Cournot competition under uncertainty. Conservative and optimistic equilibria.

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Abstract

In this paper we analyze competition between firms with uncertain demand functions. A duopoly model is considered in which two identical firms producing homogeneous commodities compete in quantities. They face uncertain market demand in a context in which two different future scenarios are possible, and no information about the probability distribution of occurrence of the scenarios is available.

This decision-making situation is formalized as a normal-form game with vector-valued utility functions for which the notion of Pareto equilibrium is adopted as a natural extension of that of Cournot equilibrium. Under standard assumptions about the demand functions, we characterize the complete set of Pareto equilibria. In the second part of the paper, we analyse the equilibria to which the agents will arrive depending on their attitude to risk. We find that equilibria always exist if both agents are simultaneously pessimistic or optimistic. In the non-trivial cases, for pessimistic firms, infinitely many equilibria exist, whereas when firms act optimistically, only those pairs of strategies corresponding to the Cournot equilibria in each scenario can be equilibria.

Keywords: Pareto equilibria, Cournot games, Uncertainty, Attitude to risk.
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1 Introduction

Uncertainty plays an important role in oligopolistic models since it is difficult or impossible to assure that random events will not influence the outcomes of oligopolistic competition. For instance, firms often face major uncertainty about demand, competitors' costs, distribution of consumers' reservation prices, and other market features relevant in decision-making by the firms. Therefore, when managers make choices or decisions in these situations, they must somehow incorporate this uncertainty into their decision-making process.

This paper examines the effect of introducing uncertainty about demand in duopoly models into a non-probabilistic framework. To this end, it is assumed that there are several possible scenarios or states of nature, of which only one will be realized as the true state, and the probabilities of occurrence of the different scenarios are unknown. Specifically, we analyse the extension of a Cournot duopoly (Cournot, 1838) in which two firms producing homogeneous products face a different market demand in each one of two scenarios and they have to decide their strategies on quantities before uncertainty is resolved.

Situations where firms face uncertain demand which depends on the final scenario often occur in areas affected by political decisions that modify the initial demand level. For instance, the demand in the solar energy market is highly influenced by the presence or absence of financial support from the Government to consumers for the installation of equipment. In the case of Government support, the demand changes because new consumers with a different pattern of consumption enter the market. Thus, companies involved in the market may face uncertain demand because there are two possible future scenarios: with and without support. However, the firms' strategic decisions about the level of supply they will offer often must be adopted before the uncertainty about support is resolved.

Several oligopoly models under uncertainty have previously been addressed in the literature. Lagerlöf (2007) and Grimm (2008) analyzed Cournot competition under demand uncertainty for risk-neutral firms. They showed that multiple equilibria may exist and provided a plausible setting in which uniqueness of the Cournot equilibrium under demand uncertainty is guaranteed. The effects of introducing attitudes to risk on oligopoly competition have been considered by Asplund (2002), who analyzed competition in prices and quantities between risk-averse firms. Fontini (2005) studied the impact that optimistic and pessimistic attitudes play in a Cournot oligopoly when each firm is uncertain about whether the other firms will act as

Cournot competitors.

In the aforementioned papers, uncertainty is introduced into the model through randomness in the corresponding function and expected utility theory is applied to make decisions. As a consequence, the results heavily depend on the probability distributions considered. However, a range of situations exists for which no information about the probability distribution of the random variable is available, or in which inaccuracy of the information about the probability distribution or about the distribution parameters may yield unrealistic predictions. In such situations, subjective expected theory (Savage, 1954), in which the decision-makers have beliefs in the forms of probability distributions, is frequently applied. It has been argued that a single probability distribution is insufficient to describe the decision-maker's beliefs in some situations of uncertainty. An alternative direction, in which uncertainty aversion is taken into account is maxmin expected utility (Gilboa and Schmeidler, 1989), where the decision-maker considers a range of possible probabilities, and preferences are represented by the minimum of all the expected utilities.

The case of complete absence of information about probability distributions can be considered as an extreme case in the setting of Gilboa and Schmeidler (1989), which leads to the two well-known maxmin and maxmax criteria for choice under uncertainty. The present research is developed in this decisional context.

The analysis presented in the paper contributes to the existing literature in several ways. First, we assume that the profits of the firms depend on the future scenarios, and no information exists on the probability of occurrence of each scenario. As a consequence, unlike previous literature, the results about the equilibria in the extended Cournot model do not rely on the assumptions or on the beliefs about probability distributions, and our analysis covers situations where information is not available. Second, firms are uncertain both about the reservation price and about the number of consumers in the market. This way of modeling uncertainty allows us to go beyond the cases discussed in the literature, where uncertainty is considered only for one of the two features mentioned. And third, once the set of equilibria for the Cournot duopoly with no assumption about the firms' attitudes towards risk is obtained, we analyse strategic competition for risk-averse firms and also for firms that show preference for risk. In this decisional context, in which firms have to decide their strategies on quantities before uncertainty is resolved, we present an *ex-ante* analysis which permits the identification of the equilibria to which risk-averse firms and risk-preference firms will eventually arrive.

To begin our analysis, we establish a general framework for the study of duopolistic competitions under uncertainty, where, as is usual in oligopoly models, for reasons of analytical tractability, we assume the demand functions are linear. As Vives (1984) pointed out, such an assumption presents the unappealing feature that negative prices and outputs can be obtained. However, the non-negativity of outputs is included in our analysis, and even though negative prices are possible, we prove that the prices in any non-degenerate equilibrium are positive.

The Cournot duopoly under uncertainty is formalized as a game with vector-valued utility functions. The natural extension of the concept of Nash equilibrium was introduced by Shapley (1959) for the class of multicriteria matrix games with the name of Pareto equilibrium. Bade (2005) studied the existence of Pareto equilibria in games with vector payoffs in which agents' preferences are incomplete in different economic models. She precedes us in showing the applicability in oligopoly situations of the theoretical results which characterize equilibria in the framework of vector-valued utilities.

Our first result is the characterization of the set of Pareto equilibria of the Cournot game, with and without non-negative conditions on demand realizations. This set is symmetric and only depends on the quantities of perfect competition in the two scenarios, thus generalizing the classic Cournot equilibrium. On the negative side, the set of Pareto equilibria contains an infinity of demand pairs which renders this concept insufficient for conclusions to be drawn about the equilibria at which agents will arrive in real-world duopolistic markets.

However, this general setting permits the analysis of a range of situations in which firms exhibit different attitudes towards risk. In the second part of the paper, specific attitudes towards risk are introduced into the model. When agents are pessimistic, exhibiting extreme risk aversion, the maxmin principle is applied. In this context, we show that in non-trivial cases multiple equilibria exist. An important fact is that the total quantity offered in all these equilibria is a constant, and all equilibria yield the same price. That is, the price at equilibria is unique.

On the other hand, for optimistic agents, who only value the best results they can obtain, a criterion consisting of maximizing the maximum benefit is applied. In this case, we also prove that equilibria always exist. Moreover, we show that any game has either a unique optimistic equilibrium, which coincides with the Cournot equilibrium of one of the scenarios, or that both Cournot equilibria of the scenarios are optimistic equilibria.

The relevance of the results we present herein for conservative and optimistic equilibria lies in the understanding of the effects of uncertainty in the final equilibrium outcome. If firms are conservative, the uncertainty with respect to both the total quantity to be offered and the price that reaches such a quantity disappears. That is, regardless of the state of nature that finally occurs, all conservative equilibria provide the same output and price level. However, if firms show preference for risk, the price depends on the state of nature that will take place, and since different equilibria may exist, the quantity that the firms will eventually produce at equilibrium can not be predicted.

The rest of the paper is organized as follows. In Section 2 the Cournot model under uncertainty is presented and the corresponding set of Pareto equilibria is characterized. In Section 3, the attitude towards risk of the agents is introduced into the analysis. We obtain results which permit the identification of equilibria when agents show extreme risk aversion and when the agents are optimistic. Section 4 is devoted to the conclusions of our research. In order to ease the presentation, proofs are included in an Appendix.

2 Pareto Equilibria in the Cournot model under demand uncertainty

In this section a Cournot model with an uncertain linear demand function is analysed. In previous literature uncertainty in the demand function is usually modelled either as an uncertain intercept, which is applied in cases where firms are uncertain about the reservation price, or as an uncertain slope, which can represent situations in which firms are uncertain about the number of consumers in the market.

We address a general model where uncertainty affects both the intercept and the slope which allows us to represent situations of uncertainty in terms of type and number of consumers. For example, firms trying to introduce a new product in the market are unsure whether they are going to be successful in a small market with consumers of high income with a high reservation price or in a bigger market with consumers of low income with a lower reservation price.

We consider a duopoly model in which two identical firms producing homogeneous commodities compete in quantities and face uncertain market demand since two different future scenarios are possible.

The inverse demand function at scenario k, $k = 1, 2$, is given by $p = \alpha_k - \gamma_k q$, with α_k, $\gamma_k > 0$. For simplicity, it is assumed that firms have no fixed costs and their marginal costs are equal to zero. In our setting, firms make their output decision,

q^1, q^2, before the uncertainty is resolved. For $i = 1, 2$, the benefit for firm i at scenario k is

$$\Pi^i_k(q^1, q^2) = q^i(\alpha_k - \gamma_k(q^1 + q^2)).$$

Without loss of generality, throughout the paper it is assumed that $\frac{\alpha_1}{\gamma_1} < \frac{\alpha_2}{\gamma_2}$, that is, the quantity of perfect competition in the first scenario is lower than that of the second scenario.

If the two scenarios are considered separately, then a Cournot equilibrium exists in each, given by

$$(q^{1*}_k, q^{2*}_k) = (\frac{\alpha_k}{3\gamma_k}, \frac{\alpha_k}{3\gamma_k}).$$

The situations we are going to investigate are those in which no information exists about the probability of occurrence of the scenarios or the existing information is not held by the agents. In such cases, the game the firms face is formally a normal-form game with vector-valued utility function $G = \{(A^i, u^i)_{i=1,2}\}$, where A^i is the set of strategies that agent i can adopt and u^i is his vector-valued utility function,

$$u^i := (\Pi^i_1(q^1, q^2), \Pi^i_2(q^1, q^2)).$$

We refer to these games as Cournot games under uncertainty, and denote the Cournot game under uncertainty in which firms are allowed to select any non-negative quantity as $G^{UC}_+ = \{(\mathbb{R}_+, \Pi^i)_{i=1,2}\}$.

We will adopt the term Pareto Equilibrium (PE) to refer to the natural extension of the concept of Nash equilibrium for these games with vector-valued utilities.

Definition 2.1. (q^{*1}, q^{*2}) is a Pareto Equilibrium for the game $G = \{(A_i, u^i)_{i=1,2}\}$ if $\nexists\ q^1 \in A_1$ such that $u^1_k(q^1, q^{*2}) \geq u^1_k(q^{*1}, q^{*2})$ for $k = 1, 2$ (with a strict inequality for some k), and $\nexists\ q^2 \in A_2$ such that $u^2_k(q^{*1}, q^2) \geq u^2_k(q^{*1}, q^{*2})$ for $k = 1, 2$ (with a strict inequality for some k).

The set of Pareto Equilibria for $G = \{(A_i, u^i)_{i=1,2}\}$ is denoted as $PE(G)$.

Note that in our context of complete uncertainty about the occurrence of the scenarios, a Pareto equilibrium consists of a pair of strategies of the agents such that neither firm can raise its benefit in both scenarios by deviating from the equilibrium strategy.

Our focus is on the game G^{UC}_+ in which $A_i = \mathbb{R}_+$. However, as a previous step we analyse the relaxed Cournot game $G^{UC} = \{(\mathbb{R}, \Pi^i)_{i=1,2}\}$ in which no non-negativity constraint is imposed.

It is worth noting that at each scenario, each firm's benefit function, $\Pi_k^i(q^1, q^2) = q^i(\alpha_k - \gamma_k(q^1 + q^2))$, is strictly concave in its own action. As a consequence, given the action of one of the agents, the benefit of the other attains its maximum where its derivative is null. For $i, j = 1, 2$ with $i \neq j$, denote $r_k^i : A_j \to \mathbb{R}$ as the function which represents the best response of agent i to the actions of agent j at scenario k,

$$r_k^i(q^j) = \frac{\alpha_k - \gamma_k q^j}{2\gamma_k}.$$

The following result characterizes the whole set of Pareto Equilibria of the Cournot game under uncertainty, G^{UC}. It establishes that the set of equilibria is bounded by the graphs of the best responses of each agent to the others' action at each scenario. This result can be extended to more general benefit functions provided that they fulfill adequate concavity requirements.

Theorem 2.2. *The set of Pareto Equilibria for the game G^{UC} is*

$$PE(G^{UC}) = \{(q^1, q^2) \,|\, r_1^1(q^2) \leq q^1 \leq r_2^1(q^2),\, r_1^2(q^1) \leq q^2 \leq r_2^2(q^1)\}.$$

The linearity of the response functions enables the set of equilibria, G^{UC}, to be represented as the convex hull[1] of its extreme points. These points correspond to the equilibria of certain associated games. For $k, l \in \{1, 2\}$, the *component game* $G_{k,l}^C$ is defined as a game with complete preferences consisting of $G_{k,l}^C = \{(A_1, \Pi_k^1), (A_2, \Pi_l^2)\}$. The equilibrium of game $G_{k,l}^C$ is obtained as the best mutual response, that is, the point solving $q^1 = r_k^1(q^2)$ and $q^2 = r_l^2(q^1)$:

$$(q^{1*}, q^{2*}) = (\frac{2}{3}\frac{\alpha_k}{\gamma_k} - \frac{1}{3}\frac{\alpha_l}{\gamma_l}, \frac{2}{3}\frac{\alpha_l}{\gamma_l} - \frac{1}{3}\frac{\alpha_k}{\gamma_k}).$$

Denote as $C_k^* = \dfrac{\alpha_k}{3\gamma_k}$ the Cournot equilibrium quantity at scenario k. With this notation, the equilibrium for game $G_{k,l}^C$ becomes

$$(2C_k^* - C_l^*, 2C_l^* - C_k^*).$$

The following result establishes that the set of Pareto equilibria coincides with the convex hull of the equilibrium points obtained when each of the firms considers each one of the possible scenarios. This fact is a consequence of the linearity of the response functions, and does not extend to the case of general benefit functions.

[1]The convex hull of $S \subseteq \mathbb{R}^2$, is $conv(S) = \{z \in \mathbb{R}^2 : z = \alpha x + (1 - \alpha)y,\, x, y \in S,\, \alpha \in [0, 1]\,\}$.

Corollary 2.3. *The set of Pareto Equilibria for the game G^{UC} is*

$$PE(G^{UC}) = conv\{(C_1^*, C_1^*), (C_2^*, C_2^*), (2C_1^* - C_2^*, 2C_2^* - C_1^*), (2C_2^* - C_1^*, 2C_1^* - C_2^*)\}.$$

In the game we want to investigate, firms can only select non-negative quantities. We prove that the set of Pareto equilibria of the game G_+^{UC} coincides with those Pareto equilibria of G^{UC} with nonnegative components. Formally,

Proposition 2.4. *The set of Pareto Equilibria for the game G_+^{UC} is*

$$PE(G_+^{UC}) = \{(q^1, q^2) \in PE(G^{UC}) : q^1, q^2 \geq 0\}.$$

In the following result, part a) is a necessary and sufficient condition for the coincidence of the set of equilibria of the game with non-negative strategies and that of the relaxed game. Part b) describes the equilibria of G_+^{UC} when this condition does not hold. Recall that $\frac{\alpha_1}{\gamma_1} < \frac{\alpha_2}{\gamma_2}$.

Corollary 2.5. *a) $PE(G_+^{UC}) = PE(G^{UC})$ if and only if $\frac{\alpha_2}{2\gamma_2} \leq \frac{\alpha_1}{\gamma_1}$.*

b) Otherwise,

$$PE(G_+^{UC}) = conv\{(\tfrac{\alpha_1}{3\gamma_1}, \tfrac{\alpha_1}{3\gamma_1}), (\tfrac{\alpha_2}{3\gamma_2}, \tfrac{\alpha_2}{3\gamma_2}), (\tfrac{\alpha_1}{\gamma_1}, 0), (\tfrac{\alpha_2}{2\gamma_2}, 0), (0, \tfrac{\alpha_1}{\gamma_1}), (0, \tfrac{\alpha_2}{2\gamma_2})\}.$$

Note that the condition in case a) means that the quantity of perfect competition in the first scenario has to be at least as much as the monopolistic quantity in the second scenario.

Figure 1 illustrates the sets of Pareto equilibria in the two different cases.

It is interesting to point out that, as for standard Cournot duopoly, the set of Pareto Equilibria in Cournot games under uncertainty only depends on the perfect competition quantity of each scenario.

Another consequence of Proposition 2.4 refers to the extension of the classic result on the symmetry of standard Nash equilibria. For these Cournot games under uncertainty where firms face identical benefit functions, Pareto equilibria are not necessarily symmetric. However, the set of Pareto equilibria is a symmetric set[2]. Formally,

Corollary 2.6. *The set $PE(G_+^{UC})$ is symmetric.*

The symmetry of the set of equilibria means that for each non-symmetric equilibrium pair, another equilibrium exists in which each firm offers the quantity offered by the opponent in the first equilibrium.

[2] A set $S \subseteq \mathbb{R}^2$ is symmetric if for all $(x_1, x_2) \in S$ then $(x_2, x_1) \in S$.

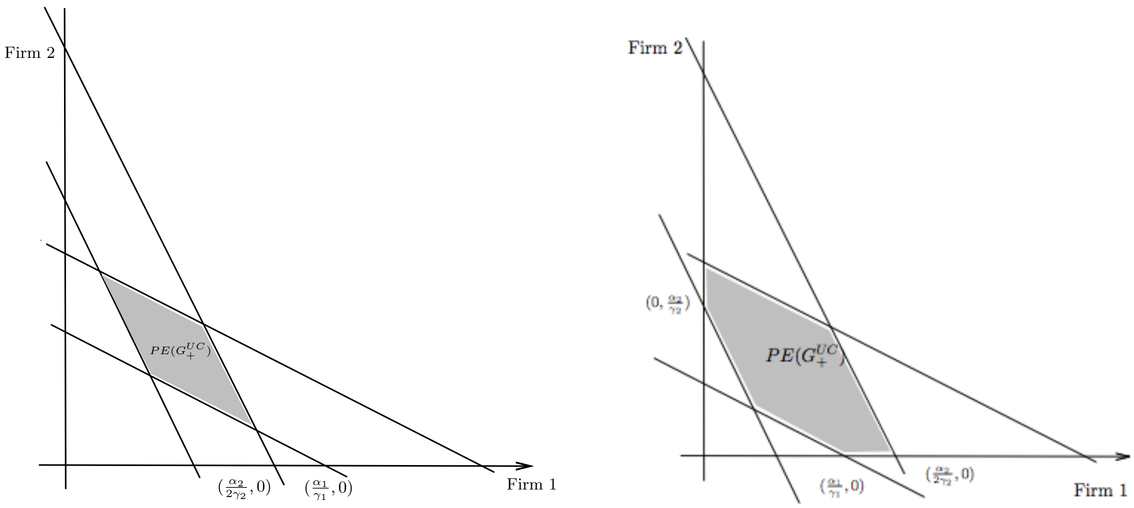

Figure 1: Sets of Pareto equilibria.

3 Conservative and optimistic equilibria

In the general model considered in the section above, firms are assumed to lack any information about the occurrence of the scenarios, nor is any assumption made on the firms' attitude towards risk. However, firms can show different attitudes to risk, from extreme risk-aversion to extreme preference for risk, due to several reasons, such as the presence of liquidity constraints or costly financial distress.

The rest of the paper is devoted to the identification of the equilibria to which the agents will arrive when they exhibit different attitudes towards risk. In our model such attitudes are explained by the importance that each firm gives to the realization of profits: a firm is risk-averse if it gives relatively greater importance to the scenario with low profits, the reverse is true for firms with preference for risk. More precisely, we consider the extreme cases, i.e., firms present extreme risk-aversion (or preference) when they only take into account the scenario that implies the lowest (or highest) profits. The former are named conservative firms and the latter, optimistic firms.

In the previous section, the set of Pareto equilibria for the Cournot game under uncertainty only depends on the quantities of perfect competition in both scenarios. However, it is when we seek to refine the set of equilibria by including risk attitudes in the model, that the other parameters of the demand function and the relationships between them become relevant.

Given the inverse demand functions of the scenarios $p = \alpha_k - \gamma_k q$, for $k = 1, 2$, if $\gamma_1 \neq \gamma_2$, then there is a unique value of the demand for which prices coincide at both scenarios, $q = \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$. The relative position of this value with respect to the Cournot quantities and to the perfect competition quantities at the two scenarios plays a key role in the analysis of the conservative equilibria and optimistic equilibria of the Cournot game. We first observe that when the reservation price in the first scenario is higher than that of the second, then this value is smaller than the total quantity offered in perfect competition in the first scenario. Otherwise, this value is either non-positive or greater than or equal to the perfect competition quantity in the second scenario. These facts are stated formally in the following lemma which is easy to prove by using a geometric argument.

Lemma 3.1. *Provided that* $\frac{\alpha_1}{\gamma_1} < \frac{\alpha_2}{\gamma_2}$,

a) If $\alpha_1 > \alpha_2$ *then* $0 < \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} < \frac{\alpha_1}{\gamma_1}$.

b) If $\alpha_1 \leq \alpha_2$

b1) $\gamma_2 < \gamma_1$, *then* $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} \leq 0$.

b2) $\gamma_2 > \gamma_1$, $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} > \frac{\alpha_2}{\gamma_2}$.

3.1 Conservative equilibria

In order to model situations in which agents exhibit extreme risk aversion, we consider a standard normal-form game in which the utility of the firms is represented by the worst benefit obtained in the scenarios. For $i = 1, 2$, the real-valued utility function of firm i is:

$$\Pi_c^i(q^1, q^2) = Min\{\Pi_1^i(q^1, q^2), \Pi_2^i(q^1, q^2)\}.$$

In accordance, the definition of conservative equilibrium is:

Definition 3.2. (q^{*1}, q^{*2}) is a conservative equilibrium for the Cournot game $G = \{(A_i, \Pi^i)_{i=1,2}\}$ if for each $q^1 \in A_1$, $q^1 \neq q^{*1}$, $\Pi_c^1(q^1, q^{*2}) < \Pi_c^1(q^{*1}, q^{*2})$ holds, and for each $q^2 \in A_2$, $q^2 \neq q^{*2}$, $\Pi_c^2(q^{*1}, q^2) < \Pi_c^2(q^{*1}, q^{*2})$ holds.

The set of conservative equilibria of a game G is denoted by $E^c(G)$.

In an equilibrium the strategy of each firm is the best response to the strategy of the other firm, and therefore, conservative firms obtain quantities such that no individual deviation produces an improvement in the minimum benefit. The strategies adopted by the firms in a conservative equilibrium can be seen as robust strategies, in the sense that whichever scenario finally materializes each firm maximizes its assured benefit.

We first establish that, for the Cournot game under uncertainty, the set of conservative equilibria is a subset of the set of Pareto equilibria. A self-contained proof of this fact is presented in the Appendix. The result can also be obtained as a consequence of Theorem 1 in Bade (2005).

Proposition 3.3. $E^c(G_+^{UC}) \subseteq PE(G_+^{UC})$.

We are now interested in the existence and in the identification of the conservative equilibria for Cournot games under uncertainty G_+^{UC}. The following theorem is a result that is central to our analysis. It shows that if the firms exhibit extreme risk aversion, then equilibria always exist. Moreover, it states the conditions on the parameters of the demand functions for a unique equilibrium to exist. In this case the equilibrium coincides with the Cournot equilibrium in one of the scenarios. We also prove that when multiple equilibria exist, the total quantity offered by the agents is the value of the demand for which prices coincide at both scenarios, $\frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2}$. The proof of the results rely on the concavity of the conservative utility of the agents with respect to their own action.

Theorem 3.4. *Assume that* $\frac{\alpha_1}{\gamma_1} < \frac{\alpha_2}{\gamma_2}$. *The set of conservative equilibria for the Cournot game* G_+^{UC} *is*

a) If $\alpha_1 \leq \alpha_2$, *then* $E^c(G_+^{UC}) = \{(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})\}$.

b) If $\alpha_1 > \alpha_2$ *and* $\frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2} \leq \frac{2\alpha_1}{3\gamma_1}$, *then* $E^c(G_+^{UC}) = \{(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})\}$.

c) If $\alpha_1 > \alpha_2$ *and* $\frac{2\alpha_1}{3\gamma_1} < \frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2} < \frac{2\alpha_2}{3\gamma_2}$, *then*

$$E^c(G_+^{UC}) = PE(G_+^{UC}) \cap \{(q^1, q^2) \mid q^1 + q^2 = \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}\}.$$

d) If $\alpha_1 > \alpha_2$ *and* $\frac{2\alpha_2}{3\gamma_2} \leq \frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2}$, *then* $E^c(G_+^{UC}) = \{(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})\}$.

Figures 2 and 3 illustrate cases b) and c) respectively. In the case of Figure 2, the whole set of Pareto equilibria lies on the region in which the conservative utility coincides with the benefit in scenario 1, and therefore the conservative equilibrium is the Cournot equilibria in scenario 1. The best response functions of risk-averse

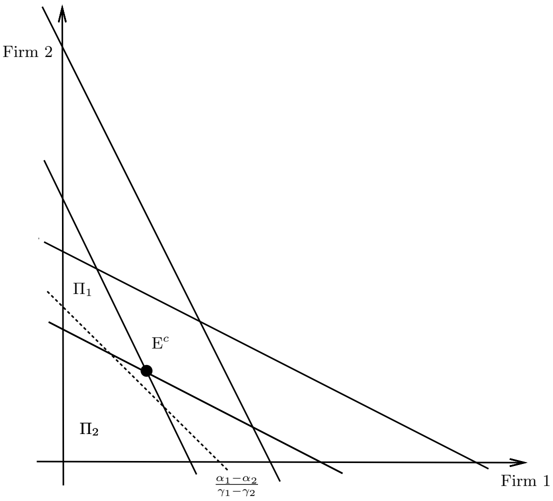

Figure 2: A unique conservative equilibrium

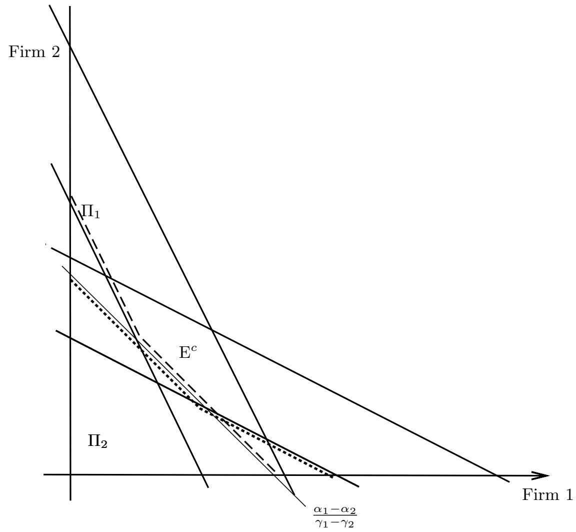

Figure 3: Multiple conservative equilibria

firms in case c) are represented in Figure 3. The points located on both lines are the conservative equilibria.

Remarks: A first remark is that only in the cases in which the Cournot equilibria quantities in the two scenarios are very similar, does the conservative equilibrium correspond to the equilibria in the second scenario. Note that $\frac{\alpha_1}{\gamma_1} < \frac{\alpha_2}{\gamma_2}$ and $\frac{2\alpha_2}{3\gamma_2} < \frac{\alpha_1}{\gamma_1}$ must hold so that $\{(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})\}$ is the conservative equilibrium.

It is also worth remarking that in the case in which a multiplicity of conservative equilibria exists (case c)), all of them correspond to the same price: $p = \dfrac{\alpha_2\gamma_1 - \alpha_1\gamma_2}{\gamma_1 - \gamma_2}$, and the total quantity offered is $q = \dfrac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$. That is, the equilibrium price is unique. In addition, given one of these equilibria, each agent obtains the same benefit in both scenarios.

Moreover, the explicit expressions of the total quantity and the price at the conservative equilibria enables us to make predictions about the equilibrium price. If the reservation prices of the markets (α_k) approach each other (move away from each other) or the sizes of the markets (γ_k) differ more (become more similar), then the total equilibrium quantity decreases (increases) and therefore the equilibrium price increases (decreases).

If the changes are such that the conditions in c) are no longer fulfilled, the prices at the equilibria then depend on the final scenario. A detailed study of comparative statics for this model would involve keeping track of the changes in the corresponding parameters for each case, and no general results can be obtained.

3.2 Optimistic Equilibria

The other extreme case in terms of risk attitude of the firms is the situation when the two firms select their strategies by only taking into account the best of the results they can obtain. The utility of the firms is now given by:

$$\Pi^i_{op}(q^1, q^2) = Max\{\Pi^i_1(q^1, q^2), \Pi^i_2(q^1, q^2)\}.$$

This optimistic utility function coincides with Π^i_1 when $(\gamma_1 - \gamma_2)(q^1 + q^2) \leq \alpha_1 - \alpha_2$, and with Π^i_2 otherwise.

Definition 3.5. (q^{*1}, q^{*2}) is an optimistic equilibrium for the Cournot game $G = \{(A_i, \Pi^i)_{i=1,2}\}$ if for each $q^1 \in A_1$, $q^1 \neq q^{*1}$, $\Pi^1_{op}(q^1, q^{*2}) < \Pi^1_{op}(q^{*1}, q^{*2})$ holds, and for each $q^2 \in A_2$, $q^2 \neq q^{*2}$, $\Pi^2_{op}(q^{*1}, q^2) < \Pi^2_{op}(q^{*1}, q^{*2})$ holds.

We denote by $E^{op}(G)$ to the set of optimistic equilibria of game G.

Optimistic equilibria are also Pareto Equilibria, as established below.

Proposition 3.6. *$E^{op}(G^{UC}_+) \subseteq PE(G^{UC}_+)$.*

In contrast to the case of the conservative utility function, the optimistic utility does not exhibit desirable concavity properties. This fact increases the complexity of the analysis of the existence and identification of equilibria. In the following result Cournot games under uncertainty are classified depending on the relative position of the value $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$ and the Cournot quantities. It establishes that in certain cases (a), b) d)) a unique optimistic equilibrium exists which coincides with the Cournot equilibrium of one of the scenarios.

For case c), the relative position of these quantities does not permit the existence of equilibria to be concluded. However, it permits us to prove that only the Cournot equilibria associated to the scenarios can be optimistic equilibria for the uncertain Cournot game.

Theorem 3.7. *Assume that $\frac{\alpha_1}{\gamma_1} < \frac{\alpha_2}{\gamma_2}$. The set of optimistic equilibria for the game G^{UC}_+ is*

a) If $\alpha_1 \leq \alpha_2$, then $E^{op}(G^{UC}_+) = \{(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})\}$.

b) *If* $\alpha_1 > \alpha_2$ *and* $\frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2} \le \frac{2\alpha_1}{3\gamma_1}$, *then* $E^{op}(G_+^{UC}) = \{(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})\}$.

c) *If* $\alpha_1 > \alpha_2$ *and* $\frac{2\alpha_1}{3\gamma_1} < \frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2} < \frac{2\alpha_2}{3\gamma_2}$, *then* $E^{op}(G_+^{UC}) \subseteq \{(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1}), (\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})\}$.

d) *If* $\alpha_1 > \alpha_2$ *and* $\frac{2\alpha_2}{3\gamma_2} \le \frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2}$, *then* $E^{op}(G_+^{UC}) = \{(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})\}$.

It remains to analyse the situation in case c). In order to do this, it is important to identify at which values of the strategy of his opponent, an optimistic agent switches from reacting with the best response at one scenario to reacting with the best response at the other scenario. The values of q for which the benefit obtained in scenario 1 with the best response in scenario 1 coincides with the benefit in scenario 2 with the best response in scenario 2 are the values for which agent 2 will change from one of the best responses to the other. The relative positions of one of these values and the Cournot quantities of the scenarios, determine the optimistic equilibria of the uncertain Cournot game.

Recall that for $k = 1, 2$, given an action of agent i, the best response of agent j in his feasible set of actions ($q^j \ge 0$) in scenario k is $r_k^j(q^i) = \frac{\alpha_k - \gamma_k q^i}{2\gamma_k}$ if $q^i \le \frac{\alpha_k}{\gamma_k}$, and $r_k^j(q^i) = 0$ if $q^i \ge \frac{\alpha_k}{\gamma_k}$.

The Lemma below states that for optimistic agents there is at most one point, q_m, in the set of actions in which the optimistic best response switches from the best response of scenario 1 to that of scenario 2.

Lemma 3.8. *Assume that* $\frac{\alpha_1}{\gamma_1} < \frac{\alpha_2}{\gamma_2}$ *in the game* G_+^{UC}. *Let* $q_m = \frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2} - \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{\alpha_2\gamma_1-\alpha_1\gamma_2}{\gamma_1-\gamma_2}$. *For* $j = 1, 2$ *the best response function of an optimistic agent* j, r_{op}^j, *is given by* $r_{op}^j(q^i) = r_1^j(q^i)$ *for* $q^i \le q_m$, *and* $r_{op}^j(q^i) = r_2^j(q^i)$ *for* $q^i > q_m$.

The following result establishes that, also in case c) of Theorem 3.7, optimistic equilibria exist and identifies the optimistic equilibria in the various cases.

Theorem 3.9. *Assume that* $\frac{\alpha_1}{\gamma_1} < \frac{\alpha_2}{\gamma_2}$. *If* $\alpha_1 > \alpha_2$ *and* $\frac{2\alpha_1}{3\gamma_1} < \frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2} < \frac{2\alpha_2}{3\gamma_2}$, *then*

a) *If* $q_m < \frac{\alpha_1}{3\gamma_1}$, *then* $E^{op}(G_+^{UC}) = \{(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})\}$.

b) $\frac{\alpha_1}{3\gamma_1} \le q_m \le \frac{\alpha_2}{3\gamma_2}$, *then* $E^{op}(G_+^{UC}) = \{(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1}), (\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})\}$.

c) $q_m > \frac{\alpha_2}{3\gamma_2}$, *then* $E^{op}(G_+^{UC}) = \{(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})\}$.

As a conclusion, in relation to the optimistic equilibria of the Cournot game under uncertainty, any one of three situations is possible: the Cournot equilibrium for the

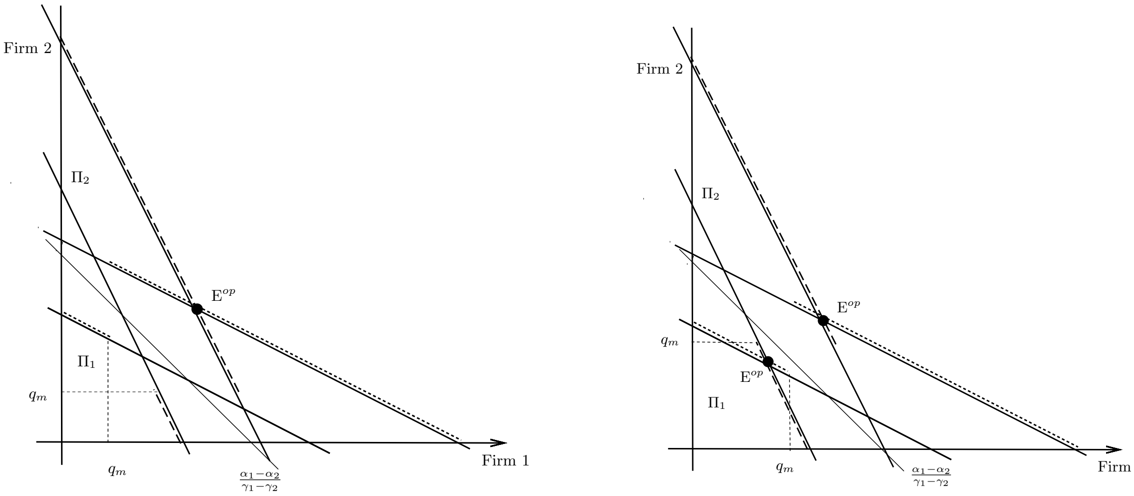

Figure 4: Optimistic Equilibria.

first scenario is the optimistic equilibrium; the Cournot equilibrium for the second scenario is the optimistic equilibrium; or both of them are optimistic equilibria.

Figure 4 illustrates two of these situations. Dotted lines represent the optimistic best response of Firm 1, and dashed lines represent the optimistic best response of Firm 2.

Two examples follow. In one of them, a unique optimistic equilibrium exists which coincides with the Cournot equilibrium in scenario 2. In the other example, the Cournot equilibria at the two scenarios are optimistic equilibria.

Example 3.10. Consider the Cournot game under uncertainty in which the demand functions at scenario 1 and 2 are respectively: $p = 10 - 100q$ and $p = 1 - q$. In this case $\alpha_1 = 10$, $\gamma_1 = 100$, $\alpha_2 = 1$, $\gamma_2 = 1$. As $\frac{\alpha_2}{2\gamma_2} > \frac{\alpha_1}{\gamma_1}$, according to Corollary 2.5 b) the set of Pareto equilibria is

$$PE(G_+^{UC}) = conv\{(1/30, 1/30), (1/3, 1/3), (1/10, 0), (1/2, 0), (0, 1/10), (0, 1/2)\}.$$

Since $q_m = 0$, the condition in Theorem 3.9 case a), holds and $(1/3, 1/3)$ is the optimistic equilibrium.

Note that the Cournot equilibrium in the first scenario is (1/30,1/30) which will yield a profit equal to 1/9 for each firm at scenario 1. However, if one of the firms produces 1/30, its optimistic opponent will adopt its best response in scenario 2 and produce 29/60, which will yiel higher profits if eventually scenario 2 is realized: 841/3600. Thus, $(1/30, 1/30)$ is not an optimistic equilibrium. In contrast, if one of the firms produces 1/3, the best response of the other firm in scenario 1 is 0, and thus its profit equals 0. In scenario 2 the best response is 1/3, giving a profit of 2/3. Therefore, $(1/3, 1/3)$ is the optimistic equilibrium and the profit of each firm in the optimistic equilibrium will be either 0 or 2/3.

In this uncertain Cournot game, the conservative equilibria are those Pareto equilibria (q^1, q^2), such that $q^1 + q^2 = 1/11$. The quantity produced by each firm in the set of conservative equilibria varies from 11/1210 to 9/110, and their profits (which coincide in the two scenarios) varies accordingly between 1/121 and 9/121.

Example 3.11. Consider the Cournot game under uncertainty in which the demand functions at scenario 1 and 2 are respectively: $p = 300 - 150q$ and $p = 100 - 30q$. In this case $\alpha_1 = 300$, $\gamma_1 = 150$, $\alpha_2 = 100$, $\gamma_2 = 30$. Therefore, $\frac{\alpha_2}{2\gamma_2} \leq \frac{\alpha_1}{\gamma_1}$ and according to corollary 2.5 a) the set of Pareto equilibria is

$$PE(G_+^{UC}) = conv\{(2/3, 2/3), (10/9, 10/9), (2/9, 14/9), (14/9, 2/9)\}.$$

The Cournot equilibrium in scenario 1 is $(2/3, 2/3)$, and the Cournot equilibrium in scenario 2 is $(10/9, 10/9)$. Since $q_m = \frac{5-\sqrt{5}}{3}$, then $(2/3, 2/3)$ and $(10/9, 10/9)$ are the optimistic equilibria for this uncertain Cournot game.

In this case, if both firms adopt the strategy corresponding to the Cournot equilibria in scenario 1, then they obtain either 200/3 or 40. If they adopt the strategy corresponding to scenario 2, they obtain, either 0 or 1000/27.

In this case, as can be expected, given the quantity of Cournot equilibrium in scenario 1, the best response of the opponent in scenario 2 is 4/3 with a profit of 160/3, that is, less than 200/3, and therefore (2/3,2/3) is an equilibrium. And given the quantity of Cournot equilibrium in scenario 2, the best response of the opponent in scenario 1 is 4/9 with a profit of 800/27(less than 1000/27), and thus (10/9,10/9) is an equilibrium.

In this uncertain Cournot game, the conservative equilibria are those Pareto equilibria (q^1, q^2), such that $q^1 + q^2 = 5/3$. The quantity to produce by each firm in the set of conservative equilibria varies from 1/3 to 4/3, and their profits (which coincide in the two scenarios) varies accordingly between 50/3 and 200/3.

4 Concluding remarks

An alternative analysis of the Cournot duopoly under demand uncertainty, which differs from those existing in the literature, is presented in this paper.

The situations considered are formalized as normal-form games with vector-valued utility functions. When no assumption about the firms' attitude towards risk can be made, then the set of equilibria to which the agents will arrive solely depends on the quantities of perfect competition of the scenarios as is the case in the classic Cournot equilibrium. In addition, the set of Pareto equilibria is a symmetric set, although Pareto equilibria are not necessarily symmetric.

The introduction into the model of the attitude of the firms towards risk carries a major implication on the equilibria that the firms will attain. The analysis of the particular cases of pessimistic and optimistic firms provides interesting results. The existence of equilibria when both firms are simultaneously pessimistic is established, together with conditions on the parameters of the demand functions for the unicity of the equilibrium. When there are multiple equilibria, a significant property is that all of them yield the same price. In relation with situations in which both firms are simultaneously optimistic, we also prove that equilibria always exist and we show that any uncertain Cournot game has either a unique optimistic equilibrium, which coincides with the Cournot equilibrium of one of the scenarios, or both Cournot equilibria of the scenarios are optimistic equilibria.

The results presented in this paper constitute the starting point for a complete study of the equilibria in the cases in which the risk-attitude of each agent is different. This analysis may help to explain some real-world situations. For instance, the recent financial crisis has highlighted the differences between firms with respect to attitude towards risk and the potential equilibria should be investigated in this framework.

5 Appendix: proofs

Proof of Theorem 2.2: First note that since $\frac{\alpha_1}{\gamma_1} < \frac{\alpha_2}{\gamma_2}$, then $r_1^i(q^j) < r_2^i(q^j)$ for all $q^j \in A_j$.

Consider a point $(\bar{q}^1, \bar{q}^2)$ such that $\bar{q}^1 < r_1^1(\bar{q}^2)$. Since each firm's objective function, Π_k^i, is strictly concave in the firm's own quantity, both $\Pi_1^1(q^1, \bar{q}^2)$ and $\Pi_2^1(q^1, \bar{q}^2)$ are increasing for $q^1 \leq r_1^1(\bar{q}^2)$, therefore, it follows that if agent 1 moves to $\bar{q}^1 + \varepsilon$ then his benefit will increase in both scenarios. Hence, $(\bar{q}^1, \bar{q}^2) \notin PE(G^{UC})$. Analogously, this holds for $\bar{q}^1 > r_2^1(\bar{q}^2)$.

On the other hand, if $r_1^1(\bar{q}^2) \leq \bar{q}^1 \leq r_2^1(\bar{q}^2)$, $r_1^2(\bar{q}^1) \leq \bar{q}^2 \leq r_2^2(\bar{q}^1)$, then any individual movement of one of the agents produces an increase of the benefit in one of the scenarios and a decrease in the other, and therefore $(\bar{q}^1, \bar{q}^2) \in PE(G^{UC})$.

Proof of Theorem 2.4: The first inclusion ($\subseteq$) follows from the definition of Pareto equilibrium.

The other inclusion is a consequence of the strict concavity of the benefit functions. To proof it, consider $q^* = (q^{*1}, q^{*2}) \in PE(G_+^{UC})$. We will distinguish the following cases:

a) $q^{*1}, q^{*2} > 0$.

Suppose that the contrary is true: $q^* \notin PE(G^{UC})$. It follows that for an agent, say agent 1, there exists an alternative, $q^1 < 0$ such that $u_1^1(q^1, q^{*2}) \geq u_1^1(q^{*1}, q^{*2})$ and $u_2^1(q^1, q^{*2}) \geq u_2^1(q^{*1}, q^{*2})$ (with a strict inequality). Let $\bar{q} = (0, q^{*2})$, $0 = \lambda q^1 + (1-\lambda)q^{*1}$ with $\lambda \in (0,1)$. It follows from the strict concavity of u_k^1 that $u_k^1(\bar{q}) > \lambda u_k^1(q^1, q^{*2}) + (1-\lambda)u_k^1(q^{*1}, q^{*2}) \geq u_k^1(q^{*1}, q^{*2})$. This contradicts $q^* \in PE(G_+^{UC})$.

b) $q^{*1} > 0$, $q^{*2} = 0$ (or $q^{*1} = 0$, $q^{*2} > 0$).

Suppose that the contrary is true: $q^* \notin PE(G^{UC})$. Hence $q^{1*} < \frac{\alpha_1}{\gamma_1}$ or $q^{1*} > \frac{\alpha_2}{2\gamma_2}$.

If $q^{1*} < \frac{\alpha_1}{\gamma_1}$, then for a fixed q^{1*}, the benefit of agent 2, $\Pi_k^2(q^{1*}, q^2)$, is strictly increasing for both $k = 1, 2$ at $q^2 = 0$, and therefore, a strategy of agent 2 exists, $q^2 = \varepsilon$ with $\varepsilon > 0$ such that $\Pi_k^2(q^{1*}, \varepsilon) > \Pi_k^2(q^{1*}, 0)$ for $k = 1, 2$. This contradicts $q^* \in PE(G_+^{UC})$.

If $q^{1*} > \frac{\alpha_2}{2\gamma_2}$, then the benefit of agent 1 when q^2 fixed at 0, $\Pi_k^1(q^1, 0)$ is strictly decreasing for both $k = 1, 2$, and therefore, $\varepsilon > 0$ exists, such that, for $q^1 = q^{1*} - \varepsilon > 0$, $\Pi_k^2(q^{1*} - \varepsilon, 0) > \Pi_k^2(q^{1*}, 0)$ holds for $k = 1, 2$. This contradicts $q^* \in PE(G_+^{UC})$.

c) $q^{*1} = 0$, $q^{*2} = 0$. The reasoning is analogous to that above.

Proof of Proposition 3.3: Let (q^{*1}, q^{*2}) be a conservative equilibrium for the Cournot game G_+^{UC}, and suppose to the contrary that it is not a Pareto equilibrium. It follows that, for a firm i, a strategy $q^i \in \mathbb{R}_+$ exists such that $\Pi_k^i(q^i, q^{*j}) \geq \Pi_k^i(q^{*i}, q^{*j})$ for $k = 1, 2$ (with a strict inequality).

Therefore, since $\Pi_k^i(q^{*i}, q^{*j}) \geq \Pi_c^i(q^{*i}, q^{*j})$ for $k = 1, 2$, then $\Pi_k^i(q^i, q^{*j}) \geq \Pi_c^i(q^{*i}, q^{*j})$ for $k = 1, 2$ and $\Pi_c^i(q^i, q^{*j}) \geq \Pi_c^i(q^{*i}, q^{*j})$. Thus $q^i \in R_+$ exists such that $\Pi_c^i(q^i, q^{*j}) \geq \Pi_c^i(q^{*i}, q^{*j})$, which is a contradiction with (q^{*1}, q^{*2}) being a conservative equilibrium.

Proof of Theorem 3.4: To prove our result we consider several cases which depend on the relative positions of the demand functions in the two scenarios. Recall that $\frac{\alpha_1}{\gamma_1} < \frac{\alpha_2}{\gamma_2}$, and note that the conservative utility function, Π_c^i coincides with Π_1^i for (q^1, q^2) such that $(\gamma_1 - \gamma_2)(q^1 + q^2) \geq \alpha_1 - \alpha_2$, and coincides with Π_2^i otherwise.

a1) $\alpha_1 \leq \alpha_2$ and $\gamma_1 > \gamma_2$.

It is easy to see that in this case $\Pi_c^i(q^1, q^2) = \Pi_1^i(q^1, q^2)$ if and only if $q^1 + q^2 \geq \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$. However, since $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} \leq 0$, it follows that $\Pi_c^i(q^1, q^2) = \Pi_1^i(q^1, q^2)$ for all $q^1, q^2 \geq 0$ and therefore the conservative equilibria coincide with that of scenario 1, $E^c(G_+^{UC}) = \{(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})\}$.

a2) $\alpha_1 \leq \alpha_2$ and $\gamma_1 < \gamma_2$.

Here $\Pi_c^i(q^1, q^2) = \Pi_1^i(q^1, q^2)$ if and only if $q^1 + q^2 \leq \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$, and $\Pi_c^i(q^1, q^2) = \Pi_2^i(q^1, q^2)$ if and only if $q^1 + q^2 \geq \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$. Since, by Lemma 3.1, $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} > \frac{\alpha_2}{\gamma_2}$, then the whole set of Pareto Equilibria lies in the region where the conservative function coincides with the benefit in scenario 1. It is easy to prove that in this case the conservative equilibrium also coincides with that of scenario 1, $E^c(G_+^{UC}) = \{(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})\}$.

a3) $\alpha_1 \leq \alpha_2$ and $\gamma_1 = \gamma_2$.

It is straightforward that in this case $\Pi_c^i(q^1, q^2) = \Pi_1^i(q^1, q^2)$ for all (q^1, q^2), and hence $E^c(G_+^{UC}) = \{(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})\}$.

We will now analyse the cases in which $\alpha_1 > \alpha_2$.
For these values, $\gamma_1 > \gamma_2$, by Lemma 3.1 $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} < \frac{\alpha_1}{\gamma_1}$ holds, and

$$\Pi_c^i(q^1, q^2) = \begin{cases} q^i(\alpha_2 - \gamma_2(q^1 + q^2)) & \text{if } q^1 + q^2 \leq \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} \\ q^i(\alpha_1 - \gamma_1(q^1 + q^2)) & \text{if } q^1 + q^2 \geq \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} \end{cases}$$

Several subcases are now determined:

b) $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} \leq \frac{2\alpha_1}{3\gamma_1}$.

For these values of the parameters, the whole set of Pareto equilibria lies in the region where the conservative function coincides with the benefit in scenario 1, and the conservative equilibria coincide with that of scenario 1, $E^c(G_+^{UC}) = \{(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})\}$.

c) $\frac{2\alpha_1}{3\gamma_1} < \frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2} < \frac{2\alpha_2}{3\gamma_2}$.

To prove that a point $(q^{1*}, q^{2*}) \in PE(G_+^{UC})$ with $q^{1*} + q^{2*} = \dfrac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$ is a conservative equilibrium, we rely on the strict concavity of $\Pi_k^i(q^i, q^{j*})$.

Suppose that agent 1 deviates from (q^{1*}, q^{2*}) by adopting strategy q^1. If $q^1 > q^{1*}$, then since $\Pi_1^1(q^1, q^{2*})$ is decreasing, then her utility decreases since $\Pi_c^1(q^1, q^{2*}) = \Pi_1^1(q^1, q^{2*}) < \Pi_1^1(q^{1*}, q^{2*}) = \Pi_c^1(q^{1*}, q^{2*})$. If $q^1 < q^{1*}$ then $\Pi_c^1(q^1, q^{2*}) = \Pi_2^i(q^1, q^{j*})$. However, in this region, $\Pi_2^1(q^1, q^{2*})$ is increasing and therefore $\Pi_2^1(q^1, q^{2*}) < \Pi_2^1(q^{1*}, q^{2*}) = \Pi_c^1(q^{1*}, q^{2*})$.

Analogous reasoning with the deviations of agent 2, leads us to the result.

Consider now a point $(q^{1*}, q^{2*}) \in PE(G_+^{UC})$ with $q^{1*} + q^{2*} < \dfrac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$.

In this region, $\Pi_c^i(q^1, q^2) = \Pi_2^1(q^1, q^2)$ and given the action of one of the firms, the benefit at scenario 2 is strictly increasing in its own action. Therefore, any of the firms will improve its utility by increasing its quantity, and the point is not a conservative equilibrium.

d) $\frac{2\alpha_2}{3\gamma_2} \leq \frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2}$.

In this case, the set of Pareto equilibria lies in the region where the conservative utility coincides with the benefit at scenario 2 and the conservative equilibria coincides with that of scenario 2.

Proof of Proposition 3.6: Let (q^{*1}, q^{*2}) be an optimistic equilibrium for the Cournot game G_+^{UC}, and suppose to the contrary that it is not a Pareto equilibrium. It follows that, for a firm i, a strategy $q^i \in \mathbb{R}_+$ exists such that $\Pi_k^i(q^i, q^{*j}) \geq \Pi_k^i(q^{*i}, q^{*j})$ for $k = 1, 2$ (with a strict inequality). As a consequence, since $\Pi_{op}^i(q^i, q^{*j}) \geq \Pi_k^i(q^i, q^{*j})$ for $i = 1, 2$, then $\Pi_{op}^i(q^i, q^{*j}) \geq \Pi_k^i(q^{*i}, q^{*j})$ for $k = 1, 2$, and therefore $\Pi_{op}^i(q^i, q^{*j}) \geq \Pi_{op}^i(q^{*i}, q^{*j})$. This contradicts (q^{*1}, q^{*2}) being an optimistic equilibrium.

Proof of Theorem 3.7: Recall that $\frac{\alpha_1}{\gamma_1} < \frac{\alpha_2}{\gamma_2}$, and note that the optimistic utility function Π_{op}^i coincides with Π_1^i for those (q^1, q^2) such that $(\gamma_1 - \gamma_2)(q^1 + q^2) \leq \alpha_1 - \alpha_2$, and coincides with Π_2^i otherwise. Several subcases can be determined:

a1) $\alpha_1 \leq \alpha_2$ and $\gamma_1 > \gamma_2$.

In this case, $\Pi_{op}^i(q^1, q^2) = \Pi_2^i(q^1, q^2)$ if and only if $q^1 + q^2 \geq \frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2}$. However, since $\frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2} \leq 0$, it follows that $\Pi_{op}^i(q^1, q^2) = \Pi_2^i(q^1, q^2)$ for all $q^1, q^2 \geq$

0 and therefore the optimistic equilibria coincides with that of scenario 2, $E^{op}(G_+^{UC}) = \{(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})\}$.

a2) $\alpha_1 \leq \alpha_2$ and $\gamma_1 < \gamma_2$.

In this case $\Pi_{op}^i(q^1, q^2) = \Pi_2^i(q^1, q^2)$ if and only if $q^1 + q^2 \leq \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$, and $\Pi_{op}^i(q^1, q^2) = \Pi_1^i(q^1, q^2)$ if and only if $q^1 + q^2 \geq \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$. Since, by Lemma 3.1, $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} > \frac{\alpha_2}{\gamma_2}$ holds, then the whole set of Pareto Equilibria lies in the region where the optimistic function coincides with the benefit in scenario 2 and the optimistic equilibria coincide with that of scenario 2, $E^{op}(G_+^{UC}) = \{(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})\}$.

a3) $\alpha_1 \leq \alpha_2$ and $\gamma_1 = \gamma_2$.

It is straightforward that in this case $\Pi_c^i(q^1, q^2) = \Pi_2^i(q^1, q^2)$ for all (q^1, q^2), and hence $E^c(G_+^{UC}) = \{(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})\}$.

We now analyse the cases in which $\alpha_1 > \alpha_2$.
For these values, $\gamma_1 > \gamma_2$. By Lemma 3.1, $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} < \frac{\alpha_1}{\gamma_1}$ holds and

$$\Pi_{op}^i(q^1, q^2) = \begin{cases} q^i(\alpha_1 - \gamma_1(q^1 + q^2)) & \text{if } q^1 + q^2 \leq \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} \\ q^i(\alpha_2 - \gamma_2(q^1 + q^2)) & \text{if } q^1 + q^2 \geq \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} \end{cases}$$

We distinguish several sub-cases.

b) $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} < \frac{2\alpha_1}{3\gamma_1}$.

It is easy to see that since the whole set of Pareto equilibria lies in the region where Π_{op} coincides with Π_2, then the only candidate to be an optimistic equilibria is $(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})$. To prove that this point is the optimistic equilibrium, consider the possible deviations of firm 1 from $(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})$. If, by deviating, $(q^1, \frac{\alpha_2}{3\gamma_2})$ remains in the region where Π_{op} coincides with Π_2, then its benefit decreases. On the other hand, if a negative deviation takes $(q^1, \frac{\alpha_2}{3\gamma_2})$ outside this region then $\Pi_{op}(q^1, \frac{\alpha_2}{3\gamma_2}) = \Pi_1(q^1, \frac{\alpha_2}{3\gamma_2})$. Note that, since Π_1 is strictly increasing in q^1, then $\Pi_1(q^1, \frac{\alpha_2}{3\gamma_2}) < \Pi_1(\bar{q}^1, \frac{\alpha_2}{3\gamma_2}) = \Pi_2(\bar{q}^1, \frac{\alpha_2}{3\gamma_2})$, where $\bar{q}^1$ is such that $\bar{q}^1 + \frac{\alpha_2}{3\gamma_2} = \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$. Now note that $\Pi_2(\bar{q}^1, \frac{\alpha_2}{3\gamma_2}) < \Pi_2(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2}) = \Pi_{op}(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})$. Therefore, any deviation from $(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})$ yields a strict decrease of the firm's optimistic utility. As a consequence $(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})$ is the optimistic equilibrium.

c) $\frac{2\alpha_1}{3\gamma_1} < \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} < \frac{2\alpha_2}{3\gamma_2}$.

We will prove that only $(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})$ and $(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})$ can be optimistic equilibria.

1) Given a point $(q^1, q^2) \in PE(G_+^{UC})$ in the interior of $PE(G_+^{UC})$, any of the agents can deviate to his best response corresponding to the scenario in which the optimistic function coincides with the benefit.

2) Consider now $(q^1, q^2) \in PE(G_+^{UC}) \backslash \{(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1}), (\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})\}$ which lies on some of the best response lines. Assume without loss of generality that $q^2 = r_1^2(q^1)$ or $q^2 = r_2^2(q^1)$:

-If $q^2 = r_1^2(q^1)$ and $q^1 + q^2 \leq \dfrac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$, then, since q^1 is not the best response of agent 1 to this q^2, then agent 1 will improve his benefit by moving to his best response line in scenario 1.

- If $q^2 = r_1^2(q^1)$ and $q^1 + q^2 \geq \dfrac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$, then agent 2 will improve its benefit by moving to his best response line in scenario 2.

-If $q^2 = r_2^2(q^1)$ and $q^1 + q^2 \leq \dfrac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$, then agent 2 will improve his benefit by moving to his best response line in scenario 1.

- If $q^2 = r_2^2(q^1)$ and $q^1 + q^2 \geq \dfrac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$, then agent 1 will improve his benefit by moving to his best response line in scenario 2.

3) If $(q^1, q^2) \in PE(G_+^{UC})$ with $q^2 = 0$ and lies on the boundary of $PE(G_+^{UC})$, then $q^1 \geq \frac{\alpha_1}{\gamma_1}$. As a consequence of Lemma 3.1, $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} \leq \frac{\alpha_1}{\gamma_1}$ holds and it follows that at this point the optimistic utility coincides with the benefit in scenario 2. In this situation, firm 2 can improve its optimistic utility by adopting a strategy $\bar{q}^2 > 0$. Therefore, $(q^1, 0)$ is not an optimistic equilibrium.

Analogous reasoning is valid if If $(q^1, q^2) \in PE(G_+^{UC})$ with $q^1 = 0$ and lies on the boundary of $PE(G_+^{UC})$.

d) $\frac{2\alpha_2}{3\gamma_2} < \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$.

The reasoning is analogous to that of sub-case a3).

Proof of Lemma 3.8: When the strategy of agent 1 is $q^1 \leq \frac{\alpha_1}{\gamma_1}$, agent 2 has the possibility of adopting the best response function corresponding to the first scenario or to the second scenario. If agent 2 is optimistic, then he only considers the maximum of the benefits he obtains with these best responses, that is, the maximum of the following two quantities:

$$\Pi_1^2(q^1, r_1^2(q^1)) = \frac{(\alpha_1 - \gamma_1 q^1)^2}{4\gamma_1}, \ \Pi_2^2(q^1, r_2^2(q^1)) = \frac{(\alpha_2 - \gamma_2 q^1)^2}{4\gamma_2}.$$

In other words, for $q^1 \leq \frac{\alpha_1}{\gamma_1}$, the best response of agent 2 is $r^2_{op}(q^1) = r^2_k(q^1)$ where for each q^1, k is such that

$$\frac{(\alpha_k - \gamma_k q^1)^2}{4\gamma_k} = \max\{\frac{(\alpha_1 - \gamma_1 q^1)^2}{4\gamma_1}, \frac{(\alpha_2 - \gamma_2 q^1)^2}{4\gamma_2}\}.$$

On the other hand, when $q^1 \geq \frac{\alpha_1}{\gamma_1}$, then the best response of agent 2 at scenario 1 is $r^2_1(q^1) = 0$ and therefore $r^2_{op}(q^1) = r^2_2(q^1)$.

Note that, as a consequence of the symmetry of our model, for each $q \leq \frac{\alpha_1}{\gamma_1}$, $\Pi^2_1(q, r^2_1(q)) = \Pi^1_1(r^1_1(q), q)$ and $\Pi^2_2(q, r^2_1(q)) = \Pi^1_2(r^1_2(q), q)$. Hence, for $q \leq \frac{\alpha_1}{\gamma_1}$, the best response of agent 1, $r^1_{op}(q)$, is attained for the same scenario as the best response for agent 2, $r^2_{op}(q)$. On the other hand if $q \geq \frac{\alpha_1}{\gamma_1}$, then $r^1_{op}(q) = r^1_2(q)$.

It is important to identify at which values of the strategy of his opponent, an optimistic agent switches from reacting with the best response at one scenario to reacting with the best response at the other scenario. The values of q for which the benefit obtained in scenario 1 with the best response in scenario 1 coincides with the benefit in scenario 2 with the best response in scenario 2 are the values for which agent 2 will change from one of the best responses to the other. These values are obtained by solving the equation

$$\frac{(\alpha_1 - \gamma_1 q)^2}{4\gamma_1} = \frac{(\alpha_2 - \gamma_2 q)^2}{4\gamma_2}.$$

Denote these quantities as q_m and q_M:

$$q_m = \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} - \frac{1}{\sqrt{\gamma_1\gamma_2}} \frac{\alpha_2\gamma_1 - \alpha_1\gamma_2}{\gamma_1 - \gamma_2}, \quad q_M = \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} + \frac{1}{\sqrt{\gamma_1\gamma_2}} \frac{\alpha_2\gamma_1 - \alpha_1\gamma_2}{\gamma_1 - \gamma_2}.$$

A first remarkable fact is that one and only one of these switching points is below $\frac{\alpha_1}{\gamma_1}$. That is, $q_m < \frac{\alpha_1}{\gamma_1} < q_M$: Clearly $q_m < \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}$, and it follows from Lemma 3.1 that $\frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2} \leq \frac{\alpha_1}{\gamma_1}$, therefore the first inequality holds. The second inequality is obtained by taking into account that in this case $\gamma_2 < \gamma_1$, and by performing algebraic operations.

Now note that both $\Pi^2_1(q, r^2_1(q)) = \frac{(\alpha_1 - \gamma_1 q)^2}{4\gamma_1}$ and $\Pi^2_2(q, r^2_2(q)) = \frac{(\alpha_2 - \gamma_2 q)^2}{4\gamma_2}$ are convex parabolic functions attaining their minima at $\frac{\alpha_1}{\gamma_1}$ and $\frac{\alpha_2}{\gamma_2}$ respectively. Hence, for $q < \frac{\alpha_1}{\gamma_1}$ they are both decreasing. Since for $q = \frac{\alpha_1}{\gamma_1}$, $\Pi^2_2(q, r^2_2(q)) > \Pi^2_1(q, r^2_1(q))$, it follows that $\Pi^2_2(q, r^2_2(q)) > \Pi^2_1(q, r^2_1(q))$ for those q such that $q_m < q < \frac{\alpha_1}{\gamma_1}$. It also follows that $\Pi^2_1(q, r^2_1(q)) > \Pi^2_2(q, r^2_2(q))$ for all $q < q_m$.

As a consequence, given an action of one of the agents $q \geq 0$, the best response of the optimistic opponent is: For $q < q_m$, $r^2_{op}(q) = r^2_1(q) = r^1_{op}(q) = r^1_1(q)$. For all $q > q_m$, $r^2_{op}(q) = r^2_2(q) = r^1_{op}(q) = r^1_2(q)$.

Proof of Proposition 3.9: From Theorem 3.7 it is known that the only points which can be optimistic equilibria are the Cournot equilibria of the two scenarios. We will analyse whether they are or not:

a) $q_m < \frac{\alpha_1}{3\gamma_1}$.

   In this case, $r^2_{op}(\frac{\alpha_1}{3\gamma_1}) = r^2_2(\frac{\alpha_1}{3\gamma_1}) \neq \frac{\alpha_1}{3\gamma_1}$ and therefore $(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})$ is not an optimistic equilibria.

   On the other hand, since $q_m < \frac{\alpha_2}{3\gamma_2}$ also holds, then $r^2_{op}(\frac{\alpha_2}{3\gamma_2}) = r^2_2(\frac{\alpha_2}{3\gamma_2}) = \frac{\alpha_2}{3\gamma_2}$. Symmetrically, $r^1_{op}(\frac{\alpha_2}{3\gamma_2}) = r^1_2(\frac{\alpha_2}{3\gamma_2}) = \frac{\alpha_2}{3\gamma_2}$.

   It follows that $(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})$ is the unique optimistic equilibrium in this case.

b) $\frac{\alpha_1}{3\gamma_1} \leq q_m \leq \frac{\alpha_2}{3\gamma_2}$.

   In this case, $r^2_{op}(\frac{\alpha_1}{3\gamma_1}) = r^2_1(\frac{\alpha_1}{3\gamma_1}) = \frac{\alpha_1}{3\gamma_1}$ and symmetrically $r^1_{op}(\frac{\alpha_1}{3\gamma_1}) = \frac{\alpha_1}{3\gamma_1}$. Therefore $(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})$ is an optimistic equilibria.

   Analogous reasoning leads us to conclude that $(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})$ is also an optimistic equilibrium.

c) $q_m > \frac{\alpha_2}{3\gamma_2}$. By using an identical argument as in case b), we conclude that $(\frac{\alpha_1}{3\gamma_1}, \frac{\alpha_1}{3\gamma_1})$ is an optimistic equilibrium. By a reasoning to that of case a), it can be proven that $(\frac{\alpha_2}{3\gamma_2}, \frac{\alpha_2}{3\gamma_2})$ is not an optimistic equilibrium.

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