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The Path Player Game

A network game from the point of view of the network providers

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Abstract We introduce the path player game, a noncooperative network game with a continuum of mutually dependent set of strategies. This game models network flows from the point of view of competing network operators. The players are represented by paths in the network. They have to decide how much flow shall be routed along their paths. The competitive nature of the game is due to the following two aspects: First, a capacity bound on the overall network flow links the decisions of the players. Second, edges may be shared by several players which might have conflicting goals. In this paper, we prove the existence of feasible and pure-strategy equilibria in path player games, which is a non-trivial task due to non-continuity of payoff functions and the infinite, mutually dependent strategy sets. We analyze different instances of path player games in more detail and present characterizations of equilibria for these cases.

Key words Network games – Equilibria – Path player games

1 Introduction

Various types of games on networks have been studied in recent years. For instance in *routing games* [24,5] flow has to be transported from origin to destination nodes. In *load balancing games* [20,6] load is to be assigned to resources. In *facility location games* and *service provider games* [29,7], facilities have to be located and assigned to the demand points. *Network design games* [9,2] and *coordination games* [14,17,4] describe the generation of networks. In [13,15], properties of *social networks* are studied.

In this work we study a new type of routing game. Usually, in routing games the problem of sending flow in a network is considered from the point of view of the flow itself, assuming that the flow can choose a path from origin to destination. An aspect, which has not been considered yet, is the behavior of the path owners, when they are allowed to choose the amount of flow that will be sent along their paths. This new approach models systems where paths are owned by decision makers, like in public transportation, energy or information networks. The decision makers offer a certain bandwidth to be used by the flow, like a bandwidth of electricity, or a certain daily frequency of trains. Equilibria in this model describe a stable market situation among competing path owners.

A rough description of path player games is given as follows: Consider a network G and a set of players $\mathcal{P}$, one for each path $P \in \mathcal{P}$ where the flow is defined as $f : \mathcal{P} \to \mathbb{R}_+^{|\mathcal{P}|}$. Each player's strategy is to choose a nonnegative flow f_P. An upper bound on the flow is given by the flow rate r, which shall not be exceed by the overall flow in the network. The flow rate is motivated by the limited capacity of the network resources and may arise from society regulations like a limitation of traffic for ecological or security reasons. Furthermore, each single player has a lower bound, the security limit ω_P. This aspect is important, e.g. in transport optimization. The violation of both types of bounds is penalized.

More detailed, the benefit is given by three parts: First of all, a cost functions $c_e(x)$ is assigned to each edge e. It is dependent on the flow x sent along the edge. If the bounds r and ω_P are satisfied, the income of a player is given by the sum of costs over the edges that belong to his path. If one of the bounds is violated, a constant (negative) benefit is payed. Hence, the resulting benefit function is in general not continuous. Note that the benefit needs also not be strictly increasing. So a player is not necessarily interested in routing as much flow as possible. Handling too much flow could mean increasing operating costs, for instance due to over-hours or additional maintenance.

The competitive aspect of the game is given by the flow rate that has to be satisfied by the overall network flow, and by the fact that the paths may own edges shared with other paths. Hence, the flow on an edge depends on the flow on *all* paths using the edge. The players sharing an edge may have different objectives regarding that edge, which leads up to competitive situations.

Violation of the flow rate r needs to be avoided in any case, since it leads to *infeasible* solutions. This is done by introducing a high penalty. Another way is to forbid infeasible solutions. This approach results in a game in which the strategy sets of the players are mutually dependent. Such games are called generalized Nash equilibrium (GNE) games, see [16, 10]. An often studied question in GNE games is the existence of equilibria. Path player games are special instances of GNE games, see [22]. Their structure allows to prove the existence of equilibria. Note that both approaches, the approach of this paper using fixed strategy sets with a penalty, and the GNE path player game with dependent strategy sets are worthwhile to consider.

Application The following application of path player games is under consideration in the framework of the European project *ARRIVAL* [1]: To create a public transportation network, like a railway or bus network, lines have to be installed. In particular, the lines are given by their stops, for instance a railway line may go from Hamburg to Basel with intermediate stops in between. Assigned to each line is its frequency, i.e. how often the line travels within a given time horizon, for instance twice a day. The line plan has to satisfy the customers' demand and has to respect upper bounds to limit the frequencies on the edges. These bounds are usually given for security reasons, e.g. only one train is allowed to be on a block[1] at any time. Summarizing, the question of line planning is: Which lines and what frequencies shall be installed such that demand and security constraints are satisfied? This problem can be modeled as a type of path player game, called *line planning game*, in which the potential lines are the players choosing their frequencies as strategies. In contrast to cost- or customer-oriented objectives, see [25], the goal of the line planning game is to minimize the average delay of each line, and hence to obtain a line plan which is robust against delays, see [26] for first results and numerical studies using data from interregional trains in Germany.

Related games The following three types of network games are related to path player games.

We have already mentioned *routing games*, where the network flow is analyzed from the point of view of the flow. A routing game is played on a congested network, and the flow is assumed to consist of a finite or infinite number of players, acting independently and selfishly. Each of them chooses a path from the source to the destination that minimizes the cost of traveling along that path. This model can be seen as a counterpart to the path player game, as it represents the point of view of the travelers, while in the path player game the situation is analyzed from the path owners' point of view. The path player game is also related to *bandwidth allocation games*, as described in [19,18]. In bandwidth allocation games capacitated links are used by several players, sending bids to a central manager. The manager answers with prices that are proportional to the bids and cares for satisfying the capacity constraints. Each user has its own utility function that determines his payoff depending on the price and the bid. *Price taking users* just accept the price given by the manager, while *price anticipating users* may adjust their bids. In the path player game, the bids correspond to the strategies. Contrary to bandwidth games, they are not answered by a manager, but directly accepted. However, in the path player game, all con-

[1] Between two stations, a track is separated into smaller units, namely into *blocks*.

tinuous and nonnegative cost functions are allowed, while in bandwidth allocation strictly increasing, continously differentiable and concave functions (so called *elastic traffic*) are required. Note that in bandwidth allocation the existence of equilibria can not be guaranteed, but in path player games we are able to prove the existence for continuous cost functions.

Another model describing the behavior of path owners is that of *path auctions*, see e.g. [8,3]. Here each edge is owned by a player, and a central manager has the task to buy a shortest path from s to t from the edge owners. The edge owners know the price of their edge, but they are allowed to report a wrong price if they benefit from lying. The goal is to develop a payment mechanism such that every edge owner is interested to tell the truth. Such a mechanism is called *truth telling.* In a path player game in a network consisting of parallel edges from s to t, the path owners are edge owners as well, such that the path player game can be seen as a special case of path auction games.

The paper is organized as follows. In Section 2 we introduce the game model. In Section 3 we show that feasible equilibria in pure strategies exist. Further properties of path player games are discussed in Section 4. These are used in Section 5 to give necessary and sufficient conditions for equilibria in the case of strictly increasing cost functions. The paper ends with some suggestions for future work.

2 The Model

We consider a given network $G = (V, E)$ with vertices $v \in V$ and edges $e \in E$. A path P in G is given by a sequence of edges $e \in E$: $P = (e_1, ..., e_k)$. By $\mathcal{P}$ we denote the set of all paths P in G from the single source s to the single sink t, thus the set $\mathcal{P}$ is given by the structure of the network G. Each of the paths $P \in \mathcal{P}$ represents a player[2] in the path player game. Each player proposes an amount of flow f_P that he wants to be routed along his path. The complete flow is represented by a function $f : \mathcal{P} \to \mathbb{R}_+^{|\mathcal{P}|}$, while the flow on path P is denoted by f_P. For each edge $e \in E$, the flow f_e along the edge can hence be determined by the sum of the flows on paths that contain e, i.e.

$$f_e = \sum_{P:e \in P} f_P \ .$$

We assume that the demand is high enough to ensure that the players can implement the flow they proposed. Note that this is a considerable difference to bidding games, like bandwidth allocation or path auction games.

Each edge e is associated with a cost function $c_e(\cdot)$, that depends on the flow on e. The cost function represents the income of the edge owners and we

[2] In the course of this paper we will denote both, the path and the corresponding player with P, as both notations are handled equivalently.

assume these functions to be continuous and nonnegative for nonnegative flows, i.e. $c_e(x) \geq 0$ for $x \geq 0$. If the edge belongs to more than one owner, we assume that each player receives the same income. (It is possible to generalize this model by allowing the owners to share the fee in an arbitrary way.)

To calculate the cost of a path P, we sum up the costs of the edges belonging to that path, i.e.,

$$c_P(f) = \sum_{e \in P} c_e(f_e).$$

These costs are, however, not directly the benefit of player P since there are two more issues to handle:

- We require that the sum of flows in the network is bounded by a given *flow rate* r. It can be interpreted as a network capacity. We call a flow f *feasible* for a flow rate r if $\sum_{P \in \mathcal{P}} f_P \leq r$ holds. If the flow rate is exceeded, the flow is called *infeasible* and all players receive a penalty of M, with M being a large number.
- Furthermore, a security system for the players is implemented: If the flow of a player P lies below the so called *security limit* $\omega_P \geq 0$, he will receive a fixed *security payment* $\kappa_P > -M$. In this case, the path P is called *underloaded*, while we call P *loaded*, if $f_P > \omega_P$. For positive κ_P, the security limit and payment serve as an insurance that guarantees a fixed income for each player. On the other hand, if $\kappa_P < 0$, the security payment is a penalty for underloaded paths. This penalty may represent for instance additional costs for maintaining an unused resource.

Summarizing, we obtain the *benefit function* in the path player game:

Definition 1 *The* benefit function *of player $P \in \mathcal{P}$ in a path player game for $f \geq \mathbf{0}_{|\mathcal{P}|}$ is given as:*

$$b_P(f) = \begin{cases} c_P(f) & if \ \sum_{P \in \mathcal{P}} f_P \leq r \wedge f_P \geq \omega_P \\ \kappa_P & if \ \sum_{P \in \mathcal{P}} f_P \leq r \wedge f_P < \omega_P \\ -M & if \ \sum_{P \in \mathcal{P}} f_P > r \end{cases},$$

where $c_P(f) = \sum_{e \in P} c_e(f_e)$.

Some remarks about path player games should be added.

- There is a continuum of strategies as a player is allowed to choose any nonnegative real number. The benefit (or payoff) a player obtains after fixing a strategy depends on the strategies of all players.
- The path player game is noncooperative and thus it is possible that the flow created by the decisions of the players is not feasible. For instance if the benefit is a nondecreasing function, each player will try to get as much flow as possible such that the sum of all proposed flows may exceed the flow rate. Unfortunately, it turns out that even equilibria may be infeasible. Nevertheless, we will prove in Theorem 1 that feasible equilibria do exist.

3 Equilibria for General Benefit Functions

In this section we analyze equilibria in path player games for general benefit functions while, later in Section 5, we derive additional results for strictly increasing cost functions. The definition of equilibria in path player games follows the definition of a *Nash equilibrium* (see e.g. [21]): A flow f^* is an *equilibrium* in a given path player game if and only if for all players $P \in \mathcal{P}$ and for all $f_P \geq 0$ it holds that

$$b_P(f^*_{-P}, f^*_P) \geq b_P(f^*_{-P}, f_P) \ . \tag{1}$$

We will call the equilibrium *feasible* if f^* is a feasible flow, *infeasible* otherwise. An equilibrium is a game situation where none of the players is able to obtain a better outcome by changing his strategy unilaterally. Such a situation characterizes a stable state of the system.

In order to find equilibria in the path player game we have to look at the benefit of a single player who changes his own strategy, while the strategies of the competitors remain fixed. We define $f_{-P} \in \mathbb{R}_+^{|\mathcal{P}|-1}$ by deleting the component belonging to path P, such that we can fix the strategies f_{-P} of the competitors and just consider the influence of f_P. We obtain

$$c_P(f_{-P}, f_P) = \sum_{e \in P} c_e \left(f_P + \sum_{P_k \in \mathcal{P} \setminus \{P\} : e \in P_k} f_{P_k} \right) \ ,$$

with a constant term $\sum_{P_k \in \mathcal{P} \setminus \{P\} : e \in P_k} f_{P_k}$. In the following, if we want to stress the fact that f_{-P} is fixed with respect to the cost or benefit function, we will denote $c_P(f) = c_P(f_{-P}, f_P)$ or $b_P(f) = b_P(f_{-P}, f_P)$, respectively. If $c_e(f_e)$ are convex (concave) functions, then $c_P(f_{-P}, f_P)$ is also a convex (concave) function. Finally, we introduce the *decision limit* of player P as

$$d_P(f_{-P}) = r - \sum_{P_k \in \mathcal{P} \setminus \{P\}} f_{P_k} \ .$$

The set $[0, d_P(f_{-P})]$ is called the *decision interval* of player P. It contains all feasible strategies for P. From its definition we obtain the following corollary: If there is one player sending as much flow as possible (without violating the decision limit), then this is true for all players.

Corollary 1 *Any flow f satisfies: If there is a player P_k with $f_{P_k} = d_{P_k}(f_{-P_k})$ then all players $P \in \mathcal{P}$ satisfy $f_P = d_P(f_{-P})$.*

Figure 1 shows an example of a benefit function $b_P(f_{-P}, f_P)$ for fixed f_{-P}. The function depends only on the scalar f_P and is characterized by three parts: The two constant regions generated by the security payment κ_P, the infeasibility penalty $-M$, and the middle part, created by the cost function $c_P(f_{-P}, f_P)$. As the players want to maximize their benefit, we define the *best reaction set* for a player P with respect to a given flow f_{-P} as

$$f_P^{max}(f_{-P}) = \{f_P \geq 0 : f_P \text{ maximizes } b_P(f_{-P}, f_P)\} \ .$$

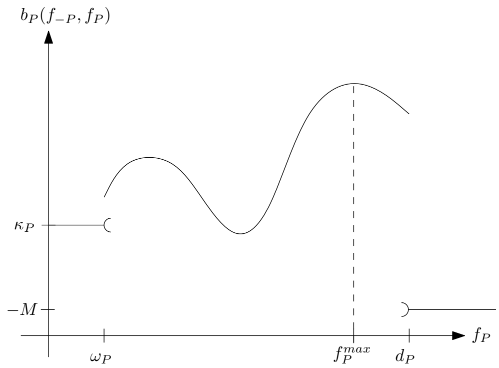

Fig. 1 Benefit function $b_P(f_{-P}, f_P)$ for fixed f_{-P}

In this paper, we assume to have continuous cost functions. As a result, we obtain nonempty best reaction sets:

Lemma 1 *Consider a path player game with cost functions $c_e(f_e)$ being continuous for all edges $e \in E$. Then, the sets $f_P^{max}(f_{-P})$ are nonempty for all $P \in \mathcal{P}$ and for all $f \in \mathbb{R}_+^{|\mathcal{P}|}$.*

Lemma 1 can be proved by considering the three parts of $b_P(f_{-P}, f_P)$ and applying Weierstrass extreme value theorem.

Example 1 $f_P^{max}(f_{-P})$ can be empty in the case of non-continuous cost functions, see Figure 2.

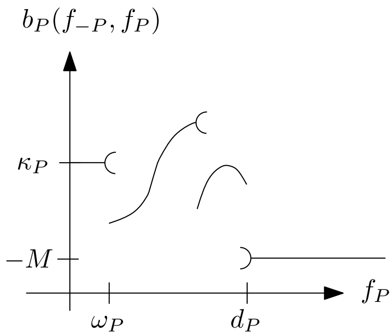

Fig. 2 Best reaction set $f_P^{max}(f_{-P})$ is empty

Best reaction sets are useful for the following characterization of equilibria. A flow is an equilibrium if each player is choosing a best reaction strategy with respect to the strategies of his opponents:

Corollary 2 *In a path player game a flow f^* is an equilibrium if and only if for all $P \in \mathcal{P}$ it is satisfied that $f_P^* \in f_P^{max}(f_{-P}^*)$.*

For infinite games with continuous benefits it is known that there exists an equilibrium in mixed strategies if the strategy spaces are nonempty and compact. Even more, if we assume continuous and quasi-concave benefit functions, there exists a pure-strategy equilibrium (see e.g., [11] for both results). In our game, we may have non-continuous benefit functions. If we just consider feasible flows, we furthermore have to deal with dependent strategy sets. Therefore, it is not evident that in path player games equilibria always exist, and if yes, it is still not clear if there is a feasible equilibria among them. We will in the following prove even more: the existence of feasible equilibria in *pure strategies*.

Theorem 1 (Existence of feasible equilibria) *In a path player game with continuous cost functions $c_e(f_e)$ for all edges $e \in E$, there is at least a feasible equilibrium in pure strategies $\hat{f}$ such that $\hat{f}_P \in f_P^{max}(\hat{f}_{-P})\ \forall\ P \in \mathcal{P}$.*

Proof Consider the set of feasible flows

$$F = \left\{ f : f_P \geq 0\ \forall\ P \in \mathcal{P}\ \wedge\ \sum_{P \in \mathcal{P}} f_P \leq r \right\}\ .$$

The set F is closed, bounded and convex. Furthermore consider the single-value function, $T : F \to \mathbb{R}^{|\mathcal{P}|}$ defined as $T(f) = f'$ whose components $f'_P = t(f_P)$ are given by

$$f'_P = f_P + \begin{cases} \min\left\{ f_P^m - f_P, \frac{f_P^m - f_P}{\sum_{P_k \in \mathcal{P}: f_{P_k} < f_{P_k}^m} (f_{P_k}^m - f_{P_k})} \cdot d \right\} & if \quad f_P < f_P^m \\ f_P^m - f_P & if \quad f_P \geq f_P^m \end{cases}\ , \tag{2}$$

where $f_P^m = \min\{f_P^{max}(f_{-P})\}$ is chosen as the smallest flow that is benefit maximizing[3] and $d = r - \sum_{P \in \mathcal{P}} f_P$ is the flow left that can be distributed among the players maintaining feasibility. Note that by definition of d_P it holds for any feasible flow $f \in F$ and all $P \in \mathcal{P}$ that

$$d = d_P(f_{-P}) + \sum_{P_k \in \mathcal{P} \setminus \{P\}} f_{P_k} - \sum_{P \in \mathcal{P}} f_P = d_P(f_{-P}) - f_P \geq 0\ .$$

Note furthermore, that by Lemma 1 f_P^m exists and that by definition of $f_P^{max}(f_{-P})$ it holds that

$$0 \leq f_P^m \leq d_P(f_{-P}) = r - \sum_{P_k \in \mathcal{P} \setminus \{P\}} f_{P_k}\ . \tag{3}$$

An interpretation of the function T is given right after this proof. In the following we prove that T is a continuous function of F into itself. Then, by Brouwer's fixed point theorem a fixed point $f = T(f)$ exists in F. Finally, we will show that each fixed point in F is representing an equilibrium in pure

[3] Note that for the proof it is not important which $f_P \in f_P^{max}$ is chosen for f_P^m as long as it is well-defined.

strategies, so that we will be able to guarantee the existence of a feasible equilibrium in pure strategies.

Part a) ($T : F \to F$)
First note that $f'_P \geq 0 \ \forall \ P \in \mathcal{P}$. Denote the sets $\mathcal{P}_1 = \{P \in \mathcal{P} : f_P < f_P^m\}$ and $\mathcal{P}_2 = \{P \in \mathcal{P} : f_P \geq f_P^m\}$.

$$\begin{aligned}
\sum_{P\in\mathcal{P}} f'_P &= \sum_{P\in\mathcal{P}_1}\left(f_P + \min\left\{f_P^m - f_P, \frac{f_P^m - f_P}{\sum_{P_k\in\mathcal{P}_1}(f_{P_k}^m - f_{P_k})}\cdot d\right\}\right) \\
&\quad + \sum_{P\in\mathcal{P}_2}(f_P + f_P^m - f_P) \\
&= \sum_{P\in\mathcal{P}} f_P + \sum_{P\in\mathcal{P}_1}\min\left\{f_P^m - f_P, \frac{f_P^m - f_P}{\sum_{P_k\in\mathcal{P}_1}(f_{P_k}^m - f_{P_k})}\cdot d\right\} \\
&\quad + \overbrace{\sum_{P\in\mathcal{P}_2}(f_P^m - f_P)}^{\leq 0} \\
&\leq \sum_{P\in\mathcal{P}} f_P + \sum_{P\in\mathcal{P}_1}\frac{f_P^m - f_P}{\sum_{P_k\in\mathcal{P}_1}(f_{P_k}^m - f_{P_k})}\cdot d \\
&= \sum_{P\in\mathcal{P}} f_P + d = \sum_{P\in\mathcal{P}} f_P + r - \sum_{P\in\mathcal{P}} f_P \\
&= r \ .
\end{aligned}$$

Therefore, $f' \in F$ since $f'_P \geq 0 \ \forall \ P \in \mathcal{P}$ and $\sum_{P\in\mathcal{P}} f'_P \leq r$.

Part b) ($T(f)$ is continuous)
We distinguish the following exhaustive cases:

i) $f_P > f_P^m$:
$f'_P = f_P^m \ \forall \ f_P > f_P^m$, i.e. $t(f_P)$ is continuous
ii) $f_P = f_P^m + 0$:
$f'_P = f_P^m$ for $f_P = f_P^m + 0$, i.e. $t(f_P)$ is continuous to the right $f_P = f_P^m + 0$
iii) $f_P < f_P^m$:
Consider $g(f) = f_P^m - f_P$ and $h(f) = \frac{f_P^m - f_P}{\sum_{P_k\in\mathcal{P}_1}(f_{P_k}^m - f_{P_k})}\cdot d$. The functions $g(f)$ and $h(f)$ are continuous and so the minimum of both functions is continuous too. It follows that $t(f_P)$ with $f'_P = f_P + \min\{g(f); h(f)\}$ is continuous.
iv) $f_P = f_P^m - 0$:
Consider the following marginal value of the mapping that we take for each flow f where $f_P \to f_P^m - 0$:

$$\lim_{f: f_P \to f_P^m - 0}\left(\overbrace{f_P}^{\to f_P^m} + \min\left\{\overbrace{f_P^m - f_P}^{\to 0}, \overbrace{\frac{f_P^m - f_P}{\sum_{P_k\in\mathcal{P}_1}(f_{P_k}^m - f_{P_k})}}^{\geq 0}\cdot \overbrace{d}^{\geq 0}\right\}\right) = f_P^m \ .$$

Thus, $t(f_P)$ is continuous to the left at $f_P = f_P^m - 0$.

Hence, T is continuous.

Part c) ($\hat{f} = T(\hat{f}) \Rightarrow \hat{f}$ is a pure strategy equilibrium)
Since T is a continuous mapping of F into itself, a fixed point $\hat{f} = T(\hat{f})$ exists by Brouwer's fixed point theorem, and $\hat{f} \in F$.
Moreover, we can explicitly describe the form of such an fixed point. Indeed, as $\hat{f} = T(\hat{f})$ then $\hat{f}'_P = \hat{f}_P$ for each path $P \in \mathcal{P}$ which in turns implies that the bracket in (2), that we will denote by K_P, equals zero. Hence $K_P = 0$ for all $P \in \mathcal{P}$.
First note that $\hat{f}_P < \hat{f}_P^m$ can not occur since from $K_P = 0$ and $\hat{f}_P^m - \hat{f}_P > 0$ it follows that $d = 0$. Then, from (3) we get

$$0 = d = r - \sum_{P \in \mathcal{P}} \hat{f}_P \geq \hat{f}_P^m - \hat{f}_P \ .$$

This implies that $\hat{f}_P \geq \hat{f}_P^m$, which means by (2) and as $K_P = 0$ that $\hat{f}_P = \hat{f}_P^m \in f_P^{max}(\hat{f}_{-P})$.
In conclusion, the fixed point satisfies $\hat{f}_P \in f_P^{max}(\hat{f}_{-P}) \ \forall \ P \in \mathcal{P}$, and hence is an equilibrium in pure strategies according to Corollary 2. $\square$

The mapping T can be interpreted as a simple auction where the players bid the flow they want to route along their paths. In particular, each player asks to receive the flow f_P^m. Then, each player receives a flow f'_P which depends on all bids and on the amount of flow that can be distributed without exceeding the flow rate r. If the current flow of a player P is greater than or equal to f_P^m, then he is given exactly $f'_P = f_P^m$, as reducing flow will not violate the flow rate. If $f_P < f_P^m$ holds, i.e. P will ask for a larger flow, we have to distinguish two cases: For the first case, $\sum_{P_k \in \mathcal{P}_1} (f_{P_k}^m - f_{P_k}) > d$ holds, that means the players want to increase their flow, but ask for more flow than available. Then, the flow rate would be violated if each player received his bid. Thus, each player receives a fraction of d proportional to his bid and smaller than his bid. In the second case, the sum of the players' bids is not exceeding r: $\sum_{P_k \in \mathcal{P}_1} (f_{P_k}^m - f_{P_k}) \leq d$. Hence, each player will receive exactly his bid. Note that in a path player game infeasible equilibria may occur. They are fully characterized in the next lemma.

Lemma 2 *In a path player game a flow f is an infeasible equilibrium if and only if for all paths P in $\mathcal{P}$ the following is satisfied:*

$$\sum_{P \in \mathcal{P}} f_P \geq r + \max_{P \in \mathcal{P}} f_P \ .$$

Proof ($\sum_{P \in \mathcal{P}} f_P \geq r + \max_{P \in \mathcal{P}} f_P \ \Rightarrow \ f$ infeasible equilibrium)
Consider a flow f such that $\sum_{P \in \mathcal{P}} f_P \geq r + \max_{P \in \mathcal{P}} f_P$ holds. This flow

is infeasible as $\max_{P\in\mathcal{P}} f_P > 0$ holds. In addition for all paths P in $\mathcal{P}$ the following is true:

$$d_P(f_{-P}) = r - \sum_{P_k\in\mathcal{P}\setminus\{P\}} f_{P_k} = r - \left(\sum_{P_k\in\mathcal{P}} f_{P_k} - f_P\right) \le r - \overbrace{\left(\sum_{P_k\in\mathcal{P}} f_{P_k} - \max_{P\in\mathcal{P}} f_P\right)}^{\ge r} \le 0$$

$$\Rightarrow \quad b_P(f_{-P}, f_P) = -M \;\forall\; f_P \quad \Rightarrow \quad f_P^{max}(f_{-P}) = [0,\infty) \;.$$

Therefore, we conclude that $f_P \in f_P^{max}(f_{-P}) \;\forall\; P \in \mathcal{P}$ and thus, as a result of Corollary 2, f is an equilibrium.

(f infeasible equilibrium $\Rightarrow\; \sum_{P\in\mathcal{P}} f_P \ge r + \max_{P\in\mathcal{P}} f_P$)
Consider a flow f such that $\sum_{P\in\mathcal{P}} f_P > r$ and $f_P \in f_P^{max}(f_{-P}) \;\forall\; P \in \mathcal{P}$, i.e. f is an infeasible equilibrium. Assume that the claim is not true, i.e. $\sum_{P\in\mathcal{P}} f_P < r + \max_{P\in\mathcal{P}} f_P$. Let $\bar{P}$ be such that $\max_{P\in\mathcal{P}} f_P = f_{\bar{P}}$. Then,

$$d_{\bar{P}}(f_{-\bar{P}}) = r - \sum_{P\in\mathcal{P}\setminus\{\bar{P}\}} f_P = r - \left(\sum_{P\in\mathcal{P}} f_P - f_{\bar{P}}\right) = r - \overbrace{\left(\sum_{P\in\mathcal{P}} f_P - \max_{P\in\mathcal{P}} f_P\right)}^{<r}$$

$$\Rightarrow \quad d_{\bar{P}}(f_{-\bar{P}}) > 0 \quad \Rightarrow \quad \exists\, f'_{\bar{P}} : b_{\bar{P}}(f_{-\bar{P}}, f'_{\bar{P}}) > -M \quad \Rightarrow \quad f_{\bar{P}} \notin f_{\bar{P}}^{max}(f_{-\bar{P}}) \;,$$

which contradicts the assumption and thus the claim follows. □

As a result, infinitely many infeasible equilibria exist in path player games.

4 Properties of Path Player Games

In this section we describe properties of path player games that will be needed for the characterization of equilibria.

Path-disjoint Network A set of paths $\bar{\mathcal{P}}$ is called *disjoint* if for all pairs $P_1, P_2 \in \mathcal{P}$ with $P_1 \ne P_2$ it holds that $P_1 \cap P_2 = \emptyset$. We call a network *path-disjoint* if the set $\mathcal{P}$ of all paths from s to t is disjoint. In a path disjoint network, $c_P(f)$ only depends on f_P and is independent from f_{-P}. In the literature, cost functions c_P with $c_P(f) = c_P(\,\cdot\,, f_P)$ are also known as *separable functions* (e.g. see [12]).

Trivial Games We will call a game with flow rate r and security limits ω_P *trivial*, if $\sum_{P\in\mathcal{P}} \omega_P > r$ holds, and *nontrivial* otherwise. In trivial games, it is possible that the entire flow rate r is used, even if all players route $f_P < \omega_P$ for all $P \in \mathcal{P}$, which cannot happen in nontrivial games.

Lemma 3 *Let f be a feasible flow in a nontrivial path player game. Then there exists at least a $P \in \mathcal{P}$ such that $d_P(f_{-P}) \ge \omega_P$.*

Proof Consider a nontrivial path player game, i.e. $\sum_{P\in\mathcal{P}} \omega_P \leq r$ and a feasible flow f. It holds for all $P \in \mathcal{P}$ that

$$\sum_{P\in\mathcal{P}} d_P(f_{-P}) = |\mathcal{P}| \cdot r - (|\mathcal{P}|-1) \cdot \sum_{P\in\mathcal{P}} f_P$$

$$= r + (|\mathcal{P}|-1) \cdot \overbrace{\left(r - \sum_{P\in\mathcal{P}} f_P\right)}^{\geq 0} \ .$$

$$\Rightarrow \quad \sum_{P\in\mathcal{P}} d_P(f_{-P}) \geq r \geq \sum_{P\in\mathcal{P}} \omega_P \quad \Rightarrow \quad \exists\, P \in \mathcal{P} : d_P(f_{-P}) \geq \omega_P \ .$$

□

Non-Compensative Security Property A path player game is called a game with *non-compensative security (NCS) property* if for all paths $P \in \mathcal{P}$ and for all flows f_{-P} with $d_P(f_{-P}) \geq \omega_P$ there exists a flow $f_P \geq \omega_P$ such that

$$b_P(f_{-P}, f_P) > \kappa_P.$$

In games with NCS property, no player P will choose the security payment κ_P when a flow $f_P \geq \omega_P$ is possible. If a player has the possibility to earn benefit by receiving income by his "productivity", he has no reason to take advantage of the security limit. The security payment shall only be used if the player has no other choice due to the strategies of his competitors, i.e. if $d_P(f_{-P}) < \omega_P$. The NCS property is an interesting attribute of games as it will enable the characterization of equilibria for strictly increasing costs (see Section 5). Note that as we assume nonnegative costs, a game where $\kappa_P < 0$ holds for all P in $\mathcal{P}$ has NCS property. In all other cases, it is not so easy to recognize if a game has NCS property. However, in some cases, the NCS property of a game follows from the following property of the benefit function. A benefit function $b_P(f)$ with $\omega_P < r$ has the *non-compensative security (NCS) property* if

$$\kappa_P < c_P(0, \ldots, 0, \omega_P, 0, \ldots, 0) =: c_P(\mathbf{0}_{-P}, \omega_P) \tag{4}$$

holds. If κ_P is sufficiently small, a player on an underloaded path gets a benefit which is lower than the income he would get if he were able to route a flow of value ω_P along that path, while no other player routes anything. The idea is that no player should have an incentive to choose his path to be underloaded if he is able to route a flow $f_P \geq \omega_P$.

To illustrate benefit functions with NCS property, consider a benefit function $b_P(f)$, where all players apart from P are routing a zero-flow, i.e. $b_P(f) = b_P(\mathbf{0}_{-P}, f_P)$. A function $b_P(f)$ as shown in Figure 3 does not have NCS property since player P will choose the security payment instead of the income obtained by routing ω_P. In general, that does not mean, that the player always prefers the benefit κ_P. It may happen (like in this illustration)

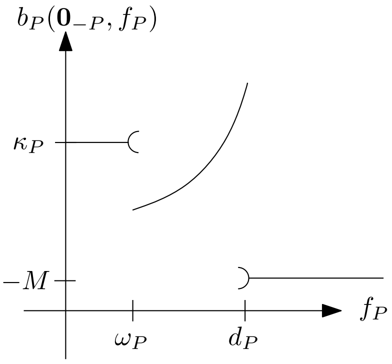

Fig. 3 No NCS property

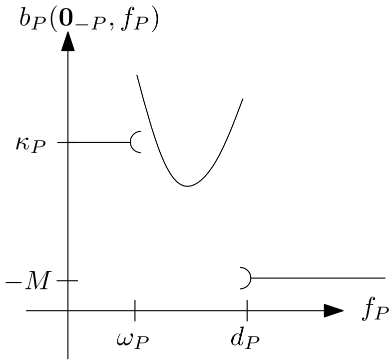

Fig. 4 NCS property

that there is a flow $f_P > \omega_P$ with $b_P(\mathbf{0}_{-P}, f_P) > \kappa_P$. However, for a benefit function without NCS property, we can not guarantee that there will be a flow f_P, that provides a higher benefit than κ_P. On the contrary, a benefit function as the one shown in Figure 4 allows the player to obtain a benefit higher than κ_P when routing $f_P = \omega_P$.

Let us now consider the relation between games with NCS property and benefit functions with NCS property. Unfortunately, a game that possesses benefit functions with NCS property is not necessarily a game with NCS property. Consider a path P with $d_P(f_{-P}) \geq \omega_P$, whose benefit functions possess NCS property. It does not necessarily hold that P is in any case able to obtain a benefit greater than κ_P. In general networks, players may share edges. It is possible that on an edge e with decreasing benefit some of the players sharing e have incentive to raise the flow f_e even if edge e induces a loss (if they can compensate that loss by gains on other edges). Consequently, $b_P(f_{-P}, f_P) \leq \kappa_P \ \forall\ f_P \geq \omega_P$ could hold, i.e. the game would possess no NCS property. We call this effect of influencing the benefit of the competitors *edge sharing effect*.

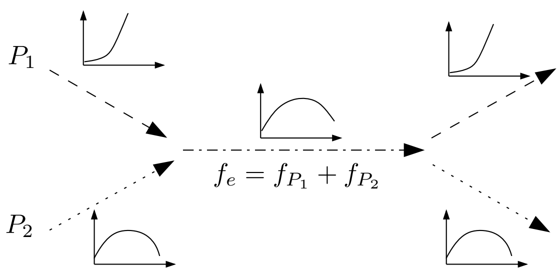

Fig. 5 Edge sharing effect

For instance see Figure 5, where P_1 would accept a decreasing income from edge e, as this loss is compensated by the remaining edges. At the same time, P_2 does not want to increase f_e too much, as at a certain point his benefit $b_P(f)$ will decrease. Nevertheless, P_2 can not avoid that P_1 increases the flow, i.e. he is forced into a situation where sending flow can create loss. Note that in this situation the name "security payment" is justified for player P_2.

As the edge sharing effect may destroy the NCS property of games, we investigate additional assumptions which prevent the edge sharing effect. The following proposition describes two situations where benefit functions with NCS property induce games with NCS property and one condition that requires at least one exclusively used edge in each path, to obtain a game with NCS property.

Proposition 1

a) A path player game with benefit functions $b_P(f)$ satisfying NCS property for all $P \in \mathcal{P}$ is a game with NCS property if (i) or (ii) does hold:
(i) For all $e \in E$: $c_e(f_e)$ are monotonically increasing ,
(ii) the network G is path-disjoint.

b) Furthermore, a path player game where each path P satisfies that

$$\bar{E}_P = \{e : e \in P \ \wedge \ e \notin P_k \ \forall \ P_k \neq P\} \ \neq \ \emptyset \quad \forall \ P \in \mathcal{P} \ ,$$

possesses the NCS property if

$$\sum_{e \in \bar{E}_P} c_e(\omega_P) > \kappa_P \ \forall \ P \in \mathcal{P} \ .$$

Proof Consider a path $P \in \mathcal{P}$ and a flow f_{-P} with $d_P(f_{-P}) \geq \omega_P$.

a) To prove (i) and (ii), we need to show for both cases that there is a flow $f_P \geq \omega_P$ such that $b_P(f_{-P}, f_P) > \kappa_P$. It can be shown that for (i) all $f_P \in [\omega_P, d_P(f_{-P})]$ fulfill this condition, while for (ii) we have to set $f_P = \omega_P$.

b) Set $f_P = \omega_P$, therefore the resulting flow is feasible. Then, we obtain

$$b_P(f_{-P}, f_P) = c_P(f_{-P}, \omega_P) = \overbrace{\sum_{e \in \bar{E}_P} c_e(\omega_P)}^{>\kappa_P} + \sum_{e \in P \setminus \{\bar{E}_P\}} \overbrace{c_e(f_e)}^{\geq 0} > \kappa_P \ ,$$

and thus the proposition follows. □

5 Equilibria for Strictly Increasing Cost Functions

In this section, we present characterizations of equilibria under the assumption of strictly increasing cost functions. We will obtain a necessary condition for equilibria and a necessary and sufficient condition if the game has NCS property or if we consider a game with no security limit. The next proposition will be useful for the proofs in this section.

Proposition 2 *Consider a path player game with strictly increasing cost functions* $c_e(f_e)$*. Then, for all* $P \in \mathcal{P}$*, for fixed* f_{-P} *and for* $f_P \in [\omega_P, d_P(f_{-P})]$ *the benefit functions* $b_P(f_{-P}, f_P)$ *are strictly increasing in* f_P*.*

The proof of Proposition 2 is based on the fact that the sum of strictly increasing functions is a strictly increasing function. This can be easily verified such that details are omitted here. Our first result is the following.

Theorem 2 *Consider a game with strictly increasing cost functions* $c_e(f_e)$ *on all edges* $e \in E$*. Assume, that the game is nontrivial and it satisfies NCS property. Then a flow* f *is a feasible equilibrium if and only if* $\sum_{P \in \mathcal{P}} f_P = r$*.*

Proof (f *feasible equilibrium* $\Rightarrow$ $\sum_{P\in\mathcal{P}} f_P = r$)
Consider a feasible equilibrium f and assume that $\sum_{P\in\mathcal{P}} f_P < r$, i.e. $f_P < d_P(f_{-P})$ for all $P \in \mathcal{P}$. Due to non-triviality we can find a path $\bar{P}$ such that $d_{\bar{P}}(f_{-\bar{P}}) \geq \omega_{\bar{P}}$ (see Lemma 3). We distinguish two cases:

- Case 1: $f_{\bar{P}} \geq \omega_{\bar{P}} \Rightarrow b_{\bar{P}}(f_{-\bar{P}}, d_{\bar{P}}(f_{-\bar{P}})) > b_{\bar{P}}(f_{-\bar{P}}, f_{\bar{P}})$ (due to Proposition 2), which contradicts f being a feasible equilibrium.
- Case 2: $f_{\bar{P}} < \omega_{\bar{P}} \Rightarrow \exists\, \hat{f}_{\bar{P}} \geq \omega_{\bar{P}}$ such that $b_{\bar{P}}(f_{-\bar{P}}, \hat{f}_{\bar{P}}) > \kappa_{\bar{P}} = b_{\bar{P}}(f_{-\bar{P}}, f_{\bar{P}})$ (due to NCS property), which contradicts f being a feasible equilibrium.

The above implies that $\sum_{P\in\mathcal{P}} f_P = r$.
($\sum_{P\in\mathcal{P}} f_P = r \Rightarrow$ *f feasible equilibrium*)
Consider a flow with $\sum_{P\in\mathcal{P}} f_P = r$, i.e. $f_P = d_P(f_{-P})$ for all $P \in \mathcal{P}$. We analyze the two cases:

- Case 1: $f_P \geq \omega_P$: As there exists at least one $\hat{f}_P \geq \omega_P$ such that $b_P(f_{-P}, \hat{f}_P) > \kappa_P$ (due to NCS property), and as $b_P(f_{-P}, f_P)$ is strictly increasing for $f_P \in [\omega_P, d_P(f_{-P})]$ (see Prop.2), and in particular, $b_P(f_{-P}, f_P) \geq b_P(f_{-P}, \hat{f}_P) > \kappa_P$ it holds that $f_P^{max}(f_{-P}) = \{d_P(f_{-P})\}$.
- Case 2: $f_P < \omega_P$: As $b_P(f_{-P}, f_P)$ is constant over $[0, \omega_P)$ and $d_P(f_{-P}) < \omega_P$, it holds that $d_P(f_{-P}) \in f_P^{max}(f_{-P})$.

Using Corollary 2, we conclude that f is a feasible equilibrium as $f_P \in f_P^{max}(f_{-P})\ \forall\ P \in \mathcal{P}$. □

Consider a game with strictly increasing costs and no security limit, that means $\omega_P = 0$ holds for all $P \in \mathcal{P}$. Such a game is nontrivial as $\sum_{P\in\mathcal{P}} \omega_P = 0$. Furthermore, as κ_P will be never obtained as benefit, we can choose for instance $\kappa_P = -1\ \forall\ P \in \mathcal{P}$ and transform it into a game that satisfies NCS property. Hence, we obtain the following Corollary from Theorem 2.

Corollary 3 *In a path player game with strictly increasing cost functions* $c_e(f_e)$ *on all edges* $e \in E$ *and security limit* $\omega_P = 0$ *for all* $P \in \mathcal{P}$*, a flow* f *is a feasible equilibrium if and only if* $\sum_{P\in\mathcal{P}} f_P = r$ *.*

Unfortunately, the converse of Theorem 2 does not hold: A game that satisfies the property

$$\text{“A flow } f \text{ is a feasible equilibrium if and only if } \textstyle\sum_{P\in\mathcal{P}} f_P = r\text{”} \tag{5}$$

needs not be nontrivial nor satisfy the NCS property. For an illustration we present the following examples.

Example 2 (5) $\nRightarrow$ NCS property.
Consider a game on a network with two paths, as illustrated in Figure 6. A flow rate $r = 1$ has to be routed from s to t. On both paths the costs are $c_P(x) = x$, and we set $\omega_1 = \kappa_1 = 1$ and $\omega_2 = \kappa_2 = 0$. In this game, a flow f with $\sum_{P\in\mathcal{P}} f_P < r$ is not an equilibrium as $f_2^{max}(f_{-2}) = \{d_2(f_{-2})\}$ for all f_{-2}, i.e. player 2 would in any case use up the remaining flow rate. On the other hand, each flow f with $\sum_{P\in\mathcal{P}} f_P = r$ is an equilibrium flow. If $\sum_{P\in\mathcal{P}} f_P = r$ holds, player 2 can not find any better strategy as he will always try to get as much flow as possible, while player 1 is also not able to improve his payoff as his benefit function is anyway constant over $[0, 1]$. That means, this game fulfills condition (5). Nevertheless, the game has not NCS property. There is no $f_1 \geq \omega_1$ with $b_1(f_1, f_2) > \kappa_1$ and so path 1 is destroying the NCS property of the game.

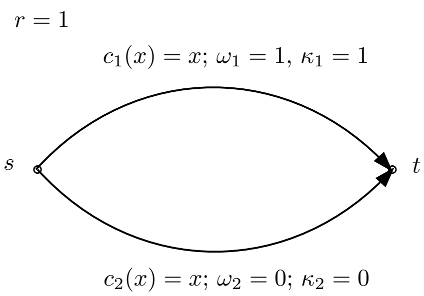

Fig. 6 Game graph of Example 2

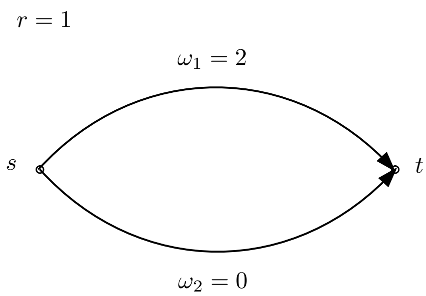

Fig. 7 Game graph of Example 3

Example 3 (5) $\nRightarrow$ non-triviality.
Consider the game illustrated in Figure 7. The graph consists of two paths, and we choose $\omega_1 = 2$ and $\omega_2 = 0$. The remaining components of the game, as cost functions and security payments may be chosen arbitrarily, but it is important that the cost functions are strictly increasing. With a similar argument as in Example 2, it is possible to show that this game fulfills (5). Nevertheless, the game is trivial, as $\sum_{P\in\mathcal{P}} \omega_P > r$.

If a game has strictly increasing cost functions and general security limit, but we can not ensure NCS property or the non-triviality of the game (and thus can not apply Theorem 2), we are still able to give a necessary condition for a profile of flows to be an equilibrium.

Lemma 4 *If a flow f in a path player game with strictly increasing cost functions $c_e(f_e)$ on all edges $e \in E$ is a feasible equilibrium then at least one of the following two cases holds:*

(i) $\sum_{P\in\mathcal{P}} f_P = r$,

(ii) $f_P < \omega_P \; \forall \; P \in \mathcal{P}$.

Proof Let f be a feasible equilibrium such that (i) and (ii) are both not true. Since (i) does not hold, $\sum_{P\in\mathcal{P}} f_P < r$. Then $f_P < d_P(f_{-P}) \; \forall \; P \in \mathcal{P}$. Since (ii) is also not true, $\exists \; \bar{P}$ with $f_{\bar{P}} \geq \omega_{\bar{P}}$. Then $b_{\bar{P}}(f_{-\bar{P}}, f'_{\bar{P}}) > b_{\bar{P}}(f_{-\bar{P}}, f_{\bar{P}}) \; \forall \; f'_{\bar{P}} \in (f_{\bar{P}}, d_{\bar{P}}(f_{-\bar{P}})]$, as according to Proposition 2, $b_P(f_{-P}, f_P)$ is strictly increasing over this domain. It follows that $f_{\bar{P}} \notin f^{max}_{\bar{P}}(f_{-\bar{P}})$. But this is a contradiction to f being an equilibrium. □

The following lemma provides a statement about the converse of Lemma 4.

Lemma 5 *Consider a path player game with strictly increasing cost functions* $c_e(f_e)$*. Let* f *be a flow with the following properties:*

(i) $\sum_{P\in\mathcal{P}} f_P = r$,
(ii) $f_P < \omega_P \; \forall \; P \in \mathcal{P}$.

Then, f *is a feasible equilibrium.*

For the proof it suffices to show that for all players P, increasing or decreasing f_P will not lead to an improvement of the benefit b_P. The following examples demonstrate that only one of the two conditions (i) and (ii) is not sufficient to guarantee a feasible equilibrium.

Example 4 ($(i) \wedge \neg(ii) \;\not\Rightarrow\; f$ *is feasible equilibrium)*
Consider a game consisting of two disjoint paths connecting s and t, i.e. $\mathcal{P} = \{1, 2\}$. Set $r = 1$, $\omega_1 = \omega_2 = 0.25$ and the security payment $\kappa_1 = \kappa_2 = 2$. With cost functions $c_P(x) = x$ for $P = 1, 2$, the flow $f = (0.5, 0.5)$ fulfills (i) but not (ii). This flow with $b_1(f) = b_2(f) = 0.5$ is not an equilibrium as $f^{max}_1(0.5) = f^{max}_2(0.5) = [0, 0.25)$.

Example 5 ($\neg(i) \wedge (ii) \;\not\Rightarrow\; f$ *is feasible equilibrium)*
Consider a game consisting of two disjoint paths, $\mathcal{P} = \{1, 2\}$. Set $r = 1$, $\omega_1 = \omega_2 = 0.5$ and $\kappa_1 = \kappa_2 = 0.1$. With cost functions $c_P(x) = x$ for $P = 1, 2$, a flow $f = (0.45, 0.45)$ with $b_1(f) = b_2(f) = 0.1$ is no equilibrium as $f^{max}_1(0.45) = f^{max}_2(0.45) = 0.55$.

We have seen that a feasible flow with property (ii) need not be an equilibrium. This does not change if we assume to have a trivial game or a game without NCS property. The following example illustrates the assertion.

Example 6 Consider again a game consisting of two disjoint paths connecting s and t, i.e. $\mathcal{P} = \{1, 2\}$. Set $r = 5$ and $\omega_P = 3$ for $P = 1, 2$, i.e. the game is trivial. Furthermore, choose $\kappa_P = 1$ for $P = 1, 2$ and $c_1(f_1) = f_1$, $c_2(f_2) = f_2/10$. This game does not satisfy NCS property as for all $f_1 \geq 0$ there is no $f_2 \in [\omega_2, d_2(f_1)]$ such that $b_2(f_1, f_2) > \kappa_2$. Consider the feasible flow $f = (0, 0)$. The flow f fulfills (ii) and $d_P(f_{-P}) = r$ for all P. Nevertheless, since $b_1(d_1(0), 0) = 5 > b_1(0, 0) = \kappa_1 = 1$ this flow is not an equilibrium.

We remark that in [28] further results for other types of cost functions have been derived. This includes a necessary condition for differentiable costs. It is also sufficient if the costs are differentiable and concave, and if the game has no security limit. Finally, for convex costs a dominating strategy set is determined.

6 Conclusion and Further Research

In this paper, we presented results for equilibria in a new network game, the path player game. We proved the existence of feasible equilibria in pure strategies. Furthermore, we presented a necessary condition for equilibria if the cost functions are strictly increasing. If the game furthermore satisfies the NCS property, we obtained even a necessary and sufficient condition.

Path player games have various aspects which are currently under research. In [23] the concept of path player games is extended to *games on polyhedra* allowing more general dependencies among strategy sets. Treating path player games as GNE games (instead of penalizing infeasible solutions) provides a simpler (continuous) payoff function. This is an advantage in the analysis of a potential function (see [22]), which turns out to exist for the path player game in both versions. Moreover, it turns out that path player games may have multiple equilibria. This motivates the analysis of non-dominated solutions in the sense of Pareto, see [27] for some first results on the relation between equilibria and Pareto solutions.

An extension for future research is to consider not paths but complete subgraphs as players. This reflects the fact that in real-world situations network providers usually own a subnetwork. Furthermore, some applications require integer solutions. Thus, the extension of the path player game to an integer version is of interest; for the line planning game it has already implemented in [28]. The results for the line planning game (see [26]) are promising and motivate further research in this field.

Moreover, repeated or stochastic versions of the game could be considered to refine the set of equilibria. Finally, it is open work to analyze the situation as an optimization problem, that means to look for minimal cost flows in the network, and to compare them with the equilibria of the game.

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