Pyramidal values[1]

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Abstract

We propose and analyze a new type of values for cooperative TU-games, which we call *pyramidal values*. Assuming that the grand coalition is sequentially formed, and all orderings are equally likely, we define a pyramidal value to be any expected payoff in which the entrant player receives a salary, and the rest of his marginal contribution to the just formed coalition is distributed among the incumbent players. We relate the pyramidal-type sharing scheme we propose with other sharing schemes, and we also obtain some known values by means of this kind of pyramidal procedures. In particular, we show that the Shapley value can be obtained by means of an interesting pyramidal procedure that distributes nonzero dividends among the incumbents. As a result, we obtain an alternative formulation of the Shapley value based on a measure of complementarity between two players. Finally, we introduce the family of *proportional* pyramidal values, in which an incumbent receives a dividend in proportion to his initial investment, measured by means of his marginal contribution.

Keywords: Game theory, TU games, pyramidal values, procedural values, Shapley value, co-values, consensus values, egalitarian Shapley values.

1 Introduction

In this paper we propose a general procedure for obtaining a broad class of solution concepts based on a pyramidal distribution of the benefits, that are sequentially obtained through a dynamic process of coalition formation, in which players successively come into play and join the current coalition until the grand coalition is formed. The well-known Shapley value (Shapley [15]) has been characterized in Weber [17] as the average over all permutations of a very extreme pyramidal distribution of the benefits, in which the entrant player receives all the just generated benefits (jointly created by the existing coalition of players and the entrant), when the grand coalition is sequentially formed, and all orderings are equally likely. However, such extreme shares

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immediately lead us to point out two questions: Why the incumbents are going to accept the deal? Why the entrant is going to stay in the coalition after receiving all his contribution?

Assuming also that all orderings are equally likely, we propose to compose values using a more general pyramidal sharing scheme in which the entrant player receives a salary and the right to get part of the benefits derived from subsequent incorporations to the just formed coalition, whereas the remaining benefit is distributed among the incumbent players. In Section 2, we first introduce some standard concepts and notation on Game Theory that will be used throughout this paper, we provide a formal definition of a *pyramidal sharing scheme*, and we establish some general properties of the class of values derived from those schemes. We also analyze the relation between the notion of pyramidal sharing schemes and the idea of "procedural" values as defined by Malawski in [10]. In Section 3 we obtain some known values by means of pyramidal sharing schemes. On the one hand, we show that the Shapley value can also be obtained as a non-extreme pyramidal value which is based on the second-order difference operator for a pair of players considered by Segal [14]; and on the other hand, we derive the family of *consensus values* introduced by Ju, Borm and Ruys [8], and also the family of *egalitarian Shapley values* introduced by Joosten [7], also described by van den Brink, Funaki and Ju [16], and more recently by Casajus and Huettner [2], as pyramidal values. Both families, which intend to reconcile marginalism with egalitarianism, arise following a egalitarian approach to determine the right to get part of the benefits derived from subsequent incorporations to the just formed coalition, and a marginalistic one when determining entrant's salary. In Section 4, we define a proportional family of pyramidal values in which the entrant player receives as salary his own value plus a fixed proportion of his added value (i.e., the jointly created benefit less his salary), whereas the remaining benefit is distributed among the incumbent players according to each player's *contribution* to the coalition previously formed. Section 5 concludes the paper.

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2 Pyramidal values

An n-person cooperative game in characteristic function form with transferable utility (TU game) is an ordered pair (N, v), where N is a finite set of n players and $v : 2^N \to \mathbb{R}$ is a map assigning a real number $v(S)$, called the value of S, to each coalition $S \subseteq N$, and where $v(\emptyset) = 0$. The real number $v(S)$ represents the reward that coalition S can achieve by itself if all its members act together. Let G_n be the space of all TU games with fixed player set N, where $n = |N|$, and identify $(N, v) \in G_n$ with its characteristic function v when no ambiguity appears. One of the main topics dealt with in Cooperative Game Theory is, given a game $(N, v) \in G_n$, to divide the amount $v(N)$ between players if the grand coalition N is formed. A *payoff vector*, or *allocation*, is any $\mathbf{x} \in \mathbb{R}^n$, which gives player $i \in N$ a payoff x_i. A payoff vector is said to be *efficient* if $\sum_{i \in N} x_i = v(N)$.

A *value* φ for TU games is an assignation which associates to each n-person game $(N, v) \in G_n$ a payoff vector $\varphi(N, v) \in \mathbb{R}^n$. The Shapley value, which we will denote by ϕ, is one of the most interesting values in Cooperative Game Theory. It can be characterized as the average of the marginal contribution vectors over all permutations (Weber [17]). Formally, let $(N, v) \in G_n$, and let $\Pi(N)$ denote the set of all permutations on the player set N, which we will represent as bijections $\pi : N \to N$. For a permutation $\pi \in \Pi(N)$, $\pi(i) \in N = \{1, \ldots, n\}$ represents agent i's position in order π. Define the set of all predecessors of i in π to be $P_\pi(i) = \{j \in N \mid \pi(j) < \pi(i)\}$, and the set of all his successors to be $S_\pi(i) = \{j \in N \mid \pi(j) > \pi(i)\}$. Moreover, the direct successor of i in the order π will be denoted by $ds_\pi(i)$. Now, the *marginal contribution vector* $m^\pi(v) \in \mathbb{R}^n$ of game v and permutation π is given by

$$m_i^\pi(v) = v(P_\pi(i) \cup \{i\}) - v(P_\pi(i)), \quad i \in N,$$

which assigns to each player $i \in N$ its marginal contribution to the worth of the coalition consisting of all his predecessors in π.[2] In that case, when player j joins coalition $P_\pi(j)$, he generates the surplus $m_j^\pi(v)$, which, according to Weber [17] characterization of the Shapley value, is distributed among the current coalition as follows:

- Entrant j's salary: $s_j^\pi(v) = m_j^\pi(v)$
- Incumbents $P_\pi(j)$'s shares: $a_{ij}^\pi(v) = 0$, for all $i \in P_\pi(j)$

In this setting, we define a class of values, which we call *pyramidal values*, that is based on a more general sharing scheme in which the entrant player receives a salary and the right to get part of the benefits derived from subsequent incorporations to the just formed coalition, whereas the remaining benefit is distributed among the incumbent players. Formally:

Definition 1. Let $\mathcal{P}$ be a value for TU games. Then, $\mathcal{P}$ is called a *pyramidal value*, if for all orders $\pi \in \Pi(N)$ with $n \geq 1$, and for every n-person TU game $(N, v) \in G_n$, there exists a *pyramidal sharing scheme* $\mathcal{S}(v) = \{(s_j^\pi(v), (a_{ij}^\pi(v))_{i \in P_\pi(j)})_{j \in N} \mid \pi \in \Pi(N)\}$ such that

$$s_j^\pi(v) + \sum_{i \in P_\pi(j)} a_{ij}^\pi(v) = m_j^\pi(v), \quad \forall\, j \in N. \tag{1}$$

and verifying:

$$\mathcal{P}_i(v) = \sum_{\pi \in \Pi(N)} \frac{1}{n!} p_i^\pi(v), \quad \forall\, i \in N, \tag{2}$$

where $p_i^\pi(v) = s_i^\pi(v)$ if $\pi(i) = n$, and

$$p_i^\pi(v) = s_i^\pi(v) + \sum_{j \in S_\pi(i)} a_{ij}^\pi(v), \text{ for all } i \in N \text{ with } \pi(i) < n. \tag{3}$$

[2] In the sequel, for convenience, we will write singleton $\{i\}$ just as i.

Note that negative salaries or shares are allowed in the previous definition. As usual, negative quantities must be interpreted as costs, penalties or investments in a broad sense. Note also that condition (1) assures that every value generated by means of a pyramidal sharing scheme is efficient. However, since we do not impose any other condition over the sharing scheme, it could be the case that the shares of the incumbents, i.e., the *dividends*[3], depend on the future, or that the salaries are non rational. Thus, Definition 1 may be too general. We will provide some conditions over a pyramidal sharing scheme in order to restrict ourselves to deal with non-anticipative sharing schemes which in addition respect common sense bounds in order to avoid salaries too low and too high.

Definition 2. Let $\mathcal{S}$ be a pyramidal sharing scheme. Then, $\mathcal{S}$ will be a $\mathcal{P}$*-rational* sharing scheme if it satisfies the following properties. Let $(N, v) \in G_n$ be any n-person game:

(i) *Salaries Rationality*. If (N, v) is superadditive, then $v(i) \leq s_i^\pi(v) \leq m_i^\pi(v)$, for all orders $\pi \in \Pi(N)$.

(ii) *Dividends Rationality (Non anticipative shares)*. If $\pi, \pi' \in \Pi(N)$ are two orders which coincide up to moment $k \in \{2, \ldots, n\}$ (i.e., $\pi^{-1}(\ell) = \pi'^{-1}(\ell)$, for all $\ell = 1, 2, \ldots, k$), then $a_{ij}^\pi(v) = a_{ij}^{\pi'}(v)$, for all $j \in N$ with $1 < \pi(j) = \pi'(j) \leq k$, and for all $i \in P_\pi(j) = P_{\pi'}(j)$.

Obviously, the properties of the sharing scheme determine the pyramidal value properties, so let us formalize some other interesting properties of a pyramidal sharing scheme. We will translate to the pyramidal sharing scheme the usual properties of additivity, dummy and symmetry, and besides them we will also translate the usual monotonicity conditions in order to provide appropriate incentives to the agents.

Definition 3. Let $(N, v) \in G_n$ be any n-person TU game, and let $\mathcal{S}(v) = \{(s_j^\pi(v), (a_{ij}^\pi(v))_{i \in P_\pi(j)})_{j \in N} \mid \pi \in \Pi(N)\}$ be a pyramidal sharing scheme. Then, $\mathcal{S}$ verifies,

(i) *Constant Salary*. If for all $j \in N$ there exists a real constant $k_j(v) \in \mathbb{R}$ such that $s_j^\pi(v) = k_j(v)$, for all $\pi \in \Pi(N)$.

(ii) $\mathcal{P}$*-Additivity*. If for all orders $\pi \in \Pi(N)$, and for all $j \in N$ it holds:

- $s_j^\pi(v + w) = s_j^\pi(v) + s_j^\pi(w)$, and
- $a_{ij}^\pi(v + w) = a_{ij}^\pi(v) + a_{ij}^\pi(w)$, for each $i \in P_\pi(j)$,

for all $(N, v), (N, w) \in G_n$, where $v + w$ is given by $(v + w)(S) = v(S) + w(S)$, for all $S \subseteq N$.

(iii) $\mathcal{P}$*-Dummy player*. If

- $s_i^\pi(v) = v(i)$, and
- $a_{ij}^\pi(v) = 0$, for every $j \in S_\pi(i)$, and all orders $\pi \in \Pi(N)$,

for all $i \in N$ being a dummy player (i.e., $v(S \cup i) = v(S) + v(i)$ for every coalition S).

[3]Note that these dividends are not the same as the well-known Harsanyi dividends, which are associated to coalitions, not only to agents.

(iv) $\mathcal{P}$-*Symmetry*. If, for all symmetric players $i, j \in N$ (i.e., $v(S \cup i) = v(S \cup j)$, for all $S \subseteq N \setminus \{i, j\}$),

- $s_i^\pi(v) = s_j^{\pi_{ij}}(v)$, and
- for all $k \in N \setminus \{i, j\}$, $a_{ik}^\pi(v) = a_{jk}^{\pi_{ij}}(v)$, for all $k \in S_\pi(i)$,

where the order π_{ij} is defined as $\pi_{ij}(k) = \pi(k)$, $\pi_{ij}(i) = \pi(j)$ and $\pi_{ij}(j) = \pi(i)$.

(v) $\mathcal{P}$-*Strong monotonicity*. If it satisfies *strong monotonic salaries* and *dividends*, defined as follows. Let $i \in N$ be any player, and let $(N, v), (N, w)$ be two n-person games for which $v(S \cup i) - v(S) \leq w(S \cup i) - w(S)$, for all $S \subseteq N \setminus i$, and being $v(T \cup i) - v(T) < w(T \cup i) - w(T)$ for some $T \subseteq N \setminus i$, then

- *Strong monotonic salaries*: $s_i^\pi(v) \leq s_i^\pi(w)$, for all orders $\pi \in \Pi(N)$ and $(s_i^\pi(v))_{\pi \in \Pi(N)} \neq (s_i^\pi(w))_{\pi \in \Pi(N)}$.
- *Strong monotonic dividends*: $a_{ij}^\pi(v) \leq a_{ij}^\pi(w)$, for all $j \in S_\pi(v)$, for all orders $\pi \in \Pi(N)$, with $a_{ij}^{\pi'}(v) < a_{ij}^{\pi'}(w)$, for some order π'

Note that the constant salary property implies that the salary is an inherent attribute of each player, and it can be related, for instance, to his personal training. Moreover, since $P_\pi(i) = \emptyset$ for all orders π such that $\pi(i) = 1$, then each player's constant salary equals his own value $v(i)$. $\mathcal{P}$-Additivity, $\mathcal{P}$-dummy player and $\mathcal{P}$-symmetry trivially lead to the same properties for the corresponding pyramidal value. Let us recall those well-known properties of values for TU games, as well as other properties which we will use later. Formally, a *value* $\varphi : G_n \to \mathbb{R}^n$:

(i) is *efficient* if $\sum_{i \in N} \varphi_i(v) = v(N)$, for all $(N, v) \in G_n$;

(ii) is *additive* if $\varphi(v + w) = \varphi(v) + \varphi(w)$, for all $(N, v), (N, w) \in G_n$;

(iii) is *relative invariant with respect to strategic equivalence* if $\varphi(N, w) = a\varphi(N, v) + b$, for every $(N, v) \in G_n$, $a > 0$ and $b \in \mathbb{R}^n$, where w is given by $w(S) = av(S) + \sum_{i \in S} b_i$, for all $S \subseteq N$;

(iv) is *symmetric* if $\varphi_i(v) = \varphi_j(v)$, for all $(N, v) \in G_n$, and for all symmetric players $i, j \in N$;

(v) *preserves desirability* [11] if $\varphi_i(v) \leq \varphi_j(v)$, for all players $i, j \in N$ such that $v(S \cup i) \leq v(S \cup j)$, for all $S \subseteq N \setminus \{i, j\}$, for all $(N, n) \in G_n$;

(vi) is *strong monotonic* [18] if $\varphi_i(v) \leq \varphi_i(w)$, for every player $i \in N$, and for all games $(N, v), (N, w) \in G_n$ for which $v(S \cup i) - v(S) \leq w(S \cup i) - w(S)$, for all $S \subseteq N \setminus i$;

(vii) is *coalitionally monotonic* [18] if for every coalition $T \subseteq N$ and every two games $(N, v), (N, w) \in G_n$ such that $v(T) > w(T)$ and $v(S) = w(S)$, for all $S \neq T$, it follows $\varphi_i(v) \geq \varphi_i(w)$, for every player $i \in T$;

$(viii)$ verifies *positivity* [9] if $\varphi_i(v) \geq 0$, for all $i \in N$, whenever the game (N, v) is *monotonic* (i.e., $v(T) \geq v(S)$, for each T and S such that $T \supseteq S$);

(ix) verifies the *dummy property* if $\varphi_i(v) = v(i)$, for all $(N, v) \in G_n$, and for every dummy player $i \in N$;

(x) verifies the *null player property* if $\varphi_i(v) = 0$, for all $(N, v) \in G_n$, and for every *null player* $i \in N$ (i.e., $v(S \cup i) = v(S)$, for all $S \subseteq N \setminus i$);

(xi) verifies the *null player out property* [4] if $\varphi_j(N, v) = \varphi_j(N \setminus i, v|_{N\setminus i})$, for all $j \neq i \in N$, for all $(N, v) \in G_n$ such that i is a null player in v. Here, $(N \setminus i, v|_{N\setminus i})$ is the restricted game given by $v|_{N\setminus i}(S) = v(S)$, for all $S \subseteq N \setminus i$;

(xii) is *standard for two-person games* if $\varphi_i(v) = v(i) + \frac{1}{2}\big(v(\{i, j\}) - v(i) - v(j)\big)$, for all $i \neq j$, for every two-person game $(\{i, j\}, v) \in G_2$.

Proposition 1. *Any additive and efficient value φ can be obtained as a $\mathcal{P}$-additive pyramidal value. Moreover, if φ verifies the null player out property, then the corresponding pyramidal sharing scheme $\mathcal{S}_\varphi$ verifies dividends rationality.*

Proof. Let φ be any additive and efficient value. Let us first recall the unanimity basis for G_n, $\{(N, u_T)\}_{T\subseteq N}$, with $T \neq \emptyset$, where

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0, & \text{otherwise.} \end{cases}$$

We will show that the value of any multiple of a unanimity game $\varphi(ku_T)$, $k \in \mathbb{R}$, can be obtained by means of a pyramidal sharing procedure. Let $\pi \in \Pi(N)$ be any given order. Let us consider the following redistribution, where $t_\pi \in T$ is the last member of T according to the order π.

- For every player $j \in P_\pi(t_\pi)$, his salary is $s^\pi_j(ku_T) = 0$, and he distributes $a^\pi_{ij}(ku_T) = 0$ among his predecessors $i \in P_\pi(j)$.
- When the last member of T arrives, he distributes k as follows:

$$s^\pi_{t_\pi}(ku_T) = \varphi_{t_\pi}(ku_T) + \sum_{j\in S_\pi(t_\pi)} \varphi_j(ku_T), \tag{4}$$

$$a^\pi_{it_\pi}(ku_T) = \varphi_i(ku_T), \text{ for all } i \in P_\pi(t_\pi). \tag{5}$$

- For all $j \in S_\pi(t_\pi)$, his salary is $s^\pi_j(ku_T) = \varphi_j(ku_T)$, which is paid by t_π. That is, $a^\pi_{ij}(ku_T) = 0$, for all $i \in P_\pi(j) \setminus \{t_\pi\}$, and $a^\pi_{t_\pi j}(ku_T) = -\varphi_j(ku_T)$.[4]

Clearly, the proposed sharing scheme gives $\varphi(ku_T)$. Now, let $(N, v) \in G_n$ be a given TU game. Then it can be expressed as (see Shapley [15]) $v = \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} \Delta(T)u_T$, where $\Delta(T)$ is the *Harsanyi dividend* of T in (N, v), given by $\Delta(T) = \sum_{\substack{S\subseteq T\\ S\neq\emptyset}} (-1)^{t-s} v(S)$, s and t being the cardinalities of S and

[4] Those negative shares can be interpreted as investments on human capital.

T, respectively. Thus, the $\mathcal{P}$-additive sharing scheme $\mathcal{S}$ defined by

$$s_j^\pi(v) = \sum_{T\subseteq N} s_j^\pi(\Delta(T)u_T),$$
$$a_{ij}^\pi(v) = \sum_{T\subseteq N} a_{ij}^\pi(\Delta(T)u_T), \text{ for all } i \in P_\pi(j),$$

for all $j \in N$, and for all $\pi \in \Pi(N)$, recovers $\varphi(v)$. Note that $\mathcal{S}$ verifies condition (1). Now, we will check that if φ verifies the null player out property, then the pyramidal sharing scheme is dividends rational:

- For every player $j \in P_\pi(t_\pi)$, $s_j^\pi(\Delta(T)u_T) = 0$, and $a_{ij}^\pi(\Delta(T)u_T) = 0$ for all $i \in P_\pi(j)$, which clearly do not depend on $S_\pi(j)$.
- For all $j \in S_\pi(t_\pi)$, note that $j \notin T$ and therefore it is a null player in the game $(N, \Delta(T)u_T)$. Then, since φ verifies the null player out property and it is efficient, it also verifies the null player property, and therefore $s_j^\pi(\Delta(T)u_T) = \varphi_j(\Delta(T)u_T) = 0$, which is paid by t_π. Thus, $s_j^\pi(\Delta(T)u_T) = 0$ and $a_{ij}^\pi(\Delta(T)u_T) = 0$ for all $i \in P_\pi(j)$, which clearly do not depend on $S_\pi(j)$.
- For the last incoming member of T, and taking into account that φ verifies null player out and null player properties, it follows:

$$s_{t_\pi}^\pi(N, \Delta(T)u_T) = \varphi_{t_\pi}(N, \Delta(T)u_T) + \sum_{S_\pi(t_\pi)} \varphi_j(N, \Delta(T)u_T) = \varphi_{t_\pi}(T, \Delta(T)u_T) + 0, \tag{6}$$
$$a_{it_\pi}^\pi(N, \Delta(T)u_T) = \varphi_i(N, \Delta(T)u_T) = \varphi_i(T, \Delta(T)u_T), \text{ for all } i \in P_\pi(t_\pi), \tag{7}$$

  which depend only on $T \subseteq P_\pi(t_\pi)$

Therefore, the pyramidal sharing scheme is dividends rational. □

It is also remarkable that two different pyramidal sharing schemes $\mathcal{S}_1$ and $\mathcal{S}_2$ may lead to the same value; far from being a drawback, this fact is an advantage. Having two different implementations of the same value enlarges the opportunities to apply it as an effective solution to a given game in a specific situation. Let us think about the *extreme* pyramidal sharing scheme which determines the Shapley value, in which the entrant player receives the whole benefits, and no dividends are distributed. Such extreme shares immediately lead us to point out two questions: Why the incumbents are going to accept the deal? Why the entrant is going to stay in the coalition after receiving all his contribution? We can avoid those questions by obtaining the Shapley value also as a non-extreme pyramidal value. For instance, Proposition 1 provides us with an alternative and non-extreme pyramidal sharing scheme for obtaining the Shapley value as a pyramidal one. Let (N, u_T) be the unanimity game with respect to coalition $T \subseteq N$, and let us consider the following pyramidal shares:

(i) Entrant j's salary: $s_j^\pi(\Delta(T)u_T) = \frac{\Delta(T)}{t}$, if $j = t_\pi \in T$ is the last member of T according to the order π; and $s_j^\pi(\Delta(T)u_T) = 0$, otherwise;

(ii) Incumbents $P_\pi(j)$'s shares: $a^\pi_{ij}(\Delta(T)u_T) = \frac{\Delta(T)}{t}$, if $j = t_\pi \in T$ is the last member of T according to the order π and $i \in P_\pi(j) \cap T$; and being $a^\pi_{ij}(\Delta(T)u_T) = 0$ otherwise,

for all $j \in N$, and for all orders $\pi \in \Pi(N)$. The final payoff that player $i \in N$ receives according to the order $\pi \in \Pi(N)$ is then given by $\frac{\Delta(T)}{t}$ if $i \in T$, and 0 if $i \notin T$. Thus, according to Proposition 1 the additive pyramidal value we obtain is given by $\sum_{\substack{T\subseteq N \\ i\in T}} \frac{\Delta(T)}{t}$, for all $i \in T$, which is precisely the expression of the Shapley value in terms of the Harsanyi dividends of the game. The pyramidal sharing scheme for the original game (N, v) is:

(i) Entrant j's salary: $s^\pi_j(v) = \sum_{T\subseteq P_\pi(j)} \frac{\Delta(T\cup j)}{t+1}$

(ii) Incumbents $P_\pi(j)$'s shares: $a^\pi_{ij}(v) = \sum_{\substack{T\subseteq P_\pi(j) \\ i\in T}} \frac{\Delta(T\cup j)}{t+1}$

Since there are at least two different pyramidal sharing schemes which result in the Shapley value, it follows that there are a continuum of them with the same property (all their linear convex combinations). However, the obtained pyramidal sharing scheme is not salaries rational in general. The question is whether the Shapley value might be obtained by means of a *rational* non-extreme pyramidal sharing scheme. Unexpectedly, we give a positive answer in Section 3.

Relation with procedural values

Pyramidal sharing schemes are closely related to the idea of *procedural values*, introduced by Malawski [10]. Procedural values are pyramidal values for which the marginal contribution of the entering player is divided among the players proportionally to a weight system which does not depend on the players' names nor on their contributions. To be specific (see Malawski [10]), let s be a *procedure* on G_n, that is, a family of nonnegative coefficients $((s_{k,j})^k_{j=1})^n_{k=1}$ such that $\sum^k_{j=1} s_{k,j} = 1$, for all k. Then, the procedural value ψ^s determined by the procedure s is the pyramidal value obtained by means of the following salaries and dividends:

- Entrant j's salary: $s^\pi_j(v) = s_{\pi(j),\pi(j)} m^\pi_j(v)$,
- Incumbents $P_\pi(j)$'s shares: $a^\pi_{ij}(v) = s_{\pi(j),\pi(i)} m^\pi_j(v)$, for all $i \in P_\pi(j)$. (8)

The class of pyramidal sharing schemes is obviously larger than the class of procedural sharing schemes. For instance, the pyramidal sharing scheme described above to derive the Shapley value is not procedural. Later, in Sections 3 and 4, we will show that also the class of pyramidal values is larger than the class of procedural values. To be specific, the class of pyramidal values contains linear values which are not procedural, such as the consensus family of values (Ju *et al.* [8]), and also there exist non-linear pyramidal values that cannot be obtained through a procedural scheme, such as the proportional family introduced in Section 4. In fact, when restricting to the sub-class of procedural values Malawski proves in [10] that efficiency, linearity, symmetry, positivity and coalitional monotonicity characterize the class of procedural values. In our context, this can be read as a stronger version of our Proposition 1. He also establishes that symmetry and coalitional monotonicity can be replaced by desirability preservation.

3 Relation with other values

In this section, we obtain some known families of values by means of pyramidal sharing schemes. Such constructions show some interesting features of the analyzed values. We first prove that the Shapley value can also be obtained as a non-extreme pyramidal value in which the entrant player receives only his own value as salary whereas the remaining benefit is distributed among the incumbent players. As a consequence, we establish a new formulation of the Shapley value which rests on the second-order difference operator used in Segal [14], which is in turn closely related to the notions of increasing differences and supermodularity (see Ichiisi [6]) and has a meaningful economic interpretation. Then, we derive the family of *consensus values* (Ju, Borm and Ruys [8]), and also the family of *egalitarian Shapley values* (Joosten [7], van den Brink, Funaki and Ju [16], Casajus and Huettner [2]), as pyramidal values. Both families arise following a egalitarian approach to determine the right to get part of the benefits derived from subsequent incorporations to the just formed coalition, and a marginalistic one when determining entrant's salary.

For the interested reader, and for the sake of completeness, we collect the formal definitions of all the known values we will analyze in this Section in a final Appendix. We also recover the characterizations results we use.

The Shapley value as a rational non-extreme pyramidal value

For a given order $\pi \in \Pi(N)$, and players $i, j \in N$ such that $\pi(i) \leq \pi(j)$, let us define the *marginal contribution of player j with respect to player i, according to order* π, to be

$$m^{\pi}_{ij}(v) = v(\{k \in N | \pi(i) \leq \pi(k) \leq \pi(j)\}) - v(\{k \in N | \pi(i) \leq \pi(k) < \pi(j)\}).$$

Note that $m^{\pi}_{ij}(v)$ can be interpreted as the marginal contribution of agent j to the group leaded by agent i according to order π. If we denote the coalition of all players who have arrived between players i and j by $S_\pi(i,j) = \{k \in N \,|\, \pi(i) < k < \pi(j)\}$, then $m^{\pi}_{ij}(v) = v(S_\pi(i,j) \cup \{i,j\}) - v(S_\pi(i,j) \cup i)$, if $i \neq j$, and $m^{\pi}_{jj}(v) = v(j)$.

Now, we define in Proposition 2 a rational non-extreme pyramidal procedure which is based on these marginal contributions and which turns out to give the Shapley value as a final payoff. In this pyramidal sharing scheme, player $i \in P_\pi(j)$ receives the marginal contribution of player j with respect to i, according to order π, at the cost of paying to his direct successor the marginal contribution of player j with respect to this direct successor.

Proposition 2. *The Shapley value can be obtained through the pyramidal sharing scheme $\mathcal{S}$ that distributes the marginal contribution of player $j \in N$ among the agents in $P_\pi(j) \cup j$ as follows:*

(*i*) *Entrant j's salary:* $s^{\pi}_{j}(v) = v(j)$

(*ii*) *Incumbents $P_\pi(j)$'s shares:* $a^{\pi}_{ij}(v) = m^{\pi}_{ij}(v) - m^{\pi}_{ds_\pi(i),j}(v)$,

for every order $\pi \in \Pi(N)$, every player $i \in N$, and every n-person TU game (N, v).

Before undertaking the proof, and for the sake of clarity, let us illustrate the sharing scheme by means of the following example.

Example 1. We will consider the concrete case of four players, when we fix the identity permutation. Then, the incumbents $P_\pi(4)$'s shares, as well as his own salary, when player 4 arrives according to our proposal, are explained in the following table:

Player	Payments on player 4's arrival
4	$v(4) - v(\emptyset)$
3	$(v(34) - v(3)) - (v(4) - v(\emptyset))$
2	$(v(234) - v(23)) - (v(34) - v(3))$
1	$(v(1234) - v(123)) - (v(234) - v(23))$
Total	$v(1234) - v(123)$

Now, we will prove Proposition 2:

Proof. We must check first that the pyramidal sharing scheme defined above satisfies condition (1) for every order $\pi \in \Pi(N)$, and every player $j \in N$.

Let (N, v) be a given TU game, and $\pi \in \Pi(N)$ be any order. Let us consider player j's arrival, who has arrived in the k-th position. Denote j as i_k. For all $i \in P_\pi(i_k)$, if he has arrived in the ℓ-th position according to π, denote i as i_ℓ and the amount he receives from i_k by $a_{\ell k}$. Then, the following holds

$$\begin{aligned} a_{1k} &= v(\{i_1, \ldots, i_k\}) - v(\{i_1, \ldots, i_{k-1}\}) - v(\{i_2, \ldots, i_k\}) + v(\{i_2, \ldots, i_{k-1}\}), \\ a_{\ell k} &= v(\{i_\ell, \ldots, i_k\}) - v(\{i_\ell, \ldots, i_{k-1}\}) - v(\{i_{\ell+1}, \ldots, i_k\}) + v(\{i_{\ell+1}, \ldots, i_{k-1}\}), \quad \ell = 2, k-2 \\ a_{k-1,k} &= v(\{i_{k-1}, i_k\}) - v(i_{k-1}) - v(i_k) + v(\emptyset). \end{aligned}$$

Thus, taking into account that the two first terms of $a_{\ell+1,k}$ get canceled with the two last terms of $a_{\ell k}$, for all $\ell = 1, \ldots, k-2$, it follows that

$$\sum_{\ell=1}^{k-1} a_{\ell k} = v(\{i_1, \ldots, i_k\}) - v(\{i_1, \ldots, i_{k-1}\}) - v(i_k) + v(\emptyset) = m_j^\pi(v) - s_j^\pi(v),$$

and therefore, we have defined a pyramidal sharing scheme. Therefore, it is enough to check that the derived pyramidal value, which is given by

$$\mathcal{P}_i(v) = \sum_{\pi \in \Pi(N)} \frac{1}{n!} p_i^\pi(v), \quad i \in N, \tag{9}$$

where $p_i^\pi(v) = v(i)$ if $\pi(i) = n$, and

$$p_i^\pi(v) = v(i) + \sum_{j \in S_\pi(i)} \left(m_{ij}^\pi(v) - m_{ds_\pi(i),j}^\pi(v) \right), \text{ otherwise,}$$

verifies the set of axioms that characterizes the Shapley value (efficiency, symmetry, dummy player and additivity, for instance). By definition, all pyramidal values are efficient. Moreover, symmetry and additivity follow trivially from $s_i^\pi(v) = v(i)$ and $a_{ij}^\pi(v) = m_{ij}^\pi(v) - m_{ds_\pi(i),j}^\pi(v)$, which are symmetric and additive. Dummy player property holds because $a_{ij}^\pi(v) = m_{ij}^\pi(v) - m_{ds_\pi(i),j}^\pi(v) = 0$, for every dummy player $i \in N$, for all $j \in S_\pi(i)$, and for all orders $\pi \in \Pi(N)$. $\square$

Beal, Remila and Solal [1] propose a *compensation method* to derive the Shapley value, which is also obtained by a sharing system. The main difference being that the sharing system they propose is *inefficient*, in the sense that condition (1) is not fulfilled for every order and every player. According to Beal, Remila and Solal's sharing system, the entering player and the incumbents receive as compensation their joint worth, which is split equally among them, and must be paid by the remaining players, which also share the debt equally among them.

Example 2. Let us consider again example 1. Then, the sharing system as well as the final payoff associated to the identity permutation, both according to the compensation method, are explained in the following tables:

Player	4's arrival	3's arrival	2's arrival	1's arrival
4	$-\frac{v(1)}{3} - \frac{v(\{1,2\})}{2} - v(\{1,2,3\}) + \frac{v(N)}{4}$			
3	$\frac{v(\{1,2,3\})}{3} + \frac{v(N)}{4}$	$-\frac{v(1)}{3} - \frac{v(\{1,2\})}{2}$		
2	$\frac{v(\{1,2\})}{4} + \frac{v(\{1,2,3\})}{3} + \frac{v(N)}{4}$	$\frac{v(\{1,2\})}{4}$	$-\frac{v(1)}{3}$	
1	$\frac{v(1)}{3} + \frac{v(\{1,2\})}{4} + \frac{v(\{1,2,3\})}{3} + \frac{v(N)}{4}$	$\frac{v(1)}{3} + \frac{v(\{1,2\})}{4}$	$\frac{v(1)}{3}$	0
Total	$v(N)$	0	0	0

Thus, the total payoffs are:

Player	Payoffs
4	$-\frac{v(1)}{3} - \frac{v(\{1,2\})}{2} - v(\{1,2,3\}) + \frac{v(N)}{4}$
3	$-\frac{v(1)}{3} - \frac{v(\{1,2\})}{2} + \frac{v(\{1,2,3\})}{3} + \frac{v(N)}{4}$
2	$-\frac{v(1)}{3} + \frac{v(\{1,2\})}{2} + \frac{v(\{1,2,3\})}{3} + \frac{v(N)}{4}$
1	$\frac{v(1)}{3} + \frac{v(\{1,2\})}{2} + \frac{v(\{1,2,3\})}{3} + \frac{v(N)}{4}$

The example states, in particular, that Beal, Remila and Solal's sharing system is inefficient. Note also that, given an order $\pi \in \Pi(N)$, the final payoff each player receives according to both sharing schemes do not coincide. In fact, for a given order $\pi \in \Pi(N)$, the payoff vector $p^\pi(v)$ which determines the non-extreme pyramidal sharing scheme defined in Proposition 2 turns out to be the marginal contribution vector of the reversed order (see Proposition 3 below).

Proposition 2 allows to give an alternative expression (12) of the Shapley value as a function of the second-order differences, which remarks the dependence of the Shapley value of player i on his complementarity with the rest of players. First, note that,

$$a_{ij}^\pi(v) = v(S_\pi(i,j) \cup \{i,j\}) - v(S_\pi(i,j) \cup i) - v(S_\pi(i,j) \cup j) + v(S_\pi(i,j)) = \Delta_{ij}^2(S_\pi(i,j)),$$

where Δ_{ij}^2 is the *second-order difference operator* for a pair of players $i, j \in N$ considered by Segal [14], which is defined as a composition of marginal contribution operators (i.e., first-order difference operators) as follows

$$\Delta_{ij}^2(S) = v(S \cup \{i,j\}) - v(S \cup j) - v(S \cup i) + v(S) = \Delta_{ji}^2(S), \quad \forall\, S \subseteq N \setminus \{i,j\},$$

and it is interpreted by Segal as a measure of complementarity of players i and j with respect to the players in S. That is, in our case, the shares that player i receives when player $j \in S_\pi(i)$ arrives and joins coalition $P_\pi(j)$ depend on their complementarity with respect to the intermediate players. Since $\Delta^2_{ij}(S_\pi(i,j))$ could be negative, the entrance of player j can be detrimental to player i in some situations (i.e., orders). However, player $i \in N$ can assume such a negative share if he expects to be globally favoured, i.e., if $s_i^\pi(v) + \sum_{j \in S_\pi(i)} a_{ij}^\pi(v) \geq 0$. This is the case if the game is superadditive and positive (see Proposition 3's proof). Let us think for instance in the European Union, to which a country joins because it is beneficial from a global point of view, in spite of the fact that some other countries' incorporation can damage its situation. If the game is convex, then $\Delta^2_{ij}(S_\pi(i,j)) \geq 0$, for every pair of players $i, j \in N$ and for all orders $\pi \in \Pi(N)$, and therefore the entrance of every player always benefits the incumbents.

Thus, expression (9) can be written as:

$$\phi_i(v) = \mathcal{P}_i(v) = v(i) + \sum_{j \neq i} I_{ij}(v), \text{ for all } i \in N, \tag{10}$$

where

$$I_{ij}(v) = \frac{1}{n!} \sum_{\substack{\pi \in \Pi(N) \\ \pi(i) < \pi(j)}} \Delta^2_{ij}(S_\pi(i,j)), \text{ for all } i \neq j \in N, \tag{11}$$

is an *interaction index* of the game $(N, v) \in G_n$ for coalition $S = \{i, j\}$. This index should be interpreted as a measure of the extent of the profitability of the cooperation among the members of $S \subseteq N$ (see Grabisch and Roubens [5]).[5] Now, counting the number of orders $\pi \in \Pi(N)$ for which $S_\pi(i,j) = S \subseteq N \setminus \{i,j\}$, it holds that

$$\phi_i(v) = \mathcal{P}_i(v) = v(i) + \sum_{j \neq i} \sum_{S \subseteq N \setminus \{i,j\}} \frac{(n-s-1)!s!}{n!} \Delta^2_{ij}(S), \text{ for all } i \in N. \tag{12}$$

The next result relates the non-extreme pyramidal sharing scheme we have defined with the extreme one which classically defines the Shapley value.

Proposition 3. *Let (N, v) be a given TU game, and let $\mathcal{S}$ be the pyramidal sharing scheme defined in proposition 2. Then, for all orders $\pi \in \Pi(N)$, and every player $i \in N$, it holds $p_i^\pi(v) = m_i^{\widetilde{\pi}}(v)$, where $\widetilde{\pi}$ is the reversed order defined by $\widetilde{\pi}(i) = n - \pi(i) + 1$.*

Proof. Fix a permutation π and a player $i \in N$ which is not in the last position. When i joins coalition $P_\pi(i)$, he receives $v(i)$, and when its immediate successor $ds_\pi(i)$ joins, i receives

$$a^\pi_{i,ds_\pi(i)}(v) = \Delta^2_{i,ds_\pi(i)}(\emptyset) = v(\{i, ds_\pi(i)\}) - v(ds_\pi(i)) - v(i).$$

[5]This concept was implicitly first considered by Owen [13] who defined the *co-value* $q_{ij}(v)$ of i and j. The interaction index (11) differs from Owen's co-value in the number of orders in which $\Delta^2_{ij}(S)$ is considered. Owen's co-value takes into account only those orders in which i, S players, and j arrive in the first places.

When adding the two terms, the individual values $v(i)$'s get canceled, and we are left with $v(\{i, ds_\pi(i)\}) - v(ds_\pi(i))$. Suppose a direct successor of $ds_\pi(i)$ joins now. Then i receives $a^\pi_{i,ds^2_\pi(i)} = \Delta^2_{i,ds^2_\pi(i)}(ds_\pi(i))$, which equals

$$v(\{i, ds_\pi(i), ds^2_\pi(i)\}) - v(\{ds_\pi(i), ds^2_\pi(i)\}) - v(\{i, ds_\pi(i)\}) + v(ds_\pi(i)).$$

If we add now the terms corresponding to the three consecutive arrivals the terms $v(i)$ and $v(\{i, ds_\pi(i)\}) - v(ds_\pi(i))$ get again canceled, and we are left with

$$v(\{i, ds_\pi(i), ds^2_\pi(i)\}) - v(\{ds_\pi(i), ds^2_\pi(i)\}). \tag{13}$$

Iterating the argument $(n - \pi(i))$ times, we obtain that $p^\pi_i(v) = v(S_\pi(i) \cup i) - v(S_\pi(i))$, for every player $i \in N$ with $\pi(i) < n$. Then, taking into account that the player who arrives in the last position would receive just $v(i) = m^{\tilde{\pi}}_i(v)$, we are done. □

Observe that, according to the previous proof (see expression (13)), if the game is superadditive and positive (i.e., it is monotonic), then for all orders $\pi \in \Pi(N)$, player i's accumulated benefits at each step along the formation of the grand coalition are always nonnegative, for every player $i \in N$.

The two rational pyramidal sharing schemes which lead to the Shapley value represent, in turn, the two extreme options to define each entrant's salary and incumbents' dividends. The classic one promotes personal productivity by means of salaries, whereas our new proposal promotes positive synergy between agents by means of dividends. The former verifies strong monotonicity in salaries, while the following dividends' strong convexity condition holds for the latter:

Definition 4. Let $\mathcal{S}(v) = \{(s^\pi_j(v), (a^\pi_{ij}(v))_{i \in P_\pi(j)})_{j \in N} \mid \pi \in \Pi(N)\}$ be a pyramidal sharing scheme, let $i \neq j \in N$ be any two distinct players, and let (N, v), (N, w) be two n-person games for which $\Delta^2_{ij}(S, v) \leq \Delta^2_{ij}(S, w)$, for all $S \subseteq N \setminus \{i, j\}$, and being $\Delta^2_{ij}(S, v) < \Delta^2_{ij}(S, w)$ for some $S \subseteq N \setminus \{i, j\}$. Then, $\mathcal{S}$ verifies *Strong convex dividends* if $a^\pi_{ij}(v) \leq a^\pi_{ij}(w)$ for all orders $\pi \in \Pi(N)$ with $\pi(i) < \pi(j)$, and moreover $a^\pi_{ij}(v) < a^\pi_{ij}(w)$ for some order π with $\pi(i) < \pi(j)$.

Egalitarian Shapley values and consensus values

The idea of pyramidal values provides a constructive approach to deal with the family of *α-egalitarian Shapley values* (Joosten [7]) and the family of *consensus values* (Ju, Borm and Ruys [8]). Those families are constructed as linear convex combinations of the Shapley value and egalitarian-type values. This approach exhibits the trade-off between marginalism and egalitarianism, which is the basic feature of both families. In fact, both pyramidal sharing schemes adopt an egalitarian approach to share the added value among the incumbents, whereas the salaries' determination adopts a marginalistic one. The main difference between the two schemes lies in rationality considerations for the salaries. The consensus family's pyramidal scheme takes each player's own value as a minimum reference point, whereas the egalitarian Shapley family's pyramidal scheme does not.

Proposition 4. *For every game $(N, v) \in G_n$, and for all $\alpha \in [0, 1]$, the α-egalitarian Shapley value can be obtained through the pyramidal sharing scheme $\mathcal{S}$ given by the following rule:*

(*i*) *Entrant j's salary: $s_j^\pi(v) = \alpha m_j^\pi(v)$, if $\pi(j) > 1$; and $s_j^\pi(v) = v(j)$, whenever $\pi(j) = 1$,*

(*ii*) *Incumbents $P_\pi(j)$'s shares: $a_{ij}^\pi(v) = \frac{(1-\alpha)m_j^\pi(v)}{|P_\pi(j)|}$,*

for all $j \in N$, and for every order $\pi \in \Pi(N)$.

Proof. Malawski shows in [10] that every α-egalitarian Shapley value is in fact a procedural value obtained as a convex combination of the Shapley value and the equal division value, which are themselves procedural. Now, taking into account both extreme procedures, it is straightforward to check that our description (8) of the dividends and salaries of the procedural values produces in this case the dividends and salaries of the theorem. □

Note that the pyramidal sharing scheme considered above to deal with the family of *α-egalitarian Shapley values* is not rational in general, since salaries can not assured to be rational. If we take for example the game (N, v) with two players and such that $v(1) = v(2) = 1$, $v(N) = 2$, $\alpha = 0.5$ and π the identity, then $s_1^\pi(v) = 0.5 < 1 = v(1)$. Observe in particular that this game is monotonic. It is also remarkable that, since the *α-egalitarian Shapley values* are procedural values, then incumbents $P_\pi(j)$'s shares admit many other definitions as long as the salaries do not change. For instance, Malawski [10] proposes three procedures leading to the equal division. The one used in Proposition 4 and two others, which can also be employed to determine the dividends $a_{ij}^\pi(v)$.

Now, we will show that the consensus family arises when the entrant player $j \in N$ receives a intermediate salary between his own value $v(j)$ and his marginal contribution $m_j^\pi(v)$, given by his own value $v(j)$ plus a fixed proportion of his *reduced marginal contribution*, which is precisely his added value once he has been paid accordingly to his own value $v(j)$, i.e. $s_j^\pi = v(j) + \alpha(m_j^\pi(v) - v(j))$, and the remaining benefit $(1-\alpha)(m_j^\pi(v) - v(j))$ is distributed equally among the incumbent players in a non-anticipative manner. Therefore, α-consensus values are not procedural in general.

Proposition 5. *For every game $(N, v) \in G_n$, and for all $\alpha \in [0, 1]$, the α-consensus value can be obtained through the pyramidal sharing scheme $\mathcal{S}$ given by the following rule:*

(*i*) *Entrant j's salary: $s_j^{\pi,\alpha}(v) = v(j) + \alpha(m_j^\pi(v) - v(j))$,*

(*ii*) *Incumbents $P_\pi(j)$'s shares: $a_{ij}^{\pi,\alpha}(v) = \frac{(1-\alpha)(m_j^\pi(v) - v(j))}{|P_\pi(j)|}$,*

for all $j \in N$, and for every order $\pi \in \Pi(N)$.

Proof. Let $\alpha \in [0, 1]$, and consider the α-dummy property introduced by Ju *et al.* [8]. Then, we will show that the value obtained through the pyramidal sharing scheme we have considered verifies it. First, we will check the following equalities, for ever player $i = 1, \dots, n$:

$$\sum_{\pi \in \Pi(N)} \sum_{j \in S_\pi(i)} \frac{m_j^\pi(v)}{|P_\pi(j)|} = (n-1)!(v(N) - v(i)), \tag{14}$$

$$\sum_{\pi \in \Pi(N)} \sum_{j \in S_\pi(i)} \frac{-v(j)}{|P_\pi(j)|} = -(n-1)! \sum_{\substack{j \in N \\ i \neq j}} v(j), \tag{15}$$

Let $i \in N$ be a fixed player, then it follows from Proposition 4 that his α-egalitarian Shapley value $\varphi_i^\alpha(v) = \alpha\phi_i(v) + (1-\alpha)ED_i(v)$, where $ED_i(v) = \frac{v(N)}{n}$, for all $i \in N$, is given by

$$\frac{(1-\alpha)}{n!} \sum_{\pi \in \Pi(N)} \sum_{j \in S_\pi(i)} \frac{m_j^\pi(v)}{|P_\pi(j)|} + \frac{1}{n!} \sum_{\substack{\pi \in \Pi(N) \\ \pi(i)=1}} v(i) + \frac{\alpha}{n!} \sum_{\substack{\pi \in \Pi(N) \\ \pi(i)>1}} m_i^\pi(v) =$$

$$\frac{\alpha}{n!} \sum_{\pi \in \Pi(N)} m_i^\pi(v) + \frac{(1-\alpha)}{n} v(i) + \frac{(1-\alpha)}{n!} \sum_{\pi \in \Pi(N)} \sum_{j \in S_\pi(i)} \frac{m_j^\pi(v)}{|P_\pi(j)|}.$$

Thus, taking $\alpha = 0$, equality (14) holds. Now, we will check equality (15). Let $i \in N$ be a fixed player. Let us consider the following arrangement, which shows $v(j)$'s contribution to the sum

$$\sum_{\pi \in \Pi(N)} \sum_{j \in S_\pi(i)} \frac{-v(j)}{|P_\pi(j)|},$$

depending on i's and j's arrivals.

ARRIVALS	$\pi(j) = 2$	$\pi(j) = 3$	$\cdots$	$\pi(j) = n-1$	$\pi(j) = n$	
$\pi(i) = 1$	$-v(j)$	$-\frac{v(j)}{2}$	$\cdots$	$-\frac{v(j)}{n-2}$	$-\frac{v(j)}{n-1}$	
$\pi(i) = 2$		$-\frac{v(j)}{2}$	$\cdots$	$-\frac{v(j)}{n-2}$	$-\frac{v(j)}{n-1}$	
			$\ddots$	$\cdots$	$\cdots$	
$\pi(i) = n-2$				$-\frac{v(j)}{n-2}$	$-\frac{v(j)}{n-1}$	
$\pi(i) = n-1$					$-\frac{v(j)}{n-1}$	
TOTAL	$-v(j)$	$-2\frac{v(j)}{2}$	$\cdots$	$-(n-2)\frac{v(j)}{n-2}$	$-(n-1)\frac{v(j)}{n-1}$	$-(n-1)v(j)$

Therefore, taking into account that the number of orders $\pi \in \Pi(N)$ for which $\pi(i) = k$ and $\pi(j) = \ell$, with $\ell > k$, is $(n-2)!$, for all $\ell = k+1, \dots, n$, and for all $k = 1, \dots, n-1$, and extending the previous reasoning to all $j \in N \setminus \{i\}$, equality (15) follows.

Now, let $i \in N$ be a dummy player with respect to v. Then, since $s_i^{\pi,\alpha} = v(i)$, for all orders $\pi \in \Pi(N)$, the α-dummy condition follows from (14) and (15). Trivially, the defined pyramidal value is also efficient, symmetric and additive. Therefore, taking into account Theorem 5 in [8], its coincidence with the α-consensus value is proved. $\square$

In particular, the pyramidal definition of α-consensus values offers an alternative constructive approach to the *standardized remainder vectors* that determine the α-consensus values, which provides solid ground for it in terms of the dynamics of economic activity.

4 α-Proportional pyramidal values for monotonic games

In the α-egalitarian Shapley and consensus families, the remaining surplus, which represents the value that entrant j's participation adds to the incumbents, is shared equally among all the incumbents. In this section we consider a non-egalitarian framework, in which a player's right to get part of the forthcoming benefits is determined according to his initial investment. We measure this initial investment as the value his incorporation have added to the incumbents, or in other words, by means of his marginal contribution, and define the family of *α-proportional pyramidal values*. That is, we also adopt a marginalistic approach to determine the dividends.

Taking into account that a proportional allocation with respect to a given weight system in which some of the weights can be strictly negative must be carefully used, we restrict the definition of α-proportional pyramidal values to the subclass of *monotonic* TU games (i.e., $v(S) \leq v(T)$, for all $S \subseteq T$). In that case, all marginal contributions $m_j^{\pi}(v)$, $j \in N$, $\pi \in \Pi(N)$ are nonnegative.

Definition 5. For every monotonic TU game $(N, v) \in G_n$, and every $\alpha \in [0,1]$, the *α-proportional pyramidal value* is the value obtained by means of the following pyramidal sharing scheme:

(i) Entrant j's salary:

$$s_j^{\pi,\alpha}(v) = \begin{cases} m_j^{\pi}(v), & \text{if } v(P_\pi(j)) = 0, \\ v(j) + \alpha(m_j^{\pi}(v) - v(j)), & \text{otherwise.} \end{cases}$$

(ii) Incumbents $P_\pi(j)$'s shares:

$$a_{ij}^{\pi,\alpha}(v) = \begin{cases} 0, & \text{if } v(P_\pi(j)) = 0, \\ (1-\alpha)\frac{m_i^{\pi}(v)}{v(P_\pi(j))}(m_j^{\pi}(v) - v(j)), & \text{otherwise.} \end{cases}$$

for all $j \in N$, and for all orders $\pi \in \Pi(N)$. Thus, the final payoff that player $i \in N$ receives according to the order $\pi \in \Pi(N)$ is given by:

$$pp_i^{\pi,\alpha}(v) = v(i) + \alpha(m_i^{\pi}(v) - v(i)) + (1-\alpha) \sum_{\substack{j \in S_\pi(i) \\ v(P_\pi(j)) \neq 0}} \frac{m_i^{\pi}(v)}{v(P_\pi(j))}(m_j^{\pi}(v) - v(j)), \tag{16}$$

if $v(P_\pi(i)) \neq 0$, and

$$pp_i^{\pi,\alpha}(v) = m_i^{\pi}(v) + (1-\alpha) \sum_{\substack{j \in S_\pi(i) \\ v(P_\pi(j)) \neq 0}} \frac{m_i^{\pi}(v)}{v(P_\pi(j))}(m_j^{\pi}(v) - v(j)), \tag{17}$$

if $v(P_\pi(i)) = 0$, for all $i = 1, \ldots, n$. Therefore, the α-proportional pyramidal value, which is the expected value under the former sharing scheme when all orders are equally likely, is given by

$$\mathcal{PP}_i^\alpha(v) = \frac{1}{n!}\Big(\sum_{\substack{\pi\in\Pi(N)\\ v(P_\pi(i))\neq 0}} \big(v(i) + \alpha(m_i^\pi(v) - v(i))\big) + \sum_{\substack{\pi\in\Pi(N)\\ v(P_\pi(i))=0}} m_i^\pi(v)\Big) +$$

$$\frac{1-\alpha}{n!}\Big(\sum_{\pi\in\Pi(N)} \sum_{\substack{j\in S_\pi(i)\\ v(P_\pi(j))\neq 0}} \frac{m_i^\pi(v)}{v(P_\pi(j))}(m_j^\pi(v) - v(j))\Big),\ i = 1, \ldots, n. \quad (18)$$

Proposition 6. *For every monotonic TU game $(N, v) \in G_n$, and every $\alpha \in [0, 1]$, it holds*

$$\mathcal{PP}^\alpha(v) = \alpha\phi(v) + (1-\alpha)\mathcal{PP}^0(v).$$

Proof. Trivially, if we express $v(i)$ and $m_i^\pi(v)$ as $\alpha v(i) + (1-\alpha)v(i)$ and $\alpha m_i^\pi(v) + (1-\alpha)m_i^\pi(v)$ in the first summand of (18), it follows that every α-proportional pyramidal value is the linear convex combination of the two extreme values for $\alpha = 0$ and $\alpha = 1$. Moreover, since the 1-proportional pyramidal value is in fact the Shapley value, then the result holds. □

When we restrict ourselves to the class of monotonic simple games, the whole family reduces to the Shapley value.

Proposition 7. *Let $(N, u) \in G_n$ be a monotonic simple game such that $u(i) = 0$ for every non veto player $i \in N$. Then $\mathcal{PP}^\alpha(u) = \phi(u)$, for all $\alpha \in [0, 1]$.*

Proof. Let $(N, u) \in G_n$ be a monotonic simple game, and let be $\pi \in \Pi(N)$ be a given order. Then, there exists a unique $i_\pi \in N$ with nonzero marginal contribution. Moreover:

- Since $u(P_\pi(j)) = 0$ for all $j \in P_\pi(i_\pi)$, then $s_j^{\pi,\alpha}(u) = m_j^\pi(u) = 0$ and $a_{ij}^{\pi,\alpha}(u) = 0$, for all $i \in P_\pi(j)$,
- $s_{i_\pi}^{\pi,\alpha}(u) = m_{i_\pi}^\pi(u) = 1$, $a_{ii_\pi}^{\pi,\alpha}(u) = 0$, for all $i \in P_\pi(i_\pi)$,
- If $j \in S_\pi(i_\pi)$, then j is a non veto player. Therefore, $s_j^{\pi,\alpha} = u(j) + \alpha(m_j^\pi(u) - u(j)) = 0$ and $a_{ij}^{\pi,\alpha} = (1-\alpha)m_i^\pi(u)(m_j^\pi(u) - u(j)) = 0$, for all $i \in P_\pi(j)$.

□

Let us analyze, by means of an example, the behavior of the extreme zero-proportional value and the α's choice effect over the final allocation of benefits.

Example 3. Let us consider the following 4-person game (N, v), with $v(1) = 1$, $v(2) = v(3) = v(4) = 0$, and:

S	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	N
$v(S)$	2	2	4	1	1	2	6	7	5	8	10

In this example, player's 1 and 2 marginal contributions lead to the same Shapley value $\phi_1(v) = \phi_2(v) = 2\frac{7}{12}$. The marginal contributions of player 4 are always greater or equal than those of player 2, and player 3 is in the weakest position:

$$\phi(v) = (2\frac{7}{12}, 2\frac{7}{12}, 2\frac{1}{12}, 2\frac{3}{4})$$

On the contrary, the roles of players 1 and 2 are distinguished by means of the proportional pyramidal values for all $\alpha \in [0,1)$, which are given by:

$$\mathcal{PP}^{\alpha}(v) = \alpha(2.5833, 2.5833, 2.0833, 2.75) + (1-\alpha)(4.4266, 1.8060, 1.7622, 2.0052)$$

Note that proportional values reward a player for his contribution to the establishment of the firm as well as for his contribution to the firm's growth; moreover, the marginal contributions of player 1 are greater for small size's coalitions than those of players 2, 3 and 4, which on the contrary are greater than the marginal contributions of player 1 for big size's coalitions. Thus, since parameter α controls to what extent a player must be compensated according to his participation at the beginning of the project rather than to his contribution to its evolution, the rewards that player 1 receive increase as α decreases to zero. The relative position among the rest of the players remains.

We end up by briefly discussing which properties of the list in Section 2 hold or not for the α-proportional pyramidal values, being our arguments heavily based on the previous description of these values as a linear combination of the Shapley value and the value $\mathcal{PP}^0(v)$. To be specific, every proportional value verifies efficiency, symmetry, positivity (when restricted to the class of supperadditive games), standardness for two person-games, null player and null player out. On the contrary, additivity, relative invariance with respect to strategic equivalence, strong monotonicity, and dummy properties are not (always) satisfied by these values. Note also that α-proportional pyramidal values are not procedural in general.

5 Conclusions and future research

In this paper we propose a general procedure for obtaining a broad class of solution concepts based on a pyramidal distribution of the benefits that are sequentially obtained through a dynamic process of coalition formation, in which players successively come into play and join the current coalition until the grand coalition is formed. In particular, we obtain some known values by means of pyramidal sharing schemes and we introduce a proportional family of pyramidal values, in which incumbents receive dividends in proportion to their initial investment.

Axiomatic characterizations for the proportional family, and also for some sub-classes of pyramidal values are left for future research, as well as a strategic analysis of this kind of solutions. It may be also interesting to generalize the notion of proportional pyramidal values to a weighted version in which the incumbents' shares depend on a general system of weights.

With respect to potential extensions, the pyramidal sharing of the current benefits idea allows to deal with those situations in which the number of final participants where not known in advance. Moreover, it should be interesting to introduce the notion of pyramidal sharing scheme in the context of games with a communication graph [12] and [1].

Finally, it must be pointed out that the complexity of the calculus of a pyramidal value relies crucially on the calculus of the pyramidal sharing scheme and, obviously, on the complexity of the characteristic function of the game. In the case of the two proposed families, if the marginal contributions can be computed (or at least approximated) in polynomial time, then any pyramidal value can also be estimated in polynomial time. In fact, following Castro, Gomez and Tejada [3], any value that can be expressed as an expectation of a polynomial function of the marginal contribution vectors over all permutations, when all orderings are equally likely, can be estimated in polynomial time, whenever the marginal contributions are computable in polinomial time.

Appendix

In this appendix we collect the formal definitions of all the known values analyzed in Section 3, as well as the characterization results we have used.

Theorem 1 (Shapley, 1953)**.** *There exists a unique value satisfying the efficiency, symmetry, dummy, and additivity axioms. It is the Shapley value, which is defined for every $(N,v) \in G_n$ as follows:*

$$\phi_i(N,v) = \sum_{\substack{S \subseteq N \\ i \notin S}} \frac{s!(n-s-1)!}{n!} \big(v(S \cup \{i\}) - v(S) \big), \quad i = 1, \ldots, n, \tag{19}$$

where $s = |S|$ denotes the cardinality of coalition $S \subseteq N$.

The Consensus value (Ju *et al.* [8]) is aimed to generalize the standard solution for 2-person TU games into n-person cases. It is based on a two-sided negotiation process that can be understood as a *standardized remainder rule* described by the following vectors. The reader is referred to Ju *et al.* [8] for a detailed exposition of this rule.

Definition 6 (Ju, Borm and Ruys, 2007)**.** Let $(N,v) \in G_n$, and $\pi \in \Pi(N)$ be a given permutation. Define $S_k^\pi = \{\pi^{-1}(1), \ldots, \pi^{-1}(k)\} \subseteq N$ and $S_0^\pi = \emptyset$. Then, the standardized remainder for coalition S_k^π, $r(S_k^\pi)$, is recursively defined as follows:

$$r(S_k^\pi) = \begin{cases} v(N), & \text{if } k = n, \\ v(S_k^\pi) + \frac{1}{2}\Big(r(S_{k+1}^\pi) - v(S_k^\pi) - v(\{\pi^{-1}(k+1)\}) \Big), & \text{if } k \in \{1, \ldots, n-1\}. \end{cases}$$

$r(S_k^\pi)$ is the value left for S_k^π after allocating surpluses to earlier leavers $N \setminus S_k^\pi$. Then, the standardized remainder vector, $sr^\pi(v)$, which corresponds to the situation where the players leave the game one by one in the order $(\pi^{-1}(n), \ldots, \pi^{-1}(1))$, is defined recursively by:

$$sr_{\pi^{-1}(k)} = \begin{cases} v(\{\pi^{-1}(k)\}) + \frac{1}{2}\Big(r(S_k^\pi) - v(S_{k-1}^\pi) - v(\{\pi^{-1}(k)\}) \Big), & \text{if } k \in \{2, \ldots, n\}, \\ r(S_1^\pi), & \text{if } k = 1. \end{cases}$$

Definition 7 (Ju, Borm and Ruys, 2007)**.** For every $(N, v) \in G_n$, the *consensus value* $\Psi(v)$ is defined as the average, over the set of all permutation $\Pi(N)$, of the individual standardized remainder vectors, i.e.,

$$\Psi(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} sr^{\pi}(v).$$

Definition 8 (Ju, Borm and Ruys, 2007)**.** For every $(N, v) \in G_n$ and $\alpha \in [0, 1]$, the *α-consensus value* $\Psi^{\alpha}(v)$ is defined as the average, over the set of all permutation $\Pi(N)$, of the individual α-remainder vectors, i.e.,

$$\Psi^{\alpha}(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} (sr^{\pi})^{\alpha}(v).$$

Here, the α-remainder $r^{\alpha}(S_k^{\pi})$ and the individual α-remainder vector $(sr^{\pi})^{\alpha}(v)$ are defined as follows:

$$r^{\alpha}(S_k^{\pi}) = \begin{cases} v(N), & \text{if } k = n, \\ v(S_k^{\pi}) + (1-\alpha)\Big(r^{\alpha}(S_{k+1}^{\pi}) - v(S_k^{\pi}) - v(\{\pi^{-1}(k+1)\})\Big), & \text{if } k \in \{1, \dots, n-1\}. \end{cases}$$

and

$$(sr_{\pi^{-1}(k)})^{\alpha} = \begin{cases} v(\{\pi^{-1}(k)\}) + \alpha\Big(r^{\alpha}(S_k^{\pi}) - v(S_{k-1}^{\pi}) - v(\{\pi^{-1}(k)\})\Big), & \text{if } k \in \{2, \dots, n\}, \\ r^{\alpha}(S_1^{\pi}), & \text{if } k = 1. \end{cases}$$

The authors introduce the following property in order to characterize the family of consensus values. The next theorem corresponds to Theorem 5 in Ju, Borm and Ruys [8]. We make use of (a) characterization.

Definition 9 (Ju, Borm and Ruys, 2007)**.** A *value* $\varphi : G_n \to \mathbb{R}^n$ verifies the *α-dummy property* if $\varphi_i(v) = \alpha v(i) + (1-\alpha)\Big(v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}\Big)$, for all $(N, v) \in G_n$, and every dummy player $i \in N$ with respect to v.

Theorem 2 (Ju, Borm and Ruys, 2007)**.** *(a) The α-consensus value Ψ^{α} is the unique one-point solution concept on G_n that satisfies efficiency, symmetry, the α-dummy property and additivity.*

(b) The α-consensus value Ψ^{α} is the unique function that satisfies efficiency, symmetry, the α-dummy property and the transfer property over the class of TU games.

(c) For any $v \in G_n$, it holds that

$$\Psi^{\alpha}(v) = \alpha\alpha\phi(v) + (1-\alpha)E(v),$$

where $E(v)$ is the equal surplus solution of v, i.e., $E_i(v) = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}$.

(d) The α-consensus value Ψ^{α} is the unique function that satisfies efficiency and the α-equal welfare loss property over the class of TU games.

The Egalitarian Shapley values (Joosten [7]) make the trade-off between marginalism and egalitarianism by means of convex combinations of the Shapley value and the equal division solution.

Definition 10 (Joosten, 1996)**.** For every $(N, v) \in G_n$ and $\alpha \in [0, 1]$, the α-egalitarian Shapley value $\varphi^{\alpha}(v)$ is given by

$$\varphi^{\alpha}(v) = \alpha\phi(v) + (1 - \alpha)ED(v),$$

where $ED(v)$ is the equal division value which distributes the worth $v(N)$ equally among all players: $ED(v) = (\frac{v(N)}{n}, \ldots, \frac{v(N)}{n})$.

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