Lambda(t-t_{k_2+i})}\right]\Upsilon\right)\alpha^i I_{s1}(\alpha^N I_{s1})^{k_1}\xi_0, \tag{5.8}$$

*for $0 \le i \le N-1$, with $t_{k_1} < t_{k_2+i} \le t_{k_1+1}$, where*

$$\begin{aligned}I_{s1} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \\ \alpha &= e^{\Lambda h_2} - \varphi\Upsilon, \\ \varphi &= \Lambda^{-1}[I - e^{\Lambda h_2}].\end{aligned}$$

**Proof.** The proof is divided in two parts. First, starting from an update instant $t_{k_1}$ the evolution of the system is found. Giving some steps back in time, the evolution is written with respect to the initial state. Second, the evolution for a generic interval between two consecutive update times $t_{k_2+i}$ and $t_{k_2+i+1}$ is given.

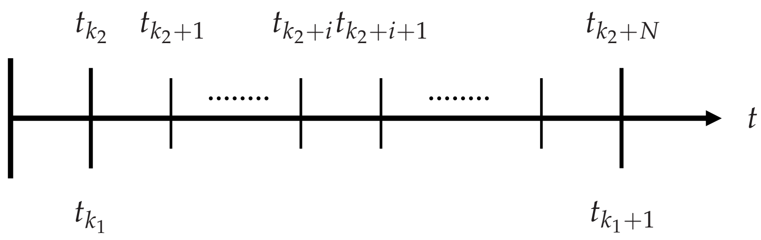



Figure 5.3: Sampling times in both links

Consider an update time $t_{k_1}$. As Figure 5.3 shows and considering Assumption 5.1, it holds $t_{k_2} \equiv t_{k_1}$. In the interval $t \in [t_{k_2}, t_{k_2+1}) \equiv [t_{k_1}, t_{k_2+1})$ the system response is

$$\xi(t) = e^{\Lambda(t-t_{k_2})}\xi(t_{k_1}) - \Lambda^{-1}[I - e^{\Lambda(t-t_{k_1})}]\Upsilon\xi(t_{k_2}). \tag{5.9}$$

The state error becomes zero at the instant $t_{k_1}$. This can be represented by $\xi(t_{k_1}) = I_{s1}\xi(t_{k_1}^-)$. Furthermore, it is verified that $\xi(t_{k_2}) = \xi(t_{k_1}) = I_{s1}\xi(t_{k_1}^-)$. Substituting in (5.9) it yields

$$\begin{aligned} \xi(t) &= \left(e^{\Lambda(t-t_{k_1})} - \Lambda^{-1}[I - e^{\Lambda(t-t_{k_1})}]\Upsilon\right) I_{s1}\xi(t_{k_1}^-) \\ &= F\xi(t_{k_1}^-),\ t \in [t_{k_2}, t_{k_2+1}). \end{aligned}$$

The evolution given in (5.9) still holds for a generic interval $t \in [t_{k_2+i}, t_{k_2+i+1})$. Therefore, if one uses (5.9) it is possible to find $\xi(t_{k_1}^-)$:

$$\begin{aligned} \xi(t_{k_1}^-) &= e^{\Lambda h_2}\xi(t_{k_2-1}) - \Lambda^{-1}[I - e^{\Lambda h_2}]\Upsilon\xi(t_{k_2-1}) \\ &= e^{\Lambda h_2}\xi(t_{k_2-1}) - \varphi\Upsilon\xi(t_{k_2-1}). \end{aligned}$$

At instant $t_{k_2-1}$ the augmented vector does not experiment any change. This can be represented by $\xi(t_{k_2-1}) = \xi(t_{k_2-1}^-)$. Hence,

$$\xi(t_{k_1}^-) = [e^{\Lambda h_2} - \varphi\Upsilon]\xi(t_{k_2-1}^-) = \alpha\xi(t_{k_2-1}^-).$$

Repeating this procedure $N$ times it is obtained

$$\xi(t) = F\alpha^N I_{s1}\xi(t_{k_1-1}^-),\ t \in [t_{k_2}, t_{k_2+1}),$$

since $t_{k_2-N} = t_{k_1-1}$. Give now $k_1$ steps back to $t_0$, so that

$$\xi(t) = F(\alpha^N I_{s1})^{k_1}\xi_0,\ t \in [t_{k_2}, t_{k_2+1}).$$

Now, the evolution of the system depends on the initial state. Finally, by moving $i$ steps forward to the generic interval $t \in [t_{k_2+i}, t_{k_2+i+1})$,

$$\xi(t) = \left(e^{\Lambda(t-t_{k_2+i})} - \Lambda^{-1}[I - e^{\Lambda(t-t_{k_2+i})}]\Upsilon\right)\alpha^i I_{s1}(\alpha^N I_{s1})^{k_1}\xi_0,\ t \in [t_{k_2+i}, t_{k_2+i+1}),$$

for $0 \leq i \leq N-1$. This proves the proposition. $\square$

It is worth noting that, without the terms in brackets, equation (5.8) is equivalent to the evolution of a discrete system with sampling time $h_2$ that jumps each $N$ instants. This fact inspires the following theorem, which gives sufficient conditions to ensure the global exponential stability of the equilibrium point $\xi_e = [0 \;\; 0]^T$.

**Theorem 5.1.** *The equilibrium point $\xi_e$ of the system described by (5.7) is globally exponentially stable if the eigenvalues of $\alpha^N I_{s1}$ lie inside the unit circle, where $\alpha$ and $I_{s1}$ where defined in Proposition 5.1.*

**Proof.** The proof is similar to that in [160], but with differences to adapt it to the case in which the network also mediates the connection of controller and actuator. Take the norm of the response given in Proposition 5.1, that is

$$\|\xi(t)\| = \|\{e^{\Lambda(t-t_{k_2+i})} - \Lambda^{-1}[I - e^{\Lambda(t-t_{k_2+i})}]\Upsilon\}\alpha^i I_{s1}(\alpha^N I_{s1})^{k_1}\xi_0\|, \tag{5.10}$$

which can be bounded as

$$\|\xi(t)\| \leq \|\{e^{\Lambda(t-t_{k_2+i})} - \Lambda^{-1}[I - e^{\Lambda(t-t_{k_2+i})}]\Upsilon\}\alpha^i I_{s1}\| \|(\alpha^N I_{s1})^{k_1}\| \|\xi_0\|. \tag{5.11}$$

The triangle inequality is used to analyze the first term on the right-hand side of (5.11):

$$\begin{aligned} \|\cdot\| &\leq \left\|\left(e^{\Lambda(t-t_{k_2+i})} - \Lambda^{-1}[I - e^{\Lambda(t-t_{k_2+i})}]\Upsilon\right)\right\| \left\|\alpha^i\right\| \|I_{s1}\| \\ &\leq \left\|\left(e^{\Lambda(t-t_{k_2+i})} - \Lambda^{-1}[I - e^{\Lambda(t-t_{k_2+i})}]\Upsilon\right)\right\| K_a, \end{aligned}$$

where $K_a \triangleq \|\alpha^i\| \|I_{s1}\|$. Now,

$$\begin{aligned} \|\cdot\| &\leq \left(\left\|e^{\Lambda(t-t_{k_2+i})}\right\| + \left\|\Lambda^{-1}[I - e^{\Lambda(t-t_{k_2+i})}]\Upsilon\right\|\right) K_a \\ &\leq \left(\left\|e^{\Lambda(t-t_{k_2+i})}\right\| + \left\|\Lambda^{-1}\right\| \left\|[I - e^{\Lambda(t-t_{k_2+i})}]\right\| \|\Upsilon\|\right) K_a \\ &\leq \left(e^{\bar{\sigma}(\Lambda)h_2} + \left\|\Lambda^{-1}\right\| (1 + e^{\bar{\sigma}(\Lambda)h_2}) \|\Upsilon\|\right) K_a \triangleq K_b, \end{aligned}$$

being $\bar{\sigma}(\Lambda)$ the largest singular value of $\Lambda$.

On the other hand, the second term on the right-hand side of (5.11) $\|(\alpha^N I_{s1})^{k_1}\|$ is bounded if the eigenvalues of $\alpha^N I_{s1}$ lie inside the unit circle. Then, $\|(\alpha^N I_{s1})^{k_1}\| \leq K_c e^{-\alpha_1 k_1}$, with $K_c, \alpha_1 > 0$. Observe that $t_{k_1+1} = (k_1+1)h_1 > t$, hence

$$K_c e^{-\alpha_1 k_1} < K_c e^{-\alpha_1(\frac{t}{h_1}-1)} = K_c e^{\alpha_1} e^{-\frac{\alpha_1}{h_1}t} = K_d e^{-\alpha t},$$

with $K_d, \alpha > 0$. Therefore $\|\xi(t)\| \leq K_b K_d e^{-\alpha t}\|\xi_0\|$, so $\xi(t)$ approaches $\xi_e$ exponentially. $\square$

### 5.3.2 Extension to uncertain systems

In the previous section, it is shown that the system is exponentially stable whether the eigenvalues of a test matrix $\alpha^N I_{s1}$ are inside the unit circle. If the test matrix is completely known, the eigenvalues can be easily found. However, a perfect knowledge of the test matrix requires a perfect system identification. This reasoning leads to a question: if the dynamics of the plant is perfectly known, why to choose a wrong model? That is, why are the pairs $(A, B)$ and $(A_c, B_c)$ different? Obviously, it would make no sense to choose a wrong model if the actual dynamics of the plant were known.

In most circumstances, some dynamics of the system cannot be perfectly modelled, appearing undesired uncertainties. In these cases, it is not possible to find the eigenvalues of a matrix whose parameters are unknown. Aiming at solving this drawback, this section presents a method to deal with uncertainties in the model.

The elements of the test matrix $\alpha^N I_{s1}$ are a nonlinear function of the uncertainties, that is,

$$\alpha^N I_{s1} \triangleq M_T = \{[m_{ij}] : m_{ij} = f(a_{ij}, b_{ij})\}, \tag{5.12}$$

with $A = \{[a_{ij}]\}, B = \{[b_{ij}]\}$.

The relation between the parameters of matrices $A, B$ with the elements of $M_T$ depends on the structure of the matrix test. In general, it is a complex relation. This problem has not been yet tackled in the literature. To solve it, this section resorts to the theory of interval systems or integral matrices [31, 139, 140, 215, 249].

Consider a discrete-time system described by

$$x(k+1) = \Phi x(k), \tag{5.13}$$

where $\Phi = \{[\phi_{ij}] : \phi_{ij}^{min} \leq \phi_{ij} \leq \phi_{ij}^{max}\}$. If the stability of system (5.13) is ensured, the eigenvalues of $\Phi$ will be inside the unit circle. Therefore, the aim of this section is to study the stability of systems as (5.13) for the whole range of uncertainties.

The results based on the Kharitonov theorem cannot be used as the stability of interval matrices is more complex than that of interval polynomials [215]. The first approximations to this problem provide sufficient conditions that needed large computation times [31]. Recently, necessary and sufficient conditions have been proposed in [139, 140]. The stability tests are based on the solution of linear matrix inequalities, so efficient interior point algorithms can be used.

The following proposition states that the application to uncertain systems can be casted as a problem of stability of interval systems.

**Proposition 5.2.** *Let* $A = \{[a_{ij}] : a_{ij}^{min} \leq a_{ij} \leq a_{ij}^{max}\}$, $B = \{[b_{ij}] : b_{ij}^{min} \leq b_{ij} \leq b_{ij}^{max}\}$ *be a pair of interval matrices including the uncertainties of the model. Then, if the interval matrix* $M_T$ *is stable, system* (5.5) *is stable using the model-based controller* (5.6) *for all the range of parametric uncertainties.*

The proof is immediate. Assuming that upper and lower bounds of the different elements $[m_{ij}]$ of $M_T$ are known for the variation range of uncertainties, the results of interval matrices in [140] can be applied to check the stability (and hence the eigenvalues) of matrix $M_T$.

This method produces conservative results, because some unfeasible combinations of uncertainties of some of the parameters may be being considered in the criterion.

### 5.3.3 Limitations of parametric uncertainties

The scheme proposed here exploits a model at the controller end in order to reduce the data rate. As the model better approximates the actual plant, better dynamical performance and larger reduction in the network traffic may be achieved.

In case the model significantly differs from the actual plant, the effect of using it at the controller end can become counterproductive. When this difference is large, the control signal obtained could even destabilize the system.

In what follows, an algorithm is proposed for finding the maximum parametric uncertainty that a model can withstand without becoming an option worse than a simple state-feedback controller. Given range of uncertainties, if a larger packet rate is needed to stabilize the system with the model-based than with the state-feedback controller, then the model is said not to tolerate the uncertainties.

Consider a continuous plant described by (5.5), where $A \in [\underline{A}, \overline{A}]$ and $B \in [\underline{B}, \overline{B}]$. Consider both control schemes:

**State-feedback controller:** The input is given by $u = Kx(t_k)$, with $t_{k+1} - t_k = h$, being $h$ the sampling time. Let $R_{SF}$ denote the minimum rate required to stabilize the system. That is, for any rate $R \geq R_{SF}$ the system with state-feedback control is stable. The relation between the rate $R$ and the sampling period $h$ depends on the networking scheme:

- *Network in both paths:* Two packets are needed each $h$ seconds, so $R = \frac{2}{h}$ packets/s.
- *Network in sensor to controller path:* One packets is needed each $h$ seconds, so $R = \frac{1}{h}$ packets/s.

**Model-based controller:** The estimated state evolves according to (5.6), where $A_c = \frac{\underline{A}+\overline{A}}{2}$ and $B_c = \frac{\underline{B}+\overline{B}}{2}$. That is, the model is chosen in the middle of the uncertainty range. Let $R_{MB}$ denote the minimum rate required to stabilize the system, so that for every rate $R \geq R_{MB}$ the system with a model-based controller is stable. The relation between the rate and the sampling period is:

- *Network in both paths:* $N+1$ packets are needed each $Nh_2$ seconds, so $R = \frac{N+1}{Nh_2}$ packets/s.
- *Network in sensor to controller path:* One packets is needed each $h_2$ seconds, so $R = \frac{1}{h_2}$ packets/s.

Let $U_r$ denote the uncertainty range defined as $U_r = [\Delta A, \Delta B] = \left[\frac{\underline{A}-\overline{A}}{2}, \frac{\underline{B}-\overline{B}}{2}\right]$. The objective of this section is to find the maximum uncertainty range tolerated by the model-based controller. To do so, the following optimization problem must be solved:

$$\max_{U_r} : \{R_{MB}(U_r) < R_{SF}\}, \tag{5.14}$$

where the notation $R_{MB}(U_r)$ refers to the fact that the minimum rate for the model-based scheme $R_{MB}$ strongly depends on he uncertainty range. The rate $R_{SF}$ also depends on the uncertainties. However, it is assumed to be constant for the optimization problem, hence avoiding the notation $R_{SF}(U_r)$.

In order to solve problem (5.14), the relation of $R_{MB}$ with the uncertainties must be known. However, it is not straightforward to find an analytical expression for any of the networking schemes under consideration. Nonetheless, the results of the previous section can be exploited to solve this optimization problem at least in a relatively conservative way. Using the techniques based on interval matrices, it will be possible to find the maximum uncertainty range defined above.

**Algorithm 5.1.**

1. Choose the sampling periods $h_1, h_2$, and a minimum uncertainty range.
2. Find the interval matrix (5.12).
3. Check the stability of the closed loop system by using the results in [140].
4. If the system is stable, enlarge the uncertainty range and return to step 2. If not, the maximum uncertainty range has been found.

**Remark 5.1.** Finding the interval matrix (5.12) is not a trivial task. In scalar systems, the limits of the interval matrix are in one of the extreme points of the uncertainty range. But this is not inherently true for higher order systems. An approximated method consists in using techniques based on numerical sweeping that, at the cost of greater computational complexity, are easily programmable and are carried out offline.

An application of this algorithm will be shown in the following.

### 5.3.4 Numerical examples

Two examples are given in this section to explore the influence of the uncertainties on the traffic reduction and stability.

**Example 5.1. NCS with parametric uncertainties**

Consider a linear unperturbed plant described by equation (5.5). The pair of matrices is given by

$$A = \begin{bmatrix} 0.05 + a_1 & 0.95 + a_2 \\ -0.2 + a_3 & 0.1 + a_4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $a_1, a_2, a_3, a_4$ are unknown parameters which belong to the interval $[-0.1, 0.1]$. The plant and the model-based controller are connected by means of a network, as Figure 5.2 shows. The model has the structure of (5.2) with

$$A_c = \begin{bmatrix} 0.05 & 0.95 \\ -0.2 & 0.1 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The controller is chosen according to $K = [-1 \quad -2]$.

If one follows the aforementioned procedure, the first step consists of finding the upper and lower bounds for the elements of the test matrix. To do this, a multi-dimensional discretization of the considered region is made. For each point of the discretization mesh, the maximum and the minimum value of the test matrix must be found. The computation complexity is high (exponential), but the calculations are made only once offline.

Two update times has been chosen, $h_2 = 0.6$ or $h_2 = 0.65$ seconds and $N = 2$, yielding

$$M_T(0.6) = \begin{bmatrix} (0.1196 & 0.6315) & (-0.0664 & 0.0456) & 0 & 0 \\ (-1.1046 & -0.7291) & (-0.7342 & -0.6760) & 0 & 0 \\ (-0.5344 & -0.0225) & (-0.4253 & -0.3132) & 0 & 0 \\ (-0.6513 & -0.2758) & (-0.6516 & -0.5933) & 0 & 0 \end{bmatrix},$$

$$M_T(0.65) = \begin{bmatrix} (-0.0157 & 0.5566) & (-0.1720 & -0.0483) & 0 & 0 \\ (-1.1664 & -0.7558) & (-0.8455 & -0.7694) & 0 & 0 \\ (-0.6301 & -0.0578) & (-0.5236 & -0.3998) & 0 & 0 \\ (-0.7224 & -0.3117) & (-0.7384 & -0.6622) & 0 & 0 \end{bmatrix}.$$

This way, it is possible to build two matrices $M_T^{\min}$ and $M_T^{\max}$ with the minimum and maximum values, respectively, of the matrix $M_T$. Let $M_o = 0.5(M_T^{\max} + M_T^{\min})$ and $\Delta M = 0.5(M_T^{\max} - M_T^{\min})$ denote the average and the increment matrices, respectively. From [140], for the system to be stable, the following LMI must be feasible:

$$\begin{bmatrix} -X & XM_o^T & U \\ * & -X + \sum_{i,j=1}^{n} \eta_{ij} \Delta m_{ij}^2 e_i e_i^T & 0 \\ * & * & -V \end{bmatrix} < 0$$

where $e_i$ $(i = 1, \ldots, n)$ are the coordinate vectors and $U, V$ are matrices defined by $U = [Xe_1 \ \ldots \ Xe_n \ \ldots \ Xe_1 \ \ldots \ Xe_n]$ and $V = diag\,\{\eta_{11}, \ldots, \eta_{1n}, \ldots, \eta_{n1}, \ldots, \eta_{nn}\}$.

The decision variables of the LMI are the positive-definite matrix $X$ and the real scalars $\eta_{ij} > 0,\ i, j = 1, 2, \ldots, n$.

With $h_2 = 0.6$ seconds the problem is feasible. However, for $h_2 = 0.65$ seconds the LMI is unfeasible, so the stability cannot be assessed.

Assume now that the actual system is described by

$$\dot{x}(t) = \begin{bmatrix} 0.1 & 0.9 \\ -0.1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

As Figure 5.4 illustrates, the system is stabilized with a sampling time $h_2 = 0.7$ seconds. The conclusion is that the proposed algorithm may be conservative as it ensures the stability for the whole family of systems belonging to the uncertainty range.

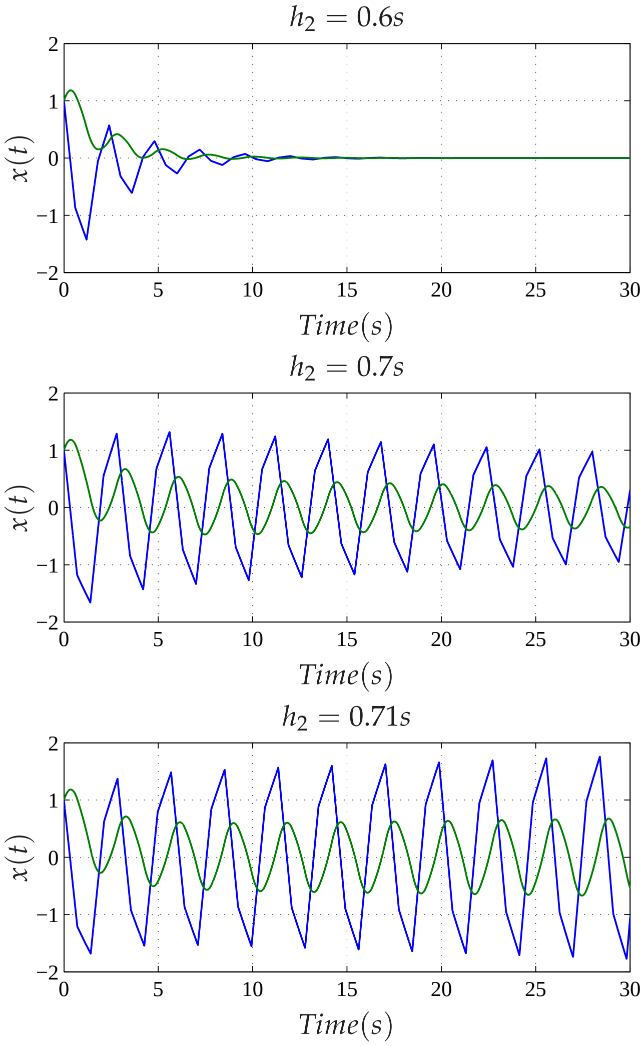



Figure 5.4: Evolution of the states for different values of the sampling rate

**Example 5.2. Limitations of parametric uncertainties**

This example explores the uncertainties that the model tolerates. In order to stabilize the previous nominal plant with a state-feedback controller, the maximum sampling period is $h = 0.991$ seconds.

If the network only mediates the sensor-controller path, the minimum packet rate required is $R = 1/h \approx 1.01$ packets/s. Assume that a model is introduced in the controller-side of the communication. Now, let us express the uncertainties as a percentage of each element of matrix $A$. Table 5.1 lists the maximum uncertainty tolerated for different values of the sampling period.

For the case in which the network is present in both paths, sensor-controller and controller-actuator, the sampling period $h = 0.991$ requires a packet rate of $R = 2/h \approx 2.018$ packets/s. Table 5.2 lists the maximum uncertainty tolerated for

| $h$ ($s$) | R ($packets/s$) | Uncertainty (%) |
|---|---|---|
| 0.991 | 1.01 | 65 |
| 1 | 1 | 65 |
| 3 | 0.33 | 45 |
| 5 | 0.2 | 35 |
| 8 | 0.125 | 30 |
| 10 | 0.1 | 25 |

Table 5.1: Sampling period, rate and admissible uncertainty

| $N$ | $h_2$ ($s$) | R ($packets/s$) | Uncertainty (%) |
|---|---|---|---|
| 2 | 0.743 | 2.018 | 70 |
| 2 | 0.8 | 1.875 | 65 |
| 2 | 0.9 | 1.667 | 65 |
| 2 | 1 | 1.5 | 0 |
| 3 | 0.66 | 2.018 | 60 |
| 3 | 0.7 | 1.905 | 60 |
| 3 | 0.8 | 1.667 | 60 |
| 3 | 0.9 | 1.480 | 55 |
| 4 | 0.62 | 2.018 | 55 |
| 4 | 0.7 | 1.786 | 55 |
| 4 | 0.8 | 1.562 | 50 |
| 4 | 0.9 | 1.389 | 50 |

Table 5.2: $N$, sampling period $h_2$, rate $R$ and admissible uncertainty

different values of $h_2$ and $N$.

For instance, with a 55% of uncertainties the traffic over the network could be reduced by 25% with a model-based controller, choosing $N = 3$ and $h_2 = 0.9$ seconds.

## 5.4 Aperiodic solutions

Section 5.3 has shown that the traffic over the network is reduced if a model-based controller is used. It has been possible to enlarge the sampling periods in both links while preserving the system stability. However, in some situations it might not be necessary to send periodic updates of the state of the system, as the model performs an effective estimation of the actual state of the plant. Then, it could be more interesting to use the network only when it is really required, despite the fact that a lower control performance in terms of rate of convergence or optimality is attained.

In these situations, self-triggered or event-triggered sampling strategies are of interest. These sampling policies are somehow more efficient from the point of view of bandwidth use, as communications are invoked only when significant information requires to be transmitted [47, 133, 142, 234]. The main difference between both approaches is given by the device that triggers the communication events, namely, the transmitter in the event-based case; and the receiver in the self-triggered case (see Figure 5.5).

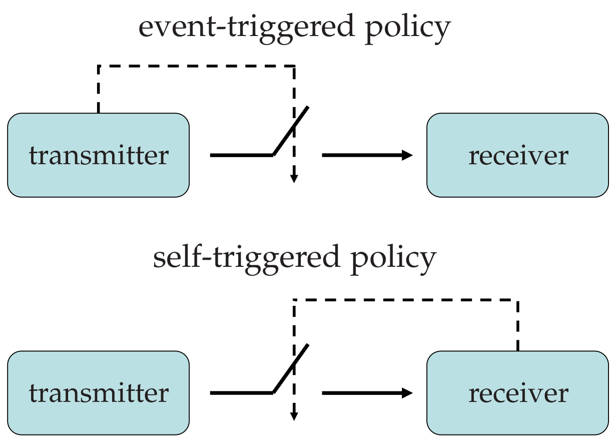



Figure 5.5: Event and self-triggered sampling policies

In the former, the decision of the transmitter is based on the information measured directly from the plant, so event-triggered solutions require a continuous monitoring of the system. Furthermore, the receiver must be 'listening' the channel all time as it does not know the exact transmission time. On the other hand, in self-triggered policies the receiver chooses the next sampling time based on the received information and on a model of the plant dynamics. It informs the transmitter when the following sample is needed. Therefore, both elements could remain asleep between two consecutive transmission. Roughly speaking, self-triggered solutions are able to reduce the energy consumption and the traffic, with an adequate triggering of the sampling instants based on model-based predictions.

In this section, a simplification of the general scheme presented in Figure 5.2 is made, as it is assumed that there exists a direct connection between the controller and the actuator. Additionally, an uncertain perturbed system is considered. The model-based controller chooses the time-instant in which the sensor must send the state of the plant following a self-triggered sampling policy (see Figure 5.6).

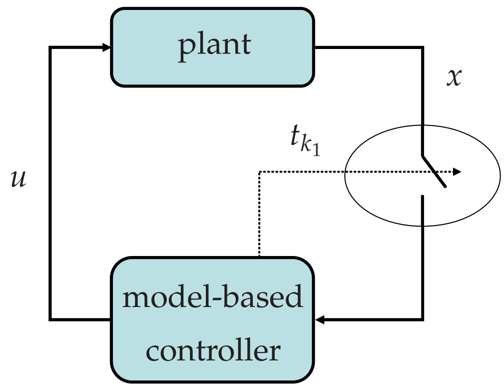



Figure 5.6: Model-based controller with self-triggered sampling policy

The plant and the model are given by (5.1) and (5.2) with $A_c \triangleq A$ and $B_c \triangleq B$:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + w(t), & (5.15)\\ \dot{x}_c(t) &= Ax_c(t) + Bu(t), \quad \forall t \in [t_{k_1}, t_{k_1+1}), & (5.16)\\ u(t) &= Kx_c(t), & (5.17)\\ x_c(t_k) &= x(t_{k_1}), \quad k_1 \in \mathbb{N}. & (5.18) \end{aligned}$$

Vector $w(t)$ is a disturbance process belonging to the set $\mathcal{W}$, which is defined by

$$\mathcal{W} = \{w : \|w(t)\|_\infty \leq \gamma,\ \gamma > 0\}. \tag{5.19}$$

It is assumed that a feedback local controller $K$, associated with a continuous Lyapunov function $V(t) = x^T(t)Px(t)$ has been designed for system (5.15) so that the control law $u_{SF}(t) = Kx(t)$ ensures the practical stability of the closed-loop system.

Based on all these considerations, the following section proposes a Lyapunov-based method to manage the sampling policy.

### 5.4.1 Lyapunov-based sampling policy

This section describes the proposed procedure to minimize the access to the network while preserving the closed-loop practical stability.

As in the periodic case, the model error is defined by $\delta(t) \triangleq x(t) - x_c(t)$, where $\delta(t_{k_1}) = 0,\ \forall k_1$. The dynamics of $\delta(t)$ between two consecutive sampling times is given by

$$\dot{\delta}(t) = A\delta(t) + w(t), \quad \forall t \in [t_{k_1}, t_{k_1+1}). \tag{5.20}$$

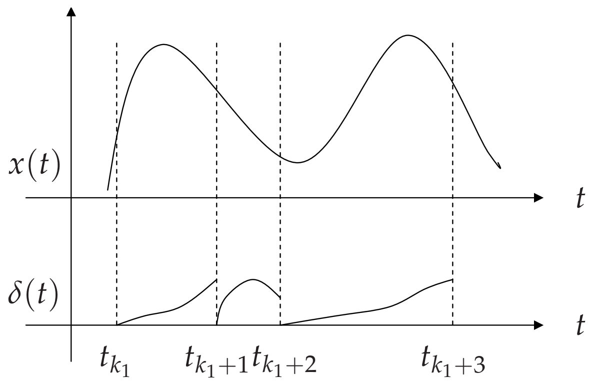



Figure 5.7: Possible evolution of the state and the model error

A possible evolution of the state of the system and the error is depicted in Figure 5.7. Given equations (5.16) and (5.20), it is possible to compute the values of $x_c(t)$ and $\delta(t)$ for a given instant $t \in [t_{k_1}, t_{k_1+1})$:

$$x_c(t) = e^{(A+BK)(t-t_{k_1})}x_c(t_{k_1}), \quad \forall t \in [t_{k_1}, t_{k_1+1}), \tag{5.21}$$

$$\delta(t) = \cancel{e^{A(t-t_{k_1})}\delta(t_{k_1})} + \int_{t_{k_1}}^{t} e^{A(t-\tau)}w(\tau)d\tau, \quad \forall t \in [t_{k_1}, t_{k_1+1}). \tag{5.22}$$

The following proposition establishes a bound on the model error.

**Proposition 5.3.** *The model error, whose dynamics is given by* (5.20) *with* $w \in \mathcal{W}$*, is bounded by*

$$\|\delta(t)\|_\infty \leq \gamma\phi(t, t_{k_1}), \; \forall t \in [t_{k_1}, t_{k_1+1}), \tag{5.23}$$

*where* $\phi(t, t_{k_1}) = \frac{1}{\|A\|_\infty}\left(e^{\|A\|_\infty(t-t_{k_1})} - 1\right)$ *and* $\|A\|_\infty$ *is the infinite norm of* $A$*.*

**Proof.** From equation (5.22), the norm of the error can be bounded as follows:

$$\begin{aligned}\|\delta(t)\|_\infty &= \left\|\int_{t_{k_1}}^{t} e^{A(t-\tau)}w(\tau)d\tau\right\|_\infty \leq \int_{t_{k_1}}^{t} \left\|e^{A(t-\tau)}\right\|_\infty \|w(\tau)\|_\infty d\tau \\ &\leq \gamma\int_{t_{k_1}}^{t} e^{\|A\|_\infty(t-\tau)}d\tau = \gamma\frac{1}{\|A\|_\infty}\left(e^{\|A\|_\infty(t-t_{k_1})} - 1\right).\end{aligned}$$

□

In what follows, the Lyapunov-based sampling procedure is developed. The controller goal is to maximize the next sampling instant $t_{k_1+1}$, while preserving the practical stability of the system. To do so, the sampling instants are chosen guaranteeing that the Lyapunov function decrease ($\dot{V} < 0$), except when the system is close to the equilibrium point.

Taking the time derivative of the Lyapunov function for $t \in [t_{k_1}, t_{k_1+1})$, it yields

$$\frac{d}{dt}V(t) = x^T(t)P\dot{x}(t) + \dot{x}^T(t)Px(t) = 2x^T(t)P\dot{x}(t). \tag{5.24}$$

Now, substitute $x(t)$ by $\delta(t) + x_c(t)$ and use their dynamics in (5.21) and (5.22):

$$\begin{aligned}\dot{V}(t) &= 2(\delta^T(t) + x_c^T(t))P(\dot{\delta}(t) + \dot{x}_c(t)) \\ &= 2(\delta^T(t) + x_c^T(t))P(A\delta(t) + w(t) + Ax_c(t) + Bu(t)), \ \forall t \in [t_{k_1}, t_{k_1+1}).\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{V}(t) &= \delta^T(t)(PA + A^TP)\delta(t) + 2\delta^T(t)Pw(t) + 2x_c^T(t)Pw(t) \\ &+ 2\delta^T(t)(PA + A^TP + PBK)x_c(t) + \\ &+ x_c^T(t)\Big(P(A+BK) + (A+BK)^TP\Big)\,x_c(t), \ \forall t \in [t_{k_1}, t_{k_1+1}).\end{aligned} \tag{5.25}$$

The objective of the controller is to maximize $t_{k_1+1}$ while guaranteeing that the upper bound on the time derivative of the Lyapunov function is negative for all possible disturbance trajectories, that is,

$$\begin{aligned}&\max \ t_{k_1+1} \\ &\text{subject to: } \frac{d}{dt}V(t) < 0, \quad \forall t \in [t_k, t_{k+1})\end{aligned} \tag{5.26}$$

This optimization problem is difficult to solve. The parameter to be optimized, $t_{k_1+1}$, is involved in a nonlinear equation and there are an infinite number of constraints, because they must be satisfied for all $t \in [t_{k_1}, t_{k_1+1})$.

In order to obtain the next sampling time, a mesh of discrete values of $t_{k_1+1}$ is studied. Let $t_{k_1+1} = T_{\min} + n\Delta$, where $T_{\min}$ and $\Delta$ are two positive known constants. The objective changes to find the maximum $n$ such that the time derivative of the Lyapunov function is negative for all possible disturbances at those time instants of the mesh.

The next iterative algorithm provides an approximate solution to (5.26).

**Algorithm 5.2.**

1. Set $t_{k_1+1} = t_{k_1} + T_{min}$.

2. Solve the problem

$$\begin{aligned}&\min_{\delta(t_{k_1+1}), w(t_{k_1+1})} \ -\frac{d}{dt}V(t_{k_1+1}) \\ &\text{subject to:} \\ &\quad \|w(t_{k_1+1})\|_\infty \le \gamma \\ &\quad \|\delta(t_{k_1+1})\|_\infty \le \gamma\phi(t_{k_1+1}, t_{k_1})\end{aligned} \tag{5.27}$$

3. If $\dot{V}(t_{k_1+1}) < 0$, increase $t_{k_1+1} = t_{k_1+1} + \Delta$ and go to Step 2. Otherwise, choose $t_{k_1+1}$.

The value of $\Delta$ must be chosen small enough in such a way that the dynamics of the controller state, and hence of the Lyapunov function, are smooth between two consecutive sampling instants. This avoids multiple sign changes of the derivative of the Lyapunov function from $t_{k_1}$ to $t_{k_1+1}$. In general $T_{\min}$ is chosen according with the minimum sampling time of the sensors. However, aiming at reducing the computational time, it may be selected to be larger.

It is worth noting that, as the system approaches to the equilibrium point, the effects of the disturbances become more apparent and tend to increase the Lyapunov function. In this case the sampling instant is that of the corresponding periodic controller that guarantees practical stability.

### 5.4.2 On-line computation based on quadratic programming

In order to find the sampling instants the controller has to follow Algorithm 5.2, which implies the resolution of several optimization problems. All those steps must be solved on-line, which may require high computational time. Therefore, it is of crucial importance to pose the optimization problem in a way such that the algorithm can be solved in an efficient and fast manner.

This section proposes the use of quadratic programming to find the optimal solution to (5.27). Before, it is worth giving some remarks about QP problems [173].

**Definition 5.1. Quadratic programming problem.** Assume the vectors $\xi \in \mathbb{R}^p$, $f \in \mathbb{R}^p$, and the symmetric matrix $H \in \mathbb{R}^{p\times p}$. The QP problem is stated as

$$\min_{\xi} \; g(\xi) = \frac{1}{2}\xi^T H\xi + f^T\xi + c, \tag{5.28}$$

subject to:

$$D\xi \leq b \tag{5.29}$$

Next proposition shows that problem (5.27) can be stated as a QP.

**Proposition 5.4.** *Problem (5.27) can be formulated as a QP if the elements of equations*

*(5.28)-(5.29) are chosen as*

$$\begin{aligned}
\xi &= \begin{bmatrix} \delta(t) \\ w(t) \end{bmatrix}, \\
H &= -2\begin{bmatrix} PA + A^T P & P \\ P & 0 \end{bmatrix}, \\
f^T &= -2x_c^T(t_{k_1+1})\begin{bmatrix} PA + A^T P + K^T B^T P & P \end{bmatrix}, \\
c &= -x_c^T(t_{k_1+1})\left(P(A+BK) + (A+BK)^T P\right) x_c(t_{k_1+1}),
\end{aligned}$$

*and for the constraint*

$$D = \begin{bmatrix} I_n & 0 \\ -I_n & 0 \\ 0 & I_n \\ 0 & -I_n \end{bmatrix}, \quad b = \begin{bmatrix} \gamma\phi(t_{k_1+1}, t_k)\bar{1}_n \\ \gamma\phi(t_{k_1+1}, t_k)\bar{1}_n \\ \gamma\bar{1}_n \\ \gamma\bar{1}_n \end{bmatrix},$$

*where $\bar{1}_n \in \mathbb{R}^n$ is a column vector whose components are ones and $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix.*

**Proof.** From equation (5.25), and after some mathematical manipulation, it can be seen that the minimization of $-\dot{V}(t)$ is equivalent to the minimization of $G(t)$, defined as

$$\begin{aligned}
G(t) = -\delta^T(t)(PA + A^T P)\delta(t) - 2\delta^T(t)Pw(t) - \\
- 2x_c^T(t)Pw(t) - 2\delta^T(t)(PA + A^T P + PBK)x_c(t). \quad (5.30)
\end{aligned}$$

Constant terms in (5.25) have been suppressed as they do not affect the minimization problem. Therefore, by using the previously defined augmented variable $\xi^T(t) = [\delta^T(t)\ w^T(t)]$, the same matrices of (5.28) are obtained for the QP problem. Finally, constraints of (5.29) can be easily described with given matrices $D, b$. □

It is well known that QP problems can be easily solved as there exist appropriate software packages, as `quadprog` in Matlab, that tackle these optimization problems in a computationally effective way. Furthermore, the resulting QP problem is, indeed, a multi-parametric QP (mpQP), for which the explicit solution can be obtained [13]. In particular, the parameter $\theta \in R^{n+1}$ of the mpQP problem is:

$$\theta(t_{k_1}, j) = \begin{bmatrix} x_c(t_{k_1} + j\Delta) \\ \phi(t_{k_1} + j\Delta, t_{k_1}) \end{bmatrix}.$$

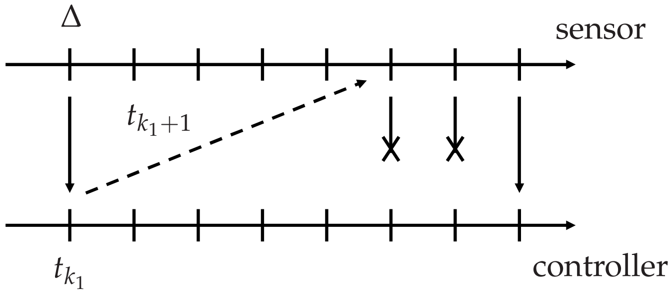



Figure 5.8: Data exchange between sensor and controller under unreliable communication

Assuming that the sign of the time derivative of the Lyapunov function does not experiment multiple changes between $t_{k_1+i}$ and $t_{k_1+i+1}$, Algorithm 5.2 provides a suboptimal solution to problem (5.26). Note that this assumption will be satisfied for a sufficiently small $\Delta$.

#### 5.4.2.1 Extension to unreliable channels

Up to this point perfect channels have been assumed, since no delays, packet dropouts or quantization effects have been introduced. However, in the NCS framework is quite common the use of non-reliable protocols, such as User Datagram Protocol (UDP), because of the real-time requirements.

To extend the previous results for a scenario in which packet dropouts are present, the controller follows again Algorithm 5.2 to obtain the next sampling time without losses $t^{wl}_{k_1+1}$. A possible data exchange between sensor and controller is depicted in Figure 5.8. However, to ensure the stability of the system, it will demand the sensor to send the state at sampling instant:

$$t_{k_1+1} = t^{wl}_{k_1+1} - n_p\Delta, \tag{5.31}$$

where $n_p$ is the maximum contemplated consecutive number of package dropouts. It is assumed that the sensor can track the correct packet delivery at the controller by using standard acknowledgment strategies.

### 5.4.3 Numerical example

In this section, the previous algorithm is applied to an unstable plant in order to show how the controller manages to reduce the traffic load while maintaining the practical stability of the system.

**Example 5.3.** Consider the following LTI system:

$$\dot{x}(t) = \begin{bmatrix} 1 & 0.99 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + w(t),$$

where the disturbances $w(t)$ are supposed to verify $\|w(t)\|_\infty \leq 0.01$. The initial condition for the system and the controller is $x_0^T = [10 \quad -5]$.

With a sampling period of $T_{\min} = 1s$ and the following state-feedback controller:

$$K = \begin{bmatrix} -4.5263 & -4.4110 \end{bmatrix},$$

the asymptotic stability of the system without disturbances is ensured.

The associated Lyapunov function is

$$V(t) = x^T(t) \begin{bmatrix} 0.2161 & 0.1156 \\ 0.1156 & 0.1083 \end{bmatrix} x(t).$$

Assume now that this system is controlled over a shared network as Figure 5.6 shows. The objective is to reduce the traffic through the shared medium. No packet dropouts are considered.

If the disturbances are assumed to be zero, the evolution of the system and the error between the state of the system and of the controller are shown in Figure 5.9, together with the asynchronous sampling instants.

It is worth noting that only the first three sampling times are bigger than 1 second. As explained before, when the system is evolving near the equilibrium point, the optimization problem is not able to find any sampling time larger than $T_{\min}$. In other works, as [142], the sampling instants are only enlarged when the system is close to the equilibrium. If one combines both approaches, it will be possible to enlarge the inter-sampling times for both situations, transient and steady state.

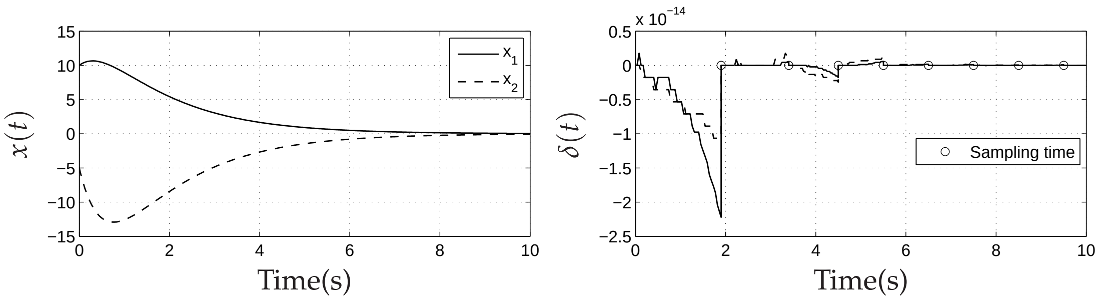



Figure 5.9: Evolution of the state (left) and the model error (right) without disturbances

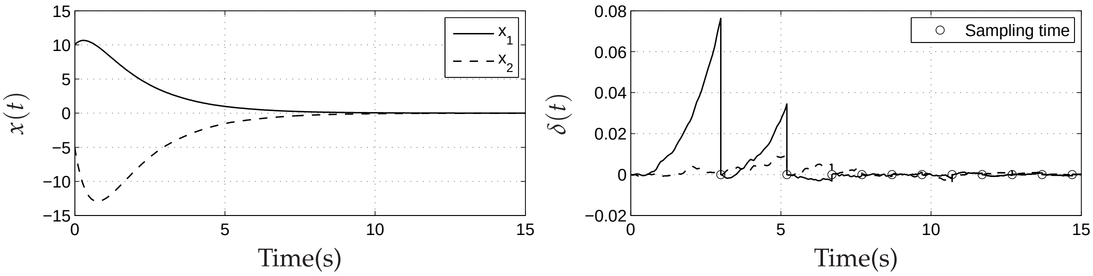



Figure 5.10: Evolution of the state (left) and the model error (right) with disturbances

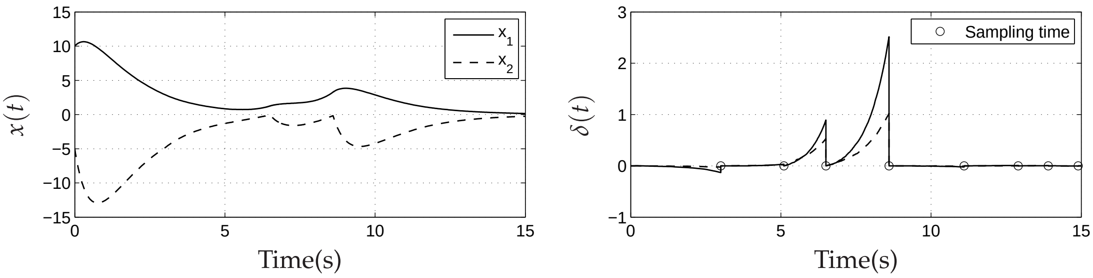



Figure 5.11: Evolution of the state (left) and the model error (right) when the system evolves far from the equilibrium

Consider now the perturbed case. The evolution of the state and the error is depicted in Figure 5.10. Again, the sampling times are bigger than $T_{\min}$ only when the system is far from the equilibrium.

Finally, the method is tested with Gaussian disturbances with different variances: from 7 to 8 seconds, the variance is 0.5; and for the rest of the experiment the variance is 0.01. The asynchronous sampling periods for this case are shown in Figure 5.11.

## 5.5 Chapter summary

This chapter presents two solutions regarding the use of model-based controllers in networked systems. In both cases, the objective has been the reduction of the communication through the network, looking for a better use of the available bandwidth in shared channels. The problem is tackled from two different points of views, depending on the sampling policy between plant and controller, periodic or aperiodic.

For the periodic case, a natural extension of the work of Montestruque and Antsaklis [160, 161] is presented, in which the network is introduced for the first time between controller and actuator. The dynamics of the closed-loop system is studied and the conditions to ensure the stability are derived.

Additionally, the method is extended to deal with uncertain systems. Using the available theory of interval matrices, it is possible to consider parametric uncertainties, although the method suffers from excessive conservatism. In future works, the structure of matrix $M$ may be exploited to obtain less conservative results. It is also necessary to pay attention to the limits of uncertainties that the model tolerates, since in some situations the inclusion of the model could become counterproductive.

The second part of the chapter presents an aperiodic sampling policy, in which the asynchronous sampling times between sensor and controller are decided on-line using a self-triggered approach. The proposed method is based on the Lyapunov stability theory. It is shown that, by using open-loop predictions, adequate asynchronous sampling times can be found by solving several optimization problems.

Finally, the section proves that those optimization problems can be written as standard QP problems (indeed multi-parametric QP) which can be easily and quickly solved on-line with the available software. Future works may include the consideration of uncertainties in the plant model and of transmission delays in the communications. Moreover, the problem of designing a suitable value for $\Delta$ is a stimulating future work. It is worth commenting that this drawback disappears in discrete-time systems [151].

# Chapter 6

# Scheduled communication for state estimation and control

## 6.1 Introduction

In recent years, the advances in wireless communication, micro-electro-mechanical systems and digital electronics have boosted the emergence of a newborn branch in control systems. Now, the elements of the control loop are connected by means of a network, so problems as limited bandwidth, congestion, delays and dropouts need to be considered.

Although some researchers have proposed solutions robust against unreliable network conditions, see Chapter 4 and references therein, the control community has shifted towards the paradigm of co-design, in which both control and communications are taken into account at the same level of importance.

Sensor scheduling for state estimation is one of the co-design problems that has received more attention in the literature in the last years. The objective is to achieve an adequate estimation of the state of the system, when it is being observed by means of a set of sensors. Those sensors share the same communication network, so an appropriate medium access control becomes necessary.

Other issues that arise in this kind of problems are those related to the bandwidth usage and energy consumption. On the one hand, the available transmission rate must be divided and, if the sensors transmit much information, contention problems arise producing undesirable delays and packet dropouts. Furthermore, wireless devices are urged to optimize the energy usage to increase their lifetime without battery replacements.

Wireless sensor networks (WSNs) are now technically and economically afford-

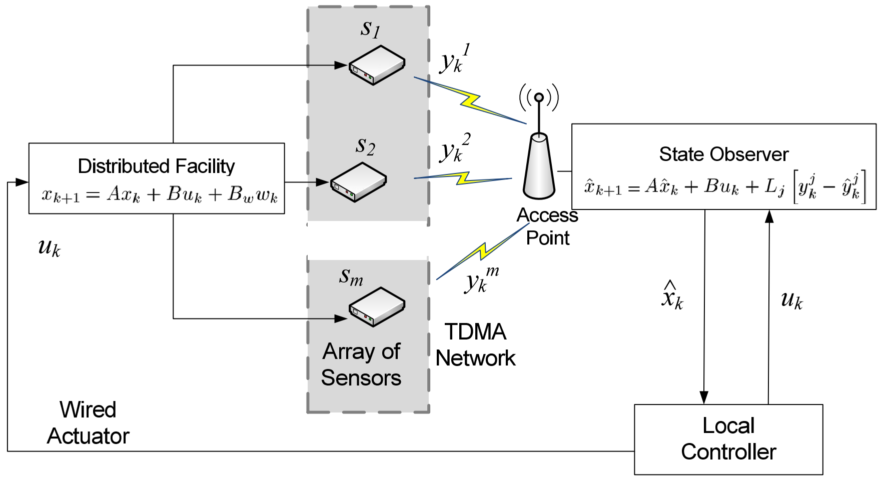



Figure 6.1: Networked control scheme.

able with the advent of protocols 802.11x, 802.15.4 and technologies such as ZigBee. Although most of wireless networks protocols implement contention-based protocols, many control applications require regular delivery of data samples at fixed rate, with minimum or at least predictable delay. With these constraints in mind, medium access protocols that reserve transmission slots for each node, such as TDMA methods, have regained interest, specially in WSNs [245, 267]. Note that the energy-aware WSN-specific 802.15.4 protocol (the basis for ZigBee) also offers contentionless access through reserved slots for fixed-bandwidth traffic.

With regard to the scheduling, there are mainly two ways of transmitting information: following a periodic or an aperiodic scheme. In the former, signals are sent through the network following a concrete pattern, which is defined statically or dynamically. This strategy has been applied in the controller-plant loop [206], and in the observation framework [97, 238, 263, 264]. In these works, the study of periodic systems [15] provides a strong and widely used basis. Ideas of Kalman filtering have also been modified to design periodic observers [89, 221, 262]. Related to the co-design, the communication sequences are chosen so that the observability of the system is preserved [71, 75, 89, 261, 262, 263]. However, none of these works solve the problem of finding an optimal pattern.

For aperiodic scheduling, some authors have proposed the co-design of the bus scheduling and the controller/observer gains. A model predictive framework has been employed in [7, 198]. Some authors propose to solve finite-horizon optimization problems to get a suboptimal sensor scheduling [104, 158]. The work of Gupta et al. [76] proposes a stochastic scheduling which is close to the optimal solution.

However, extra traffic over the network is needed to coordinate all sensors.

A quite different approach is given in [43, 44]. The objective is the co-design of the observer and a special communication protocol, named TOD, to asymptotically reconstruct the plant states. Although this protocol seems to outperform the classical Round Robin protocol implemented in periodic schemes [82], it requires two conditions that make it unappropriated in our scheme: on the one hand, all sensors must have a local observer, which implies more energy consumption and complexity; while a more important fact is that each sensor receives signals from the rest, exponentially increasing the traffic over the network as the number of sensors grows. Moreover, this architecture becomes more vulnerable to the risk of hidden-node effects.

The main advantage of the aperiodic approaches is that they use information of the measured output and past estimates to optimally select the output to be sent and the gain, purveying more degrees of freedom to perform the optimization. However, it incurs in more complicated and heavy mathematical operations. Moreover, an extra traffic must be added to the network to select the corresponding sensor. On the other hand, periodic approaches can be initially configured, in such a way that each sensor knows when it must send its measurement. This implies better energy usage, as the devices may remain asleep between two consecutive transmission instants.

Recently, some authors have noticed that the periodic phenomenon appears in optimal aperiodic schedules [88, 184]. If this fact is proved to be true, this would imply that the positive features of both approaches could be inherited, providing an optimal solution for sensor scheduling which is, at the same time, mathematically and energy efficient.

This chapter is interested in the effect of the TDMA scheduling on the stability of feedback loops, and how it should be devised to minimize steady state errors. It contributes to the field of sensor scheduling in different aspects:

**Periodic scheduling:** An $H_\infty$ periodic observer is proposed. It may increase its inner rate, which could further reduce the communication through the network. Additionally, new insights are given in the optimality of patterns, at least from a numerical point of view. Finally, the co-design of the observer and the scheduling is carried out aiming for pole placement in the dynamics of the observation error.

**Aperiodic scheduling:** An aperiodic Kalman-based filter is proposed. It has several points of similarity with the one of [76], as the sensor selection is made by solving an *one-step-ahead* optimization problem. Both the observation matrices and the transmitted output are chosen to minimize the expected value of the error.

These co-designs can be directly implemented under 802.15.4, ZigBee and other TDMA protocols, so their interest in real-world applications is undeniable. Later, by making use of a powerful result for switching systems given by Ramadge in [205], we will give the conditions that must be verified for a system in order that the optimal scheduling eventually converges to a periodic pattern. Those conditions, namely, nonsingular system matrix, trajectory of the covariance matrix bounded and some restrictions on the limit points, are not strict, as it will be shown.

With respect to the boundedness on the covariance matrix, some interesting cases are studied, i.e., the diagonal and block-diagonal situation. Furthermore, a geometrical study is given to give insights into the general situation. Finally, some additional remarks about the optimal period are given.

It is worth mentioning that the periodic phenomenon in *a priori* aperiodic systems has been observed in other areas, as the periodicity in the inter-execution times in self-trigger control [142].

## Related publications

1. L. Orihuela, F. Gómez-Estern, F. R. Rubio. *Stability and performance of networked control systems with time-multiplexed sensors and oversampled observer*. 18th IFAC World Congress. Milano, Italy. pp:9200-9205, 2011. [183]

2. L. Orihuela, A. Barreiro, F. Gómez-Estern, F. R. Rubio. *¿La gestión óptima del canal de comunicaciones para la estimación implica un patrón de muestreo periódico?*. XXXIII Jornadas de Automática. Vigo, Spain. 2012. [181]

3. L. Orihuela, F. Gómez-Estern, F. R. Rubio. *Scheduled communication in sensor networks*. IEEE Transactions on Control Systems Technology. Under review. [184]

4. L. Orihuela, A. Barreiro, F. Gómez-Estern, F. R. Rubio. *Periodicity of the optimally scheduled distributed Kalman filter*. Automatica. Submitted. [182]

## 6.2 Problem statement

Consider a discrete-time large-scale plant, as the one presented in Section 2.2.2, whose dynamics is given by[1]

$$x_{k+1} \quad = \quad Ax_k + Bu_k + B_w w_k. \tag{6.1}$$

As Figure 6.1 suggests, the plant outputs are being measured by a set of $m$ sensors, possibly spatially distributed. At each sampling period $k$, only one sensor can access the network to send its packet to the remote observer. At instant $k$ the sensor $j$ sends the output

$$y_k^j = C_j x_k + v_k^j, \tag{6.2}$$

where $C_j$, $j = 1, \dots, m$, are matrices of appropriate dimensions. Processes $v_k^j$ represent noises in the measurement. A necessary assumption for the rest of the chapter is that the pair $(A, C)$ is detectable, where $C = \begin{bmatrix} C_1^T & C_2^T & \dots & C_m^T \end{bmatrix}^T$.

Assume that the dynamics of the observer located at the other end of the network is given by

$$\hat{x}_{k+1} \quad = \quad A\hat{x}_k + Bu_k + L_j(y_k^j - \hat{y}_k^j), \tag{6.3}$$

$$\hat{y}_k^j \quad = \quad C_j \hat{x}_k. \tag{6.4}$$

Element $L_j$ represents the observer gain for each different received output. With this estimation, the controller computes a control signal defined by

$$u_k = K\hat{x}_k, \tag{6.5}$$

where $K$ is the controller matrix of appropriate dimensions.

The estimation error is defined as

$$e_k = x_k - \hat{x}_k. \tag{6.6}$$

Finally, let $\xi_k$ denote an augmented vector stacking the system state and the estimation error, that is,

$$\xi_k = \begin{bmatrix} x_k \\ e_k \end{bmatrix}. \tag{6.7}$$

An observer is completely determined if the sequences $\{C_j\}$ and $\{L_j\}$ are defined. Given the plant (6.1)-(6.2) and the observer (6.3)-(6.4), a **scheduling law** is any mechanism that implements some choice of the active sensor $j$ (the one that

[1]This chapter uses the notation $x_k$ instead $x(k)$ to facilitate the reading of some sections.

uses the network) and the gain $L_j$ to be applied to the filter. The scheduling law can be time-based or state-based. A **time-based scheduling law** is of the form $j = j(k)$ and $L_j = L_j(k)$, and includes as particular cases the periodic scheduling of the sensors [89, 262].

A **state-based scheduling law** decides the sensor $j$ and the gain $L_j$ as functions of the state variables, yielding an aperiodic solution.

The main advantage of the state-based approach is that it uses information of the observer to optimally select the output and the gain. However, it incurs in more complicated and heavy mathematical operations. Moreover, there must exist an extra traffic over the network to select the corresponding sensor.

The purpose of this chapter is twofold. First, an appropriate scheduling law must be defined. Second, the observer gains in (6.3) must be suitably designed according to that law. A time-based and a state-based scheduling law are provided:

**Time-based (periodic scheduling):** Given a periodic pattern, an $H_\infty$ observer is proposed. Some considerations related to pattern design are given.

**State-based (aperiodic scheduling):** A Kalman-based filter is proposed in which both the sensor and the gain are chosen to minimize the variance of the observation error.

Then it will be shown that, under some mild conditions, this aperiodic observer results in a periodic scheduling of the outputs. That is, a state-based scheduling law produces an equivalent time-based law, providing the observer with the benefits of both approaches.

## 6.3 Periodic scheduling

In this situation, the sensors are activated following a previously predefined periodic pattern. This solution is computational efficient, as the selection of the sensors requires no extra calculation. Furthermore, regarding energy consumption the devices may be turned off between two consecutive measurements.

An $H_\infty$ periodic observer is proposed to estimate the state of the plant. However, before formally introducing the problem statement, some preliminar definitions must be introduced. First, consider the concept of *measurement pattern*, the time-based law that rules the scheduling in a periodic scheme.

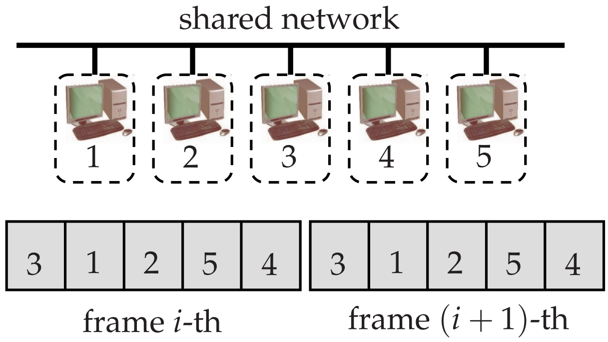



Figure 6.2: Time slots and frames in TDMA communication

**Definition 6.1.** A **measurement pattern** $\varphi_N \in \mathbb{R}^N$ is a vector whose components indicate which sensor is active. That is,

$$\varphi_N(i) \equiv \{j : j = 1, \ldots, m\}, \quad i = 1, \ldots, N. \tag{6.8}$$

Hence, $\varphi_N(i) = j$ implies that sensor $j$ uses the network in the $i$-th position of the pattern. In a TDMA context, $\varphi_N(i) = j$ indicates that sensor $j$ must send its data in the $i$-th time slot of the frame (see Figure 6.2). Note that it is possible to grant priority to some outputs over the rest. For instance, $\varphi_N = [1 \ \ 2 \ \ 1]$ is a pattern in which sensor 1 uses the network twice in a pattern.

After a certain number of sampling instants (the length of the periodic pattern) the same sensor is activated again.

**Example 6.1. Motivating example.** Consider the discrete-time system from [95]:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & 0 & -2 \end{bmatrix} x_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u_k, \\ y_k &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} x_k. \end{aligned}$$

The rank of the observability matrix is lower than three for each one of the outputs $y_k^1 = [1 \ \ 0]\, y_k$, $y_k^2 = [0 \ \ 1]\, y_k$ separately. None of them can be used to observe the system. In fact, neither output can stabilize the observation error, as the unobservable subspace is unstable for both cases. Nevertheless, a simple pattern such as $\varphi_2 = [1 \ \ 2]$ can be used to stabilize the observation error [95]. ▼

Previous example shows that some systems cannot be observed by measuring only certain outputs, but a pattern of them may be enough. With this motivation in mind, the problem considered in this section is formally defined.

**Definition 6.2.** $H_\infty$ **periodic observation problem.** Let $z_k = D\xi_k$ be the controlled output. The $H_\infty$ periodic observation problem consists in finding a measurement pattern $\varphi_N \in \mathbb{R}^N$ and a set of observer gains $L_j$, $j = 1, \ldots, N$, in such a way that:

- The dynamics of the system and the estimation error are asymptotically stable for $w_k \equiv v_k \equiv 0$.
- Under the assumption of zero initial conditions for the state of the system and the estimation error, the effects of the disturbances and the sensor noises in the controlled output are attenuated below $\gamma$, such that $\|z_k\|_{L_2} \leq \gamma \left\| \begin{bmatrix} w_k \\ v_k^k \end{bmatrix} \right\|_{L_2}$.

In order to present a solution for this problem it is necessary to study the dynamics of the system and of the observation error during a complete measurement pattern. In the following the evolution of the augmented state vector is going to be described for a situation in which the observer updates the estimate at a rate faster than the outputs transmission rate. By letting the controller and the observer run at a higher frequency than the transmission of partial outputs, the proposed scheme involves open-loop estimation intervals. This *oversampled* observer allows to reduce the network traffic and also the energy consumption. Ensuring the stability of the oversampled observer is harder with respect to a typical observer.

Therefore, two different periods can be defined: during a *observation period* (OP) all the sensors are turned off; in a *measurement period* (MP) a sensor sends a plant output. Therefore, given a certain measurement pattern $\varphi_N$, all the sensors remain asleep for $P_o$ observation periods between two consecutive measurement periods. Thus the dynamics of the observer differs between both types of periods:

$$\begin{aligned} (\text{MP}): \ \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + L_j\left[y_k^j - \hat{y}_k^j\right]; \\ (\text{OP}): \ \hat{x}_{k+1} &= A\hat{x}_k + Bu_k. \end{aligned}$$

The structure of the observer is similar to a Luenberger-like observer with gain $L_j$ for each measurement period. On the other hand, the observer evolves in open loop during an OP. Figure 6.3 illustrates the complete scheduling pattern for a case with two observation periods.

In the following, the dynamics of the disturbances are assumed to be slower than that of the oversampled observer, that is, they remain constant between two consecutive measurement periods:

$$w_k = w_{k+1} = \ldots = w_{k+P_o}.$$

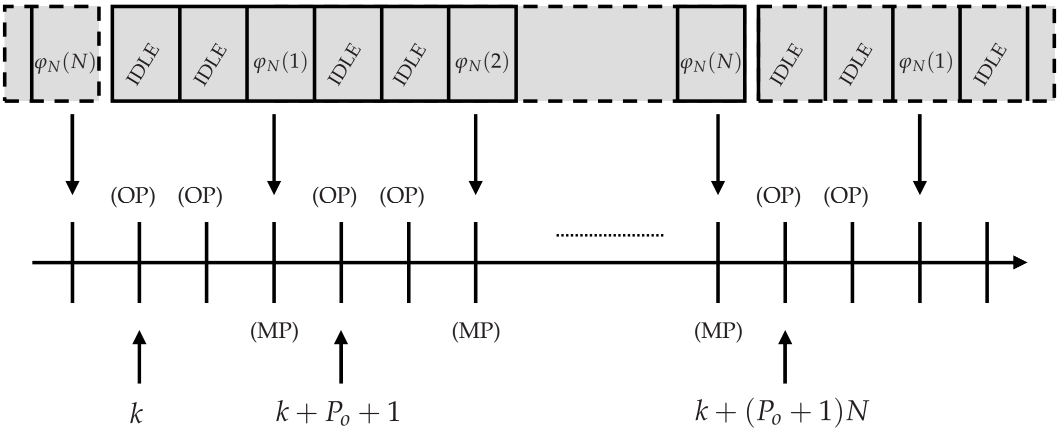



Figure 6.3: Time scheduling for the oversampled observer with $P_o = 2$

The following proposition states the complete evolution of the system along a measurement period.

**Proposition 6.1.** *Given a measurement pattern $\varphi_N \in \mathbb{R}^N$ and a constant $P_o \in \mathbb{N}$, the evolution of the augmented state $\xi_k$ from $k$ to $k+(P_o+1)N$ is given by*

$$\begin{aligned} \xi_{k+(P_o+1)N} &= \Xi(P_o, N)\xi_k \\ &+ \sum_{i=1}^{N} \Xi(P_o, i-1)\Omega(P_o, \varphi(N+1-i)) \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_{k+(P_o+1)(N-i)} \\ &- \sum_{i=1}^{N} \Xi(P_o, i-1) \begin{bmatrix} 0 \\ L_{\varphi_N(N+1-i)} \end{bmatrix} v_{k+(P_o+1)(N-i)}^{\varphi_N(N+1-i)}, \end{aligned} \tag{6.9}$$

*where*

$$\Omega(P_o, i) = I + \sum_{j=1}^{P_o} \begin{bmatrix} A+BK & -BK \\ 0 & (A-L_iC_i) \end{bmatrix} \begin{bmatrix} (A+BK)^{j-1} & \Psi_{j-1} \\ 0 & A^{j-1} \end{bmatrix}, \tag{6.10}$$

$$\Xi(P_o, i) = \begin{bmatrix} (A+BK)^{(P_o+1)i} & \Xi_{12}(P_o, i) \\ 0 & \prod_{j=\varphi_N(i)}^{\varphi_N(1)} [\vartheta_j] \end{bmatrix}, \tag{6.11}$$

*being $\vartheta_j = (A - L_jC_j)A^{P_o}$, $\Psi_i = \sum_{j=1}^{i} (A+BK)^{P_o - j} BKA^{j-1}$ and $\Xi_{12}(P_o, i)$ a matrix whose structure is given in the proof.*

**Proof.** The measurement pattern $\varphi_N$ indicates the activation of the sensors. The one-step evolution for an observation period is

$$\begin{aligned}\xi_{k+1} &= \begin{bmatrix} A+BK & -BK \\ 0 & A \end{bmatrix} \xi_k + \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_k \\ &= \Delta \xi_k + \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_k.\end{aligned}$$

From instant $k$ to $k+P_o-1$, no output is received from the system. After $P_o$ observation periods, the following relation holds

$$\xi_{k+P_o} = \Delta^{P_o}\xi_k + \sum_{i=1}^{P_o} \Delta^{i-1} \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_k, \tag{6.12}$$

where it has been assumed that dynamics of the oversampled observer are faster than that of the disturbances. It is easy to check that

$$\Delta^i = \begin{bmatrix} (A+BK)^i & \Psi_i \\ 0 & A^i \end{bmatrix},$$

where $\Psi_i$ has been defined previously.

After $P_o$ observation periods the observer receives a new measurement (see Figure 6.3). The one-step evolution of the augmented state within a measurement period is given by

$$\begin{aligned}\xi_{k+1} &= \begin{bmatrix} A+BK & -BK \\ 0 & (A-L_jC_j) \end{bmatrix} \xi_k + \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_k - \begin{bmatrix} 0 \\ L_j \end{bmatrix} v_k^j, \\ &= \Lambda_j \xi_k + \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_k - \begin{bmatrix} 0 \\ L_j \end{bmatrix} v_k^j. \end{aligned} \tag{6.13}$$

Hence the complete evolution after $P_o$ observation periods plus one measurement period can be obtained by combining (6.13) and (6.12):

$$\xi_{k+P_o+1} = \Lambda_j \Delta^{P_o}\xi_k + \Lambda_j \sum_{i=1}^{P_o} \Delta^{i-1} \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_k + \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_k - \begin{bmatrix} 0 \\ L_j \end{bmatrix} v_k^j,$$

or in a more compact way,

$$\xi_{k+P_o+1} = \Lambda_j \Delta^{P_o}\xi_k + \Omega(P_o, j) \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_k - \begin{bmatrix} 0 \\ L_j \end{bmatrix} v_k^j, \tag{6.14}$$

where $\Omega(P_o, j)$ was defined in (6.10) and

$$\Lambda_j \Delta^i = \begin{bmatrix} (A+BK)^{i+1} & (A+BK)\Psi_i - BKA^i \\ 0 & (A-L_jC_j)A^i \end{bmatrix}.$$

Recall that index $j$ refers to the output which is being sent according to the measurement pattern, that is $\varphi_N(i) = j$. This structure is repeated every $P_o$ observation periods and one measurement period. Hence, the following dynamic equation can be obtained:

$$\begin{aligned} \xi_{k+N} &= \prod_{j=\varphi_N(N)}^{\varphi_N(1)} (\Lambda_j \Delta^{P_o}) \xi_k \\ &+ \sum_{i=1}^{N} \left[ \prod_{j=\varphi_N(i-1)}^{\varphi_N(1)} (\Lambda_j \Delta^{P_o}) \right] \Omega(P_o, \varphi(N+1-i)) \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_{k+N-i} \\ &- \sum_{i=1}^{N} \left[ \prod_{j=\varphi_N(i-1)}^{\varphi_N(1)} (\Lambda_j \Delta^{P_o}) \right] \begin{bmatrix} 0 \\ L_{\varphi_N(N+1-i)} \end{bmatrix} v_{k+N-i}^{\varphi_N(N+1-i)}. \end{aligned}$$

After some mathematical manipulations, it is not difficult to see that

$$\prod_{j=\varphi_N(i)}^{\varphi_N(1)} (\Lambda_j \Delta^{P_o}) = \Xi(P_o, i),$$

where $\Xi(P_o, i)$ has been defined previously in the hypotheses of the proposition (6.11). Block $(1,2)$ of $\Xi(P_o, i)$, that is, $\Xi_{12}(P_o, i)$ can be obtained from the above multiplications. □

Proposition 6.1 encompasses the classical Luenberger observer given in Section 2.3.1. Assume $w_k \equiv v_k \equiv 0, \forall k$. When $N = 1$ and $P_o = 0$, and there is only one output, the dynamics (6.9) matches that of the Luenberger observer.

### 6.3.1 Asymptotic stability of the unperturbed system

This section is intended to clarify the concepts introduced so far. The dynamics of the unperturbed system is given in the following corollary.

**Corollary 6.1.** *Given a measurement pattern $\varphi_N \in \mathbb{R}^N$ and assuming that $v_k \equiv w_k \equiv 0$, $\forall k$, the evolution of the augmented state $\xi_k$ from $k$ to $k + (P_o + 1)N$ is given by*

$$\xi_{k+(P_o+1)N} = \begin{bmatrix} (A+BK)^{(P_o+1)N} & \Xi_{12}(P_o, i) \\ 0 & \prod_{j=\varphi_N(i)}^{\varphi_N(1)} [\vartheta_j] \end{bmatrix} \xi_k,$$

*where $\vartheta_j = (A - L_j C_j) A^{P_o}$.*

The stability of the unperturbed system with this feedback strategy can be studied using fairly standard results of discrete-time systems. Notice that the dynamics given in Corollary 6.1 is equivalent to that of an autonomous discrete-time system.

**Lemma 6.1.** *The discrete-time system with the evolution given by Corollary 6.1 is asymptotically stable if and only if the eigenvalues of the following matrices are inside the unit circle:*

- $(A+BK)^{(P_o+1)N}$,
- $\prod_{j=\varphi_N(N)}^{\varphi_N(1)}[\vartheta_j]$.

The proof is immediate if extended eigenvalue properties of triangular matrices are used. It is worth mentioning the implication of this lemma. As in the classical Linear Quadratic Gaussian control (LQG), the separation principle also holds in the problem under study, so the controller matrix and the observation gains could be designed in separated steps. Since the chapter is mainly interested in the observation problem, it will assumed that the controller is known.

**Example 6.2.** Consider the system of Example 6.1 and the pattern $\varphi_2 = [1\ \ 2]$. Assume that the observer gains are $L_1^T = [0\ \ -2.2\ \ -0.8]$ and $L_2^T = [0.5\ \ 0\ \ 0]$.

Through Lemma 6.1, the stability of the observation error without observation periods is ensured. Let

$$\Xi_{22} = \prod_{j=\varphi_N(i)}^{\varphi_N(1)}[\vartheta_j].$$

The eigenvalues of $\Xi_{22}$ are $\lambda(\Xi_{22}) = \{0, 0.4, 0.5\}$. However, with $P_o = 1$, the eigenvalues of $\Xi_{22}$ are $\lambda(\Xi_{22}) = \{0, -2, 0.5\}$, so the observation error is unstable.

Nevertheless, if $L_1^T = [0\ \ -2\ \ -0.67]$ and $L_2^T = [1\ \ 0\ \ 0]$ are chosen with $P_o = 1$, the eigenvalues are $\lambda(\Xi_{22}) = \{0, 0.83, 0.09\}$. Hence, the oversampled observer maintains the stability and achieves a reduction of traffic through the network. ▼

### 6.3.2 Observer Design

In previous sections, the dynamics of the augmented state and stability conditions for the unperturbed system have been given. However, a method to design the observers remains unspecified, which is the goal of this section. From the dynamics of the augmented vector (6.9), it is straightforward to see that the nonlinear relations between the different elements hinder the design of the observers. It seems impracticable to tackle the synthesis problem by using standard techniques. Instead, an LMI-based design method is proposed. It resorts to periodic systems theory. Some preliminaries on periodic systems are presented next.

A $p$-periodic discrete-time system is described by $x_{k+1} = A_k x_k + B_k u_k$, where $A_k, B_k$ are $p$-periodic matrices, that is, $A_{k+p} = A_k, B_{k+p} = B_k$, $\forall k$. This type of system has been widely studied from the eighties. There are numerous results in the literature about stability, control, optimality, etc. See [240] and the references therein.

An important result concerning periodic systems is the so-called Periodic Lyapunov Lemma, an extension of the Lyapunov lemma for this sort of systems [15].

**Lemma 6.2.** ***Periodic Lyapunov Lemma [15].*** *The p-periodic system $x_{k+1} = A_k x_k$ is asymptotically stable if and only if there exists a p-periodic matrix $P_k > 0$ such that*

$$A_k^T P_{k+1} A_k - P_k < 0, \quad \forall k \in \{1, \dots, p\}. \tag{6.15}$$

The key idea of this section is that the evolution of the system given previously can be written as the evolution of a periodic system. Then the vast theory existing in this field will be of invaluable help in the design procedure.

**Proposition 6.2.** *Given a measurement pattern $\varphi_N \in \mathbb{R}^N$ and a constant $P_o \in \mathbb{N}$, the dynamics of the augmented state given in Proposition 6.1 is equivalent to the following* N*-periodic system:*

$$\xi_{k+1} = \mathcal{A}_k \xi_k + \bar{B}_{w,k} w_k - \bar{B}_{v,k} v_k^k, \ \forall k \in \{1, \dots, N\},$$

*with periodic matrices*

$$\begin{aligned} \mathcal{A}_k &= \begin{bmatrix} (A+BK)^{P_o+1} & (A+BK)\Psi_{P_o} - BKA^{P_o} \\ 0 & (A - L_k E_k C) A^{P_o} \end{bmatrix}, \\ \bar{B}_{w,k} &= \Omega(P_o, k) \begin{bmatrix} B_w \\ B_w \end{bmatrix}, \\ \bar{B}_{v,k} &= \begin{bmatrix} 0 \\ L_k \end{bmatrix}. \end{aligned}$$

**Proof.** The proof follows immediately from that of Proposition 6.1. First, it was obtained in equation (6.14) that

$$\xi_{k+P_o+1} = \Lambda_{j_1} \Delta^{P_o} \xi_k + \Omega(P_o, j_1) \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_k - \begin{bmatrix} 0 \\ L_{j_1} \end{bmatrix} v_k^{j_1},$$

where $j_1$ is used instead of the general notation $j$ to denote that the output received is the first element in the measurement pattern, that is, $\varphi_N(1) = j_1$.

Obviously, for the following $P_o + 1$ periods, it holds

$$\xi_{k+2(P_o+1)} = \Lambda_{j_2}\Delta^{P_o}\xi_{k+P_o+1} + \Omega(P_o, j_2)\begin{bmatrix} B_w \\ B_w \end{bmatrix} w_{k+P_o+1} - \begin{bmatrix} 0 \\ L_{j_2} \end{bmatrix} v^{j_2}_{k+P_o+1}.$$

For the last $P_o + 1$ periods in the pattern:

$$\begin{aligned}\xi_{k+N(P_o+1)} &= \Lambda_{j_N}\Delta^{P_o}\xi_{k+(N-1)(P_o+1)} + \Omega(P_o, j_N)\begin{bmatrix} B_w \\ B_w \end{bmatrix} w_{k+(N-1)(P_o+1)} \\ &- \begin{bmatrix} 0 \\ L_{j_N} \end{bmatrix} v^{j_N}_{k+(N-1)(P_o+1)}.\end{aligned}$$

In the following measurement period (MP) the observer receives again the same output that was received in the first MP. Therefore,

$$\begin{aligned}\xi_{k+(N+1)(P_o+1)} &= \Lambda_{j_1}\Delta^{P_o}\xi_{k+N(P_o+1)} + \begin{bmatrix} B_w \\ B_w \end{bmatrix} w_{k+N(P_o+1)} \\ &- \begin{bmatrix} 0 \\ L_{j_1} \end{bmatrix} v^{j_1}_{k+N(P_o+1)}.\end{aligned}$$

It is easy to see that there exists an $N$-periodic relation in the evolution of the error, hence the proof is concluded. □

The following lemma presents a method to solve the $H_\infty$ periodic observation problem stated in Definition 6.2.

**Lemma 6.3.** *Let the controlled output be $z_k = D\xi_k$, with $D$ a matrix of appropriate dimensions, and let $\gamma > 0$ be a positive scalar. Then, the dynamics of the augmented state given by Proposition 6.2 is asymptotically stable in the absence of disturbances if there exists an* N*-periodic positive definite matrix $P_k$ such that the following matrix inequalities are satisfied:*

$$\begin{bmatrix} -P_{k-1} + D^T D & 0 & 0 & A_k^T P_k \\ * & -\gamma^2 I & 0 & \bar{B}^T_{w,k} P_k \\ * & * & -\gamma^2 I & -\bar{B}^T_{v,k} P_k \\ * & * & * & -P_k \end{bmatrix} < 0, \quad \forall k \in \{1, \ldots, N\}, \tag{6.16}$$

*where $P_0 = P_N$. Moreover, under zero initial condition the $H_\infty$ norm is bounded below $\gamma$, so that $\|z_k\|_{L_2} \leq \gamma \left\| \begin{bmatrix} w_k \\ v_k^k \end{bmatrix} \right\|_{L_2}$.*

**Proof.** The proof is based on the periodic Lyapunov lemma. Choose the following $N$-periodic Lyapunov function:

$$V_k = \xi_k^T P_k \xi_k, \; P_{k+N} = P_k, \forall k.$$

From Proposition 6.2, the forward difference of the Lyapunov function $\Delta V_k \triangleq V_{k+1} - V_k$ is given by

$$\begin{aligned}\Delta V_k &= \xi_k^T (A_k^T P_{k+1} A_k - P_k)\xi_k + 2\xi_k^T A_k^T P_{k+1}(\bar{B}_{w,k} w_k - \bar{B}_{v,k} v_k^k) \\ &+ (\bar{B}_{w,k} w_k - \bar{B}_{v,k} v_k^k)^T P_{k+1}(\bar{B}_{w,k} w_k - \bar{B}_{v,k} v_k^k).\end{aligned}$$

Now, some null terms are added:

$$\Delta V_k = \Delta V_k \pm \gamma^2 \begin{bmatrix} w_k \\ v_k^k \end{bmatrix}^T \begin{bmatrix} w_k \\ v_k^k \end{bmatrix} \pm z_k^T z_k.$$

The controlled output is $z_k = D\xi_k$. The forward difference can be written in the following quadratic manner:

$$\begin{aligned}\Delta V_k &= \begin{bmatrix} \xi_k \\ w_k \\ v_k^k \end{bmatrix}^T \begin{bmatrix} -P_k + D^T D & 0 & 0 \\ * & -\gamma^2 I & 0 \\ * & * & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \xi_k \\ w_k \\ v_k^k \end{bmatrix} \\ &+ \begin{bmatrix} \xi_k \\ w_k \\ v_k^k \end{bmatrix}^T \begin{bmatrix} A_k^T \\ \bar{B}_{w,k}^T \\ -\bar{B}_{v,k}^T \end{bmatrix} P_{k+1} \begin{bmatrix} A_k & \bar{B}_{w,k} & -\bar{B}_{v,k} \end{bmatrix} \begin{bmatrix} \xi_k \\ w_k \\ v_k^k \end{bmatrix} \\ &+ \gamma^2 \begin{bmatrix} w_k \\ v_k^k \end{bmatrix}^T \begin{bmatrix} w_k \\ v_k^k \end{bmatrix} - z_k^T z_k.\end{aligned}$$

The first two terms on the right-hand side can be grouped, yielding

$$\Delta V_k = \begin{bmatrix} \xi_k \\ w_k \\ v_k^k \end{bmatrix}^T M_k \begin{bmatrix} \xi_k \\ w_k \\ v_k^k \end{bmatrix} + \gamma^2 \begin{bmatrix} w_k \\ v_k^k \end{bmatrix}^T \begin{bmatrix} w_k \\ v_k^k \end{bmatrix} - z_k^T z_k.$$

Note that matrix $M_k$ is $N$-periodic provided that $P_k$ is $N$-periodic. Assume now that $M_k$ is negative definite for all $k$, i.e. $M_k < 0, \; \forall k$. For $w_k \equiv v_k^k \equiv 0$, one can easily see that $V(k)$ decreases in each step $k$. As the Lyapunov function is periodic, it will always decrease, which ensures the asymptotic stability of the system.

Consider now the case $w_k \neq v_k^k \neq 0$ and zero initial conditions. Since $M_k < 0,\ \forall k$, it holds

$$\Delta V_k < \gamma^2 \begin{bmatrix} w_k \\ v_k^k \end{bmatrix}^T \begin{bmatrix} w_k \\ v_k^k \end{bmatrix} - z_k^T z_k. \tag{6.17}$$

The summation of both sides of (6.17) from $k_0$ to $k$ is

$$V_k - V_{k_0} < \gamma^2 \sum_{j=k_0}^{k} \begin{bmatrix} w_j \\ v_j^j \end{bmatrix}^T \begin{bmatrix} w_j \\ v_j^j \end{bmatrix} - \sum_{j=k_0}^{k} z_j^T z_j.$$

Observe that $\sum_{j=k_0}^{k} \Delta V_j = \sum_{j=k_0}^{k} (V_{j+1} - V_j) = V_{k+1} - V_{k_0}$. Under zero initial condition, it holds $V_{k_0} = 0$. Recall that the Lyapunov function is positive definite. Let $k \to \infty$, that is,

$$0 < \gamma^2 \sum_{j=k_0}^{\infty} \begin{bmatrix} w_j \\ v_j^j \end{bmatrix}^T \begin{bmatrix} w_j \\ v_j^j \end{bmatrix} - \sum_{j=k_0}^{\infty} z_j^T z_j,$$

thus $\|z_k\|_{L_2} < \gamma^2 \left\| \begin{bmatrix} w_k \\ v_k^k \end{bmatrix} \right\|_{L_2}$.

Finally, it will be shown that conditions (6.16) imply that $M_k < 0,\ \forall k$. Applying Schur complements to $M_k$, it can be obtained that $M_k < 0$ is equivalent to

$$\begin{bmatrix} -P_k + D^T D & 0 & 0 & A_k^T \\ * & -\gamma^2 I & 0 & \bar{B}_{w,k}^T \\ * & * & -\gamma^2 I & -\bar{B}_{v,k}^T \\ * & * & * & -P_{k+1}^{-1} \end{bmatrix} < 0.$$

Conditions (6.16) are obtained if the above inequality is pre- and post- multiplied by $diag\{I, I, I, P_{k+1}\}$ and its transpose. □

Note that previous inequalities are nonlinear. Given a controller matrix $K$, to obtain LMI conditions, choose diagonal matrices $P_k = diag\{P_k^1, P_k^2\}$ and define $Y_k \triangleq P_k^2 L_k$ in the nonlinear terms $A_k^T P_k$, $\bar{B}_{w,k}^T P_k$ and $\bar{B}_{v,k}^T P_k$. With variables $P_k^1, P_k^2$ and $Y_k$ the inequalities above become LMIs. The observation matrices are synthesized as $L_k = (P_k^2)^{-1} Y_k$.

**Remark 6.1.** In some situations it is of interest to proportionate different observation gains to the same sensor when it appears repeatedly in the pattern. To do so, Lemma 6.3 can be used by introducing different matrices $Y_k$ for each element in the pattern.

Besides designing the observation matrices, Lemma 6.3 may also be used to find the maximum number of observation periods (OP) that can be introduced between two consecutive samplings.

As a final remark, it is possible to solve the LMIs derived from Lemma 6.3 in order to the matrices that minimize the $H_\infty$ gain $\gamma$. Define $\delta = \gamma^2$ and solve the problem

$$\begin{array}{l} \min\limits_{\delta} \\ s.t. \quad (6.16). \end{array}$$

This optimization problem can be efficiently solved with available software, as `mincx` in Matlab.

### 6.3.3 Pattern design

After presenting the design procedure for the $H_\infty$ periodic observation problem, the next step is the following: among all possible patterns, which ones stabilize the system? Which one is optimal? Obviously, a first idea would be to test different patterns by using the results of previous section. However, as the dimension of the system and the length of the pattern grow, this problem becomes computationally impractical.

In a first approach to the problem, it is obvious that those patterns which do not observe the dynamics of the observation error can be discarded. The research on controllability and observability of generic periodic system do not apply to the case under study, as only the output matrix $C_j$ is periodic [71, 75, 263]. Next, an appropriate observability criterion is introduced.

**Lemma 6.4.** *System* (6.1) *is observable at instant $k$ if and only if matrix*

$$O = \begin{bmatrix} C' \\ C'A' \\ \vdots \\ C'A'^{\,n-1} \end{bmatrix} \tag{6.18}$$

*has rank equal to $n$, where $A' \triangleq A^N$ and*

$$C' \triangleq \begin{bmatrix} C_k \\ C_{k+1}A \\ \vdots \\ C_{k+N-1}A^{N-1} \end{bmatrix}.$$

**Proof.** From an initial state $x_0$, the future outputs of the system will be calculated. The property of observability does not depend on the inputs, so that the autonomous system is considered. The first $N$ outputs are:

$$\begin{aligned} y_k &= C_k x_0, \\ y_{k+1} &= C_{k+1} A x_0, \\ &\vdots \\ y_{k+N-1} &= C_{k+N-1} A^{N-1} x_0. \end{aligned}$$

Let define $Y_1 \triangleq \begin{bmatrix} y_k^T & y_{k+1}^T & \cdots & y_{k+N-1}^T \end{bmatrix}^T$. It is immediate that $Y_1 = C' x_0$. After that, the pattern is repeated, so that the following $N$ outputs are

$$\begin{aligned} y_{k+N} &= C_k A^N x_0, \\ y_{k+N+1} &= C_{k+1} A A^N x_0, \\ &\vdots \\ y_{k+2N-1} &= C_{k+N-1} A^{N-1} A^N x_0. \end{aligned}$$

In a similar way, let define $Y_2 \triangleq \begin{bmatrix} y_{k+N}^T & y_{k+N+1}^T & \cdots & y_{k+2N-1}^T \end{bmatrix}^T$, so $Y_2 = C' A' x_0$. This procedure is repeated for the next $N$ steps, obtaining $Y_3 = C' A'^2 x_0$. Therefore, the set of all the future outputs $Y \triangleq \begin{bmatrix} y_k^T & y_{k+1}^T & \cdots \end{bmatrix}^T = \begin{bmatrix} Y_1^T & Y_2^T & \cdots \end{bmatrix}^T$ can be described by

$$Y = \begin{bmatrix} C' \\ C'A' \\ C'A'^2 \\ \vdots \end{bmatrix} x_0.$$

Using the Cayley-Hamilton theorem, $A'^l$ can be expressed as a linear combination of $A', A'^2, \ldots, A'^{n-1}$, for all $l \geq n$. Hence, $rank(O) = n$ is a necessary and sufficient condition to uniquely reconstruct $x_0$ from the future outputs. □

The definition of complete observability complements the previous lemma.

**Definition 6.3.** System (6.1) is completely observable if and only if is observable for all $k = 1, \ldots, N$.

Therefore, those patterns that do not achieve the complete observability of system (6.1) can be discarded. The observability test is easier, computationally speaking, than the LMI-based stability tests.

Nevertheless, this section seeks the optimal pattern that minimizes a given cost index. The index can be defined as

$$J = \sum_{j=0}^{\infty} e_j^T Q e_j, \tag{6.19}$$

where $Q$ is a positive definite matrix. A similar idea was firstly introduced in [79], but for optimal feedback stabilization of a system with only one unstable pole. The dependence of the index (6.19) with the chosen pattern is very complex, but it is possible to compare them, at least, in a numerical way. In the following, it is assumed that no disturbances affect the system and $P_o = 0$.

**Proposition 6.3.** *Given a pattern of length N and a positive definite matrix Q, the cost index J can be calculated as*

$$J = e_0^T \left[ \sum_{j=0}^{\infty} (\alpha_N^T)^j \Phi_N (\alpha_N)^j \right] e_0, \tag{6.20}$$

*where*

$$\begin{aligned} \alpha_N &= \theta_N \theta_{N-1} \dots \theta_2 \theta_1, \\ \Phi_N &= \sum_{i=1}^{N} \alpha_i^T Q \alpha_i, \\ \theta_k &= \vartheta_k(P_o = 0) = A - L_k C_k. \end{aligned}$$

The proof is immediate by substituting the evolution of the system (6.9) with $w_k = v_k = 0,\ \forall k$, in the cost index (6.19).

In order to use Proposition 6.3, the initial state of the system must be known, which seems unrealistic in some situations. A more natural situation considers a known bound on the observation error $\|e_0\|^2$. For such cases, an upper bound of the cost index is searched. To do so, the following relation is used: $x^T A x \leq \lambda_{\max}\{A\}\|x\|^2$. Hence, the cost index (6.20) can be bounded by $J \leq \lambda_{\max}\{\beta_N\}\|e_0\|^2$, with $\beta_N \triangleq \sum_{j=0}^{\infty} (\alpha_N^T)^j \Phi_N (\alpha_N)^j$.

For a periodic pattern, there are two features to be designed: the length and the pattern itself. The length is critical, in such a way that, if it is not fixed *a priori*, the number of combinations grows to infinity. Among all possible patterns, this section looks for the one that minimizes the maximum eigenvalue of matrix $\beta_N$.

Despite the fact that the cost index has points of similarity with a geometric series, to the best of our knowledge, it is not possible to obtain an analytical expression for $\beta_N$. However, if the pattern verifies the condition in Lemma 6.3, matrix $\alpha_N$ has all its eigenvalues inside the unit circle. Then the infinite sum (6.20) can be replaced by a finite one without incurring in practical errors, as it is shown in the following example. Finally, this finite sum can be calculated numerically.

Once the optimal pattern has been found, the design of the observation matrices is carried out by using the techniques described in the previous section.

**Example 6.3.** Consider the discrete-time system in [262]:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1 & 0.1 & 0 & 0 \\ 0.1 & 1.25 & 0 & 0 \\ 1 & 0.1 & 1/6 & 0.5 \\ 0 & 0 & 0 & 1.25 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u_k, \\ y_k &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} x_k. \end{aligned} \tag{6.21}$$

There exist two possible outputs to send through the network $y_k^1 = [1\ 0]y_k$ and $y_k^2 = [0\ 1]y_k$. The weighting matrix is chosen as $Q = diag\{1, 1\}$. It is assumed that $e_0 = x_0$, that is, the initial condition for the observer is exactly zero. In that paper the choice of the pattern was made based only on the observability criterion.

The objective here is to choose and to design the optimal pattern among all possible ones. To calculate the cost index, a finite horizon is needed. Here, this horizon is chosen larger enough to neglect the errors in the approximation of $\beta_N$.

Figure 6.4 depicts the minimum cost index reached for each pattern length and for two different initial conditions:

$$x_{0a}^T = \begin{bmatrix} 2 & -1 & 0.5 & -0.3 \end{bmatrix}; \quad x_{0b}^T = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

As it can be seen, the cost depends on the initial condition. Good patterns for some conditions could not be adequate for others.

The analysis can be carried out to minimize the maximum eigenvalue of $\beta_N$. Recall that the cost is bounded by $J < \lambda_{\max}\{\beta_N\} e_0^T e_0$.

As Figure 6.5 shows, the larger values of the pattern length does not always imply the lower costs. For this example, a length of 7 is appropriate. The optimal pattern is $\varphi_7 = [1\ 1\ 2\ 1\ 2\ 2\ 2]$. ▼

### 6.3.4 Mixed design with pole placement

In this section, a co-design method for both the observer matrices and the pattern at the same time is proposed. The idea is similar to that presented in [56]. The method

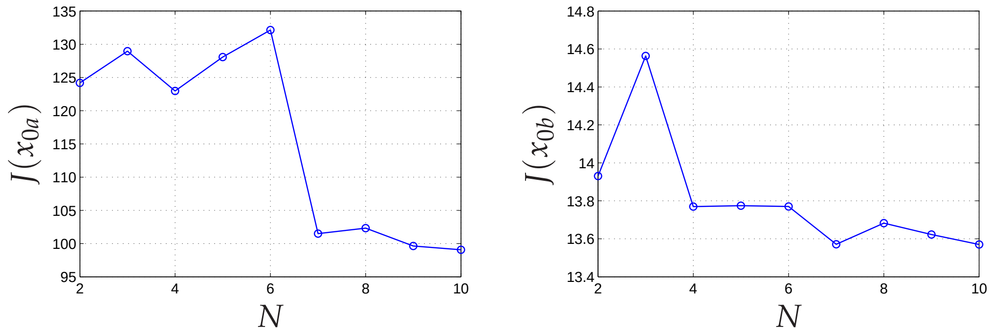



Figure 6.4: Cost index for various pattern lengths and two different initial conditions

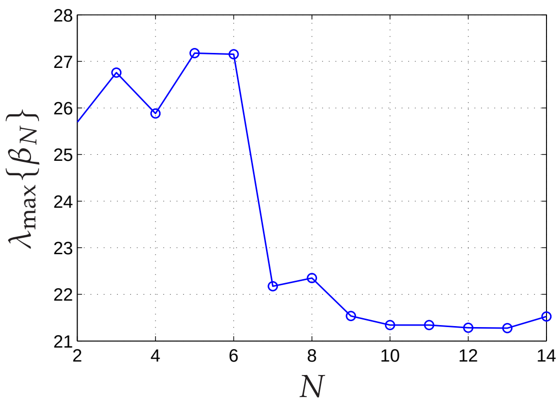



Figure 6.5: Maximum eigenvalue of $\beta_N$ for various pattern lengths

is given for the unperturbed system with $P_o = 0$, but it can be easily extended for $P_o > 0$. The dynamics of the observation error given in Proposition 6.1 can be viewed as a switched system $e_{k+1} = \Theta_k e_k$, where $\Theta_k = (A - L_k C_k)$. The following algorithm describes the idea.

**Algorithm 6.1.**

1. Find a nonsingular matrix $T$ such that $A = T^{-1}\hat{A}T$, where $\hat{A}$ is an upper (or lower) triangular matrix.

2. Define a new set of coordinates as $\varrho_k = Te_k$. Then, $\varrho_{k+1} = T\Theta_k T^{-1}\varrho_k = \hat{\Theta}_k\varrho_k$. The switched matrix $\hat{\Theta}_k$ is defined by $\hat{\Theta}_k = \hat{A} - TL_kC_kT^{-1}$.

3. Choose $L_k$ so that matrix $TL_kC_kT^{-1}$, and hence $\hat{\Theta}_k$, are upper (or lower) triangular.

4. Using properties of the product of upper (lower) triangular matrices, design the measurement pattern $\varphi_N$ in order to place the poles of the dynamics of the observation error after $N$ instants.

Steps 1 and 2 are trivial. The key step is 3, as not only do we lose degrees of freedom in the choice of $L_k$, but the triangularization of $\hat{\Theta}_k$ may also be impossible. This is the main disadvantage of the algorithm, which can only be applied to certain systems. If it is possible to triangularize $\hat{\Theta}_k$, there are certain freedom to place the eigenvalues of the observation error.

**Example 6.4.** Consider the discrete-time system from [95], but with a variable element in position $(3,3)$:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & 0 & a \end{bmatrix} x_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u_k, \\ y_k^1 &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x_k, \\ y_k^2 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_k, \end{aligned}$$

where $a \in \mathbb{R}$. Matrix $A$ is upper triangular, so no change of coordinates must be done. The observer gains are $L_1 = [l_{11} \;\; l_{12} \;\; l_{13}]^T$ ; $L_2 = [l_{21} \;\; l_{22} \;\; l_{23}]^T$. Hence, matrices $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are given by

$$\hat{\Theta}_1 = \begin{bmatrix} 1 & -l_{11} & 0 \\ 0 & -1-l_{12} & -3 \\ 0 & -l_{13} & a \end{bmatrix}, \; \hat{\Theta}_2 = \begin{bmatrix} 1-l_{21} & 0 & 0 \\ -l_{22} & -1 & -3 \\ -l_{23} & 0 & a \end{bmatrix}.$$

To make them upper triangular, some degrees of freedom are lost, as $l_{13} = l_{22} = l_{23} = 0$. If one chooses, for instance, the pattern $\varphi_3 = [1 \;\; 1 \;\; 2]$, the eigenvalues of the complete observation error are:

$$\begin{aligned} \lambda_1 &= 1 - l_{21}, \\ \lambda_2 &= -(1+l_{12})^2, \\ \lambda_3 &= a^3. \end{aligned}$$

Eigenvalue $\lambda_3$ cannot be placed, as it does not depend on any of the observation gains. To preserve the stability a necessary condition is that $|a| < 1$. On the other hand, eigenvalues $\lambda_1$ and $\lambda_2$ can be freely placed by choosing adequately $l_{21}$ and $l_{12}$, respectively. ▼

Note that this method is not restricted to periodic patterns. Some eigenvalues can be prioritized online with respect to others. For instance, in the previous example, output 1 (output 2) could be sent when the dynamics related to eigenvalue $\lambda_1$ (eigenvalue $\lambda_2$) needs to be modified.

## 6.4 Aperiodic scheduling

In contrast to the previous section, in which the scheduling followed an strict pattern, here an aperiodic use of the shared medium is proposed. This solution is able to obtain better results, as the choice of the output is made following some kind of optimal decision. The cost is an increasing necessity of calculus and the existence of an external agent that manages the activation of the different sensors.

As explained before, the proposed observer is a Kalman-based filter, that is, it tries to minimize the variance of the error $E\left[e_k^T e_k\right]$. It is assumed that $w_k$ and $v_k^j$ are both i.i.d. Gaussian processes with

$$\begin{aligned} E[w_k] &= 0,\ E[w_k w_k^T] = Q, & (6.22)\\ E[v_k^j] &= 0,\ E[v_k^j v_k^{jT}] = R_j,\ \forall j. & (6.23)\end{aligned}$$

An observer is completely determined if the sequences $\{C_j\}$ and $\{L_j\}$ are defined. Similar approaches have studied the optimal design of the observation gains $L_j$ for a given pattern [89, 262]. However, in this section both families $C_j$ and $L_j$ will be designed to obtain an optimal observer that minimizes the variance of the observation error. Let $P_k = E\left[e_k e_k^T\right]$ denote the covariance matrix at time $k$. Making an analysis similar to that of optimal Kalman filtering, the evolution of $P_k$ can be calculated as

$$\begin{aligned} P_{k+1} &= AP_kA^T + B_wQB_w^T - AP_kC_k^TL_k^T - C_kE_kP_kA^T \\ &+ L_kC_kP_kC_k^TL_k^T + L_kR_kL_k^T. \end{aligned} \tag{6.24}$$

Using known properties of the trace of a matrix, the derivative of equation (6.24) with respect to $L_k$ is

$$\frac{\partial tr\left\{P_{k+1}\right\}}{\partial L_k} = -2C_kP_kA^T + 2\left(C_kP_kC_k^T + R_k\right)L_k^T.$$

The optimal observer gain $L_k^*$ can be found by imposing $\frac{\partial tr\{P_{k+1}\}}{\partial L_k} = 0$:

$$L_k^* = AP_kC_k^T\left[C_kP_kC_k^T + R_k\right]^{-1}. \tag{6.25}$$

Using this value for the observer gain in equation (6.24), the matrix covariance at instant $k+1$ is

$$P_{k+1} = AP_kA^T + B_wQB_w^T - AP_kC_k^T\left[C_kP_kC_k^T + R_k\right]^{-1}C_kP_kA^T. \tag{6.26}$$

Other authors assume that a pattern of outputs is given and they solve a periodic Riccati equation to obtain a steady-state periodic covariance matrix [89, 262]. On the contrary, in this thesis the activation of the sensor is also made in an optimal way. That is, the activated sensor at instant $k$ is determined by

$$\begin{aligned} C_k^* &= arg \min \; tr\left\{P_{k+1}\right\}, \\ &= arg \max \; tr\left\{AP_kC_k^T\left[C_kP_kC_k^T + R_k\right]^{-1} C_kP_kA^T)\right\}. \end{aligned} \tag{6.27}$$

The previous optimization problem can be easily solved since only $m$ different matrices $C_k$ are defined, so that it is straightforward to find the optimal one.

Assuming that $P_k$ is bounded $\forall k$, the state of the plant will be estimated with a estimation error with zero mean and bounded variance.

It can be seen that the choice of the output $j$ and the gain $L$ are based on the covariance matrix, that is, $j = j(P_k)$ and $L = L(P_k, j(P_k))$. It is obviously state based. Furthermore, it is also an *one-step-ahead* scheduling law because it optimizes $tr\ P_{k+1}$. Other *N-steps-ahead* laws, based on the traces of $P_{k+1}, \ldots, P_{k+N}$, might be subject of future research.

**Example 6.5.** Consider the system proposed in [89] with the following parameters:

$$A = \begin{bmatrix} 1.1 & 0 & 0 & 0 & 0 & 0 \\ -1.5 & 0 & -0.75 & -1.5 & 0.75 & -0.75 \\ -1.1 & 0 & 0 & -1.1 & 0 & 0 \\ 0 & 0 & 0 & 1.1 & 0 & 0 \\ 1.1 & 0.75 & 0 & 1.1 & 0 & -0.75 \\ -0.75 & 0 & -0.75 & -0.75 & 0 & -0.75 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \left\{ \begin{aligned} C_1 &= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\ C_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix} \\ C_3 &= \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \end{aligned} \right. .$$

The disturbances and noises in (6.1) and (6.2) are described by $Q = 0.35I_{6\times6}$ and $R_i = 1$, $i = 1, 2, 3$. The initial conditions for the system and the observer are $x(0) = [1\ \ 50\ \ 7\ \ 6\ \ 1\ \ 2]^T$, $\hat{x}(0) = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]^T$. Finally, the initial covariance matrix is $P_0 = 0.2I_{6\times6}$. The same periodic controller as in [89] is employed.

Figures 6.6 shows the evolution of the observation error and which sensor is activated in each instant.

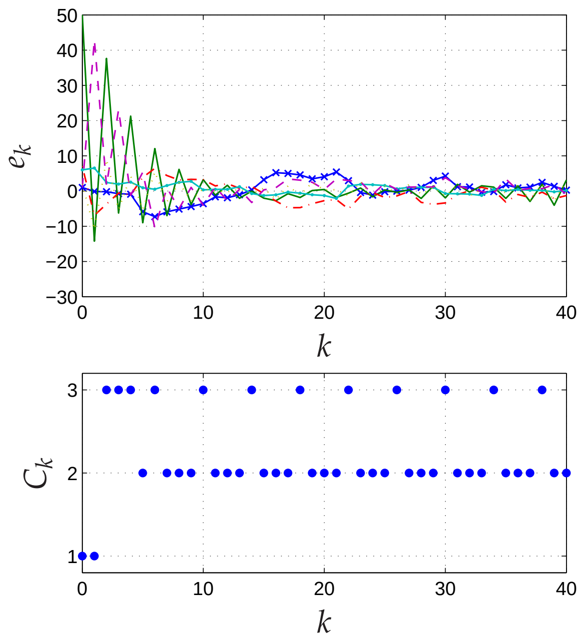



Figure 6.6: Evolution of the observation error (up) and sensors activated (bottom) with the proposed observer design.

All states entered a small 'band' around the origin in about 8 steps, whilst 20 steps were needed in [89]. The key of the improvement is the optimal choice of the sensors. At the beginning, different outputs are chosen to achieve a better transient. However, it is worth noting the periodic phenomenon that appears when the system evolves near the practical steady-state. Pattern 1, 2 and 2 was proposed in [89].

Figure 6.7 depicts the cost measured as the trace of the covariance matrix. This cost is plotted for three situations. The steady-state value of the trace of the covariance matrix is smaller for the proposed method.

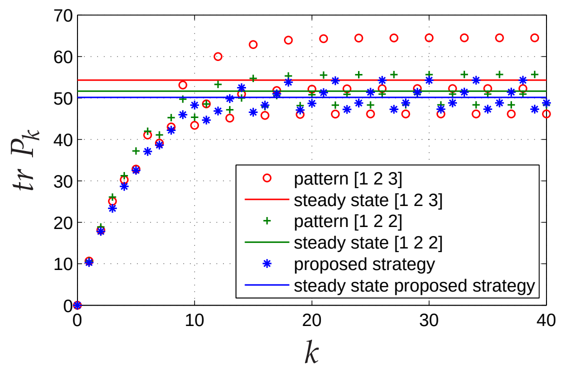



Figure 6.7: Traces of the covariance matrix for different strategies.

▼

The previous example shows that, for some particular systems, the activated sensors under the proposed optimal policy follows a concrete periodic pattern. If this was true, the sensors could be properly initialized and to obtain some benefits: reduced computational burden and minimum energy consumption, as the sensors could remain asleep between two consecutive measurements.

The following sections study this problem and proves that, under some conditions in particular cases, the intuition is true and the optimal scheduling is eventually periodic.

## 6.5 Periodicity of the optimal scheduling

Some preliminaries are needed before proceeding with the section.

### 6.5.1 Preliminaries

Recall the dynamics of the covariance matrix when the optimal Kalman gain is chosen:

$$P_{k+1} = AP_kA^T + B_wQB_w^T - AP_kC_k^T\left[C_kP_kC_k^T + R_k\right]^{-1}C_kP_kA^T.$$

Using well-known matrix inversion lemmas, this equation can be rewritten as

$$P_{k+1} = A\left(P_k^{-1} + C_k^TR_k^{-1}C_k\right)^{-1}A^T + B_wQB_w^T. \tag{6.28}$$

This formulation will be used in the developments that follow. Now, the Riemannian distance for positive definite matrices is introduced.

**Definition 6.4.** For any pair of matrices $P, Q > 0$, the Riemannian distance $\delta$ is defined by

$$\delta(P,Q) = \left(\sum_{i=1}^{d}\log^2\lambda_i\right)^{\frac{1}{2}},$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of the matrix $PQ^{-1}$.

This distance has some interesting properties.

**Proposition 6.4.** *[17] Let $P, Q > 0$ and $R \geq 0$. For any nonsingular matrix $A$,*

$$\begin{aligned}\delta(APA^*, AQA^*) &= \delta(P,Q),\\ \delta(P^{-1}, Q^{-1}) &= \delta(P,Q),\\ \delta(P+R, Q+R) &\leq \frac{\alpha}{\alpha+\beta}\delta(P,Q),\end{aligned}$$

*where $\alpha = \max(\|P\|, \|Q\|)$ and $\beta = \inf\{\langle Rx, x\rangle, \|x\| = 1\}$.*

To proceed, the notation given in [205] is borrowed. Let $h_i : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $i = 1, \ldots, m$, and $s : \mathbb{R}^{n\times n} \to \mathbb{N}$ be fixed maps. Consider the discrete-time system

$$X_{k+1} = h_{s(X_k)}(X_k), \quad X_0 \in \mathbb{R}^{n\times n}. \tag{6.29}$$

The system can be thought of as a switched system. The state-based scheduling law $s$ partitions the state space $\mathbb{R}^{n\times n}$ into disjoint sets $\{U_j, j = 1, \ldots, m\}$ with $U_j = s^{-1}(j)$. When the state belongs to $U_j$ for some instant $k$, the system evolves according to the transition map $h_j$.

Since the system (6.29) is deterministic, each initial state $X_0$ produces a different trajectory $\{X_k\}$ and a unique sequence $\{s(X_k)\}$. The observation sequence $\{s(X_k)\}$ is **eventually periodic** if there exist integers $N, M > 0$ such that

$$s(X_{k+N}) = s(X_k), \quad \forall k \geq M.$$

Then, $N$ would be the **eventual period** of $\{s(X_k)\}$.

Let $int(U_j)$ denote the interior of the set $U_j$. If $Z \in U_j$ but $Z \notin int(U_j)$, then $Z$ is a **boundary point**. The interior of the partition induced by $s$ is the open set $G = \cup_j\, int(U_j)$, and the **boundary** of the partition is the closed set $H = \mathbb{R}^{n\times n} - G$. In the remaining, we refer to $H$ as the **switching boundary**.

The following theorem is a modification of Theorem 2.1 proposed by Ramadge in [205]. It employs the distance $\delta$ instead of the norm $\|.\|$, as in [205]. Moreover, it trivially extends the results of the theorem to sequences of $n \times n$ matrices instead of vectors.

**Theorem 6.1.** *Let $\delta$ be distance in the metric space $(\mathbb{R}^{n\times n}, \delta)$, and let $X_0 \in \mathbb{R}^{n\times n}$ be a fixed initial state for (6.29). Assume that:*

**i)** *For each $i = 1, \ldots, m$, the map $h_i$ is $C^1$ on $int(U_i)$, with $\delta(h_i(X), h_i(Y)) \leq \delta(X, Y)$, $\forall X, Y \in int(U_i)$.*

**ii)** *The trajectory $\{X_k\}$ is bounded and all its limit points are contained in $G$, i.e., there are no limit points on the switching boundary.*

*Then the observation sequence $\{s(X_k)\}$ is eventually periodic.*

#### 6.5.1.1 Proof of Theorem 6.1

The proof follows the same steps than that of Theorem 2.1 in [205], so only the parts that have been modified will be detailed.

Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the map

$$F(X) = h_{s(X)}(X).$$

A subset $\mathcal{X} \subset \mathbb{R}^{n\times n}$ is $F$-invariant if $X \in \mathcal{X}$ implies $F(X) \in \mathcal{X}$.

For the remainder of the proof, consider a distance $\delta$ in the metric space $(\mathbb{R}^{n\times n}, \delta)$, and assume that the $h_i$ satisfy the assumptions of the theorem. Note that for all $i = 1, \dots, m$, if $X, Y \in U_i$, with $Y \in B(X, \epsilon) \subseteq U_i$, some $\epsilon > 0$, then $\delta(h_i(X), h_i(Y)) \leq \delta(X, Y)$.

The proof of the theorem makes use of the following results.

**Lemma 6.5.** *Let $\{X_k\}$ be a bounded state trajectory of* (6.29) *with a infinite number of distinct terms. If the set of limit points $\Omega$ of $\{X_k\}$ has no element on the switching boundary, then $\Omega$ is a nonempty compact $F$-invariant subset of $G$.*

**Proof.** The proof is similar to that of Lemma 2.1 in [205], but using the distance instead of the norm.

As Ramadge showed, the set of limit points $\Omega$ is nonempty and compact. The $F$-invariant part must be adapted. Let $X \in \Omega$, with $X \in U_j$. As no limit points lies on the switching boundary, then $X \in int(U_j)$. Let $\epsilon > 0$. Since $h_j$ is continuous on $int(U_j)$, there exists a $\sigma > 0$ such that $\delta(X, Y) < \sigma$ implies that $\delta(h_j(X), h_j(Y)) < \epsilon$. Select $k > 0$ so that $\delta(X, X_k) < \sigma$. Then

$$\delta(F(X), F(X_k)) = \delta(h_j(X), h_j(X_k)) < \epsilon.$$

Since $\epsilon$ can be chosen arbitrarily small, if follows that $F(X)$ is a limit point of $\{X_k\}$, i.e., $F(X) \in \Omega$. □

The following lemma focuses on the nonempty compact $F$-invariant subsets of $G$. The proof is not included as it can be trivially extended to elements in the space $\mathbb{R}^{n\times n}$.

**Lemma 6.6.** *[205]. Let $\mathcal{X}$ be a nonempty compact subset of $G$. Then:*

**i)** *There exists $\mu > 0$ such that for each $X \in \mathcal{X}$, $B(X, \mu) \subseteq U_{s(X)}$.*

**ii)** *If $X, X' \in \mathcal{X}$, $0 < \epsilon \leq \mu$, and $B(X, \epsilon) \cap B(X', \epsilon) \neq \emptyset$, then $s(X) = s(X')$.*

The following proposition is the core of the proof. It shows that any nonempty compact $F$-invariant subset contained in $G$, i.e. in the interior of the cosets of $s$, is the 'preimage' of a finite state automaton.

**Proposition 6.5.** *Let $\mathcal{X}$ be a nonempty compact F-invariant subset of G. Then, there exits an open F-invariant set $G_o$, with $\mathcal{X} \subseteq G_o \subseteq G$, a finite set Q, functions $\alpha : Q \to Q$, and $\beta : Q \to N$, and a surjection $\pi : G_o \to Q$, such that*

$$\begin{array}{ccccc} G_o & \stackrel{F}{\to} & G_o & \stackrel{s}{\to} & N \\ \pi\downarrow & & \pi\downarrow & \beta\nearrow & \\ Q & \stackrel{\alpha}{\to} & Q & & \end{array}$$

*i.e., such that for each $X \in G_o$, $\pi(F(X)) = \alpha(\pi(X))$, and $s(X) = \beta(\pi(X))$.*

**Proof.** The proof is again a modification of that of Proposition 2.1 in [205] to use distances.

Let $0 < \epsilon \leq \mu$ and set

$$G_o = \bigcup_{X \in \mathcal{X}} B(X, \epsilon).$$

Clearly, $G_o$ is an open set and $\mathcal{X} \subseteq G_o \subseteq G$. If $W \in G_o$, then there exists $X \in \mathcal{X}$ with $W \in B(X, \mu)$. By Lemma A.2, we have $s(X) = s(W)$, and hence,

$$\delta(F(W), F(X)) = \delta(h_{s(X)}(W), h_{s(X)}(X)) \leq \delta(X, W) < \epsilon,$$

i.e., $F(W) \in B(F(X), \epsilon)$. Since $\mathcal{X}$ is $F$-invariant, $F(X) \in \mathcal{X}$, and hence $F(W) \in G_o$. Thus, $G_o$ is $F$-invariant.

The rest of the proof, that is, the existence of the functions $\alpha, \beta$, the surjection $\pi$, and the finite set $Q$, follows the same steps that the work of Ramadge. □

Now, all the preliminaries needed to prove Theorem 6.1 have been given.

**Proof of Theorem 6.1. [205]** If the state trajectory contains only a finite number of distinct terms, then the theorem is trivially true.

If $\{X_k\}$ contains an infinite number of distinct terms, then it follows from Lemma 6.5 that the set of limit points $\Omega$ is a nonempty compact $F$-invariant subset of $G$. Then an application of Proposition 6.5 shows that there exists an open set $G_o$ containing $\Omega$ and a finite automaton $(Q, \alpha)$ with observation map $\beta$ that is a quotient of the dynamic system (6.29) restricted to $G_o$. Since $\Omega$ is the set of limit points of $\{X_k\}$, the state trajectory eventually enters and thereafter remains in $G_o$. From this point on, the evolution of the index sequence is governed by a finite state deterministic system, and thus it is eventually periodic. □

### 6.5.2 Periodicity of the Kalman-based filter

Using the notation introduced above, the evolution of the covariance matrix $P_k$ given in (6.28) corresponds to that of the switched system given in (6.29). The transmitted outputs $j$ are determined by the scheduling law $s(P_k)$. It is initially considered that $s$ is any scheduling law producing a bounded sequence of $\{P_k\}$.

The following theorem states the periodicity of the proposed filter.

**Theorem 6.2.** *Let the evolution of the covariance matrix be given by (6.28) or by any other finite combination of inversion, conjugacy and sum, and let $P_0 \in \mathbb{R}^{n\times n}$ be a fixed initial state for* (6.28)*. Assume that*

**i)** *The system matrix $A$ is nonsingular.*

**ii)** *The trajectory $\{P_k\}$ is bounded and there are no limit points on the switching boundary.*

*Then the sequence of outputs $\{j\}$ is eventually periodic.*

**Proof.** It will be proved that the assumptions $i)$ and $ii)$ imply that the conditions of Theorem 6.1 are verified.

First of all, consider the evolution of the covariance matrix (6.28), which can be rewritten as

$$P_{k+1} = h_{s(P_k)}(P_k),$$

where the maps $h_i,\ i = 1,\dots,m$, and $s$ are defined by

$$\begin{aligned} h_i(P_k) &= A\left(P_k^{-1} + C_i^T R_i^{-1} C_i\right)^{-1} A^T + B_w Q B_w^T, \\ s(P_k) &= arg\min_i tr\ P_{k+1}. \end{aligned}$$

Maps $h_i,\ i = 1,\dots,m$ are $C^1$ on $int(U_i)$. Furthermore, using Proposition 6.4 and the fact that $A$ is nonsingular, it can be verified that $\delta(h_i(X), h_i(Y)) \leq \delta(X,Y)$, $\forall X, Y \in int(U_i)$, where $\delta$ is the Riemannian distance described in Definition 6.4.

As the trajectory $\{P_k\}$ is bounded and there are no limit points of $\{P_k\}$ on the switching boundary, then all the points in Theorem 6.1 are verified and the sequence $\{j\}$ is eventually periodic. $\square$

The assumptions of the theorem deserve some comments. Assumption $i)$ is not severe. If the discrete system is obtained by sampling a continuous system, matrix $A$ is nonsingular.

As explained in the previous section, assumption $ii)$ is necessary. First of all, if the trajectory $\{P_k\}$ is not bounded, the observer will not be able to estimate the state of the system, as the variance of the estimation error is unbounded. Therefore, it is uninteresting that the scheduling eventually results in a periodic selection. However, it does have interest to know whether the scheduling law produces a bounded trajectory. This is subject of study in the following section.

With respect to the presence of limit points in the switching boundary, Ramadge shows in [205] that chaotic behaviours (on both the state and the observation sequence) are observed in systems whose limit points belong to the switching boundary. Therefore, this assumption seems to be indispensable. The problem is that the location of the limit points of $\{P_k\}$ is unknown *a priori*.

The reference [29] suggests that the sets of limit points of intermittent Kalman Filters (KFs) have a fractal nature, similar to a Cantor set, inherited from the interpretation of intermittent sequences as binary codes of numbers in $[0, 1]$. Notice that a Cantor Set has zero measure since it is obtained by recursively removing 'middle thirds' of the interval $[0, 1]$. Very likely, this fractal nature might be shared by switched KFs in networked operation, and thus condition $ii)$ on limit points outside the boundary could be assured by moving the boundary off the fractal, by adequate modifications of the KF parameters (matrices $Q$,$R$).

## 6.6 Bounded trajectories

Theorem 6.2 assumes that the scheduling law yields a bounded trajectory $\{P_k\}$. The optimal policy presented in Section 6.4 can be described by the following state-based scheduling law $s(P_k)$:

$$s(P_k) = arg \min_j \; tr \; P_{k+1}|_{C_j} \,. \tag{6.30}$$

In the following subsections it will be proved that the optimal scheduling proposed in (6.30) yields a bounded trajectory of $\{P_k\}$ for some particular and interesting cases.

The proof that the proposed Kalman observer is bounded if $(A, C)$ is detectable, cannot be presented here in its general form. It seems that it would require a very involved and lengthy discussion of many combination cases regarding (a) the stable-antistable subspaces of $A$; (b) the unobservable subspaces for $(A, C_j)$; and (c) the way in which the $C_j$ share (or distribute) the output information, that is, the subspaces spanned by $C_j$.

Instead, a deep analysis will be presented at the end of this section about the implications of choosing the scheduling law (6.30) in a general case.

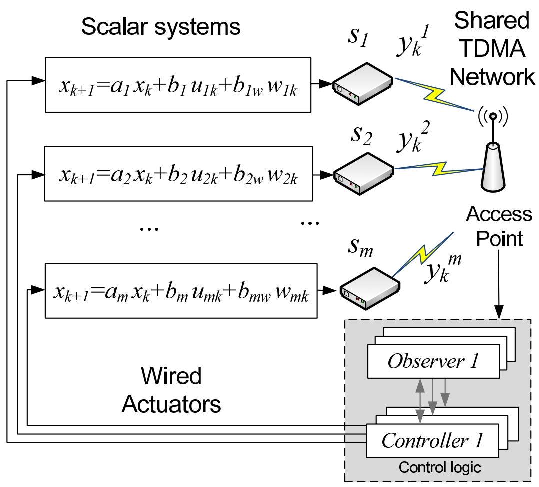



Figure 6.8: Diagonal case: a set of scalar systems sharing a communication network

### 6.6.1 The diagonal case

Suppose that matrices $A, B_w$ are diagonal,

$$\begin{aligned} A &= diag\{a_1, a_2, \ldots, a_n\}, \\ B_w &= diag\{1, 1, \ldots, 1\}, \end{aligned}$$

and the disturbances and noises are described by

$$\begin{aligned} Q &= diag\{q_1, q_2, \ldots, q_n\}, \\ R_j &= r_j, \ \forall j, \end{aligned}$$

with positives $q_j, r_j, j = 1, \ldots, n$. Further suppose that the set of output matrices $\{C_j\}$ is $\{e_j\}$, the canonical basis, with $e_j$, ($j = 1, \ldots, n$) the row vector containing a one '1' in the $j$th position and 0's elsewhere. This situation represents the case of $n$ independent first-order subsystems, coupled by the restriction that only one measurement $y_k^j$ can be transmitted at a time (see Figure 6.8).

The following two propositions study the scheduling law and the evolution of the covariance matrix for this particular case.

**Proposition 6.6.** *Suppose that the covariance matrix is $P_k = diag\{p_1, p_2, \ldots, p_n\}$ at instant k. Then, using the proposed optimal policy (6.30) the activated sensor j is*

$$j = arg \max_j \frac{a_j^2 p_j^2}{p_j + r_j}.$$

**Proposition 6.7.** *Suppose that the covariance matrix is $P_k = diag\{p_1, p_2, \ldots, p_n\}$ at instant $k$. Then, given that output $j$ ($j = 1, \ldots, n$) is transmitted at instant $k$, the covariance matrix at $k+1$ is given by*

$$P_{k+1} = \begin{array}{cl} \begin{bmatrix} g(p_1) & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & f(p_j) & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & g(p_n) \end{bmatrix} & \begin{matrix} \\ \\ \textit{row } j \\ \\ \\ \end{matrix} \end{array}$$

*where scalar functions $f(p_i), g(p_i)$ are defined by*

$$\begin{aligned} f(p_i) &= a_i^2 r_i \frac{p_i}{p_i + r_i} + q_i, \\ g(p_i) &= a_i^2 p_i + q_i. \end{aligned}$$

The proofs are straightforward by substituting matrices $A, B_w, C, Q$ and $R_j$, ($j = 1, \ldots, n$) in equation (6.26).

The main conclusion of these propositions is that the covariance matrix remains diagonal for all $k$. It is worth noting the importance of the functions $f(p_i)$ and $g(p_i)$. While function $g$ makes the covariance increase with $|a_i| \geq 1$ or decrease with $|a_i| < 1$, function $f$ is always a map that makes that the covariance tends to a stable value. Figure 6.9 illustrates both functions. It is also important to remark that function $f(p_i)$ is upper bounded by

$$f(p_i) < a_i^2 r_i + q_i, \ \forall p_i.$$

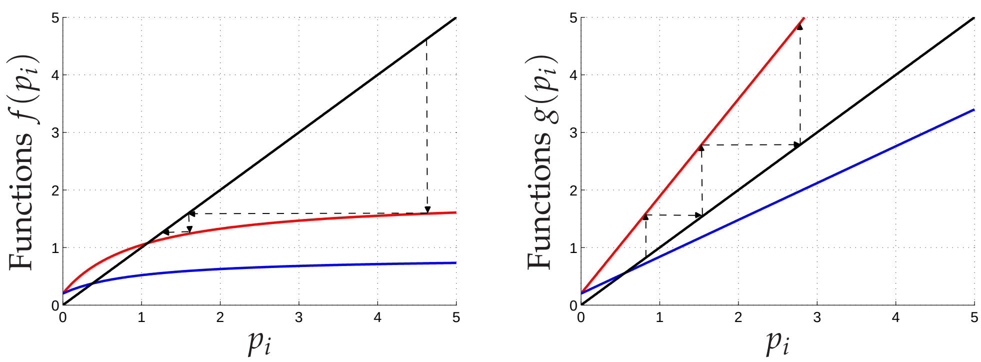



Figure 6.9: Functions $f(p_i), g(p_i)$ for $|a| > 1$ (red line) and $|a| < 1$ (blue line). Bisector of the first quadrant (black line).

These features are the key of the following lemma, which states that the proposed optimal policy yields a bounded sequence of $\{P_k\}$. The name 'stabilizing' map or function is used to denote a function whose application results in a stable trajectory.

**Lemma 6.7.** *In the diagonal case, the sequence of $\{P_k\}$ is bounded if the scheduling law (6.30) is used.*

**Proof.** Assume first that $n = 2$, i.e., the dimension of the system is 2, and consider three situations: (a) $|a_1| < 1, |a_2| < 1$; (b) $|a_1| \geq 1, |a_2| < 1$; (c) $|a_1| \geq 1, |a_2| \geq 1$.

**(a)** $|a_1| < 1, |a_2| < 1$**:** This case is trivial. Both families of functions $f(p_1), g(p_1)$ and $f(p_2), g(p_2)$ are stabilizing, so $p_i$ is bounded for $i = 1, 2$.

**(b)** $|a_1| \geq 1, |a_2| < 1$**:** In this case, functions $f(p_2), g(p_2)$ and $f(p_1)$ are stabilizing, but $g(p_1)$ is not. Let $\bar{p}_2$ denote the maximum $p_2(k), \forall k$. Assume that for some $k$,

$$\frac{a_2^2 p_2(k)^2}{p_2(k) + r_2} > \frac{a_1^2 p_1(k)^2}{p_1(k) + r_1},$$

that is, sensor 2 is to be activated at $k+1$, as Proposition 6.6 states. The left-hand side of the equation above is upper bounded by $\frac{a_2^2 \bar{p}_2^2}{\bar{p}_2 + r_2}$ for all $k$. Furthermore, at instant $k+1$ it holds $p_1(k+1) = g(p_1(k))$. As $g$ makes $p_1$ increase, there must exist some $k^* > k$ such that

$$\frac{a_2^2 \bar{p}_2^2}{\bar{p}_2 + r_2} < \frac{a_1^2 p_1(k^*)^2}{p_1(k^*) + r_1},$$

as function $\frac{a_1^2 p_1(k)^2}{p_1(k) + r_1}$ is increasing in $p_1(k)$ due to $|a_1| > 1$. It turns out that sensor 1 will transmit at instant $k^* + 1$. Then, it has been proved that there exists a maximum number of consecutive instants in which sensor 2 remains activated before one sample of sensor 1 is sent. Let $n_1$ denote that maximum number. As function $f$ is stabilizing, $p_1(k^* + 1)$ is upper bounded by

$$p_1(k^* + 1) < a_1^2 r_1 + q_1.$$

Therefore, the maximum value of $p_1(k)$ is given by

$$\bar{p}_1 \triangleq \max p_1(k) = g^{n_1}(a_1^2 r_1 + q_1),$$

where $g^{n_1}$ denotes that function $g$ is applied $n_1$ consecutive times. Then, both $p_1, p_2$, and hence matrix $P$, are bounded.

**(c)** $|a_1| \geq 1, |a_2| \geq 1$**:** In this case, functions $f(p_1)$ and $f(p_2)$ are stabilizing, and $g(p_1)$ and $g(p_2)$ are not. The same idea than before can be used: after a finite consecutive number of instants the activated sensor switches. Assume that for some instant $k$ sensor 1 is transmitting. Then, it turns out that $p_1(k+1) < a_1^2 r_1 + q_1$. On the other hand, $p_2(k)$ will increase. There must exist some $k_2$ in which the following holds:

$$\frac{a_2^2 p_2(k_2)^2}{p_2(k_2) + r_2} > \frac{a_1^2 (a_1^2 r_1 + q_1)^2}{(a_1^2 r_1 + q_1) + r_1}.$$

Then, sensor 2 will be activated at instant $k_2 + 1$. Taking into account that function $f$ is upper bounded, it is verified that $p_2(k_2+1) < a_2^2 r_2 + q_2$. Now, $p_1(k)$ is increasing every $k$ while $p_2(k)$ remains bounded. There must exist some $k_1$ in which

$$\frac{a_2^2 (a_2^2 r_2 + q_2)^2}{(a_2^2 r_2 + q_2) + r_2} < \frac{a_1^2 p_1(k_1)^2}{p_1(k_1) + r_1}.$$

Again, sensor 1 will be transmitting at instant $k_1 + 1$. It has been proved that the activated sensor shifts after a finite number of instants. Let $n_i,\ i = 1, 2$, denote the maximum number of consecutive instants that sensor $i$ remains active before the other sensor occupies the shared medium. Then, both covariance diagonal elements will be bounded by

$$\begin{aligned} \bar{p}_1 \triangleq \max_k p_1(k) &= g^{n_2}(a_1^2 r_1 + q_1), \\ \bar{p}_2 \triangleq \max_k p_2(k) &= g^{n_1}(a_2^2 r_2 + q_2). \end{aligned}$$

So far it has been demonstrated that the covariance is bounded for systems with dimension 2.

The extension for higher dimensional system is trivial. The key idea is again the same: the maximum number of consecutive instant that any sensor (measuring an unstable dynamics) remains without being activated is bounded. Consider the general case that $|a_i| < 1$ for $1 \leq i < d$ and $|a_i| \geq 1$ for $d \leq i \leq n$. The components $1 \leq i < d$ of the covariance matrix are bounded for the same reasons argued before.

The components $p_d, \ldots, p_n$ will increase if they are not transmitted. But after some instants, condition given in Proposition 6.6 will be verified for every sensor $d \leq i \leq n$. Then, as $f(p_i) < a_i^2 r_i + q_i$, $(1 \leq i \leq d)$, these components will also be bounded. Hence the proof is ended. □

Lemma 6.7 shows that the proposed optimal scheduling produces a bounded trajectory of the covariance matrix and, hence, a periodic scheduling of the outputs

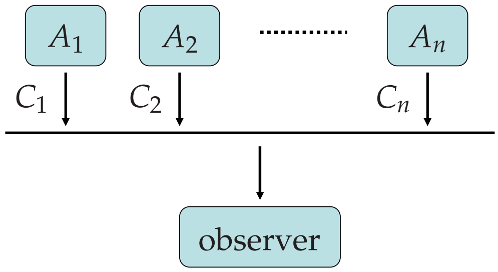



Figure 6.10: Block diagonal case: a set of systems sharing a communication network

through Theorem 6.2. It could seem that no extra assumptions are needed, but this is not really true. Since the complete matrix $C$ is the identity matrix, the observability of the pair $(A, C)$ is achieved. That assumption can not be relaxed.

### 6.6.2 The block diagonal case

Suppose that matrices $A, B_w$ are block diagonal,

$$\begin{aligned} A &= diag\{A_1, A_2, \ldots, A_n\}, \\ B_w &= diag\{I, I, \ldots, I\}, \end{aligned}$$

and disturbances and noises are described by

$$Q = diag\{Q_1, Q_2, \ldots, Q_n\},$$

with positive definite $Q_i,\ i = 1, \ldots, n$. The output matrix is also a block diagonal matrix defined by $C = diag\{C_1, C_2, \ldots, C_n\}$. This situation is the natural extension of the previous one, but the systems that are sharing the communication network are of higher dimension.

As before, two propositions are presented to study the scheduling law and the evolution of the covariance matrix for this case.

**Proposition 6.8.** *Suppose that the covariance matrix is $P_k = diag\{P_1, P_2, \ldots, P_n\}$ at instant k. Then, using the proposed optimal policy (6.30) the activated sensor j is*

$$j = arg \max_j tr\ A_j P_j C_j^T M_j C_j P_j A_j^T,$$

*with $M_j = (C_j P_j C_j^T + R_j)^{-1}$.*

**Proposition 6.9.** *Suppose that the covariance matrix is $P_k = diag\{P_1, P_2, \ldots, P_n\}$ at instant $k$. Then, given that output $j$ ($1 \leq j \leq n$) is transmitted at instant $k$, the covariance matrix at $k+1$ is given by*

$$P_{k+1} = \begin{bmatrix} g(P_1) & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & f(P_j) & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & g(P_n) \end{bmatrix} \quad \textit{block } j$$

*where matrix functions $f(P_i), g(P_i)$ are defined by*

$$\begin{aligned} f(P_i) &= A_i P_i A_i^T + Q_i - A_i P_i C_i^T M_i C_i P_i A_i^T, \\ g(P_i) &= A_i P_i A_i^T + Q_i. \end{aligned}$$

The proofs are immediate and hence omitted. Note that the covariance matrix remains block diagonal for all $k$. As before, the importance of functions $f(P_i), g(P_i)$ is crucial. If all the eigenvalues of $A_i$ lie inside the unit circle both functions are stabilizing maps. However, if matrix $A_i$ is unstable different behaviours are observed, as the following lemma states.

**Lemma 6.8.** *Assume that matrix $A_i$ is unstable and the pairs $(A_i, C_i)$ are detectable $\forall i$. Then,*

**i)** $g(P_i) > P_i$, *for all* $P_i > 0$;

**ii)** $\lim_{k\to\infty} f^k(P_i) = \bar{P}_i$,

*where $\bar{P}_i$ is the unique solution of the Riccati equation $\bar{P}_i = f(\bar{P}_i)$. Furthermore, if $C_i$ is a full-rank matrix then,*

**iii)** $f(P_i) \leq A_i(C_i R_i^{-1} C_i)^{-1} A_i^T + Q_i$.

**Proof.**

i) Condition $g(P_i) > P_i$ is equivalent to $g(P_i) - P_i > 0$, that is,

$$A_i P_i A_i^T + Q_i - P_i > 0. \tag{6.31}$$

Since matrix $A_i$ is unstable, there is no positive definite $P_i$ that verifies the Lyapunov equation $A_i P_i A_i^T - P_i = -Q_i$. Hence, for every $P_i$, condition (6.31) holds and then $g(P_i) > P_i$.

ii) The proof is trivial, because $P_i(k+1) = f(P_i(k))$ represents a Riccati recursion. As the pair $(A_i, C_i)$ is detectable, the recursion tends to the unique solution of the Riccati equation $\bar{P}_i = f(\bar{P}_i)$.

iii) Consider the following positive definite matrix $Y$:

$$Y = \lim_{y \to \infty} \begin{bmatrix} y & 0 & \cdots & 0 \\ 0 & y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y \end{bmatrix}.$$

It is trivially true that $X \leq Y$ holds for any $X$ positive definite. Furthermore, if $X \leq Y$, then $f_i(X) \leq f_i(Y)$ [226]. Hence, $f(Y)$ is an upper bound of $f(X)$, $\forall X \geq 0$. Using the well known matrix inversion lemma, it yields

$$f(Y) = A_i(Y^{-1} + C_i^T R_i^{-1} C_i)^{-1} A_i^T + Q_i.$$

As $C_i$ is full-rank, it turns out

$$f(Y) = \{Y^{-1} \to 0\} = A_i(C_i^T R_i^{-1} C_i)^{-1} A_i^T + Q_i.$$

□

At this point all the preliminaries have been introduced, so the corresponding result to that of Lemma 6.7 for block diagonal matrices can be given.

**Lemma 6.9.** *Assume that matrices $A, B_w, C, Q$ are block diagonal. If the pairs $(A_i, C_i)$ are detectable with full-rank $C_i$, then the scheduling law (6.30) produces a bounded sequence of $\{P_k\}$.*

**Proof.** The proof follows the same idea as before. Firstly, it will be proved that the optimal scheduling law produces alternation in the activated sensor. And then, this alternation will be the key to ensure the boundedness of the sequence $\{P_k\}$.

It will only be considered the situation with two blocks in which every $A_i$, $i = 1, 2$, is unstable. The extension for general $A_i$, $i = 1, \ldots n$, can be trivially settled.

From Proposition 6.8, note that the scheduling law (6.30) can be rewritten as

$$s(P) = arg \min_j tr \, \{f_j(P_j) - g_j(P_j)\}.$$

Assume that sensor 1 is active at instant $k_1$. While this sensor uses the network, it will be verified

$$\begin{aligned} P_1(k+1) &= f_1(P_1(k)), \forall k \geq k_1, \\ P_2(k+1) &= g_2(P_2(k)), \forall k \geq k_1. \end{aligned}$$

Since the pair $(A_1, C_1)$ is detectable, function $f_1(\cdot)$ makes the covariance $P_1$ to tend to the unique solution of the Riccati equation, i.e., $P_1(k) \to \bar{P}_1$, where $\bar{P}_1 = f_1(\bar{P}_1)$. Therefore, it yields

$$tr\ \{f_1(P_1(k)) - g_1(P_1(k))\} \to tr\ \{f_1(\bar{P}_1) - g_1(\bar{P}_1)\},$$

which is obviously a bounded value. At instant $k_1$ the sensor 1 is active, hence

$$tr\ \{f_1(P_1(k_1)) - g_1(P_1(k_1))\} < tr\ \{f_2(P_2(k_1)) - g_2(P_2(k_1))\}. \tag{6.32}$$

Let us move the attention to the terms on the right-hand side of the equation. The first one $f_2(P_2(k))$ is bounded $\forall k$ by Lemma 6.8. The second one $g_2(P_2(k))$ is increasing at each instant.

Using well known properties of positive definite matrices, if $X > Y$, then $tr\{X\} > tr\{Y\}$. Therefore, $tr\{g_2(X)\} > tr\{X\}$, for all $X > 0$. Since all the terms in (6.32) are bounded except $g_2(P_2(k))$ that increases each instant, there must exist a finite $k_2$ such that

$$tr\ \{f_1(P_1(k_1+k_2)) - g_1(P_1(k_1+k_2))\} > tr\ \{f_2(P_2(k_1+k_2)) - g_2(P_2(k_1+k_2))\},$$

and, hence, sensor 2 is activated at the following instant. This proves the alternation of the activated sensor in finite time.

Denote by $n_i$ the maximum number of consecutive instants that sensor $i$ remains active before the other sensor is activated. Both $n_i$, $i = 1, 2$, are finite. As Proposition 6.9 claims, the covariance matrix remains block diagonal. Then, both elements of the covariance matrix will be bounded by

$$\begin{aligned} P_1(k) &\leq g_1^{n_2}(A_1(C_1^T R_1^{-1} C_1)^{-1} A_1^T + Q_1), \forall k, \\ P_2(k) &\leq g_2^{n_1}(A_2(C_2^T R_2^{-1} C_2)^{-1} A_2^T + Q_2), \forall k. \end{aligned}$$

Hence, the sequence of $\{P_k\}$ remains bounded. □

The hypothesis of $C_i$ being a full-rank matrix is hard. If this condition is removed, the term $f_i(P_i(k))$ in (6.32) could grow when the evolution of $P_i(k)$ is driven by function $g_i(\cdot)$. Neither the alternation of sensors nor the boundedness of the trajectory could be proved with this method. An analysis of the general case is presented in the next section.

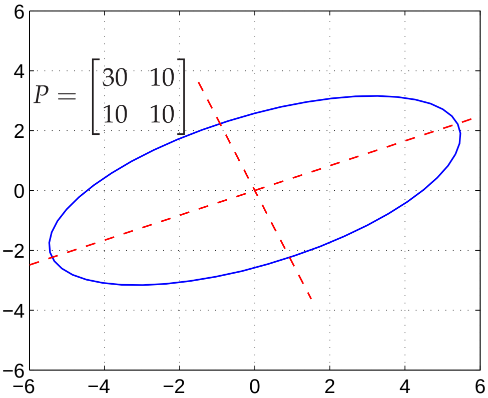



Figure 6.11: Uncertainty ellipsoid $\mathcal{E}(x, P)$

### 6.6.3 Analysis of the general case

This section tries to introduce the reader in the general case. It does not provide conclusive proofs, but attempts to give a different point of view for this problem. More precisely, a geometric analysis of the evolution of the covariance matrix is presented in this section.

Let $P$ be a covariance matrix. The uncertainty ellipsoid (see Figure 6.11) is defined as

$$\mathcal{E}(x, P) = \{x \in \mathbb{R}^n : x^T P^{-1} x \leq 1\}. \tag{6.33}$$

Assume that sensor $i$ is sending its measurements. Then, there exists a linear change of coordinates $T_i$ such that the tuple $(A, B_w Q B_w^T, C_i)$ can be written as[2]

$$A = \begin{bmatrix} A_{no} & A_{12} \\ 0 & A_o \end{bmatrix}, \; B_w Q B_w^T = \begin{bmatrix} Q_{no} & Q_{12} \\ Q_{12}^T & Q_o \end{bmatrix}, \; C_i = \begin{bmatrix} 0 & C_o \end{bmatrix}, \tag{6.34}$$

where the subindexes '$no$' and '$o$' stand for the unobservable and observable modes, respectively. From each output, it is possible to observe only a subspace of the complete state of the system.

The covariance matrix can also be partitioned equivalently as

$$P = \begin{bmatrix} P_{no} & P_{12} \\ P_{12}^T & P_o \end{bmatrix}. \tag{6.35}$$

Introducing these definitions in the evolution of the covariance matrix given in

[2]Matrices $A_{no}, A_o, A_{12} \ldots$ depend on the sensor $i$. However, with some abuse of notation the subindex $i$ has been removed when there is no ambiguity.

(6.26) it turns out that

$$P(k+1) = \begin{bmatrix} P_{no}(k+1) & P_{12}(k+1) \\ P_{12}^T(k+1) & P_o(k+1) \end{bmatrix},$$

where

$$\begin{aligned} P_o(k+1) &= A_o P_o(k) A_o^T + Q_o - A_o P_o(k) C_o^T M_o(k) C_o P_o(k) A_o^T, \\ P_{no}(k+1) &= A_{no} P_{no}(k) A_{no}^T + \Lambda_{no}(A, Q, P_o(k), P_{12}(k)), \\ P_{12}(k+1) &= \Lambda_{12}(A, Q, P_o(k), P_{12}(k)), \end{aligned}$$

with $M_o(k) = (C_o P_o(k) C_o^T + R_i)^{-1}$ and $\Lambda_{no}, \Lambda_{12}$ two matrix functions that do not depend on $P_{no}(k)$.

The pair $(A_o, C_o)$ is observable, so the evolution of $P_o(k)$ tends to the unique solution of the Riccati equation denoted by $\bar{P}_o$. On the other hand, and assuming an unstable $A_{no}$, the evolution of $P_{no}(k)$ diverges.

Geometrically, the ellipsoid evolves in such a way that some of its semi-axes are aligned with the unobservable space. Furthermore, these semi-axes grow and the other semi-axes (perpendicular to the former) remain bounded.

**Example 6.6.** Consider a system described by

$$A = \begin{bmatrix} 1.2 & 1 \\ 0 & -1.1 \end{bmatrix}, \; C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

with $Q = I$ and $R_1 = R_2 = 1$. The initial condition for the covariance matrix is $P(0) = \begin{bmatrix} 30 & 10 \\ 10 & 10 \end{bmatrix}$. The evolution of the uncertainty ellipsoid is shown in Figure 6.12 for two situations. From sensor 1 the complete state is observable, so the ellipsoid tends to the solution of the Riccati equation, which is obviously bounded. Using sensor 2, the unobservable space is defined by

$$\mathcal{N}_2 = \{x \in \mathbb{R}^2 : x = \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda \in \mathbb{R}\}.$$

Note that in Figure 6.12 the ellipsoids tend to align with $\mathcal{N}_2$. One of its semi-axes remains bounded and the other one grows with time. ▼

When one sensor is active for some time, the ellipsoid evolves in such a way that it grows in the direction of the unobservable space from that sensor and remains bounded in the perpendicular direction.

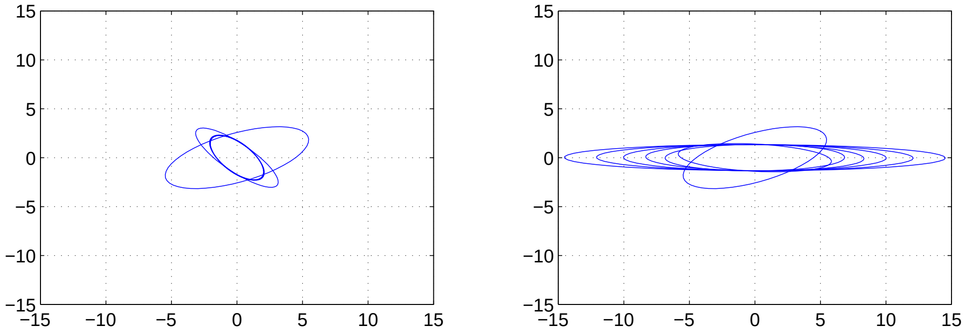



Figure 6.12: Evolution of the uncertainty ellipsoid $\mathcal{E}(x,P)$ using different sensors

Consider now the scheduling law:

$$\begin{aligned} s(P_k) &= arg \min_j \ tr \ P_{k+1}, \\ &= arg \max_j \ tr \ AP_kC_k^T \left[C_kP_kC_k^T + R_k\right]^{-1} C_kP_kA^T, \\ &= arg \max_j \ tr \ P_kC_k^T \left[C_kP_kC_k^T + R_k\right]^{-1} C_kP_k. \end{aligned}$$

Recall that system matrix $A$ is nonsingular. Then the last equality is verified due to the property

$$X \geq Y \rightarrow tr\{AXA^T\} \geq tr\{AYA^T\}, \text{for } X, Y \geq 0.$$

Let define $\varsigma_{C,R} : \mathbb{R}^{n\times n} > 0 \rightarrow \mathbb{R}^{n\times n} > 0$ as the map

$$\varsigma_{C,R}(P) = PC^T \left[CPC^T + R\right]^{-1} CP. \tag{6.36}$$

Please observe the similarities between the map $\varsigma_{C,R}(\cdot)$ and the scheduling law $s(\cdot)$. The application of the map $\varsigma_{C,R}$ to an ellipsoid given by $P$ results in a degenerated ellipsoid whose dimension is equal to the rank of $C$. Element $R$ introduces an scaling factor. Next example illustrates this idea.

**Example 6.7.** Consider the same system as in Example 6.6. Assume that sensor 2 has remained active for some time, yielding the ellipsoid $\mathcal{E}(x,P)$ that is depicted on the right-hand side of Figure 6.12. Now, the maps $\varsigma_{C_1,R_1}$ and $\varsigma_{C_2,R_2}$ are applied to $\mathcal{E}(x,P)$. The obtained degenerated ellipsoids are drawn in Figure 6.13.

In this case, both ellipsoids are transformed to intervals. Note that the application of the map $\varsigma_{C,R}$ is similar to a projection of the ellipsoid. The size of the interval

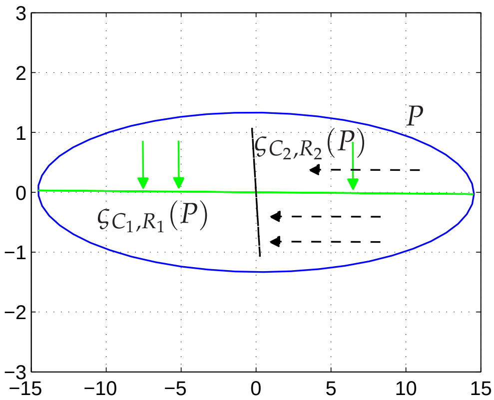



Figure 6.13: Application of the maps $\varsigma_{C_i,R_i}$ to get degenerated ellipsoids

is bounded in one case and grows in the other case (as the uncertainty ellipsoid $\mathcal{E}(x, P)$ grows).

▼

In this 2-dimensional example, the trace of matrix $P_{k+1}$ is directly related to the length of the interval generated by the application of $\varsigma_{C,R}$. In the general $n$-dimensional case, the trace depends on the size of the non degenerated semi-axes. The reason is clear, as the trace is equal to the sum of the eigenvalues, and the semi-axes of the ellipsoid are directly related with the eigenvalues of $P$.

Furthermore, when the covariance matrix is evolving driven by a sensor $i$, it seems that its associated degenerated ellipsoid is bounded as Figure 6.13 shows. On the other hand, the ellipsoids associated with the other sensors (the ones that observe unstable poles) are growing with time, so in finite time the channel will be occupied by a different sensor.

Therefore, the intuition says that the application of the scheduling law (6.30) makes that in finite time all the sensors measuring unstable poles must be activated.

The next question is: does the alternation of sensors imply a bounded trajectory of the covariance matrix? Answering this question is difficult. The intuition and simulation results motivate that thought. However, it has only been proved in the diagonal and block diagonal cases.

This problem has been studied from a stochastic point of view in [76], where each sensor is given a probability of being active at instant $k$. It was shown that, if these steady-state probability were different than zero, then the covariance matrix resulting from applying a stochastic scheduling law was indeed bounded. The forced alternation of sensors in finite time implies somehow that the probability of choosing each sensor is different than zero. Hence, this might be a sufficient condition to ensure a bounded trajectory of $\{P_k\}$ in the general case.

## 6.7 Characterization of the optimal pattern

Section 6.5 presented the conditions that must be verified to ensure that the optimal scheduling law presented gives rise to a periodic scheduling.

But a question arises: is the optimal pattern unique? Does it depend on the initial condition? As Ramadge showed [205], the eventual period of $\{s(x_k)\}$ will depend on the initial condition in the general case. When dealing with covariances, the evolution of $P_k$ is deterministic, so each initial condition $P_0$ gives rise to a unique trajectory $\{P_k\}$. However, despite producing different trajectories, the uniqueness of the optimal pattern can be ensured, as is proved next.

The next step within this development is: is it possible to know some features of the optimal pattern? Or the optimal pattern itself? As Example 6.5 showed, the system and covariance equations can be simulated to eventually get the optimal pattern. Choosing a large enough simulation time, the pattern could be discovered.

However, in this section two results are given to characterize the pattern *a priori*. Firstly, the conditions are given to have a pattern consisting of only one element. Secondly, the maximum length of it is studied. In the following it is assumed that an optimal pattern exists, that is, all the conditions in Theorem 2 are satisfied.

The next lemma establishes the uniqueness.

**Lemma 6.10.** *Assume that all the conditions in Theorem 6.2 are verified. Then, the steady-state trajectory $\{P_k\}$ and the sequence of sensors $\{j\}$ obtained by the application of the scheduling law (6.30) is unique for enough large $k$ regardless of the initial condition $P_0$.*

**Proof.** The proof is based on that of Lemma 3 in [16]. From Theorem 6.2, it can be assured that for each initial condition $P_0$ a periodic sequence of sensors $j$ is obtained. Denote by $\{P_k^1\}$ and $\{P_k^2\}$ two different trajectories of the covariance matrix for different initial conditions. By Theorem 6.2, both trajectories are bounded. Consider the optimal Kalman gain given in (6.25) for the two trajectories, namely $\{L_k^{1*}\}$ and $\{L_k^{2*}\}$. As the covariance is bounded, the corresponding closed-loop matrix $\hat{A}_k^i \triangleq A - L_k^i C_k$ is exponentially stable.

The difference $P_k^{12} \triangleq P_k^1 - P_k^2$ satisfies the equation $P_{k+1}^{12} = \hat{A}_k^1 P_k^{12} (\hat{A}_k^2)^T$. Therefore, $\lim_{k\to\infty} P_k^1 - P_k^2 = 0$. □

Based on the uniqueness, the following proposition studies the case of one-element patterns.

**Proposition 6.10.** *If any of the solutions of the Riccati equations*

$$\bar{P}_j = A\bar{P}_jA^T + B_wQB_w^T - A\bar{P}_jC_j^T\left[C_j\bar{P}_jC_j^T + R_j\right]^{-1}C_j\bar{P}_jA^T \tag{6.37}$$

*for* $j \in \{1, \dots, m\}$, *satisfies*

$$arg\min_i trP_{k+1} = j, \text{ for } P_k = \bar{P}_j, \tag{6.38}$$

*then the optimal pattern consists of only one element, and this element is exactly* $j$.

**Proof.** Let $\bar{P}_j$ be a solution of (6.37) such that verifies (6.38). Further consider that the initial value of the covariance matrix for the optimal Kalman filter is $P_0 = \bar{P}_j$. Hence, the sensor $j$ will be active for all $k > 0$. □

**Corollary 6.2.** *Consider the set of all the solutions of the equations* (6.37)*. Then, only one solution of the set verify* (6.38)*.*

**Proof.** Assume that two solutions of (6.37), that is $\bar{P}_i$ and $\bar{P}_j$, $i \neq j$, satisfy condition (6.38). Then, by choosing as initial condition $\bar{P}_i$ and $\bar{P}_j$, two different optimal periods will be obtained, which is impossible due to Lemma 6.10. □

In some situations, it is beneficial to know the length of the pattern. The following result establishes a relation between the length and the limit points of the trajectory.

**Proposition 6.11.** *Let* $N$ *be the length of the optimal pattern. Then, it holds*

$$N \leq n_{lp},$$

*where* $n_{lp}$ *denotes the number of limit points of the trajectory* $\{P_k\}$.

**Proof.** The proof is based on Proposition 6.5. The finite set $Q$ has a number of elements lower than the number of limit points. From the definition of function $\beta$, the length of the pattern is bounded by the number of elements in set $Q$. □

The usefulness of Proposition 6.11 is based on the knowledge of the number of limit points, which in most cases is difficult for a given system.

## 6.8 Practical implementation

This section presents a small guide for the control engineer that faces the problem of sensor scheduling.

If the system belongs to one of the categories studied in Section 6.6, the boundedness of the matrix covariance can be ensured using the proposed scheduling. Then, by Theorem 6.2, one concludes that the optimal selection of the sensor gives rise to a periodic pattern of activated sensors.

In case that the system cannot be described as one of the cases given in Section 6.6, the engineer can simulate the system with the proposed scheduling, as in Example 6.5. If it is observed that the scheduling eventually shifts to a periodic pattern, some results of periodic patterns may be used, as the ones presented Section 6.3. In that case, the periodic scheduling would produce a bounded covariance matrix, but maybe not the optimal solution attained with the proposed Kalman-based filter.

The control engineer decides whether to implement the non optimal periodic scheduling or the optimal aperiodic scheduling.

## 6.9 Chapter summary

The problem of designing partial observers and scheduling the communication over a shared network is tackled in this chapter. A contentionless TDMA scheduling algorithm is designed to satisfy closed-loop stability conditions and to minimize the steady state error, while reducing the data rate. Two new approaches are proposed: the periodic observer, with the new features of including pure of observation periods, $H_\infty$ disturbance attenuation, pattern optimality, and co-design of the pattern and observer for pole placement. On the other hand, in the aperiodic Kalman-based filter, both the observation matrices and the activated sensor are chosen to minimize the variance of the observation error.

Furthermore, the chapter investigates the periodic phenomenon that emerges in an *a priori* aperiodic scheduling, as it has been observed in [88, 184]. Concretely, Theorem 6.2 gives the conditions that must be verified by the system to obtain a periodic scheduling when the optimal Kalman-based filter is used. It is shown that those conditions ($A$ nonsingular, boundedness of the trajectory, and absence of limit point on the switching boundary) do not impose hard restrictions.

A deep study on whether the proposed filter gives rise to a bounded trajectory of the covariance matrix is made. Although some interesting cases are studied, there is only an analysis for the general case. The intuition says that the trajectory is bounded in all cases, but this must be proved in the future.

Further improvements should be aimed at including delays and dropouts in the communication channel, as well as studying the case when the optimization problem is *N-steps-ahead*.

# Chapter 7

# Distributed control and estimation

## 7.1 Introduction

Wireless sensor networks (WSNs) are an emerging technology that has received a great deal of attention in the past decade due mainly to its current and future envisioned applications in both military and civilian fields [1, 21, 41, 54, 129, 248].

A sensor network is composed of a large number of geographically distributed sensor nodes (agents in a broader sense) with communication capabilities among them. Sensor networks present a number of advantages over centralized architectures: physical distribution over large regions providing diverse perspectives orviewing angles of the observed phenomenon; increased robustness to failures; lower deployment and maintenance costs, etc. Due to various design considerations, such as small battery size, bandwidth and cost, WSNs typically exhibit low-power constraints and limited computation and communication capabilities. Despite these limitations, the high connectivity and sensor cooperation capabilities of WSNs provide the potential to build powerful networks that accomplish complex high-level tasks that are difficult or even impossible to attain with classical centralized or hierarchical approaches.

Distributed estimation and control, which is the topic of this chapter, naturally arises in the context of systems with multiple distributed components intended to jointly meet a system-wide objective. Broadly speaking, the objective is to induce collective behaviors through the actions taken by individual agents in a distributed way. The primary salient feature of this approach is the distribution of information. As opposed to centralized solutions (such as the ones presented in Chapter 6), no single unit has access to the whole data set gathered by all the agents. A secondary distinguishing feature is complexity. As the number of agents grow, conventional decentralized estimation schemes are also unattractive in many situations, provided that point-to-point communications are involved and scalability for a high number of nodes can be compromised [175].

The state of the art concerning distributed control strategies comprises a vast number of techniques, taking different approaches depending on the problem nature.

First attempts to formulate the problem date back to the late 70's, with the works of Davison [45, 46] and Anderson [4]. Predictive control has shown itself very prolific in this field [26, 49, 135, 168, 209, 210, 241]. Some other remarkable breakthroughs to the problem can be found in [134], based on semi-active control with applications to large-scale civil structures, and the completely innovative idea proposed in [42], where the plant and the controllers are modeled using modular blocks that communicate with their neighbors. The modules can be stacked building large-scale distributed systems. Most of the above mentioned works decompose the plant in decoupled or weakly coupled subsystems controlled by independent nodes. A closely related line of research is the so-called decentralized overlapping control, where different controllers are allowed to share control inputs of the plant [91, 92, 225].

In this chapter, a new distributed control and estimation framework is discussed. It is a continuation of the works in [155] and [149] of the same author, tackling this time the broader problem of joint distributed estimation and control.

The proposed problem considers a discrete LTI process, being controlled by a network of agents that may both collect information about the evolution of the plant and apply control actions to achieve a given goal. The problem makes full sense for geographically distributed processes where the agents have access only to partial information and actuate, possibly, only on specific control channels. In other words, no agent has the information, neither the control capabilities, to estimate and drive the overall process on its own. In this context, the networked structure of the agents plays an essential role as neighboring agents are allowed to share information and cooperate to achieve the system-wide goal.

The aim of the approach is to provide a fully distributed scheme so that the joint cooperative action of all agents drives the system to asymptotic stability, providing a cost-guaranteed solution with respect to a given quadratic index. Each node implements an observer & controller structure based on local Luenberger-like observers in combination with consensus strategies, the first part being responsible for updating the node estimation based on local sensed information, while the former takes into account the data transmitted from neighboring nodes.

Two different control & estimation schemes are presented. First, a periodic time-driven approach, where the nodes are assumed to communicate at every sampling time; then a more efficient event-driven scheme that triggers the agent communications only when significant information must be transmitted. Event-driven approaches, see Chapter 5, are specially beneficial in the context of WSNs as a reduction in the transmission frequency implies bandwidth savings but also an improvement in average transmission delays and packet collisions, for back-off retransmission are reduced. Moreover, in WSNs the battery life span is of great importance, and it is mainly related to the number of transmissions of the device.

As it will be shown, the separation principle does not hold, this forcing to a joint design of control and estimation. As a first extension of previous works in [155] and [149] to tackle both, distributed control and estimation, this chapter neglects the effects of transmission delays and dropouts in the network. Although both are admittedly relevant phenomena in real-world applications, these have been dropped for the present work in favor of obtaining a tractable mathematical design method.

The stability of the solution is guaranteed through discrete-time Lyapunov functions, from which the design problem is cast into the solution of a set of matrix inequalities. Asymptotic stability and Global Ultimately Uniformly Boundedness (GUUB) of solutions are proved for, respectively, the time-driven and the event-driven approaches. To the best of the author's knowledge, this is the first approach that considers distributed estimation and control in a cost-guaranteed scheme using an event-based sampling policy.

## Related publications

1. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Control y observación distribuida en sistemas de control a través de redes*. XXXII Jornadas de Automática. Sevilla, Spain. 2011. [187]

2. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio. *Distributed consensus-based estimation considering network induced delays and dropouts*. Automatica. 48(10):2726-2729, 2012. [155]

3. P. Millán, L. Orihuela, I. Jurado, C. Vivas, F. R. Rubio. *Distributed estimation in networked systems under periodic and event-based communication policies*. International Journal of Systems Science. Accepted. [149]

4. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Computationally efficient distributed*

*$H_\infty$ observer for sensor networks*. International Journal of Control. Under review. [191]

5. P. Millán, L. Orihuela, C. Vivas, F. R. Rubio, D. V. Dimarogonas, K. H. Johansson. *Sensor-network-based robust distributed control and estimation*. Control Engineering Practice. Submitted. [156]

6. L. Orihuela, P. Millán, C. Vivas, F. R. Rubio. *Suboptimal distributed control and observation: application to a four coupled tanks system*. Journal of Process Control. Submitted. [190]

## 7.2 System description. Initial considerations

Consider the distributed control and estimation scheme depicted in Figure 7.1. The large-scale process is monitored and controlled through a network of agents, each one exhibiting all or part of the following capabilities: sensing plant outputs, estimating the state of the plant, applying control actions, and communicating to neighboring agents.

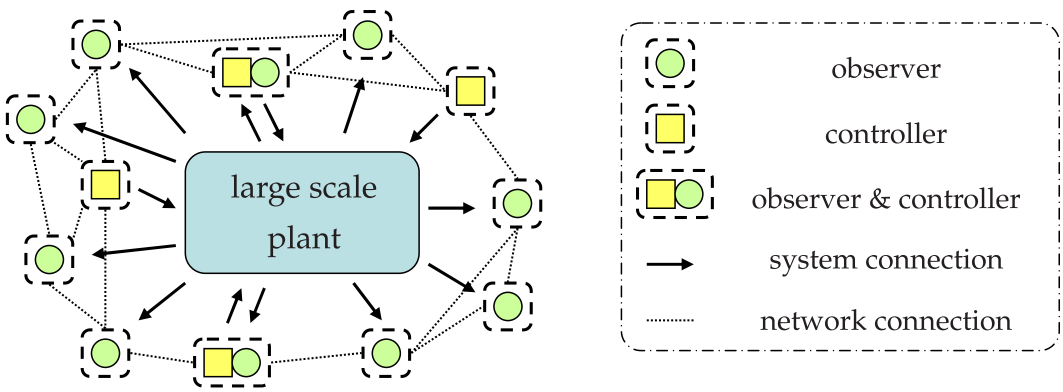



Figure 7.1: Distributed scheme for the control of a large-scale plant

In the following, the different elements composing the distributed system are described in detail.

### 7.2.1 Plant

Consider a discrete LTI system described in the state-space representation. As Figure 7.1 illustrates, the plant is being controlled and/or sensed by a set of $p$ agents, each one possibly managing different control signals. The dynamics of the system can be described as

$$x(k+1) = Ax(k) + \sum_{i=1}^{p} B_i u_i(k), \tag{7.1}$$

where $u_i \in \mathbb{R}^{r_i}$ is the control signal that agent $i$ applies to the system.

Let us define an augmented control matrix

$$B \triangleq \begin{bmatrix} B_1 & B_2 & \dots & B_p \end{bmatrix},$$

and an augmented control vector

$$\mathcal{U}(k) \triangleq \begin{bmatrix} u_1^T(k) & u_2^T(k) & \dots & u_p^T(k) \end{bmatrix}^T. \tag{7.2}$$

Then, equation (7.1) can be compactly rewritten as

$$x(k+1) = Ax(k) + B\mathcal{U}(k), \tag{7.3}$$

where $\mathcal{U}(k) \in \mathbb{R}^r$, with $r = \sum_{i=1}^{p} r_i$. It is assumed that the pair $(A, B)$ is stabilizable.

### 7.2.2 Network

Using the notation introduced in Section 2.2.2.2, the network in Figure 7.1 is topologically defined by its graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $l$ links between $p$ agents. The graph $\mathcal{G}$ is directed, with nodes $\mathcal{V} = \{1, 2, \dots, p\}$ and links $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The set of agents connected to node $i$ is named the *neighborhood* of $i$ and is denoted by $\mathcal{N}_i \equiv \{j : (i,j) \in \mathcal{E}\}$. Link $(i,j)$ implies that agent $i$ receives information from agent $j$.

### 7.2.3 Agents

The agents are assumed to behave as any or both, observers and controllers. The approach adopted in this work is an observer-based scheme in which every agent is assumed to build its own estimate of the plant state based on the information locally collected by the agent (plant input) and on that shared with neighboring agents.

If acting as a sensor, agent $i$ measures a specific plant output, $y_i$, such that

$$y_i(k) = C_i x(k) \in \mathbb{R}^{m_i}, \tag{7.4}$$

where matrices $C_i \in \mathbb{R}^{m_i \times n}$ are known. Let $C$ denote an augmented output matrix defined as

$$C \triangleq \begin{bmatrix} C_1^T & C_2^T & \dots C_p^T \end{bmatrix}^T.$$

The pair $(A, C)$ is assumed to be detectable.

If agent $i$ has control capabilities, the control counterpart generates an estimation-based control input to the plant, $u_i(k)$, in the form

$$u_i(k) = K_i \hat{x}_i(k) \in \mathbb{R}^{r_i}, \tag{7.5}$$

where $\hat{x}_i \in \mathbb{R}^n$ denotes the estimation of agent $i$, and $K_i \in \mathbb{R}^{r_i \times n}$ $(i \in \mathcal{V})$ are the local controllers to be designed.

The estimation of the plant state are obtained at every agent $i$ from a local observer structure that takes the form

$$\hat{x}_i(k+1) = A\hat{x}_i(k) + B\hat{\mathcal{U}}_i(k) + M_i(y_i(k) - C_i\hat{x}_i(k)) + \sum_{j \in \mathcal{N}_i} N_{ij}(\hat{x}_j(k) - \hat{x}_i(k)), \tag{7.6}$$

where $\hat{\mathcal{U}}_i(k) \in \mathbb{R}^r$ is the augmented control locally estimated by agent $i$ defined as

$$\mathcal{U}_i(k) \triangleq \left[ \begin{array}{ccccc} \hat{u}_{1i}^T(k) & \dots & u_i^T(k) & \dots & \hat{u}_{pi}^T(k) \end{array} \right]^T,$$

where $\hat{u}_{ji}(k) = K_j \hat{x}_i(k), \forall j \neq i$.

Looking at equation (7.6), each node has two different sources of information to correct its estimations. The first one consists of the output measured from the plant $y_i(k)$, which is used similarly to a classical Luenberger observer, $M_i(y_i(k) - C\hat{x}_i(k))$, being $M_i$ an observer gain to be designed.

The second source of information comes from the estimates received from neighboring nodes, which is also used to correct estimations with the terms $N_{ij}(\hat{x}_j(k) - \hat{x}_i(k))$, where $N_{ij}$, $(i, j) \in \mathcal{E}$, are consensus matrices to be designed.

Observe that the individual nodes have no information about the exact control signal being applied to the plant, as each actuator node applies a different control signal based on its particular state estimation (7.5), that is, $\mathcal{U}(k) \neq \mathcal{U}_i(k)$. Ideally, equation (7.6) should be implemented using the augmented control vector $\mathcal{U}(k)$ that the network, as a whole, applies to the plant. However, this information is not available at the nodes. To circumvent this difficulty and make equation (7.6) realizable, the proposed solution consists, roughly speaking, of letting each node to run its observer with the augmented control vector obtained from its particular estimate. That is,

$$B\hat{\mathcal{U}}_i(k) = \sum_{j=1}^{p} B_j K_j \hat{x}_i(k) = BK\hat{x}_i(k),$$

where the augmented control matrix satisfies $K^T = \begin{bmatrix} K_1^T & K_2^T & \dots & K_p^T \end{bmatrix}$.

The actual control vector applied to the plant is however built on the estimates of each node

$$B\mathcal{U}(k) = \sum_{j=1}^{p} B_j K_j \hat{x}_j(k).$$

In general, estimated and actual augmented control vectors differ, but if the observers are properly designed and the node estimations converge to the plant states, these discrepancies progressively vanish.

The observation error of agent $i$ is defined as

$$e_i(k) = x(k) - \hat{x}_i(k). \tag{7.7}$$

Finally, let $\mathcal{M}, \mathcal{N}, \mathcal{K}$ denote the sets of the observer and controller gains to be designed:

$$\begin{aligned} \mathcal{M} &= \{M_i, i \in \mathcal{V}\}, \\ \mathcal{N} &= \{N_{ij}, (i,j) \in \mathcal{E}\}, \\ \mathcal{K} &= \{K_i, i \in \mathcal{V}\}. \end{aligned}$$

**Remark 7.1.** Matrices $C_i$ or $B_i$ may be equal to zero for some agents. In those cases, the agents will become pure controllers or pure observers, respectively.

The following sections present design methods to obtain these sets in order to stabilize the plant and the observation errors.

## 7.3 Problem formulation

This section formally states the problem to be solved.

**Definition 7.1. Suboptimal distributed control and observation problem.** Consider a discrete LTI plant with dynamics given by (7.1). The plant is sensed and controlled by a set of $p$ agents in a network whose topology can be represented by the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The dynamics of the agents are given by (7.6), each of them receiving a measurement from the plant defined by (7.4), and applying a control signal defined by (7.5). Then, given the cost function

$$J = \sum_{j=k_0}^{\infty} x^T(j) Q_x x(j) + \sum_{j=k_0}^{\infty} \sum_{i \in \mathcal{V}} \left[ e_i^T(j) Q_i e_i(j) + u_i^T(j) R_i u_i(j) \right], \tag{7.8}$$

the suboptimal distributed control and observation problem consists of finding observers $M_i$, $i \in \mathcal{V}$ and $N_{ij}$, $(i,j) \in \mathcal{E}$ and controllers $K_i$, $i \in \mathcal{V}$, such that:

- The system is asymptotically stable.
- The dynamics of the observation errors are asymptotically stable $\forall i \in \mathcal{V}$.
- A cost guaranteed solution is obtained by minimizing the upper bound of the cost function (7.8).

## 7.4 Periodic sampling case

Let us consider first the case of periodic communication between agents. That is, each node receives information from neighboring agents each instant $k$.

### 7.4.1 Dynamics of the state and observation error

This section study the dynamics of the system state and the observation error. Define the error vector $e^T(k) = [e_1^T(k) \quad e_2^T(k) \quad \dots \quad e_p^T(k)] \in \mathbb{R}^{np}$ and the augmented vector $\xi^T(k) = \left[x^T(k) \ \ e^T(k)\right]$.

**Proposition 7.1.** *The dynamics of the state of the plant $x(k)$ is given by*

$$x(k+1) = (A + BK)\, x(k) + \Upsilon(\mathcal{K})e(k), \tag{7.9}$$

*where*

$$\Upsilon(\mathcal{K}) = \begin{bmatrix} -B_1K_1 & -B_2K_2 & \dots & -B_pK_p \end{bmatrix}.$$

The proof is immediate from equations (7.3)-(7.7) and the definition of the error vector.

**Proposition 7.2.** *The dynamics of the error vector $e(k)$ is given by*

$$e(k+1) = \left(\Phi(\mathcal{M}) + \Psi(\mathcal{K}) + \Lambda(\mathcal{N})\right) e(k), \tag{7.10}$$

*where*

$$\begin{aligned} \Phi(\mathcal{M}) &= diag\{(A - M_1C_1), \dots, (A - M_pC_p)\}, \\ \Psi(\mathcal{K}) &= diag\{BK, \dots, BK\} + \begin{bmatrix} \Upsilon(\mathcal{K}) \\ \vdots \\ \Upsilon(\mathcal{K}) \end{bmatrix}, \\ \Lambda(\mathcal{N}) &= \sum_{(i,j)\in\mathcal{E}} \Theta(N_{ij}), \end{aligned}$$

*with*

$$\Theta(N_{ij}) = \begin{array}{cc} \begin{array}{ccccccc} col. & & i & & j & & \end{array} & \\ \left[\begin{array}{ccccccc} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & -N_{ij} & \cdots & N_{ij} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{array}\right] & \begin{array}{c} \\ \\ row\ i \\ \\ \\ \end{array} \end{array}.$$

**Proof.** The observation error of agent $i$ at instant $k+1$ can be obtained using equation (7.7) and Proposition 7.1:

$$\begin{aligned} e_i(k+1) &= x(k+1) - \hat{x}_i(k+1) \\ &= (A+BK)x(k) + \Upsilon(\mathcal{K})e(k) - (A+BK)\hat{x}_i(k) \\ &\quad -M_i(y_i(k) - C\hat{x}_i(k)) - \sum_{j\in\mathcal{N}_i} N_{ij}(\hat{x}_j(k) - \hat{x}_i(k)). \end{aligned} \tag{7.11}$$

We can write $e_i(k+1) = (tr1) - (tr2)$, where $(tr1)$ are the terms of (7.11) which do not depend on the neighbors and $(tr2)$ are the rest. Consider first the terms $(tr1)$.

$$\begin{aligned} (tr1) &\triangleq (A+BK)e_i(k) - M_iC_ie_i(k) + \Upsilon(\mathcal{K})e(k) \\ &= (A - M_iC_i + BK)e_i(k) + \Upsilon(\mathcal{K})e(k). \end{aligned} \tag{7.12}$$

Consider now $(tr2)$:

$$\begin{aligned} (tr2) &\triangleq \sum_{j\in\mathcal{N}_i} N_{ij}(\hat{x}_j(k) - \hat{x}_i(k)) \\ &= \sum_{j\in\mathcal{N}_i} N_{ij}(e_i(k) - e_j(k)). \end{aligned} \tag{7.13}$$

Using equations (7.12)-(7.13) the observation error at instant $k+1$ can be written as

$$e_i(k+1) = (A - M_iC_i)\, e_i(k) + BKe_i(k) + \Upsilon(\mathcal{K})e(k) - \sum_{j\in\mathcal{N}_i} N_{ij}(e_i(k) - e_j(k)).$$

Finally, since the error vector was defined as $e^T(k) = \begin{bmatrix} e_1^T(k) & e_2^T(k) & \ldots & e_p^T(k) \end{bmatrix}$, it is immediate that the dynamics of $e(k)$ is (7.10). $\square$

**Proposition 7.3.** *The evolution of the augmented vector $\xi(k)$ is given by*

$$\xi(k+1) = \Omega(\mathcal{M},\mathcal{N},\mathcal{K})\xi(k), \tag{7.14}$$

*where*

$$\Omega(\mathcal{M},\mathcal{N},\mathcal{K}) = \left[\begin{array}{c|c} A+BK & \Upsilon(\mathcal{K}) \\ \hline 0 & \Phi(\mathcal{M}) + \Psi(\mathcal{K}) + \Lambda(\mathcal{N}) \end{array}\right].$$

The structure of (7.14) reveals that the separation principle does not hold in this case, because matrix $\Psi(\mathcal{K})$ depends on the controllers to be designed. This can be easily justified if we recall that the nodes ignore the actual control signal being applied to the plant, and resort to estimations based on the knowledge of the distributed controllers. However, despite this drawback, it will be shown that it is possible to propose an unified design in which all the elements, namely controllers and observers, can be designed to guarantee the overall system stability.

Before proceeding with the main results, a last proposition is introduced.

**Proposition 7.4.** *The cost function* (7.8) *can be written as*

$$J = \sum_{j=k_0}^{\infty} \xi^T(j)(Q + \bar{K}^T R \bar{K})\xi(j), \tag{7.15}$$

*where*

$$\begin{aligned}
Q &= diag\{Q_x, Q_1, Q_2, \dots, Q_p\}, \\
R &= diag\{R_1, R_2, \dots, R_p\}, \\
\bar{K} &= \begin{bmatrix} K_1 & -K_1 & 0 & \dots & 0 \\ K_2 & 0 & -K_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_p & 0 & 0 & \dots & -K_p \end{bmatrix}.
\end{aligned}$$

This result can be easily proved by substituting the matrices $Q, R, \bar{K}$ and the augmented vector $\xi(k)$ in (7.15).

### 7.4.2 Controller and observer design

The design method resorts to a Lyapunov-based approach to prove asymptotic stability of the system and of the dynamics of all observation errors. It is a centralized design in which both, controllers and observers are designed together.

Consider the following Lyapunov function:

$$V(k) = \xi^T(k) P \xi(k), \tag{7.16}$$

where

$$P = \begin{bmatrix} P_x & 0 \\ 0 & P_e \end{bmatrix},$$

being $P_x \in \mathbb{R}^{n\times n}$ a positive definite matrix and $P_e\mathbb{R}^{np\times np}$ a block diagonal matrix defined by

$$P_e = \begin{bmatrix} P_1 & 0 & \dots & 0 \\ 0 & P_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_p \end{bmatrix},$$

with $P_i \in \mathbb{R}^{n\times n}$ $(i \in \mathcal{V})$ positive definite matrices.

The following theorem proposes a centralized design method through a nonlinear matrix inequality.

**Theorem 7.1.** *Given matrices $Q, R$ of the cost function* (7.15)*, the suboptimal distributed controller and observer given by the sets $\mathcal{M}, \mathcal{N}, \mathcal{K}$ can be obtained by solving the following optimization problem:*

$$\min_{P,\mathcal{M},\mathcal{N},\mathcal{K}} \alpha, \tag{7.17}$$

*subject to*

$$\begin{bmatrix} -P & \Omega^T & I & \bar{K}^T \\ * & -P^{-1} & 0 & 0 \\ * & * & -\alpha Q^{-1} & 0 \\ * & * & * & -\alpha R^{-1} \end{bmatrix} < 0. \tag{7.18}$$

**Proof.** The proof is similar to that of Theorem 4.5, but it uses the Lyapunov function (7.16) instead of a Lyapunov-Krasovskii functional. The forward difference of the Lyapunov function is given by

$$\begin{aligned} \Delta V(k) &= V(k+1) - V(k) \\ &= \xi^T(k+1)P\xi(k+1) - \xi^T(k)P\xi(k). \end{aligned}$$

Using Proposition 7.3, the forward difference can be expressed in the following way:

$$\begin{aligned} \Delta V(k) &= \xi^T(k)\Omega^T P\Omega\xi(k) - \xi^T(k)P\xi(k) \\ &= \xi^T(k)\left(\Omega^T P\Omega - P\right)\xi(k). \end{aligned}$$

If matrix $\Omega^T P\Omega - P$ is negative definite, then the state of the system and the observation errors will asymptotically approach to zero. Using Schur complements (see Property B.4), the following inequalities are equivalent:

$$\Omega^T P\Omega - P < 0 \leftrightarrow \begin{bmatrix} -P & \Omega^T \\ * & -P^{-1} \end{bmatrix} < 0.$$

Previous inequality is ensured through condition (7.18).

Up to this point, the stability properties of the distributed scheme have been proved. Let us move to the optimality of the method. Note that condition (7.18) implies that

$$\Omega^T P \Omega - P + \frac{1}{\alpha} Q + \frac{1}{\alpha} \bar{K}^T R \bar{K} < 0.$$

Using Property (B.2), it is verified that

$$\xi^T(k) \left( \Omega^T P \Omega - P \right) \xi(k) < -\xi^T(k) \frac{1}{\alpha} (Q + \bar{K}^T R \bar{K}) \xi(k).$$

Taking into account the previous considerations, it holds

$$\Delta V(k) = \xi^T(k) \left( \Omega^T P \Omega - P \right) \xi(k) < -\frac{1}{\alpha} \xi^T(k) (Q + \bar{K}^T R \bar{K}) \xi(k). \tag{7.19}$$

Calculating the summation of both sides of (7.19) from $k_0$ to $k$, it yields

$$\sum_{j=k_0}^{k} \Delta V(j) < -\frac{1}{\alpha} \sum_{j=k_0}^{k} \xi^T(j) (Q + \bar{K}^T R \bar{K}) \xi(j).$$

Observe that $\sum_{j=k_0}^{k} \Delta V(j) = \sum_{j=k_0}^{k} (V(j+1) - V(j)) = V(k+1) - V(k_0)$. When $k \to \infty$, the asymptotic stability of the system implies that $V(k+1) \to 0$, hence

$$\begin{aligned} -V(k_0) &< -\frac{1}{\alpha} \sum_{j=k_0}^{\infty} \xi^T(j) (Q + \bar{K}^T R \bar{K}) \xi(j), \\ \Rightarrow J &< \alpha V(k_0), \end{aligned}$$

where Proposition 7.4 has been used. The value of $V(k_0)$ depends on the initial condition. Therefore, minimizing $\alpha$ an upper bound of the cost function $J$ is minimized regardless of the initial conditions. □

Note that inequality (7.18) is nonlinear in the decision variables because of the presence of the term $P^{-1}$. As in Chapter 4, two standard solutions can be used in order to deal with it. Appendix C gives details of both methods.

**Remark 7.2.** The design method that stems from Theorem 7.1 can be performed off-line prior to the implementation, and requires some sort of centralized information: the network topology, the information that every node collects from the plant, what control channels they have access to, etc. Nonetheless, once the observers and controllers have been designed, their implementation is fully distributed, and only requires information locally available for the nodes.

**Remark 7.3.** Casting the problem as an LMI condition provides a numerically efficient design method and allows to exploit all degrees of freedom in the design. This flexibility comes however at the cost of increasing the computational burden, that scales with the number of nodes and the dimension of the system. Finding decentralized design methods constitutes still a challenging open problem in large-scale distributed frameworks.

## 7.5 Event-based sampling case

As briefly discussed in the introduction, event-based control is a means to reduce network use by invoking a transmission among the nodes only if significant information deserves to be communicated [47]. Furthermore, event-based schemes are usually more efficient in terms of energy consumption, as most of the energy expended in distributed tasks is associated with transmissions, specially in the case of wireless communications.

For these reasons, the event-based sampling is an interesting approach with relevant practical implications. The idea is simple, instead of sampling at equidistant time instants, sampling is triggered by an event. The different definitions of *event* yields a plethora of published results. One of the most used, yet most intuitively appealing, is the one that triggers an event whenever some variable of interest has exceeded a tolerance bound. This concept has been adapted to the problem at hand as it is discussed in the following section.

### 7.5.1 Triggering rule

In the proposed control architecture, most of the energy consumption is due to the transmissions between agents, that periodically exchange the estimated state. In this section, the energy expenditure is reduced by triggering the transmissions only at specific time instants, when an event occurs. Let $l_j(k)$ denote the last time instant when node $j$ sent its estimated state to its neighbors. Next, a norm-based rule to trigger the communication events is defined.

**Definition 7.2. Triggering rule.** Given a threshold $\delta_w$, at instant $k$ agent $j$ broadcasts its estimates to every neighbor $i$ if

$$\|\hat{x}_j(l_j(k)) - \hat{x}_j(k)\|_\infty \geq \delta_w, \quad \text{for } k > l_j(k). \tag{7.20}$$

A possible evolution of $l_j(k)$ is depicted in Figure 7.2.

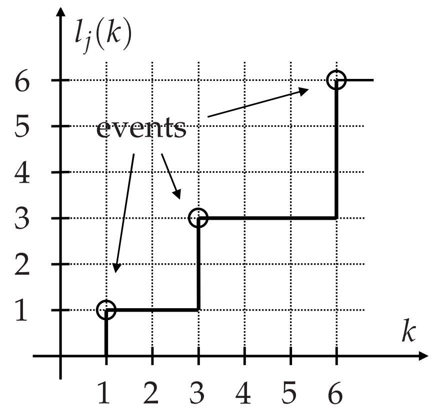



Figure 7.2: Possible evolution of $l_j(k)$

### 7.5.2 Remodeling the system dynamics

From a modeling point of view, the main difference between the time-driven and the event-driven paradigm described here is the non-uniform pattern of transmission of information. This modifies the behavior of the agents, whose dynamics can be now described as follows:

$$\begin{aligned} \hat{x}_i(k+1) &= A\hat{x}_i(k) + B\mathcal{U}_i(k) + M_i(y_i(k) - C_i\hat{x}_i(k)) \\ &+ \sum_{j\in\mathcal{N}_i} N_{ij}(\hat{x}_j(l_j(k)) - \hat{x}_i(k)). \end{aligned} \tag{7.21}$$

Equation (7.21) takes into consideration the aperiodic communication through the variable $l_j(k)$, which can be different for each agent $j \in \mathcal{N}_i$. Equation (7.21) can be rewritten as

$$\begin{aligned} \hat{x}_i(k+1) &= A\hat{x}_i(k) + B\mathcal{U}_i(k) + M_i(y_i(k) - C_i\hat{x}_i(k)) \\ &+ \sum_{j\in\mathcal{N}_i} N_{ij}(\hat{x}_j(k) - \hat{x}_i(k)) + \sum_{j\in\mathcal{N}_i} N_{ij}w_j(k), \end{aligned} \tag{7.22}$$

where

$$w_j(k) = \hat{x}_j(l_j(k)) - \hat{x}_j(k). \tag{7.23}$$

Based on (7.22), the evolution of the agents with event-based communication is equivalent to the evolution with periodic communication, difference being in the terms $w_j(k)$, given by equation (7.23). The term $w_j(k)$ can be interpreted as an external perturbation due to the discontinuous flow of information between neighbors that is reset to zero at every transmission time. This way, whenever agent $j$ broadcasts its state, it holds that $w_j(k) = 0$. It is worth pointing out that the disturbance that each agent $j$ induces on its neighbors is unknown to them, but can be tracked by agent $j$, which has access to its own local estimations.

The following result is the counterpart of Propositions 7.1-7.2 for event-based communication. Due to their similarities, its proof is omitted.

**Proposition 7.5.** *Let $w^T(k) = \begin{bmatrix} w_1^T(k) & w_2^T(k) & \dots & w_p^T(k) \end{bmatrix}$. Then, for the event-based sampling case, the evolution of the state $x(k)$ is given by Proposition 7.1, and the dynamics of the estimation error $e(k)$ is given by*

$$e(k+1) = (\Phi(\mathcal{M}) + \Psi(\mathcal{K}) + \Lambda(\mathcal{N}))\, e(k) + \Gamma(\mathcal{N})w(k), \tag{7.24}$$

*where the functions $\Phi(\mathcal{M}), \Psi(\mathcal{K}), \Lambda(\mathcal{N})$ are defined as in Proposition 7.2, and*

$$\Gamma(\mathcal{N}) = \sum_{(i,j)\in\mathcal{E}} \Delta(N_{ij}),$$

*with*

$$\Delta(N_{ij}) = \begin{array}{cc} \begin{array}{ccccccc} col. & & i & & j & & \end{array} & \\ \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & N_{ij} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} & \begin{array}{c} \\ \\ row\ i \\ \\ \\ \end{array} \end{array}.$$

### 7.5.3 Stability and trade-off between network traffic reduction and ultimate boundedness

The event-based sampling approach makes more difficult to analyze the system stability. In fact, it will be shown that the presented approach will only allow to prove the system to be globally uniformly ultimately bounded (GUUB), that is, the system is attracted and restricted to lie within an arbitrarily small region around the equilibrium point.

Theorem 7.1 allows the design of suboptimal distributed controllers and observers for periodically sampled systems. A question that naturally arises is: is it possible to use the designs obtained from Theorem 7.1 in an event-sampling context preserving the stability? And in this case, how much does the performance deteriorate with respect to periodic sampling?

In the event-based sampling policy considered here, the nodes communicate only in case that the difference between their current estimates and the last transmitted estimates exceed a given threshold. The price to be paid in this case is that asymptotic stability of the estimation errors and system are no longer guaranteed. However, GUUB stability of both dynamics can be proved.

Furthermore, it will be shown that performance degradation can be traded off with respect to transmission reductions.

Note that this section does not develop a new design method, but applies the results of Theorem 7.1 with a different sampling policy. Henceforth, the notation $\Upsilon, \Phi, \Psi, \Lambda, \Gamma$ instead of $\Upsilon(\mathcal{K}), \Phi(\mathcal{M}), \Psi(\mathcal{K}), \Lambda(\mathcal{N}), \Gamma(\mathcal{N})$ will be used to remark that sets $\mathcal{M}, \mathcal{K}, \mathcal{N}$ are assumed to be designed.

**Theorem 7.2.** *Consider the sets $\mathcal{M}, \mathcal{K}, \mathcal{N}$ obtained through Theorem 7.1, and assume a communication policy where each node $j$ broadcasts its estimate $\hat{x}_j$ to its neighbors following the triggering rule given in Definition 7.2. Then, the estimation error $e(k)$ and the state of the system $x(k)$ are GUUB with bounds*

$$\begin{aligned}
\|e(k)\|_2 &\leq \delta_w \sqrt{\frac{np\lambda_{\max}(P_e)}{\lambda_{\min}(P_e)}}(\|\Phi+\Psi+\Lambda\|_\infty \alpha_e + \|\Gamma\|_\infty), \\
\|x(k)\|_2 &\leq \delta_e \sqrt{\frac{\lambda_{\max}(P_x)}{\lambda_{\min}(P_x)}}(\|A+BK\|_2 \alpha_x + \|\Upsilon\|_2),
\end{aligned}$$

*being $\alpha_x, \alpha_e$ positive scalars defined by*

$$\alpha_e = \frac{k_e + \sqrt{k_e + \lambda_{\min}(X_e)\|\Gamma^T P_e \Gamma\|_\infty}}{\lambda_{\min}(X_e)}, \tag{7.25}$$

$$\alpha_x = \frac{k_x + \sqrt{k_x + \lambda_{\min}(X_x)\|\Upsilon^T P_x \Upsilon\|_2}}{\lambda_{\min}(X_x)}, \tag{7.26}$$

*with $k_x = \|\Upsilon^T P_x (A+BK)\|_2$, $k_e = \|\Gamma^T P_e (\Phi+\Psi+\Lambda)\|_\infty$ and $X_e, X_x$ the unique positive definite matrices such that*

$$(\Phi+\Psi+\Lambda)^T P_e (\Phi+\Psi+\Lambda) - P_e = -X_e, \tag{7.27}$$

$$(A+BK)^T P_x (A+BK) - P_x = -X_x. \tag{7.28}$$

**Proof.** Consider the following Lyapunov function for the observation error:

$$V_e(k) = e^T(k) P_e e(k),$$

with $P_e$ obtained from Theorem 7.1. The forward increment of the Lyapunov function is given by

$$\begin{aligned}
\Delta V_e(k) &= V_e(k+1) - V_e(k) \\
&= e^T(k+1) P_e e(k+1) - e^T(k) P_e e(k).
\end{aligned}$$

Using the dynamics of the observation error given in Proposition 7.5, it turns out that[1]

$$\begin{aligned}\Delta V_e(k) &= \left[(\Phi+\Psi+\Lambda)e+\Gamma w\right]^T P_e \left[(\Phi+\Psi+\Lambda)e+\Gamma w\right] - e^T P_e e \\ &= e^T(\Phi+\Psi+\Lambda)^T P_e(\Phi+\Psi+\Lambda)e - e^T P_e e \\ &\quad + w^T\Gamma^T P_e \Gamma w + 2w^T\Gamma^T P_e(\Phi+\Psi+\Lambda)e.\end{aligned}$$

From Theorem 7.1, the dynamics of the error $e(k)$ is asymptotically stable with periodic sampling, so there exists the positive definite matrix $X_e$ defined in (7.27). Then, it holds

$$\Delta V_e(k) = -e^T X_e e + w^T\Gamma^T P_e\Gamma w + 2w^T\Gamma^T P_e(\Phi+\Psi+\Lambda)e.$$

Using the infinity norm, the previous equation can be bounded as

$$\Delta V_e(k) \leq -\lambda_{\min}(X_e)\|e\|_\infty^2 + \|\Gamma^T P_e\Gamma\|_\infty\|w\|_\infty^2 + 2\|\Gamma^T P_e(\Phi+\Psi+\Lambda)\|_\infty\|w\|_\infty\|e\|_\infty.$$

The right hand side of the equation above is an algebraic second order equation in $\|e\|_\infty$. The roots of $\Delta V_e(k)=0$ can be obtained by imposing $a\|e\|_\infty^2 + b\|e\|_\infty + c = 0$, where

$$\begin{aligned}a &= -\lambda_{\min}(X_e), \\ b &= 2\|\Gamma^T P_e(\Phi+\Psi+\Lambda)\|_\infty\|w\|_\infty, \\ c &= \|\Gamma^T P_e\Gamma\|_\infty\|w\|_\infty^2.\end{aligned}$$

Note that $a<0$ and $b,c>0$. The unique positive root is exactly

$$\|e\|_\infty = -\frac{b+\sqrt{b^2-4ac}}{2a} = \alpha_e\|w\|_\infty,$$

where $\alpha_e$ is given in equation (7.25). Because of the sign of $a$, it is easy to see that the Lyapunov function $V_e(k)$ decreases whenever $\|e(k)\|_\infty > \alpha_e\|w(k)\|_\infty$. Please note that the triggering rule implies that $\|w(k)\|_\infty < \delta_w$. Therefore, it yields that $\Delta V_e(k) < 0$ in the region $\|e(k)\|_\infty > \alpha_e\delta_w$.

Now consider $k^*$ as the time instant when the estimation errors enters in the region $\|e(k)\|_\infty \leq \alpha_e\delta_w$. Then, taking into account the error dynamics given by (7.24), one can easily obtain that

$$\max\|e(k^*+1))\|_\infty = (\|\Phi+\Psi+\Lambda\|_\infty\alpha_e + \|\Gamma\|_\infty)\delta_w,$$

[1] Time indexes have been removed to alleviate the notation.

so the error might leave the region $\|e(k)\|_\infty \leq \alpha_e\delta_w$. After that, the Lyapunov function decreases again. Using the inequality $\|e\|_2 < \sqrt{np}\|e\|_\infty, \forall e \in \mathbb{R}^{np}$ the maximum of the 2-norm in $k^*+1$ can be obtained as

$$\|e(k^*+1))\|_2 = \sqrt{np}(\|\Phi+\Psi+\Lambda\|_\infty\alpha_e + \|\Gamma\|_\infty)\delta_w.$$

Although the Lyapunov function decreases, it is not possible to ensure the decreasing of the 2-norm nor the infinity norm of the error vector. It is the $P_e$-norm the one that decreases. The maximum of the $P_e$-norm for all instant $k > k^*$ is given by

$$\|e(k)\|_{P_e} = \sqrt{e^TP_ee} \leq \sqrt{\lambda_{\max}(P_e)}\|e(k^*+1)\|_2,$$

where inequality $\lambda_{\min}(P_e)\|e\|_2^2 \leq e^TP_ee \leq \lambda_{\max}(P_e)\|e\|_2^2$ has been used. From this $P_e$-norm the final 2-norm can be bounded as

$$\|e(k)\|_2 \leq \sqrt{\frac{\lambda_{\max}(P_e)}{\lambda_{\min}(P_e)}}\|e(k^*+1)\|_2, \forall k > k^*.$$

This way, the boundedness of the estimation error has been proved.

Consider now the following Lyapunov function for the state of the plant

$$V_x(k) = x^T(k)P_xx(k),$$

with $P_x$ obtained from Theorem 7.1. The forward increment of the Lyapunov function is given by

$$\begin{aligned}\Delta V_x(k) &= V_x(k+1) - V_x(k)\\ &= x^T(k+1)P_xx(k+1) - x^T(k)P_xx(k).\end{aligned}$$

Using the system dynamics given in Proposition 7.1, it turns out that

$$\begin{aligned}\Delta V_x(k) &= [(A+BK)x + \Upsilon e]^TP_x[(A+BK)x+\Upsilon e] - x^TP_xx\\ &= x^T(A+BK)^TP_x(A+BK)x - x^TP_xx\\ &\quad +e^T\Upsilon^TP_x\Upsilon e + 2e^T\Upsilon^TP_x(A+BK)x.\end{aligned}$$

From Theorem 7.1, the system is asymptotically stable under periodic sampling, so there exists the positive definite matrix $X_x$ defined in (7.28). Then, it holds

$$\Delta V_x(k) = -x^TX_xx + e^T\Upsilon^TP_x\Upsilon e + 2e^T\Upsilon^TP_x(A+BK)x.$$

Taking norms, the forward difference can be bounded as follows:

$$\Delta V_x(k) \leq -\lambda_{\min}(X_x)\|x\|_2^2 + \|\Upsilon^TP_x\Upsilon\|_2\|e\|_2^2 + 2\|\Upsilon^TP_x(A+BK)\|_2\|e\|_2\|x\|_2.$$

The right hand side of the equation above is again an algebraic second order equation in $\|x\|_2$. Operating as before, it is easy to see that the Lyapunov function $V_x(k)$ decreases whenever $\|x(k)\|_2 > \alpha_x \|e(k)\|_2$. Using the derivations above, it yields that $V_x(k)$ decreases if $\|x(k)\|_2 > \alpha_x \delta_e$, where $\delta_e$ is given in equation (7.25).

Now consider $k^*$ as the time instant when the state enters in the region $\|x(k)\|_2 < \alpha_x \delta_e$. Then, taking into account the system dynamics given in Proposition 7.5, one can easily obtain that

$$\|x(k^* + 1))\|_2 \leq (\|A + BK\|_2 \alpha_x + \|\Upsilon\|_2)\, \delta_e.$$

If the state leaves the region $\|x(k)\|_2 \leq \alpha_x \delta_e$, the Lyapunov function will decrease again. Hence, the $P_x$-norm of the state decreases. The maximum of the $P_x$-norm for all instant $k > k^*$ is given by

$$\|x(k)\|_{P_x} \leq \sqrt{\lambda_{\max}(P_x)} \|x(k^* + 1)\|_2.$$

Then, the final Euclidean norm can be bounded as

$$\|x(k)\|_2 \leq \sqrt{\frac{\lambda_{\max}(P_x)}{\lambda_{\min}(P_x)}} \|x(k^* + 1)\|_2, \forall k > k^*.$$

This way, the boundedness of the state is proved. □

**Remark 7.4.** It is worth pointing out that the choice of the infinity norm of $w(k)$ as triggering condition in Theorem 7.2 is not arbitrary. Given that each observer has access only to local information, the infinity norm can be practically implemented using just local information: since each node sends its information whenever $\|w_i(k)\|_\infty \geq \delta_w$, it turns out that at inter-sampling times $\|w(k)\|_\infty < \delta_w$, thus it is possible to upper bound $\|w(k)\|_\infty$ resorting only to local information at each node.

The parameter $\delta_w$ is related to the size of the ultimate bound region of the estimation error $e(k)$ and, indirectly, with the boundedness of the final region of $x(k)$. By enlarging the value of $\delta_w$, it is possible to reduce the amount of transmissions between the nodes, while by reducing it, a better control and estimation performance is achieved, since the plant state and the observation error are finally confined in a smaller region. This trade-off, typical in event-based frameworks, will be shown up in the following section.

## 7.6 Application example

The performance of the distributed control scheme is experimentally tested in the following. The experimental setup and the model are described providing all the considerations related to the distributed scheme.

### 7.6.1 Plant description

The experiments have been performed on the 33-041 Coupled Tanks System of Feedback Instruments [94] (see Figure 7.3). This plant, a variant of the quadruple-tank process originally proposed in [103], is a model of a fragment of a chemical plant. It is composed of four tanks, each one equiped with a pressure sensor to measure the water level. The couplings between the tanks can be changed using seven manual valves to modify the dynamics of the system. Water is delivered to the tanks by two independently controlled, submerged pumps. Drain flow rates can be modified using suitable orifice caps. Notation related with the plant is given in Table 7.1.


Figure 7.3: Plant of four coupled tanks.

| | Description |
|---|---|
| $h_i$ | Water level of tank $i$ |
| $h_i^0$ | Reference level of tank $i$ |
| $\Delta h_i$ | Increment of $h_i$ with respect to $h_i^0$ |
| $\Delta h_r$ | Reference level with respect to $h^0$ |
| $v_i$ | Voltage of pump $i$ |
| $v_i^0$ | Reference voltage of pump $i$ |
| $\Delta v_i$ | Increment of $v_i$ with respect to $v_i^0$ |
| $\Delta v_r$ | Reference voltage with respect to $v^0$ |
| $z$ | Controlled output |
| $r$ | Reference to be tracked |

Table 7.1: Notation related to the plant

The coupled tanks are controlled using Simulink and an Advanced PCI1711 Interface Card. For the experiments, the following configuration has been chosen:

- Input water is delivered to the upper tanks. Pump 1 feeds tank 1 and pump 2 feeds tank 3.
- Tanks 1 and 3 are coupled by opening the corresponding valve.

Although the plant is a compact educational platform, it can realistically represent all the relevant elements of a real-world distributed plant. For instance, large-scale chemical plants, where coupled processes can be located hundreds of meters away from each other.

The distributed control scheme proposed in this work can be applied to the four-tank plant considering a network with four agents, two of them being observers and the other two observers & controllers (see Figure 7.4). Each agent has been tagged from 1 to 4 according to the number of the tank whose level it is measuring. Agent 1 (respectively 3) measures the water level in tank 1 (3) and applies the control signal to pump 1 (2). Agents 2 and 4 measure the level in the tanks 2 and 4 respectively. The communication topology is: $2 \Leftrightarrow 1 \Leftrightarrow 3 \Leftrightarrow 4$.

The objective of the experiments is twofold. First, all four states of the plant must be estimated from every agent. Secondly, the water level of the two lower tanks is to be controlled. Notice that with this configuration, the agents applying the control signals (agents 1 and 3) do not have direct measurement of the variables being controlled (levels in tanks 2 and 4).

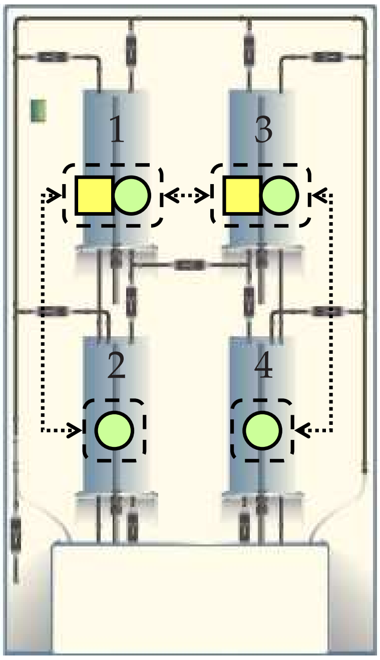



Figure 7.4: Distributed control scheme with 4 agents. Agents 1 and 3 are observers & controllers; agents 2 and 4 are observers. The dotted lines represent the communication links.

### 7.6.2 Plant modeling

The coupled tanks system admits the following nonlinear model:

$$
\begin{aligned}
\frac{dh_1(t)}{dt} &= -\frac{a_1}{S}\sqrt{2gh_1(t)} + \eta v_1(t) - \frac{a_{13}}{S}\sqrt{2g(h_1(t)-h_3(t))},\\
\frac{dh_2(t)}{dt} &= \frac{a_1}{S}\sqrt{2gh_1(t)} - \frac{a_2}{S}\sqrt{2gh_2(t)},\\
\frac{dh_3(t)}{dt} &= -\frac{a_3}{S}\sqrt{2gh_3(t)} + \eta v_2(t) + \frac{a_{13}}{S}\sqrt{2g(h_1(t)-h_3(t))},\\
\frac{dh_4(t)}{dt} &= \frac{a_3}{S}\sqrt{2gh_3(t)} - \frac{a_4}{S}\sqrt{2gh_4(t)},
\end{aligned}
$$

where $h_i(t)$ $(i = 1,\ldots,4)$ denote the water level in the corresponding tank and $v_i$ $(i = 1,2)$ are voltage applied to the pumps. $a_i$ $(i = 1,\ldots,4)$ are the outlet area of the tanks, $a_{13}$ is the outlet area between tanks 1 and 3; $\eta$ is a constant relating the control voltage with the water flow from the pump, $S$ is the cross-sectional area of the tanks, and $g$ is the gravitational constant.

This system is linearized around the equilibrium point given by $h_i^0$ and $u_i^0$, yielding

$$\Delta\dot{h}(t) = A\Delta h(t) + B\Delta v(t),$$

where $\Delta h(t) = \left[h_1(t) - h_1^0 \;\; \dots \;\; h_4(t) - h_4^0\right]^T$ and $\Delta v(t) = \left[v_1(t) - v_1^0 \;\; v_2(t) - v_2^0\right]^T$. Matrices $A$ and $B$ are obtained by using a Taylor expansion of the nonlinear equations of the model around the equilibrium point:

$$
\begin{aligned}
A &= \begin{bmatrix} -\frac{a_1 g}{S\sqrt{2gh_1^0}} - \frac{a_{13} g}{S\sqrt{2g(h_1^0 - h_3^0)}} & 0 & \frac{a_{13} g}{S\sqrt{2g(h_1^0 - h_3^0)}} & 0 \\ \frac{a_1 g}{S\sqrt{2gh_1^0}} & -\frac{a_2 g}{S\sqrt{2gh_2^0}} & 0 & 0 \\ \frac{a_{13} g}{S\sqrt{2g(h_1^0 - h_3^0)}} & 0 & -\frac{a_3 g}{S\sqrt{2gh_3^0}} - \frac{a_{13} g}{S\sqrt{2g(h_1^0 - h_3^0)}} & 0 \\ 0 & 0 & \frac{a_3 g}{S\sqrt{2gh_3^0}} & -\frac{a_4 g}{S\sqrt{2gh_4^0}} \end{bmatrix}, \\
B &= \begin{bmatrix} \eta & 0 \\ 0 & 0 \\ 0 & \eta \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Discretizing this continuous model with sampling time $T_s$, it yields

$$\Delta h(k+1) = A_D \Delta h(k) + B_D \Delta v(k),$$

where $\Delta h(k) = \left[h_1(k) - h_1^0 \; \dots \; h_4(k) - h_4^0\right]^T$ and $\Delta v(k) = \left[v_1(k) - v_1^0 \;\; v_2(k) - v_2^0\right]^T$. Matrices $A_D$ and $B_D$ are the discrete counterpart of $A$ and $B$.

The objective is not only the stabilization of the plant around the linearization point, but also to track references. To do so, the controlled output is set as $z \triangleq C_r \Delta h$, where $C_r$ is a matrix that selects the water level of tanks 2 and 4. The references are given by vector $r$. At the equilibrium points, it should be verified $z \simeq r$ and $\Delta h(k+1) \simeq \Delta h(k) \simeq \Delta h_r(k)$. To perform the tracking task, the incremental equilibrium points $(\Delta h_r, \Delta v_r)$ associated with reference $r$ are found as follows.

$$
\begin{aligned}
\Delta h_r(k) &= A_D \Delta h_r + B_D \Delta v_r, \\
r &= C_z \Delta h_r.
\end{aligned}
$$

Rewriting the equation above in blocks, it yields

$$\begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} A_D - I & B_D \\ C_z & 0 \end{bmatrix} \begin{bmatrix} \Delta h_r \\ \Delta v_r \end{bmatrix},$$

so that the incremental equilibrium point associated with $r$ can be obtained as

$$\begin{bmatrix} \Delta h_r \\ \Delta v_r \end{bmatrix} = \begin{bmatrix} A_D - I & B_D \\ C_z & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix}.$$

| | Value | Unit | Description |
|---|---|---|---|
| $h_i$ | 0-25 | cm | Water level of tank *i* |
| $v_i$ | 0-5 | V | Voltage level of pump *i* |
| S | 0.01389 | $m^2$ | Cross-sectional area |
| $a_i$ | 50.265e-6 | $m^2$ | Outlet area of tank *i* |
| $a_{13}$ | 50.265e-6 | $m^2$ | Outlet area between tanks 1 and 3 |
| $\eta$ | 2.2e-3 | $\frac{m}{V.s}$ | Constant relating voltage and flow |
| $h_1^0$ | 9.8 | cm | Reference level of tank 1 |
| $h_2^0$ | 17.4 | cm | Reference level of tank 2 |
| $h_3^0$ | 7.5 | cm | Reference level of tank 3 |
| $h_4^0$ | 13.6 | cm | Reference level of tank 4 |
| $v_1^0$ | 3.3 | V | Voltage level of pump 1 |
| $v_2^0$ | 2.6 | V | Voltage level of pump 2 |
| $T_s$ | 1 | s | Sampling time |

Table 7.2: Parameters of the plant.

It is assumed that the references are reachable by the system, that is, the inverse above does exist. Finally, to track references, we must stabilize the following system.

$$x(k+1) = A_D x(k) + B_D u(k), \tag{7.29}$$

where $x(k) \triangleq \Delta h(k) - \Delta h_r$ and $u(k) \triangleq \Delta v(k) - \Delta v_r$. Observe that this system has the same structure that the one described in (7.3).

### 7.6.3 Experimental results

The performance of the proposed distributed method is experimentally tested on the four-tank level control system, with model parameters as shown in Table 7.2.

A study and comparison of both periodic and event-based sampling possibilities is performed. First, a periodic sampling controller for the distributed four-tank system is designed taking weighting matrices in (7.8) as

$$\begin{aligned}
Q_x &= diag\{0.1, 100, 0.1, 100\}, \\
Q_1 &= diag\{1, 10, 1, 0.1\}, \\
Q_2 &= 10^{-2} \cdot diag\{1, 1, 1, 1\}, \\
Q_3 &= diag\{1, 0.1, 1, 10\}, \\
Q_4 &= 10^{-2} \cdot diag\{1, 1, 1, 1\}, \\
R &= 10^{-6} \cdot I_2.
\end{aligned}$$

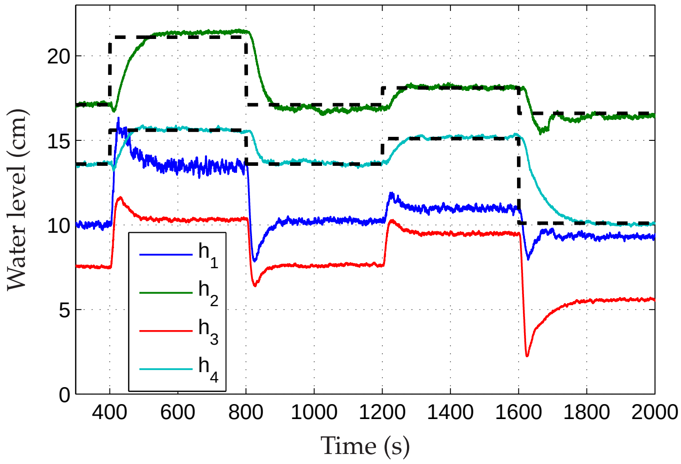



Figure 7.5: Tracking with periodic sampling. The references are shown in dashed lines

Figure 7.5 depicts the evolution of the system with the distributed periodic control scheme. A satisfactory tracking of references in tanks 2 and 4 can be observed. The effects of the chosen weighting matrices become apparent in the overshooting in tanks 1 and 3, since the objective is to perform a fast tracking of the references in the lower tanks. The average rise time of the response is around 100 seconds, about one third of the natural time constant of the open loop system.

The water level and the estimations by agent 1 are shown in Figure 7.6. It is worth pointing out that agent 1 has no direct access to level measurements in tanks 2 and 4, but it estimates these states from the information received from its neighbors.

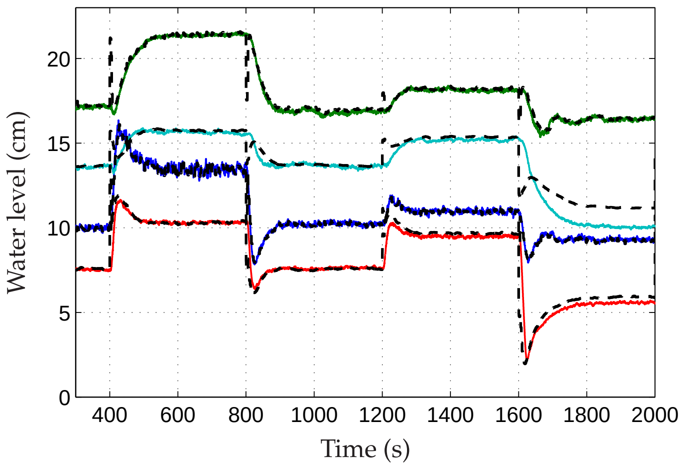



Figure 7.6: Observation in agent 1 with periodic sampling. The estimates are depicted in dashed lines

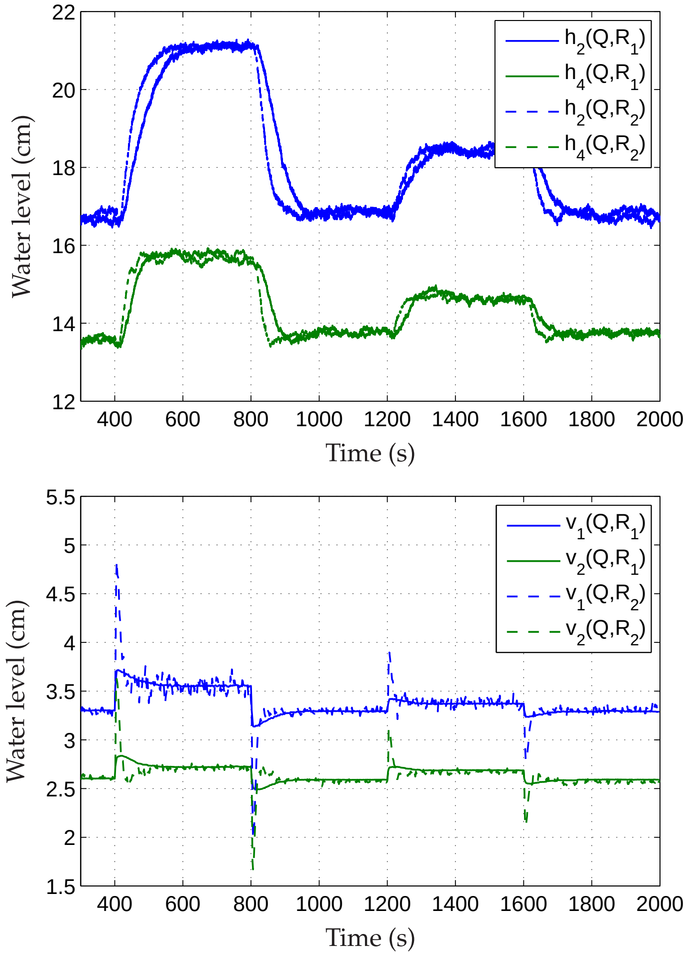



Figure 7.7: Tracking performance and control signals with periodic sampling and different weighting matrices

Next, the effect of the weighting factors in (7.8) are tested. Two experiments are conducted, both with the same weighting for matrices $Q_x$ and $Q_i$ $(i = 1, \ldots, 4)$ as in the previous experiment, and different values for $R$: $R_1 = 10^{-6} \cdot I_2$ to obtain a fast reference tracking, and $R_2 = 10^2 \cdot I_2$ to ponderate control effort in the cost functional. Figure 7.7 shows the evolution of the water levels and the control actions. As expected, a tighter tracking performance is observed for the experiment with $R_1$, at the cost of more aggressive control signals. This result shows how control performance can be traded off with respect to control actions by appropriately tuning the weighting gains.

The experiment depicted in Figure 7.8 is designed to show the decoupling capabilities of the proposed control strategy. Tank 2 is set to track references whereas the reference for tank 4 is kept constant. Coupling arises from the valve communicating

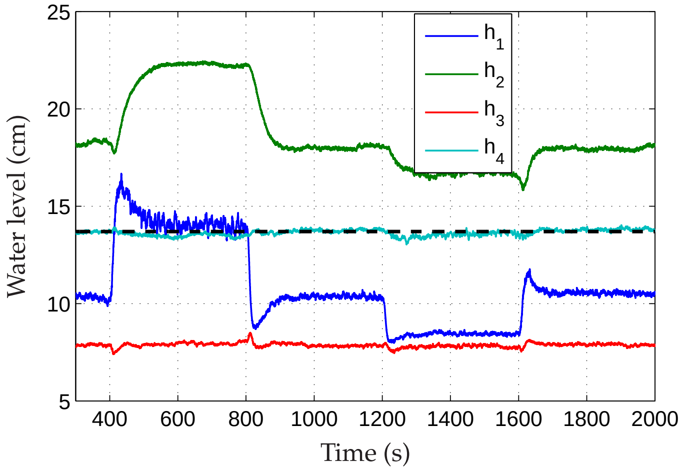



Figure 7.8: Control decoupling: a change of reference is set for tank 2 while reference of tank 4 remains constant

tanks 1 and 3. Thus, to modify the level of tank 2, tank 1 must be filled or emptied, and due to the coupling valve, tank 3 varies its level also affecting the level in tank 4. The distributed controllers achieve a remarkable decoupling of the closed-loop dynamics as can be observed from these experiments.

Lastly, the event-driven control scheme is tested. The same weighting matrices of the first experiment are considered, while different thresholds to trigger the events are used: $\delta_w = 0.1$, $\delta_w = 0.3$ and $\delta_w = 0.6$. The results for these tests are shown in Figure 7.9, where the tracking performance in tank 2 is shown.

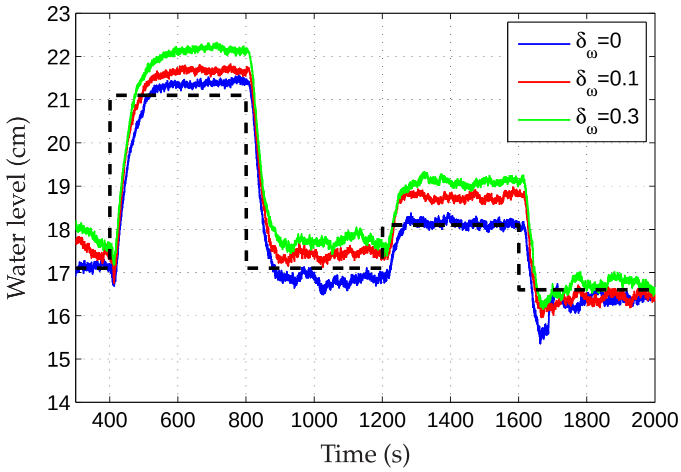



Figure 7.9: Tracking of references in tank 2 for different values of $\delta_w$

It is observed that, as expected, the larger the event threshold (larger $\delta_w$), the poorer the tracking performance becomes.

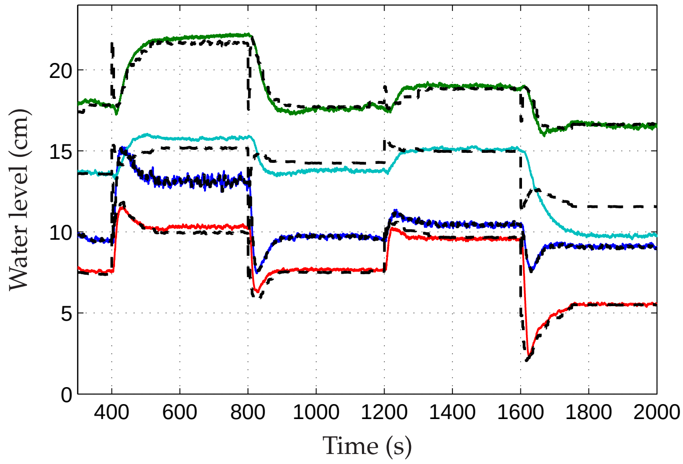



Figure 7.10: Estimation performance of agent 1 with $\delta_w = 0.6$. The estimates are shown in dashed lines

Figure 7.10 shows the observed states for node 1 with $\delta_w = 0.6$. The performance degradation due to the event-based communication scheme is now apparent when compared to the results with the periodic results. On the other hand, the event-based scheme significantly reduces the number of required transmissions, as it is depicted in Figure 7.11, that shows the evolution of the ratio of transmitted packets using an event-based policy with respect to the periodic case.

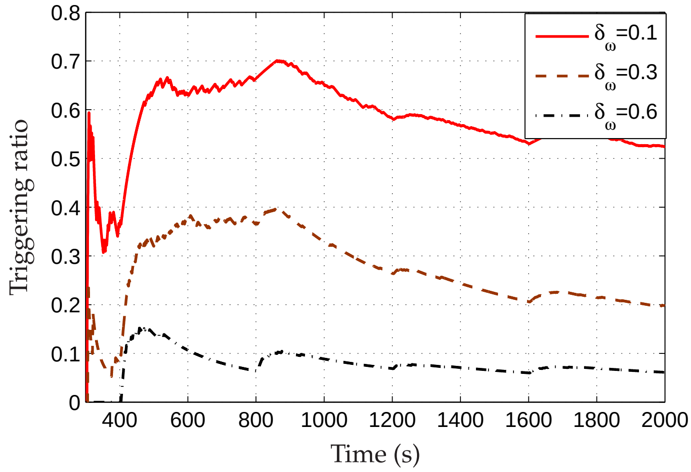



Figure 7.11: Ratio of packets sent wrt. periodic communication

## 7.7 Chapter summary

This chapter proposes a cost-guaranteed distributed estimation and control scheme for networked control systems, where the sensing and control capabilities are shared by a number of agents. The agents are assumed to collect partial information of the evolution of the plant states and have access, in general, to a subset of the control channels of the plant. The work proposes a fully distributed control and estimation scheme so that the collective behavior of all agents drive the system to stability. The result is of application for large-scale systems where both centralized or classical point-to point distributed schemes are discouraged.

Both, a periodic-sampling and an event-based scheme have been discussed. Using a four-tank level control system, experiments were conducted to show that in practice, little performance degradation is in general observed for the event-based compared to the periodic-based scheme, while the number of packets transmitted is drastically reduced. The cost-guaranteed approach adopted has been also proved to be very convenient in practical applications as allows the trade-off between control effort, performance degradation and average packet transmission rates.

# Chapter 8

# Conclusions

This final chapter of the thesis summarizes the main contributions of the work. The achievements are highlighted and a criticism on potential weaknesses of the results is included. Finally, some possible future research lines and ideas are presented.

## 8.1 Main achievements

Throughout the document, the research conducted during this thesis has been reviewed. In the following, the author would like to bring up the most important attainments from his particular point of view.

- In Chapter 3, a new stability criterion has been proposed. This criterion is applicable to time-delay systems and, indirectly, to networked control systems using the input-delay approach. It is characterized by its reduced conservatism compared with similar works in the literature, and consequently the stability of the systems can be ensured for higher bounds on the delay.

- In Chapter 4, a method to design $H_2/H_\infty$ controllers has been developed. The solution exhibits wide applicability: TDS and NCS, and different choices of the Lyapunov-Krasovskii functional. But perhaps more importantly, the impact of the solution lays in its optimality. It has been theoretically proved that this method achieves a lower bound of the optimal $H_2$ cost than comparable works in the literature, for the same $H_\infty$ index.

- Chapter 5 shows that, as hypothesized, introducing a model of the plant at the controller end of the communication alleviates the traffic over the network. Both periodic and self-triggered sampling policies have been demonstrated to maintain the stability of the closed-loop system in spite of the reduction of bandwidth usage.

- In Chapter 6, two sampling schemes regarding scheduled communication have been studied: periodic and aperiodic. Under some mild conditions, the optimal aperiodic solutions eventually converge to a periodic policy. This fact has been demonstrated for certain particular systems, although simulation results suggest that this shall be the case for any systems. The main implication is that the benefits of both schemes can be attained: reduced mathematical burden, energy efficiency, traffic reduction and optimality.

- Finally, Chapter 7 proposes a novel design approach for joint distributed estimation and control based upon Luenberger-like agents improved with consensus strategies. A remarkable feature of the approach is the significant reduction in bandwidth usage, capitalizing on event-based communication. Furthermore, as most of the energy expended in distributed tasks is associated with transmissions, the solution is more efficient in terms of energy consumption.

In summary, this thesis contains relevant elements of merit, that surpasses the current state of the art regarding the problem of control and estimation over communication networks.

## 8.2 Potential weakness and limitations

This research has formally met the stated objectives, namely it has

*(...) afford innovative solutions to some of the new problems that arise when controlling a system through a communication network. (...)*

In spite of this, it has fall short at some specific levels. Some possible issues are explained below:

- The delay-dependent stability criterion presented in Chapter 3 is without doubt of great application for time-delays systems. Furthermore, by properly using the input delay approach, it is also useful for networked control systems, as Chapter 4 demonstrated. However, it must be said that this extension to NCS is contrived.

There are a few particularities that affect NCS which have not been properly taken into account. For instance, from the stability point of view, the following two situations are regarded as equivalent with the presented formulation:

1. A system with sampling period equal to 1s and maximum communication delay of 0.5s, i.e. $\tau_M = 1 + 0.5 = 1.5$s,
2. A system with sampling period equal to 1.4s and maximum communication delay of 0.1s, i.e. $\tau_M = 1.4 + 0.1 = 1.5$s.

Intuitively, those differences must affect, not only to the control performance, but also to the stability margins. In the future, stability criteria for NCS should separate sampling periods and delays in order to get improved results.

- It has been proved that the introduction of a plant model at the controller's end reduces the traffic over the network, but still preserving the stability of the system. Therefore, it offers the possibility to improve the synthesized controllers by means of the methods in Chapter 4. However, the performance of those new model-based controllers has not been thoroughly investigated, with respect to the optimization of the cost index and disturbance rejection capabilities. There must exist a trade-off between the performance and the reduction of traffic that has not been tackled in this thesis.

- Analogously, the results presented in distributed control and estimation deserve more profound attention. The controllers and observers are designed assuming periodic communication and minimizing a cost index. Then, an event-based policy is implemented. The question is: are the pre-synthesized controllers still the optimal ones for an event-triggered scheme? Although, the simulations carried out in Chapter 7 suggest that this is indeed the case, it has not been fully demonstrated.

- The main results of this thesis are theoretical in nature. Two experimental examples have been analyzed in Chapters 4 and 7. These testbeds are adequate to test the proposed controllers and other networked solutions. Yet, one of them, the two dof robot, is an uncommon application in NCS; and in the four-coupled tank systems the communication protocol has been simulated. In this sense, a control engineer might demand more applications to experimental plants.

## 8.3 Impact of the thesis

This section aims at anticipating the impact of the thesis. This is only the opinion of the author, since the real impact will only be realised in due time.

The thesis affords three main results that shall benefit the control community. They can be organized based upon the terms:

**Short term:** Chapter 4 has presented a general method to design $H_2/H_\infty$ controllers for TDS and NCS. Theorem 4.4 demonstrates that this method outperforms existing ones with respect to minimization of the cost index. The method is applicable to different sorts of systems and functionals. As such, this result should impact the proposition of new controllers for related kinds of systems as well as more complicated choices of the functional.

**Medium term:** In the framework of sensor scheduling, Chapter 6 has shown that an optimal selection of the sensor (aiming for the minimization of the observation error variance) yields a periodic scheduling. Although this effect has been observed before in different experiments [88, 184], Theorem 6.2 is the first theoretical result concerning this issue. The impact is twofold. First, this idea can now be extended for other optimal choices of the sensors, such as other *N-step-ahead* laws. Second, the periodic response in *a priori* aperiodic systems has been observed in other areas, such as the inter-sampling time in event-based control. Therefore, an adequate modification of Theorem 6.2 may be used in different areas to explain this phenomenon.

**Long term:** The last chapter of the thesis studies the distributed estimation and control of large-scale systems. The agent structure consists of a Luenberger-like observer plus additional consensus terms. As pointed out before, this is the first time that the event-based communication has been considered in the distributed scheme. The relevance of these results is considerable. New open problems arise that shall challenge the scientific community, as will be explained in the next section.

In addition to the above, it is possible to foresee the influence of the thesis in two other aspects: bandwidth reduction and energy efficiency.

The former issue, bandwidth reduction, is pervasive to all scenarios in telecommunications. The available bandwidth in real-world applications is unavoidably limited, but new services consistently demand ever higher bit rates. Therefore, when different devices share a common medium, solutions that make an adequate use of the bandwidth, such as those in Chapters 5 and 6, are of undeniable interest.

An efficient consumption of the energy is mandatory nowadays. From smart houses to innovative vehicles, energy efficiency is becoming a necessary feature in all products. To this end, Chapters 5, 6 and 7 propose different solutions to reduce the consumption, with special emphasis in wireless sensor networks.

## 8.4 Further work

The control of systems over a communication network is a relatively mature field. Notwithstanding, there is still much room for future contributions. In light of this work, a number of stimulating challenging research lines can be suggested. In the following, some of these ideas are briefly outlined.

**Modeling of a small-size system controller over a network.** In this thesis, specifically in Chapter 4, the input delay approach has been adopted to model the unreliable communication in a NCS. As pointed out in that chapter, a number of works have employed this same model, for it incorporates delays, sampling and packet dropouts, under a unified framework. However, although the theory behind is solid and resorts to well-known time-delay systems, it suffers an unavoidable drawback from its inception: it has not been created to model NCS, but TDS. That simple, but at the same time huge disadvantage is to a great extent responsible, for instance, for the conservatism of the stability criteria.

Therefore, an interesting research line could involve exploring different models available for NCS, as the ones in [39, 239], or proposing new ones.

**Control techniques.** The thesis has presented an $H_2/H_\infty$ controller for NCS whose performance can be further boosted by means of a model. But of course, many other controllers can be implemented. For instance, output feedback control [211], model predictive control [25], or nonlinear control [243] have found application with competitive performance.

**Observer techniques.** Similarly, additional observation techniques can be adapted to the NCS framework. Sliding mode observers [50], particle filters and swarm estimation [38] appear to be adequate tools for observing these systems.

**Sensor scheduling.** Presenting a close result about the periodicity of the optimal scheduling is perhaps one of the most interesting avenues to pursue. Furthermore, other *N-steps-ahead* optimization problems can be considered. Additionally, a real implementation of those schedules under the 802.15.4 protocol would be of interest to test the effective reduction of the traffic and energy consumption.

**Distributed estimation and control.** Several elements can be subject of improvement:

- Although the presented results are applicable for distributed systems, the design step is made in a centralized way. This entails high computational costs that, for large number of agents, would become intractable. Moreover, the graph of the network must be known *a priori*, an unrealistic precondition in this context, since sensor networks are typically intended for high re-configurability, robustness to link/node failures, etc.

  For these reasons, the design method should preferably be decentralized or distributed. That is, an agent must only know what it is able to achieve on its own, and what it is able to improve with the information of its neighbours. It should be the agent the one that calculates its own controller and observer.
- The thesis does not pay particular attention on the network topology. Given a graph, different design methods are presented. But the topology of the graph has not been exploited. For instance, what is its influence on the feasibility of the problem? How does it affect to the optimality and performance? These and other questions could be subject of future research.
- Closely related to the above, there is the issue of time-varying or switching topology. In the future, the design methods must be robust or adaptive to switching topology, giving rise to more flexible and fault-tolerant sensor networks.
- Extensions to consider delays, dropouts, congestion and other drawbacks can be considered in the future.

**Multi-agent systems.** Chapter 2 presented a third kind of NCS that has not been tackled in this thesis, namely, fleets or multi-agent systems. The extensive literature dedicated to this topic suggests that it is an undeniably interesting research line (see Chapter 2 and references therein). Network-induced problems, distributed optimization, control formation and others, may also be matter of further investigation.

# Appendix A

# Stability Theorems

Two Lyapunov-based theorems are introduced in this appendix. Both results have been profusely used to study the stability of systems affected by delays. The reader may find more information about these stability theorems in [138].

In the following, $\mathbb{C}_{n,\tau} = \mathbb{C}([-\tau, 0], \mathbb{R}^n)$ denotes the banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence and designate the norm of an element $\phi$ in $\mathbb{C}_{n,\tau}$ by

$$\|\phi\|_* = \sup_{\theta \in [-\tau, 0]} \|\phi(\theta)\|.$$

## A.1 Lyapunov-Razumikhin Theorem

Consider the functional differential equation

$$\begin{aligned} \dot{x}(t) &= f(t, x_t),\ t \geq t_o, \\ x_{t_o}(\theta) &= \phi(t+\theta),\ \forall \theta \in [-\tau, 0] \end{aligned} \tag{A.1}$$

where $x_t(t),\ t \geq t_o$ denotes the restriction of $x(.)$ to the interval $[t-\tau \ \ t]$ translated to $[-\tau, 0]$, that is $x_t(\theta) = \phi(t+\theta),\ \forall \theta \in [-\tau, 0]$ with $\phi \in \mathbb{C}_{n,\tau}$.

Let the function $f(t, \theta) : \mathbb{R}^+ \times \mathbb{C}_{n,\tau} \to \mathbb{R}^n$ be continuous and Lipschitzian in $\theta$ with $f(t, 0) = 0$. Let $\alpha, \beta, \gamma, \delta : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and nondecreasing functions with

$$\begin{aligned} \alpha(r), \beta(r), \gamma(r) > 0;\ \ r \neq 0 \\ \alpha(0) = 0,\ \beta(0) = 0 \\ \delta(r) > r, \quad r > 0 \end{aligned}$$

If there exists a continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that

**(a)** $\alpha(\|x)\|) \leq V(t,x) \leq \beta(\|x\|), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$

**(b)** $\dot{V}(t,\phi) \leq -\gamma(\|x\|)$ if $V(t+\eta, x(t+\eta)) < \delta(V(t,x(t))), \forall \eta \in [-\tau, 0],$

then the trivial solution of (A.1) is uniformly stable.

## A.2 Lyapunov-Krasovskii Theorem

Consider the functional differential equation

$$\begin{aligned} \dot{x}(t) &= f(t, x_t),\ t \geq t_o, \\ x_{t_o}(\theta) &= \phi(t+\theta),\ \forall \theta \in [-\tau, 0] \end{aligned} \tag{A.2}$$

where $x_t(t),\ t \geq t_o$ denotes the restriction of $x(.)$ to the interval $[t-\tau \quad t]$ translated to $[-\tau, 0]$, that is $x_t(\theta) = \phi(t+\theta),\ \forall \theta \in [-\tau, 0]$ with $\phi \in \mathbb{C}_{n,\tau}$.

Let the function $f : \mathbb{R}^+ \times \mathbb{C}_{n,\tau} \to \mathbb{R}^n$ take bounded sets of $\mathbb{C}_{n,\tau}$ in bounded sets of $\mathbb{R}^n$ and $\alpha, \beta, \gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and nondecreasing functions with

$$\begin{aligned} \alpha(r), \beta(r) > 0;\ \ r \neq 0 \\ \alpha(0) = 0,\ \beta(0) = 0 \end{aligned}$$

If there exists a continuous function $V : \mathbb{R} \times \mathbb{C}_{n,\tau} \to \mathbb{R}$ such that

**(a)** $\alpha(\|\phi(0)\|) \leq V(t,x) \leq \beta(\|\phi\|_*), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$

**(b)** $\dot{V}(t,\phi) \leq -\gamma(\|\phi(0)\|),$

then the trivial solution of (A.2) is uniformly stable.

If $\alpha(r) \to \infty$ as $r \to \infty$, then the solutions are uniformly bounded.

If $\gamma(r) > 0$ for $r > 0$, then the solution $x = 0$ is uniformly asymptotically stable.

# Appendix B

# Linear Matrix Inequalities

This appendix contains an briefly report about Linear Matrix Inequalities (LMIs), extensively used throughout this thesis. Most contents of this appendix are based on the educational paper [2].

## B.1 Introduction

Linear matrix inequalities are no more than linear inequalities (with some particularities) in which the involved terms are matrices. The importance lies in the fact that a great variety of control problems can be formulated in a natural way using Linear Matrix Inequalities (LMIs).

In the beginning, the use of LMIs in the control context was seriously compromised due to the lack of efficient algorithms during most of the twentieth century. This situation changed dramatically with the appearance of a new generation of interior points algorithms that allowed to solve problems formulated in LMI form in a very efficient way [169]. As a result of this major breakthrough, the control community started to reinterpret previous analysis and synthesis results from the LMI point of view.

Nowadays the use of LMIs is ubiquitous in many control fields and there exists many efficient polynomial-time solvers like SeDuMi, SDPT3, etc. Moreover, there exists some parsers, like the free-distribution YALMIP, that serve as interface between the LMI formulation and the different solvers.

The interested reader may find more information in the well-referenced publications [19, 69].

## B.2 Definitions

Next, the notion of linear matrix inequality is precisely given in the following definition:

**Definition B.1.** Given the matrix variables $X_1, X_2, \ldots, X_m$, and the matrix function $H(X_1, X_2, \ldots, X_m)$ we say that the matrix inequality $H(X_1, X_2, \ldots, X_m) > 0$ (or analogously $H(X_1, X_2, \ldots, X_m) < 0$) is a linear matrix inequality on the decision variables $X_1, X_2, \ldots, X_m$ if $H(X_1, X_2, \ldots, X_m)$ is a symmetric matrix for every $X_1, X_2, \ldots,$ $X_m$ and the dependence of $H(X_1, X_2, \ldots, X_m)$ with respect to $X_1, X_2, \ldots, X_m$ is affine, where "$> 0$" and "$< 0$" stand for positive definite and negative definite, respectively.

The concept of definiteness is given next. Note that the fact that the eigenvalues are compared with zero makes that this concept is only defined over the set of symmetric matrices, which have real eigenvalues.

**Definition B.2.** A symmetric real matrix $H$ is said to be positive definite if all its eigenvalues are strictly greater than zero. Analogously, a symmetric real matrix $H$ is said to be negative definite if all its eigenvalues are strictly smaller than zero.

## B.3 Properties

The following statements, whose proof can be found in any text book on linear algebra, states important properties of a symmetric matrix.

**Property B.1.**

1. *The inverse of a nonsingular symmetric matrix is symmetric.*

2. *The eigenvalues of a symmetric real matrix are real.*

**Property B.2.** *Given a symmetric matrix $H$, $x^T H x > 0$ for every $x \neq 0$, if and only if $H > 0$. Analogously, $x^T H x < 0$ for every $x \neq 0$, if and only if $H < 0$.*

The following two properties are very useful when manipulating matrix inequalities in the context of control theory and they are repeatedly used throughout this thesis.

**Property B.3.** *Given a non singular matrix $T$: $H > 0$ if and only if $T^T H T > 0$. Analogously, $H < 0$ if and only if $T^T H T < 0$.*

**Property B.4.** ***Schur complement.*** *The following matrix inequalities:*

$$\begin{aligned} H &> 0, \\ T - S^T H^{-1} S &> 0, \end{aligned}$$

*are satisfied if and only if*

$$\begin{bmatrix} T & S^T \\ S & H \end{bmatrix} < 0.$$

The most important feature of linear matrix inequalities is that they impose convex constraints on the decision variables. That is, suppose the following set of $p$ linear matrix inequalities on the decision variables $X_1, X_2, \ldots, X_m$:

$$H_i(X_1, X_2, \ldots, X_m) < 0, \quad i = 1, \ldots, p$$

Then the set of matrices $X_1, X_2, \ldots, X_m$ that simultaneously satisfy all the linear matrix inequalities is a convex set. This stems from the fact that the inequalities $H_i(X_1, X_2, \ldots, X_m) < 0, i = 1, \ldots, p$, can be rewritten as $\lambda_{\max}(H_i(X_1, X_2, \ldots, X_m)) < 0, i = 1, \ldots, p$, where $\lambda_{\max}(\cdot)$ stands for greatest eigenvalue. As $\lambda_{\max}(\cdot)$ is a convex function in the space of symmetric matrices and $H_i(X_1, X_2, \ldots, X_m)$ is an affine function of $X_1, X_2, \ldots, X_m$, it is inferred that each LMI imposes a convex constraint on the decision variables.

The recently appeared efficient interior points algorithms [169] take advantage of the aforementioned convexity to obtain (if possible) a feasible solution for a given sets of LMIs. That is, if there exists $X_1, X_2, \ldots, X_m$ satisfying simultaneously all the LMIs, the interior points algorithm finds a solution within an affordable computational time.

## B.4 Introductory example. Writing and solving linear matrix inequalities

This section presents a very simple example to introduce the reader in the following problem: how to solve an LMI? Getting started in the resolution of LMIs is hard and sometimes obscure.

In this thesis, all LMIs have been solved using Matlab. The set of Matlab functions dealing with LMIs are part of the well-known Robust Control Toolbox [37]. The following example makes use of the functions and syntax rules described in the documentation of that toolbox.

**Example B.1.** Consider a continuous system under a state-feedback control law

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ u(t) &= Kx(t), \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, and $K \in \mathbb{R}^{m\times n}$ is the controller matrix to be designed. The objective is to minimize the upper bound of the following cost function:

$$J = \int_{t_0}^{\infty} x^T(s)Qx(s) + u^T(s)Ru(s)ds,$$

being $Q, R$ two positive definite matrices.

Let $V(t)$ denote the Lyapunov function, defined by

$$V(t) = x^T Px(t),$$

with $P \in \mathbb{R}^{n\times n}$.

Following Lemma 4.1 or any other similar design method, the controller can be synthesized by solving the following optimization problem:

$$\begin{aligned} \min_{K,P,\alpha} \quad & \alpha, \\ \text{subject to} \quad & \alpha > 0, \\ & P > 0, \\ & \alpha\left[P(A+BK) + (A+BK)^T P\right] < -Q - K^T RK. \end{aligned} \tag{B.1}$$

Note that the last constraint is not an LMI. However, using the Schur complement introduced above, it is easy to see that

$$\begin{aligned} &\alpha\left[P(A+BK) + (A+BK)^T P\right] < -Q - K^T RK \Leftrightarrow \\ &\Leftrightarrow \begin{bmatrix} P(A+BK) + (A+BK)^T P & I & K^T \\ * & -\alpha Q^{-1} & 0 \\ * & * & -\alpha R^{-1} \end{bmatrix} < 0 \end{aligned}$$

From Property B.2, previous inequality is still verified if we pre- and post- multiply it by $diag\{P^{-1}, I, I\}$ and its transpose:

$$\begin{bmatrix} P(A+BK)+(A+BK)^TP & I & K^T \\ * & -\alpha Q^{-1} & 0 \\ * & * & -\alpha R^{-1} \end{bmatrix} < 0 \Leftrightarrow$$

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ * & I & 0 \\ * & * & I \end{bmatrix} \begin{bmatrix} P(A+BK)+(A+BK)^TP & I & K^T \\ * & -\alpha Q^{-1} & 0 \\ * & * & -\alpha R^{-1} \end{bmatrix} \begin{bmatrix} P^{-1} & 0 & 0 \\ * & I & 0 \\ * & * & I \end{bmatrix} < 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} (A+BK)P^{-1}+P^{-1}(A+BK)^T & P^{-1} & P^{-1}K^T \\ * & -\alpha Q^{-1} & 0 \\ * & * & -\alpha R^{-1} \end{bmatrix} < 0.$$

Finally, defining $X_1 = P^{-1}$ and $X_2 = KP^{-1}$, the inequality can be rewritten as

$$\begin{bmatrix} AX_1 + X_1A + BX_2 + X_2^TB^T & X_1 & X_2^T \\ * & -\alpha Q^{-1} & 0 \\ * & * & -\alpha R^{-1} \end{bmatrix} < 0,$$

which is linear in the decision variables $X_1, X_2, \alpha$.

Hence the optimization problem (B.1) is equivalent to

$$\begin{aligned} \min_{X_1, X_2, \alpha} \quad & \alpha, \\ \text{subject to} \quad & \alpha > 0, \\ & X_1 > 0, \\ & \begin{bmatrix} AX_1 + X_1A + BX_2 + X_2^TB^T & X_1 & X_2^T \\ * & -\alpha Q^{-1} & 0 \\ * & * & -\alpha R^{-1} \end{bmatrix} < 0. \end{aligned} \tag{B.2}$$

The problem is now well posed and it can be solved with Matlab. First of all, the set of LMIs must be initialized with the function

```
setlmis([])
```

Next, the decision variables must be defined. To do that, the following functions are used:

```
X1    = lmivar(1,[n  1]);
X2    = lmivar(2,[m  n]);
alpha = lmivar(2,[1  1]);
```

The first argument indicates the type of variable, that is, 1 for symmetric and 2 for rectangular. The second defines the dimension.

Now, the LMIs can be defined. Two LMIs are needed for $X_1$ and $\alpha$:

$\alpha > 0$ :
```
alpha_LMI = newlmi;
lmiterm([-alpha_LMI  1  1  alpha],1,1);
```
$X_1 > 0$ :
```
X1_LMI = newlmi;
lmiterm([-X1_LMI  1  1  X1],1,1);
```

Each function `newlmi` declares a new LMI. After that, function `lmiterm` introduces terms in the LMI. Let us explain a generic call to `lmiterm`:

```
lmiterm([(-)name_LMI  posX  posY  (-)variable],left_mult,right_mult,'s');
```

`(-)name_LMI`: The name of the LMI (previously defined with `newlmi`) in which this term is included. The optional sign $-$ indicates that this term is located at the left-hand side of the $>$. If omitted, it means that the term is on the right-hand side of the inequality $>$.

`posX,posY`: The LMIs are constructed as matrices. With the pair `posX,posY` we choose the position $(X, Y)$ of this term in the matrix. As the resultant matrix must be symmetric, only the upper- or lower-triangular part of the LMI needs to be specified..

`(-)variable`: The decision variable of the term. Each term can only content one decision variable. When the sign $-$ is included before the decision variable, it means that this variable appears transposed in the corresponding term of the LMI.

`left_mult,right_mult`: These are constant elements (scalars or matrices of adequate dimensions) that multiply the decision variable at both sides. That is, the resulting term is `left_mult*variable*right_mult`.

`'s'`: This optional parameter is included if the term appears with its transpose in the same position of the LMI.

If the LMI includes a term with constant elements, that is, without decision variables, the argument `variable` must be set to 0. The null elements have not to be included.

The third LMI in (B.2) is then included with the following calls:

```
principal_LMI = newlmi;
lmiterm([principal_LMI  1  1  X1],A,1,'s');
lmiterm([principal_LMI  1  1  X2],B,1,'s');
lmiterm([principal_LMI  1  2  X1],1,1);
lmiterm([principal_LMI  1  3  -X2],1,1);
lmiterm([principal_LMI  2  2  alpha],-1,inv(Q));
lmiterm([principal_LMI  3  3  alpha],-1,inv(R));
```

So far, all the elements of the LMI have been properly included. Now, one must invoke the following function:

```
lmisys = getlmis;
```

which returns the internal description of the inequality.

Finally, two functions can be used to solve the LMIs:

$$\begin{aligned} \texttt{xopt} &= \texttt{feasp(lmisys);} \\ \texttt{[copt,xopt]} &= \texttt{mincx(lmisys,opt\_vector);} \end{aligned}$$

The first one, `feasp`, gets a possible solution if the problem is feasible. The second one, `mincx`, solves an optimization problem (as the one in (B.2)) where the minimization variable is indicated by means of `opt_vector`. The global minimum is `copt`. Output `xopt` is used to obtain the value of the decision variables by calling to function `dec2mat`:

$$\begin{aligned} X_1 &= \texttt{dec2mat(lmisys,xopt,X1);} \\ X_2 &= \texttt{dec2mat(lmisys,xopt,X2);} \\ \alpha &= \texttt{dec2mat(lmisys,xopt,alpha);} \end{aligned}$$

▼

The toolbox includes additional features to solve, for instance, generalized eigenvalue minimization problems. The interested reader is directed to [37].

Complex LMIs, as the ones presented in Theorem 3.1, can be solved following these steps. Most of non-diagonal terms are null, so they have not to be included in the code. Furthermore, by using `cells` and loops, the elements can be iteratively included.

# Appendix C

# Dealing with nonlinear terms in matrix inequalities

Sometimes, when the control problems are posed as matrix inequalities, it is inevitable that some nonlinear terms appears, so the existing methods for LMIs cannot directly be applied. This appendix proposes two different solutions for a sort of nonlinearities that is very common both in this thesis and in other approaches based in Lyapunov-Krasovskii theorem. By means of appropriate transformations and additional constraints, the nonlinear matrix inequality can be replaced by a problem with linear constraints.

Consider a nonlinear matrix inequality

$$\begin{bmatrix} f_{11}(X_1,\ldots,X_m) & \cdots & f_{1k}(X_1,\ldots,X_m) & \cdots & f_{1p}(X_1,\ldots,X_m) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{1k}^T(X_1,\ldots,X_m) & \cdots & g_{kk}(X_1,\ldots,X_m) & \cdots & f_{ip}(X_1,\ldots,X_m) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{1p}^T(X_1,\ldots,X_m) & \cdots & f_{kp}^T(X_1,\ldots,X_m) & \cdots & f_{pp}(X_1,\ldots,X_m) \end{bmatrix} < 0, \tag{C.1}$$

where $f$ are affine functions on the decision variables $X_1,\ldots,X_m$ and $g$ are nonlinear functions with the following particular structure:

$$g(X_1,\ldots,X_m) = -X_i X_j^{-1} X_i,\ i \neq j$$

Note that the nonlinear function appears in the diagonal of the inequality. In the following sections, two solutions are given to deal with the nonlinearity $X_i X_j^{-1} X_i$, $i \neq j$. The first one introduces an additional constraint which let us address the problem by means of a set of linear matrix inequalities. The second solution employs the *cone complementary algorithm* to transform the nonlinear inequality into an iterative optimization problem with linear constraints. Comparing both solutions, the former could be more conservative, but it is computationally more efficient, as the number of constraints and variables is lower.

## C.1 Direct constraint

Consider the introduction of the following additional constraint:

$$-X_i X_j^{-1} X_i < -\frac{1}{\mu} X_i,$$

being $\mu$ a positive design scalar. Note that previous condition is equivalent to $X_j < \mu X_i$. Then, the nonlinear constraint in equation (C.1) can be replaced by

$$\begin{cases} \Upsilon(X_1, \ldots, X_m) < 0, \\ X_j < \mu X_i \end{cases} \tag{C.2}$$

where $\Upsilon$ is the matrix required to be negative definite in (C.1), but substituting the terms $g(X_1, \ldots, X_m) = -X_i X_j^{-1} X_i$ by $-\frac{1}{\mu} X_i$.

It is worth comparing the proposed method with the one introduced in [257] and used in other papers to handle the same nonlinearity. While in [257] it is directly imposed $X_j$ to be $X_i$ times a given scalar, this method just restricts $X_j < \mu X_i$, which covers a much wider range of possible solutions in the space of positive definite matrices. Therefore, it leads to less conservative solutions.

## C.2 Cone complementary algorithm

Another possibility consists in using the well-known *cone complementary algorithm*. The idea is the following: firstly, the nonlinear inequality can be addressed by solving an optimization problem with linear constraints. Then, a solution for this problem can be found with an extended algorithm whose convergence is theoretically ensured.

Following the idea of [162], define a variable $T$ such that,

$$X_i X_j^{-1} X_i \geq T > 0, \tag{C.3}$$

which is equivalent to

$$\begin{bmatrix} -T^{-1} & X_i^{-1} \\ X_i^{-1} & -X_j^{-1} \end{bmatrix} \leq 0. \tag{C.4}$$

Now, introducing some new variables,

$$\hat{X}_i = X_i^{-1}, \quad \hat{T} = T^{-1}, \quad \hat{X}_j = X_j^{-1}, \tag{C.5}$$

equation (C.4) can be rewritten as,

$$\begin{bmatrix} -\hat{T} & \hat{X}_i \\ \hat{X}_i & -\hat{X}_j \end{bmatrix} \leq 0. \tag{C.6}$$

Now, instead of using the original nonlinear inequality (C.1), consider the following nonlinear minimization problem involving LMI conditions:

$$\text{Minimize } \mathbf{Tr}\left(\hat{X}_i X_i + \hat{X}_j X_j + \hat{T}T\right) \tag{C.7}$$

subject to

$$\left\{ \begin{array}{c} \mathrm{Y}(X_1, \ldots, X_m) < 0, \\ \begin{bmatrix} -\hat{T} & \hat{X}_i \\ * & -\hat{X}_j \end{bmatrix} \leq 0, \begin{bmatrix} X_i & I \\ * & \hat{X}_i \end{bmatrix} \geq 0, \begin{bmatrix} X_j & I \\ * & \hat{X}_j \end{bmatrix} \geq 0, \begin{bmatrix} T & I \\ * & \hat{T} \end{bmatrix} \geq 0, \end{array} \right. \tag{C.8}$$

where Y is as before the matrix required to be definite negative in (C.1), but substituting $X_i X_j^{-1} X_i$ by $T$. From equations (C.3), it is immediate that, if $\mathrm{Y} < 0$, then (C.1) holds. The minimization problem is introduced to force (C.5). When the LMIs in the second row of the restrictions (C.8) saturate, the optimum is reached and (C.1) holds.

In order to solve the aforementioned minimization problem (C.7) the following algorithm introduced in [51] can be implemented.

**Algorithm C.1.**

1. Set $k = 0$. Find a feasible solution under the conditions in (C.8):

   $$(X_1^0, X_2^0, \ldots, X_m^0, T^0, \hat{X}_i^0, \hat{X}_j^0, \hat{T}^0)$$

   If there is no solution, exit.

2. Solve the following optimization problem with LMI constraints with decision variables $(X_1, X_2, \ldots, X_m, T, \hat{X}_i, \hat{X}_j, \hat{T})$

   $$\min \quad \mathbf{Tr}\left(\hat{X}_i^k X_i + X_i^k \hat{X}_i + \hat{X}_j^k X_j + X_j^k \hat{X}_j + \hat{T}^k T + T^k \hat{T}\right)$$

   subject to LMIs in (C.8)

   Set $X_i^{k+1} = X_i$, $\hat{X}_i^{k+1} = \hat{X}_i$, $X_j^{k+1} = X_j$, $\hat{X}_j^{k+1} = \hat{X}_j$, $\hat{T}^{k+1} = \hat{T}$, $T^{k+1} = T$.

3. If the condition (C.1) is satisfied, exit. Otherwise, set $k = k + 1$ and return to Step 2.

The first and second steps of the algorithm are simple LMI problems, and they can be solved efficiently by using an appropriate computational software. As it is stated in Theorem 2.1 in [51], the algorithm converges and then $\hat{X}_i X_i = I$, $\hat{X}_j X_j = I$, $\hat{T}T = I$.

# Appendix D

# Robustifying controllers against uncertainties

Most results presented in this thesis assume the perfect knowledge of the model of the system. However, this assumption may be strict in real systems, which are, in general, affected by uncertainties.

This appendix presents two extended methods to deal with uncertainties that can be applied to most of the LMI-based results given throughout the thesis.

## D.1 Polytopic uncertainties

Assume that the matrices of the system $A, B$ are not exactly known[1]. Let $\Omega = [A \;\; B]$ and assume that

$$\Omega \in \mathscr{C}o\{\Omega_j, j = 1, \ldots, N\},$$

where $\mathscr{C}o\{\cdot\}$ denotes the convex hull and $\Omega_j = \begin{bmatrix} A^{(j)} & B^{(j)} \end{bmatrix}$. Hence, $\Omega = \sum_{j=1}^{N} f_j \Omega_j$ for some $0 \leq f_j \leq 1$, $\sum_{j=1}^{N} f_j = 1$, that is, the system matrix belongs to the polytope whose vertices are defined by $\Omega_j$.

Roughly speaking, the LMI-based stability conditions (and hence, the design methods proposed in LMIs) studied in this thesis can be posed as follows. Let $X_1, X_2, \ldots, X_m$ denote the decision variables. The set of linear matrix inequalities are given by:

$$H_i(X_1, X_2, \ldots, X_m, \Omega) < 0, \quad i = 1, \ldots, p. \tag{D.1}$$

[1]Although the method is explained for matrices $A, B$, is has an immediate extension for other additional system matrices with uncertainties.

The following proposition states that, if the LMIs have a common solution for all the vertices $\Omega_j$, then the stability of the system can be guaranteed over the entire polytope.

**Proposition D.1.** *[19] Assume that inequalities* (D.1) *imply that the nominal system given by* $\Omega$ *is stable. Therefore, if the set of inequalities*

$$\begin{array}{rcl} H_i(X_1, X_2, \ldots, X_m, \Omega_1) & < & 0, \quad i = 1, \ldots, p, \\ H_i(X_1, X_2, \ldots, X_m, \Omega_2) & < & 0, \quad i = 1, \ldots, p, \\ & \vdots & \\ H_i(X_1, X_2, \ldots, X_m, \Omega_N) & < & 0, \quad i = 1, \ldots, p, \end{array}$$

*are satisfied for* $X_1, X_2, \ldots, X_m$*, then the stability of the system is guaranteed over the entire polytope.*

In case of polytopic uncertainties, this proposition proposes an straightforward extension for each one of the methods given throughout the book.

## D.2 Additive uncertainties

Consider now the presence of additive uncertainties of the following form:

$$\begin{array}{rcl} A & \rightarrow & A + \Delta A(t), \\ B & \rightarrow & B + \Delta B(t), \end{array}$$

where the uncertainties are assumed to verify

$$\left[ \begin{array}{cc} \Delta A(t) & \Delta B(t) \end{array} \right] = DF(t) \left[ \begin{array}{cc} E_A & E_B \end{array} \right]. \tag{D.2}$$

Matrices $D, E_A, E_B$ are known of appropriate dimensions and $F(t)$ is an time-varying unknown matrix such that $\|F(t)\| \leq 1$.

With this new description of the system, the matrix inequalities to deal with nominal systems have to be modified. Specifically, the matrix inequalities take the form

$$H_0 + H(t) < 0,$$

where $H_0$ is any of the matrices proposed for the nominal case and $H(t)$ includes only terms related to system's uncertainties (terms of equation (D.2)).

The following lemma can be used to extend the result to systems with uncertainties.

**Lemma D.1.** *[27] Let $A, D, E$ and $F(t)$ be matrices of appropriate dimensions. If $\|F(t)\| \leq 1$, then the following holds:*

1. *Given a scalar $\epsilon > 0$,*

$$DF(t)E + E^T F^T(t)D^T \leq \epsilon^{-1}DD^T + \epsilon E^T E. \tag{D.3}$$

2. *For any matrix $P > 0$ and a scalar $\epsilon > 0$ such that $\epsilon I - EPE^T > 0$,*

$$(A + DF(t)E)P(A + DF(t)E)^T \leq APA^T + APE^T(\epsilon I - EPE^T)^{-1}EPA^T + \epsilon DD^T \tag{D.4}$$

Due to the quadratic structure of all the proposed conditions, the matrix $H(t)$ admits one of the following descriptions:

$$\begin{aligned} a) &\quad R_1F(t)R_2^T + R_2F^T(t)R_1^T, \\ b) &\quad (R_3 + R_4F(t)R_5)R_6(R_3 + R_4F(t)R_5)^T, \end{aligned}$$

where $R_i$ ($i = 1, \ldots, 5$) are constant matrices of appropriate dimensions which can depend, among others, on the matrices in (D.2), and $R_6$ is a positive definite matrix.

Taking into account relations (D.3)-(D.4), the matrix $H(t)$ can be bounded by the sum of quadratic constant terms. To obtain a compact condition, this quadratic terms are written together with $H_0$ as a single matrix, using the Schur complement. This is a well-known standard procedure. The interested reader can find an application of that procedure in [27] or [258].

# Appendix E

# Control y estimación basados en modelo para sistemas sobre redes de comunicación

**Autor:** Diego Luis Orihuela Espina
**Director:** Francisco Rodríguez Rubio
**Codirector:** Fabio Gómez-Estern Aguilar

## E.1 Resumen

En los últimos años hemos sido testigos de la introducción en el bucle de control de diferentes tecnologías de telecomunicación como las redes de datos, los sensores inteligentes, la telefonía móvil o Internet. El control de sistemas sobre redes de comunicación surge como una nueva rama dentro del control automático. La introducción de esas nuevas capacidades de comunicación trae consigo problemas adicionales que deben tenerse en cuenta. Considérese, por ejemplo, los retrasos, la comunicación basada en paquetes, las posibles pérdidas de datos, los efectos de la cuantización, el ancho de banda limitado o el consumo de energía. Muchos de estos problemas son críticos en aplicaciones de tiempo real como el control automático.

Esta tesis propone nuevas soluciones en el campo del control y estimación de sistemas sobre redes de comunicación. Aunque está principalmente enfocada a situaciones con ancho de banda limitado y restricciones energéticas y de consumo, otros problemas, como los retrasos y las pérdidas, serán tenidas en cuenta cuando sea

apropiado.

En primer lugar, le tesis estudia la estabilidad de los sistemas con retrasos -Time-Delay System (TDS)- y de los sistemas controlados a través de red -Networked Control System (NCS)- afectados por retrasos y pérdidas de paquetes. Se propone un nuevo criterio de estabilidad que consigue resultados menos conservadores que las soluciones existentes a día de hoy en la literatura.

Seguidamente, se presenta un nuevo método de diseño de controladores $H_2/H_\infty$ que puede aplicarse a TDS y NCS. Se demuestra teóricamente que el método genera controladores más óptimos que otras soluciones similares, en el sentido en que reducen el límite superior de un índice de coste.

Además, se estudia la reducción del tráfico en la red mediante el empleo de un modelo de la planta junto al controlador. Para este esquema, se proponen dos formas de comunicación: periódica y aperiódica basada en eventos.

Con respecto a los esquemas de control descentralizados para sistemas a gran escala, la gestión de la comunicación en redes de sensores parece fundamental cuando el ancho de banda disponible es limitado. La tesis ofrece dos nueva soluciones a este respecto: una gestión basada en un patrón periódico predefinido y una solución aperiódica basada en el filtro de Kalman. Aunque la primera es una solución matemáticamente menos compleja y reduce el consumo de la energía necesaria, la segunda obtiene un mejor rendimiento. Se muestra que, bajo ciertas hipótesis, una solución aperiódica *a priori* da lugar a un patrón periódico, proveyendo al sistema con los beneficios de ambas soluciones.

Finalmente, la tesis trata un problema que,a pesar de su importancia, ha recibido poca atención en la literatura: el problema conjunto de estimación y control para sistemas distribuidos. El objetivo es proponer un método de diseño que garantice la estabilidad del sistema a la vez que proporcione una solución de coste garantizado con respecto a un índice de coste dado. Además, explotando una comunicación basada en eventos, es posible reducir el ancho de banda y el consumo de energía de los dispositivos.

La mayoría de las novedades de la tesis pertenecen al campo de los resultados teóricos. No obstante, también se tienen en consideración los resultados experimentales. Se han empleado dos plantas reales para probar la eficacia de estas contribuciones de la tesis: un robot de dos grados de libertad controlado a través de red y una planta educacional de cuatro tanques acoplados.

## E.2 Organización y contribuciones de la tesis

La tesis está dividida en siete capítulos principales, además de un capítulo introductorio. El contenido de los capítulos principales se resume a continuación.

**Capítulo 2. Consideraciones iniciales sobre NCS y técnicas de observación**

En este capítulo se trata de proporcionar al lector unas ideas y un bagaje previo que serán necesarios en el resto de la tesis. Está dividido en dos partes principales. La primera de ellas desarrolla una clasificación de los sistemas de control a través de red basándose en el tipo de sistemas a controlar. Así, se diferencian los sistemas de dimensión pequeña en los que el controlador se encuentra al otro lado de la red; los sistemas de gran tamaño controlados bien de forma centralizada o descentralizada; o los llamados sistemas multi agentes. El capítulo revisa el estado del arte concerniente a estos esquemas.

La segunda parte del capítulo ofrece algunas consideraciones básicas respecto a esquemas de observación y estimación ampliamente extendidos en la literatura, a saber, el observador de Luenberger y el filtro de Kalman. A lo largo de la investigación se utilizarán diversos estimadores que estarán inspirados en algunos de ellos, por lo que se ha considerado interesante presentarlos aquí.

**Capítulo 3. Estabilidad de sistemas con retrasos**

Los resultados que se presentan en este capítulo son cruciales, ya que en la teoría de sistemas en general, y en el control automático en particular, la estabilidad de las soluciones debería estar siempre garantizada. La principal contribución de este capítulo es un nuevo criterio de estabilidad para sistemas con retrasos, menos conservador que el resto de soluciones presentes en la literatura. Es un criterio aplicable a sistemas con retrasos desconocidos, pero acotados.

El capítulo hace uso de una herramienta que será muy utilizada en el resto de la tesis: los funcionales de Lyapunov-Krasovskii. Los criterios de estabilidad presentados se pueden escribir en forma de Desigualdades Matriciales Lineales (LMIs), por lo que su verificación es sencilla con las herramientas de que se disponen hoy en día.

**Capítulo 4. Control de sistemas con retraso y a través de red**

Este capítulo se centra en el control de sistemas con retrasos. Como contribución, se presenta un nuevo método de diseño de controladores $H_2/H_\infty$ aplicable a diferentes TDS y que deja gran libertad en la elección del funcional de Lyapunov-Krasovskii. Con respecto a las soluciones propuestas por otros autores, se prueba teóricamente que el método de diseño presentado da lugar a controladores más óptimos, en el sentido de que logran una cota menor para el índice de coste $H_2$, dado un mismo índice $H_\infty$.

Seguidamente, y utilizando unas transformaciones bastantes extendidas en este campo, se muestra que este método de diseño también puede ser aplicable a los sistemas de control a través de red. Finalmente, los controladores diseñados se prueban en un robot de dos grados de libertad controlado a través de red, mostrando sus capacidades de estabilización y sus posibilidades de sintonía.

**Capítulo 5. Control basado en modelo de sistemas en red**

Con este capítulo se cierra el control de sistemas de dimensión reducida. Puede verse como una continuación del anterior, al explorar los beneficios que se alcanzan al introducir un modelo de la planta junto al controlador. Utilizando un controlador diseñado con anterioridad (por ejemplo, utilizando los métodos del capítulo anterior), este capítulo estudia el ahorro de transmisiones en los enlaces sensor-controlador y controlador-actuador, que se consigue por la introducción del modelo.

Se contribuye con dos esquemas de muestreo, uno periódico y otro asíncrono. Para ambos casos, se prueba que la estabilidad del sistema se mantiene a pesar de la reducción en el uso de ancho de banda.

**Capítulo 6. Gestión de la comunicación para la estimación y el control**

El problema de la gestión de la comunicación encuentra su aplicación en sistemas de gran escala que se controlan de forma centralizada. Además, también se puede aplicar en aquellas situaciones en los que el ancho de banda disponible esté muy limitado.

Las salidas de la planta son medidas por un conjunto de sensores que deben compartir un mismo medio de comunicación. El objetivo que se persigue es doble: por una lado se desea estimar el estado de la planta a través de estas salidas parciales; y por otro, utilizar esta estimación para controlar el sistema.

Para conseguir estos objetivos, hay que resolver un problema de co-diseño: se deben diseñar las ganancias de los observadores y los controladores y, a la vez, especificar un protocolo de comunicación adecuado. En el capítulo se proponen dos soluciones diferentes: una gestión periódica, en la que un patrón de muestras se repite constantemente; y una gestión aperiódica, en la que la elección del sensor se decide en base a un índice de coste. Se diseñan, respectivamente, un observador periódico $H_\infty$ y un filtro de Kalman aperiódico.

Una de las principales contribuciones del capítulo es que se demuestra que, bajo ciertas hipótesis suaves, la gestión aperiódica *a priori* da lugar a un comportamiento periódico en el régimen estacionario. Esto permite al ingeniero de diseño aprovechar los beneficios que ambos métodos ofrecen, tanto la optimalidad como el reducido coste energético y computacional.

**Capítulo 7. Control y estimación distribuidos**

Cuando una planta de gran tamaño está siendo controlada desde diferentes puntos, probablemente espaciados, los esquemas distribuidos son de interés. En el problema conjunto de control y estimación, la tesis propone un nuevo método que permite diseñar al mismo tiempo tanto los observadores como los controladores. Se persigue de esta forma un objetivo de optimalidad global: minimizar la cota superior de un índice de coste. Además, la reducción del uso del ancho de banda se hace posible mediante el empleo de comunicación basado en eventos. Mediante este método, los diferentes nodos o agentes sólo transmiten información cuando estiman que es necesario.

Este esquema se ha probado en una planta real de cuatro tanques acoplados mostrando, tanto una adecuada reducción del tráfico, como las posibilidades de sintonía del método de diseño.

**Capítulo 8. Conclusiones**

El último capítulo presenta los principales logros que se han alcanzado durante la tesis. De igual manera, se resaltan los puntos débiles y las limitaciones que tiene desde un punto de vista objetivo. Además se valora el posible impacto de la tesis. Finalmente, se detallan una serie de posibles líneas de investigación futuras que podrían servir como continuación a este trabajo de investigación.

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