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RELIABILITY OF NETWORKS MODELLED BY GRAPH PRODUCTS

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A ti, estés donde estés,

porque te sentirías orgullosa

ahora, como siempre.

Porque creíste en mí

y me apoyaste.

Por tu amor incondicional.

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Abstract

A general purpose in Graph Theory is to describe any graph structure and provide all the information about it as possible. The study of invariants, properties and graph families of interest has been the aim of many researches in last years. There exist several classical lines of research in Graph Theory which have been extensively investigated. The connectivity is one of them.

The connectivity of a graph represents the minimum number of vertices whose removal disconnects the graph. This notion becomes more relevant when it is applied to networks. The structure of a graph can model whichever type of network so that the reliability of the network is related to the study of the vulnerability of the graph.

We propose one graph family, called strong product graph, and several parameters of great interest as the connectivity, the superconnectivity, the average connectivity, the Menger number, the generalized connectivity and the mean distance. In the present work, we study the mentioned indices on the strong product graph in terms of known invariants of the involved factors and we give some general sharp bounds which are best possible.

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Summary

A general purpose in Graph Theory is to describe any graph structure and provide all the information about it as possible. The study of invariants, properties and graph families of interest has been the aim of many researches in last years. There exist several classical lines of research in Graph Theory which have been extensively investigated. The connectivity is one of them.

The connectivity of a graph represents the minimum number of vertices whose removal disconnects the graph. This notion becomes more relevant when it is applied to networks. The structure of a graph can model whichever type of network so that the reliability of the network is related to the study of the vulnerability of the graph.

We propose one graph family, called strong product graph, and several parameters of great interest as connectivity-type invariants. In the present work, we study the vulnerability of the strong product graph in terms of the mentioned parameters.

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De un tiempo a esta parte
el infinito
se ha encogido
peligrosamente.
Quién iba a suponer
que segundo a segundo
cada migaja
de su pan sin límites
iba así a despeñarse
como canto rodado
en el abismo.

MARIO BENEDETTI.

Introduction

A known line of research in Graph Theory is the study of the vulnerability in graphs. It is usually related to the reliability in networks. Transport and communication, physical, biologic or social networks can be modelled by a graph. For instance, a multiprocessor system, that is, processors communicated by exchanging messages, can be modelled by a graph, where every vertex represents a processor and every edge corresponds to a communication link. To study properties on the graph can provide useful information about the working efficiency of the system.

To select or design a network, many of the requirements correspond to known measures in a graph as the order, the size, the average degree, the diameter or the connectivity, among others. Since it is almost impossible to design an optimal network for all conditions, the selection criteria must be established previously.

One of the most desirable criteria to construct a large interconnection network is joining together the requirements of high reliability and small maximum transmission delay between the nodes of the network. Hence, the priority aim is to get a strong connectivity joint to a suitable diameter in such large graphs. Another important feature or fundamental principle in networks design is its extendability, that is, the possibility of building larger structures of a network preserving desirable properties.

The graphs products are useful to obtain large graphs from smaller ones whose invariants are known or can be easily calculated. It is well known that the product of graphs are an important topic of research in Graph Theory [12, 13, 59, 60, 77, 83, 49] due to the different applications that one can find in many theoretical contexts.

For instance, graphs that have a product-like structure are used in computational engineering for the formation of finite element models or construction of localized self-equilibrating systems [62, 63, 64]. They are used in a biological context [39, 46, 80, 93] for describing the relationships between genotypes and phenotypes and estimating which combinations of properties are interconvertible over short evolutionary time-scales. The graphs products are also used in computer graphics and theoretical computer science [2, 3], where it is provided a framework, called *TopoLayout*, to draw undirected graphs based on the topological features they contain. Topological features are detected recursively, and their subgraphs are collapsed into single nodes, forming a graph hierarchy. The graphs products have a well understood structure, that can be drawn in an effective way. Then, for an extension of this framework, they are of interest.

In general, for all the applications of a practical interest, the analysis of the reliability in networks pursues to guarantee a certain robustness in the network against inaccuracies and perturbations in the data. However, this analysis have to be obtained from computer simulations or they need to be estimated from measured data. In both cases, they are known only approximately. In order to deal with such inaccuracies, exact solutions based on theoretical mathematical reasonings need to be obtained.

From the point of view of Graph Theory, a natural and interesting question is what can be said about a graph invariant in a product of graphs if it is known the corresponding invariants in the factors.

There are many works treating this problem [23, 52, 53, 54, 75, 91]. With this thesis we want to contribute in this line of research presenting new results about several vulnerability parameters which have been studied on a graph product family called *strong product*.

Among the different types of graphs products that exist, the strong product of two graphs, or simply, the strong product graph, is one of the most popular products of graphs together with the cartesian product, the direct product and the lexicographic product. All of them are constructed by a similar way. Given two connected graphs, one of them has the role of main graph and the another one has the role of the copy graph. Then these products consist on taking as many copies of the latter graph as vertices the main graph has and to establish certain adjacencies between the copies (different adjacencies for each product), whenever they exist in the main graph.

The cartesian product may be the more studied product of graphs and some of a lot of papers are [25, 65, 67, 69, 71, 77, 90, 99, 100]. In last years, the researches have been also interested in the strong product graph and it is possible to find some works related to a wide range of subjects, as the connectivity [23, 91], geodetic [26], bandwidth [66] and independency [92], among others. This thesis is focused on this family of graphs. More precisely, we study the vulnerability of the strong product graph from different points of view, attending to several parameters as the connectivity, the superconnectivity, the average connectivity, the Menger number, the generalized connectivity and the mean distance, in terms of known invariants of the involved factors.

The classical measures of reliability in a network have been the *connectivity* and *edge-connectivity*, which represent the minimum number of nodes or links that must fail to disrupt the connection in the network. Namely, for any connected graph G, the *connectivity* (resp. *edge-connectivity*), denoted by $\kappa(G)$ (resp. $\lambda(G)$)

is the minimum number of vertices (resp. edges) whose removal separates the graph. These parameters have been extensively studied and there exist lots of papers treating connectivity-type problems. A well known upper bound for these parameters was given by Whitney in [95], who proved that $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G)$ represents the minimum degree of G. It comes from the fact that to disconnect a graph it is sufficient to remove the set of adjacent vertices to any vertex or the set of incident edges in any vertex of G. When the previous inequalities become equalities, the graph is called *maximally connected*. Sufficient conditions for a graph to be maximally connected are given in terms of relevant invariants in a graph, such as the number of vertices, the minimum and maximum degrees, the diameter or the girth. Chartrand [27] proved that if G is a graph with n vertices and minimum degree $\delta(G) \geq \lfloor n/2 \rfloor$ then $\lambda(G) = \delta(G)$. Soneoka *et al.* [89] proved that a graph with girth g is maximally connected if its diameter is at most $g-3$ for even girth g, and $g-2$ for odd girth g. For more information and references we remit the reader to an interesting survey by Hellwig and Volkmann in [55].

Connectivity and edge-connectivity have been frequently used to describe the reliability of a network, but these parameters present a weakness: they do not take into account what remains after the graph is disconnected. One can ask other questions as, for instance, how many components appear after disrupting the graph or what is the size of the smallest remaining connected component.

To deal with these problems, a stronger measure of connectivity is the *superconnectivity*, which was introduced in 1984 by Boesch and Tindell [20]. A graph is called *superconnected* if every minimum disconnecting set is the neighborhood of a vertex. Sufficient conditions for a bipartite graph to be superconnected in terms of the diameter and the order can be found in [7]. Meng [78] treated this parameter in line graphs, providing necessary and sufficient conditions for the line graph of a connected regular graph to be superconnected. Balbuena *et al.*

in [8] studied the superconnectivity in graphs with given girth. There exist more different works (see for instance [6, 9, 10, 11, 44]).

Connectivity and superconnectivity have been also studied in products of graphs, as the cartesian product of graphs [90], the permutation graphs [84], the product of Bermond of graphs [14], the direct product of graphs [24], the Kronecker product of graphs [47] and the matched sum graphs [15]. In most of these papers, general bounds on the index of connectivity are obtained in terms of the connectivity, the order and the minimum degree of the factors. These bounds show us that the graphs product structure generally produces a large and more reliable network without a strong increase of links.

Concerning the strong product graph, its connectivity was studied in 2010 by Špacapan [91]. The author gave a lower bound involving the connectivities and the minimum degrees of the graphs. In this thesis we show that this lower bound can be improved. Indeed, we give a general lower bound which is sharp under certain requirements on the factors. In addition, we prove that the strong product of two maximally connected graphs is not only maximally connected but also superconnected.

The connectivity, the maximal connectivity and the superconnectivity are worst-case measures on graphs and they do not always reflect what really happens throughout the graph. For instance, consider a tree and a complete graph joint to a pendant vertex. They are two graphs with connectivity one. However, the first graph is clearly more vulnerable than the second one, since in this last graph only the fail of a particular vertex may disrupt it, while in the tree, there exists several possibilities for disconnecting it removing just a vertex. Thus, it is of interest to extend the study of the connectivity between every pair of vertices of a graph, using other measures of vulnerability [4], which provide a more suitable information about the global connectedness of a graph.

Whitney in 1932 was inspired by the well-known theorems of Menger [79] and he established an equivalency [95] between the connectivity of a graph and the existence of internally disjoint paths. More precisely, he proved that a graph is r-connected, that is, the connectivity is at least r, if and only if every pair of vertices is connected by at least r pairwise disjoint paths.

Thanks to this characterization, Beineke, Oellermann and Pippert in [16] introduced the *average connectivity*, defined as the mean on the number of pairwise disjoint paths that exists between any two vertices in a graph. This index represents the expected number of vertices that must fail to separate the graph. There are two papers where the average connectivity has been treated in depth. In the first one, Beineke, Oellermann and Pippert [16] gave upper and lower bounds on the average connectivity of any connected graph in terms of its order, its size and the degree sequence of its vertices. They deduced that the average connectivity is upper bounded by the average degree just as the connectivity is upper bounded by the minimum degree, as well as they determined the maximum value of the average connectivity in a graph with given order and size. In the second one, Dankelmann and Oellermann [34] obtained new sharp bounds on the average connectivity of a graph involving its order, its size, its average degree and its chromatic number. They also obtained several bounds for some families of graphs, such as planar and outerplanar graphs and the cartesian product of two connected graphs.

The average connectivity has not been as extensively investigated as the classical connectivity, but one can find some other interesting works (see for instance [5, 56, 57, 61]). Concerning the graphs products, as far as we know, results on average connectivity are proved for the cartesian product of two graphs in [34] by Dankelmann and Oellermann. Namely, a sharp lower bound involving the average connectivity, the order and the size of the factor graphs is given. In this thesis we study the average connectivity of the strong product graph.

A general lower bound in terms of the order, the average connectivity and the average degree of the involved graphs is determined. In addiction, we prove a sufficient condition which assures that the average connectivity of the strong product graph attains its maximum value, that is, it is equal to its average degree.

Another index close to the average connectivity is the *Menger number*, which is defined as the minimum number of pairwise disjoint paths of bounded length between any two vertices in a connected graph. In a parallel computing system, the efficiency can be analyzed in terms of the number of disjoint routes of information which are able to connect two points in a short period of time. In a real-time system, the information delay must be limited since any message obtained beyond the bound may be worthless. Hence, a natural question is how many routes ensure the transmission of information in an effective time. For this aim, the Menger number can be an interesting measure of the communication efficiency in an information system modelled by a graph, since after a certain length, a route may be inefficient.

Lovász, Neumann-Lara and Plummer [74] introduced in 1978 the Menger number by proposing initially the following problem. They defined two indices on a connected graph. One index, denoted by A_ℓ, is the maximum number of vertex-disjoint paths connecting any two vertices in the graph whose lengths do not exceed a certain quantity ℓ, it is the now called Menger number between two vertices in a graph. The another one, denoted by V_ℓ, is the minimum number of vertices whose deletion disrupt all the paths of length at most ℓ joining two vertices in the graph. In general, $A_\ell \leq V_\ell$ and when $A_\ell = V_\ell$, it is the Menger's theorem properly. The authors in [74] focused on studying the ratio V_ℓ/A_ℓ, for which it is trivial that $1 \leq V_\ell/A_\ell \leq \ell - 1$. They showed that $V_\ell/A_\ell \leq \lfloor \ell/2 \rfloor$, for every pair of vertices in the graph and any positive integer ℓ, giving a family of graphs for which this bound is sharp. Since then, many Mengerian-type results have been obtained [17, 51, 86, 88, 102]. In 1983 Chung [30] deduced the lower bound $\lfloor(\ell+1)/3\rfloor \leq V_\ell/A_\ell$. In [42] the authors proved that if a graph with order n and minimum degree at least $\lfloor(n-k+2)/\lfloor(\ell+4)/3\rfloor\rfloor + k - 2$ is k-connected, then such graph has k vertex-disjoint paths of length at most ℓ between any pair of vertices. Hsu and Luczak in [58] treated a related parameter called the k-diameter of a graph, that is, the minimum on the k-distances in a connected graph, being the k-distances the minimum value on the lengths of the internally disjoint paths with length at most k, between any pair of vertices in the graph. They studied the k-diameter of k-regular and k-connected graphs, proving that every k-regular k-connected graph on n vertices has k-diameter at most $\lfloor n/2\rfloor$.

About the graphs products, in 2011 Ma, Xu, and Zhu [77] proved that the Menger number of the cartesian product of two connected graphs is lower bounded by the sum of the Menger numbers of the factor graphs. In this thesis, we study the Menger number of the strong product graph, giving two sharp lower bounds depending on the permitted length of the paths. In addition, we give the exact Menger number for strong products of paths, of cycles and of path with cycle. We complete this part by studying the *average Menger number*, which represents, in any connected graph, the expected number of pairwise disjoint paths with a prescribed length that exist between any two vertices.

It is usually interesting to know, not only how connected two vertices are in a graph but also, in general, how connected k vertices are, for every integer $k \geq 2$. The index to measure such local connectivity is he so-called *generalized k-connectivity*. This parameter is a natural extension of the classical connectivity. The generalized k-connectivity, roughly speaking, can be defined as the capability of a network to connect any set of k nodes themselves.

If one wants to connect a pair of vertices in a graph, then the minimal structure to join them is a path. However, if one wants to connect a set of k vertices, then the minimal structure that must be used is a tree. This kind of tree for connecting a set of vertices in a graph is called a *spanning tree*.

The generalized k-connectivity was introduced in 2010 by Chartrand, Okamoto and Zhang [29]. A graph is generalized k-connected if and only there exist at least k pairwise disjoint trees connecting any set of k vertices in the graph.

Some results concerning this parameter have been published recently. In [29] it was proved that the generalized k-connectivity of a complete graph on n vertices is exactly equal to $n - \lceil k/2 \rceil$ and besides, this notion is related to the rainbow trees in a connected graph. Li, Li and Zhou [72] focused on the generalized 3-connectivity and its relationship with the classical connectivity. They stated that the generalized 3-connectivity of any graph is upper bounded by its connectivity. Moreover, they proved that the generalized 3-connectivity of a planar graph G is always the value $\kappa(G)$ or $\kappa(G)-1$. The generalized k-connectivity in complete bipartite graphs was treated in [81]. The authors first obtained the number of edge-disjoint spanning trees of a complete bipartite graph and determined specifically such trees. Then, based on these results, they obtained the generalized k-connectivity of complete bipartite graphs, for any k. Recently, Li and Li [70] in 2012 analyzed the complexity of determining the generalized k-connectivity in any connected graph, proving that the corresponding computational problem of deciding if a graph is generalized k-connected is NP-complete.

The only paper treating the generalized k-connectivity on graphs products is [71], where Li, Li and Sun obtained two sharp lower bounds on the generalized 3-connectivity of the cartesian product graph, depending on a certain condition in the main graph. In any case, they proved that the generalized 3-connectivity of the cartesian product graph is lower bounded either by the sum of the generalized 3-connectivities of the factor graphs or the same sum minus one unity. In this thesis we study the generalized 3-connectivity of the strong product graph.

We obtain sharp lower bounds, depending on the considered assumptions, in terms of the classical connectivity and the generalized 3-connectivity of the involved graphs.

An essential requirement in a realtime communication network, for instance, in a travel or the sending of a message, is to have a certain control on the commuting time. For this reason, we finish this thesis by studing the *Wiener index*, parameter directly related to the *average distance*.

The diameter, the radius and the eccentricity are well studied notions in Graph Theory related to the distances in a graph. However, sometimes this concepts may not be meaningful in a graph description. For instance, the diameter of a graph represents the maximum of the distances between every pair of vertices in the graph. One could think that in a graph with a large diameter, most of the distances are large too. Nevertheless, it is not true in general. This is a reason for which it is of interest the study of other parameters which leads us to have a better idea on the distances in a graph.

The *Wiener index* or *Wiener number* of a connected graph G, denoted by $W(G)$, is defined as the total sum of the distances between every the unordered pair of vertices in a connected graph. The study of this parameter began with the chemist Wiener in his chemical work [96]. Wiener noticed that, in a molecular network, the melting point of certain hydrocarbons is proportional to the sum of the distances between the unordered pairs of vertices in the corresponding connected graph which models such network.

This graphical invariant has been studied by many researchers under different names such as *total status* or *sum of all distances*. In 1976, Entringer, Jackson and Snyder published [40], the first mathematical paper about the Wiener index, although the authors did not give it this name, they just called it *the distance of a graph*. The most relevant results were the sharp lower and upper bounds which were found. They deduced that the Wiener index of a connected graph is lower bounded by the Wiener index of a complete graph with the same order and upper bounded by the Wiener index of a path with the same order. Moreover, these bounds are just attained when the graph is exactly the complete graph and the path, respectively. This means that, fixed an order, the Wiener index are greater for sparse graphs than for dense graphs. They also gave the exact values of this parameter in paths, cycles, complete graphs and complete bipartite graphs.

In [94] several lower and upper bounds on the Wiener index of a graph are obtained in terms of the order, the size, the radius, the diameter, the independence number, the connectivity or the chromatic number. For instance, it was proved that the Wiener index of a connected graph with n vertices, m edges and diameter at most two is, exactly, $n^2 - n - m$, and if the diameter is at least three, it is at least $n^2 - n - m + 1$. Gutman and Zhang [50] determined the only graphs on n vertices and a given (vertex or edge) connectivity k having minimum Wiener index. This graph is $K_k + (K_1 \cup K_{n-k-1})$, which is the graph obtained by joining every vertex of the complete graph K_k to one isolated vertex and every vertex of the complete graph K_{n-k-1}. Wu [97] showed that the Wiener index of the line graph of G is greater than or equal to the Wiener index of G, for any connected graph with minimum degree at least two. For more information, we refer the reader to two interesting surveys [36, 73]. The first one is mainly focused on the Wiener index in trees, but it contains other general results as well as many references about this topic. The second one collects the most recent results on the Wiener index.

Considering the arithmetic mean on the total sum of the distances between all the ordered pairs of vertices in a connected graph, one can obtain the *average distance* or the *mean distance*. This parameter was introduced in 1977 by Doyle and Graver [38] and it can be seen as a natural measure of the graph compactness.

The average distance have many applications. For instance, to indicate the average delay of the messages in the processors interconnections or to compare the compactness of architectural plans. This is a extensive topic of research (see for instance [1, 31, 32, 33, 43, 68]). It is evident the direct relationship between the average distance and the Wiener index. To obtain the average distance of a graph on n vertices it is sufficient to consider twice the Wiener index (this quantity is usually called *transmission*) and to divide by $n(n-1)$, that is, the number of ordered pairs of vertices in the graph. Hence, the real difficulty in this issue is to determine the Wiener index of the considered graph.

For small graphs, one can compute directly the exact values of this parameter. Indeed, for families of graphs as the complete graphs, the paths and the cycles, the exact values of the Wiener index and the average distance were obtained, as well as general bounds [40]. Concerning graphs products, it was proved in [49] that the Wiener index of the cartesian product of two connected graphs G_1 and G_2 is exactly $|V(G_2)|^2W(G_1) + |V(G_1)|^2W(G_2)$. Concerning the strong product, in [37] it was studied relationship between the Wiener index and the clique number and the stability number of the factors. Pattabiraman and Paulraja in [83] gave several exact values of the Wiener index of the strong product of a complete r-multipartite graph and any connected graph in terms of the order and the size of the involved graphs.

In this thesis we find bounds on the Wiener index of the strong product of two connected graphs. We also give the exact values for certain strong products, as paths and cycles, only in terms of their orders. Moreover, we prove that the Wiener index of the strong product of two connected graphs is upper bounded by the Wiener index of the strong product of two paths which have the same orders as the factors.

This thesis has been structured in six chapters as follows.

Chapter 1 is devoted to introduce several basic notations and general results on Graph Theory which will be useful throughout this memory.

In *Chapter* 2 we deal with the *connectivity* and *superconnectivity* of the strong product graph. The known lower bound on the connectivity of this family of graphs is improved and we give sufficient conditions for the strong product of two connected graphs to be maximally connected and superconnected.

In *Chapter* 3 we pay attention to the *Menger number*, the *average connectivity* and the *average Menger number*. Some sharp lower bounds for these parameters are given. We also prove sufficient conditions to guarantee that the average connectivity and the average Menger number attain their maximum value.

Chapter 4 is focused on the generalized 3-connectivity of the strong product of two graphs. We give a sharp lower bound, which is best possible when a factor has generalized 3-connectivity equal to one.

In *Chapter* 5 we give general bounds for the Wiener index of the strong product of two graphs and the exact value when the factors have diameter not too large.

In *Chapter* 6 we summarize the main conclusions of this thesis and moreover, some open problems are described.

Finally, at the end of this thesis, the reader can find the bibliography which has been the source of support and inspiration for the development of this research work.

Contents

Chapter 1

Preliminaries

This chapter is devoted to introduce basic notions which will be used throughout this thesis. First, we recall some classical concepts on Graph Theory and fix the notation. Second, we make a brief introduction about the product graphs topic.

1.1 Basic notions

Those elemental notions not explicitly included here can be found in the books by Chartrand and Lesniak [28] and by Diestel [35]. Special notations and definitions will be presented where needed.

A *graph* G is a pair (V, E) where $V = V(G)$ is a nonempty set of *vertices*, $E = E(G)$ is a set of *edges* and where every edge joins a non ordered pair of vertices in G. Their cardinalities, denoted by $|V(G)|$ and $|E(G)|$, are called the *order* and the *size* of G, respectively. Two vertices of G, x and y, are *adjacent* if there exists an edge joining them. Such edge is denoted by $e = xy$ or $e = yx$ and then, it is said that e and x are *incident* (also e and y are incident). Two non adjacent vertices or edges are called *independent*.

Throughout this thesis we only consider *finite graphs*, that is, the sets $V(G)$ and $E(G)$ are finite, *simple graphs*, that means, at most one edge $e = xy$ can exist in G for every pair $x, y \in V(G)$ (we consider no loops), and also *undirected graphs*, graphs for which every edge joins an unordered pair of vertices of G.

Given a graph $G = (V, E)$, a *subgraph* of G is any graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. If $E' = \{xy \in E : x, y \in V'\}$, then G' is said to be *induced* by V' and we write $G' = G[V']$. When $V' = V$, the subgraph (not necessarily induced) is a *spanning subgraph* of G. Given two graphs G and G', their *union* is the graph $G \cup G' = (V \cup V', E \cup E')$. If $V \cap V' = \emptyset$, then G and G' are *vertex disjoint*. If $E \cap E' = \emptyset$, then G and G' are *edge disjoint*.

Let $x, y \in V(G)$ be two distinct vertices. A *path* from x to y, also called an xy-path, is a subgraph $\mathcal{P}$ with vertex set $V(\mathcal{P}) = \{x = u_0, u_1, \ldots, u_r = y\}$ and edge set $E(\mathcal{P}) = \{u_0u_1, \ldots, u_{r-1}u_r\}$, such that $u_i \neq u_j$, for all $i, j = 0, \ldots, r$ with $i \neq j$. This path is usually denoted by $\mathcal{P} : u_0u_1 \ldots u_r$ and r is the *length* of $\mathcal{P}$, denoted by $l(\mathcal{P})$. We also use $\mathcal{P}_r$ to denote such path. Vertices u_0 and u_r are called the *end vertices* and vertices of $\{u_1, \ldots, u_{r-1}\}$ are called *internal vertices*.

Two given xy-paths $\mathcal{P}$ and $\mathcal{Q}$ such that $V(\mathcal{P}) \cap V(\mathcal{Q}) = \{x, y\}$ are said to be *internally disjoint* (see Figure 1.1). A *cycle* in G of length r is a path $\mathcal{C} : u_0u_1 \ldots u_r$ such that $u_0 = u_r$. The *girth* of a graph G, denoted by $g(G)$, is the length of a shortest cycle in G. If G contains no cycles, then $g(G) = \infty$ is adopted.

The *distance* between two vertices $x, y \in V(G)$, denoted by $d_G(x, y)$, is the length of a shortest xy-path. If there is no xy-path in G, $d_G(x, y) = \infty$ is assumed. The *eccentricity* of a vertex x in G is the maximum of the distances from x to every vertex of G, that is, $ecc_G(x) = \max\{d_G(x, y) : y \in V(G)\}$.

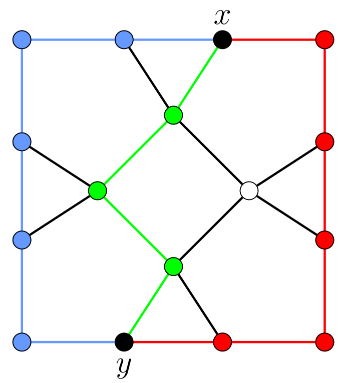

Figure 1.1: Three internally disjoint xy-paths in a connected graph.

The *diameter* of G is defined as $D(G) = \max\{d_G(x,y) : x,y \in V(G)\}$, the maximum of the distances between every pair of vertices in G. In other words, the diameter of G is the maximum value of the eccentricities of every vertex in G.

The *i-neighborhood* of x in G is $N_G^i(x) = \{y \in V(G) : d_G(x,y) = i\}$, with $i \geq 1$ (see Figure 1.2). For $i = 1$ we directly write $N_G^1(x) = N_G(x)$ and it is called the *neighborhood* of $x \in V(G)$. By $N_G[x]$ we denote the *closed neighborhood* of x in G, that is, $N_G(x) \cup \{x\}$. For any subset $W \subseteq V(G)$, $N_G(W) = \bigcup_{x \in W} N_G(x) \setminus W$ and $N_G[W] = N_G(W) \cup W$.

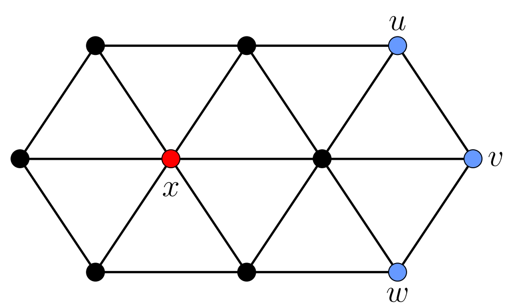

Figure 1.2: A graph G with $D(G) = 3$ and $N_G^2(x) = \{u,v,w\}$.

The *degree* of a vertex x in G is $d_G(x) = |N_G(x)|$, where the *minimum degree* of G is $\delta(G) = \min\{d_G(x) : x \in V(G)\}$ and the *maximum degree* of G is

$\Delta(G) = \max\{d_G(x) : x \in V(G)\}$. The *average degree* of G is defined as

$$\overline{d}(G) = \frac{1}{|V(G)|} \sum_{x \in V(G)} d_G(x).$$

Clearly, $\delta(G) \leq \overline{d}(G) \leq \Delta(G)$ (see Figure 1.3). When $\delta(G) = \Delta(G) = d$, the graph G is called *d-regular*.

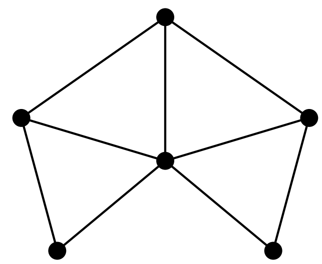

Figure 1.3: A graph G with $\delta(G) = 2$, $\Delta(G) = 5$ and $\overline{d}(G) = 3$.

It is well-known that any graph G without loops verifies the equality

$$\sum_{x \in V(G)} d_G(x) = 2|E(G)|.$$

Therefore,

$$\overline{d}(G) = \frac{2|E(G)|}{|V(G)|}.$$

Given a graph $G = (V, E)$ and a subset $S \subset V$, we denote by $G - S$ the induced subgraph $G[V \setminus S]$, that is, $G - S$ is obtained from G by removing the vertex set S and their incident edges (see Figure 1.4). Similarly, for any given subset $W \subset E(G)$, the deletion of W from G yields another graph $G - W$ obtained by removing all the edges of W (see Figure 1.5).

A graph G is said to be *connected* if for every pair of vertices there exists a path connecting them or, in other words, when G has finite diameter. A connected graph containing no cycles (*acyclic* graph) is called a *tree*. The vertices of degree

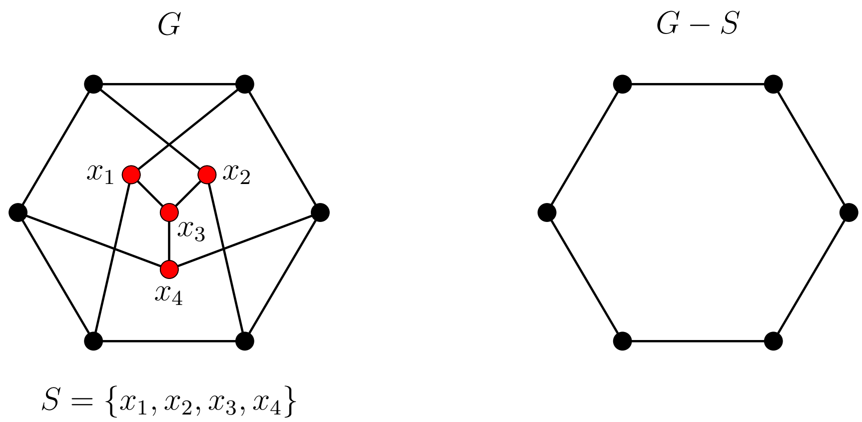

Figure 1.4: Deletion of a vertex set S in a graph G.

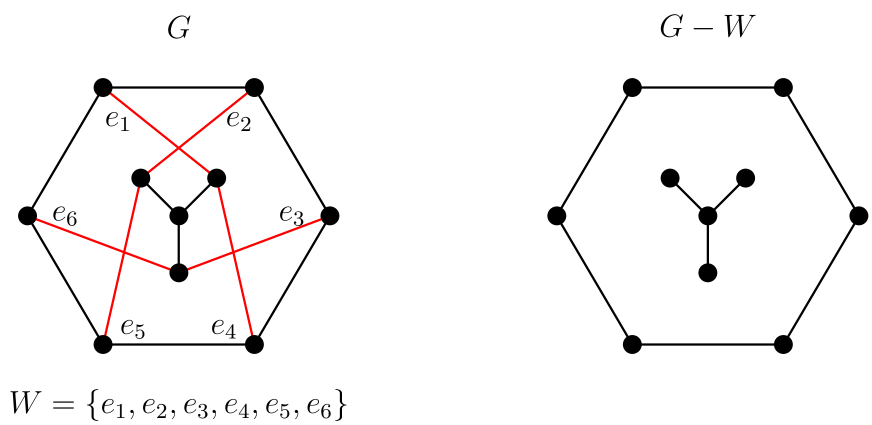

Figure 1.5: Deletion of an edge set W in a graph G.

one in a tree are called *leaves*. Observe that every tree having more than one vertex has at least two leaves. Since trees are acyclic graphs, every two vertices of a tree can be joined by an unique path. Sometimes it is interesting to consider one vertex r of a tree T as special. Such vertex is called the *root* of T and T is said to be an *r-rooted* tree.

A *cut set* of a connected graph G is a set S of vertices such that $G - S$ is not connected or is an isolated vertex. The different parts in which G is separated after removing a cut set S are called *components*. Then a *component* C of $G - S$ is a maximal connected subgraph of $G - S$, where maximal means that no other connected subgraph of $G - S$ contains C as subgraph.

A connected graph is called *k-connected* if every cut set has cardinality at least k. The *connectivity* of a graph G is the maximum integer κ such that G is κ-connected. It is denoted by $\kappa(G)$ and defined as

$$\kappa(G) = \min\{|S| : S \subseteq V(G) \text{ and } G - S \text{ is not connected or an isolated vertex}\}.$$

In other words, the connectivity of a graph G is the smallest number of vertices whose deletion from G produces a disconnected or a trivial graph (see Figure 1.6). Complete graphs, $K_n, n \geq 1$, are the only graphs which cannot be separated in two or more components after removing any set of vertices. Thus, $\kappa(K_n) = n - 1$ is adopted.

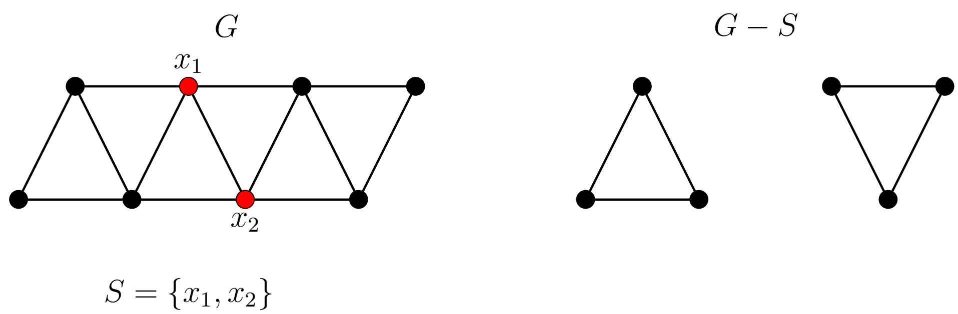

Figure 1.6: A 2-connected graph and its components after deleting S.

The *minimum cut sets* of a graph G, also called the *κ-sets*, are those having cardinality $\kappa = \kappa(G)$. A subset S of vertices of G is a *minimal cut set* if and only if $S - \{x\}$, for every $x \in S$, is not a cut set of G. If S is a minimal cut set of a graph G, then every vertex in S must be adjacent to some vertex of each component of $G - S$. Observe that every minimum cut set is a minimal cut set. Notice also that the neighborhood of any vertex $x \in V(G)$ is a cut set of G. Thus, $\kappa(G) \leq \delta(G)$ clearly holds. A graph G is called *maximally connected* if $\kappa(G) = \delta(G)$.

Going one step further, a graph G is *superconnected* if it is maximally connected and every minimum cut set consists of the neighborhood of some vertex of degree $\delta(G)$ which does not belong to the cut set (see Figure 1.7).

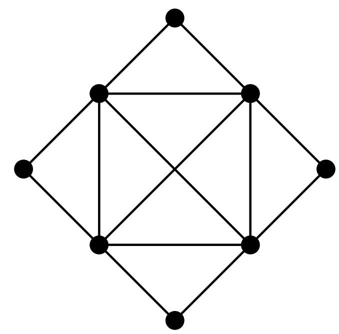

Figure 1.7: A superconnected graph.

From the theorem of Menger [79], Whitney [95] deduced the following characterization on the k-connected graphs. A graph G is called k-connected, that is, $\kappa(G) \geq k$, if and only if every pair of vertices is connected by at least k internally disjoint paths. The connectivity between two distinct vertices x and y in G, is denoted by $\kappa_G(x, y)$ and represents the maximum number of pairwise internally disjoint xy-paths in G (see Figure 1.8). Thus, the connectivity of a graph can be seen as

$$\kappa(G) = \min\{\kappa_G(x, y) : x, y \in V(G)\}.$$

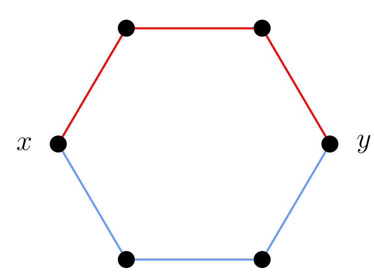

Figure 1.8: $\kappa_{\mathcal{C}}(x, y) = 2$, for any two vertices x, y in a cycle $\mathcal{C}$.

If n is the order of a connected graph G, the *average connectivity* of G is the mean of the connectivities between all the pairs of vertices in G. It is denoted by $\overline{\kappa}(G)$ and defined as

$$\overline{\kappa}(G) = \frac{1}{\binom{n}{2}} \sum_{x,y \in V(G)} \kappa_G(x,y),$$

where the pair of vertices are taken non ordered and $K(G) = \sum\limits_{x,y \in V(G)} \kappa_G(x,y)$ is called the *total connectivity* of G.

Given two distinct vertices x, y of G, the *xy-Menger number* with respect to a positive integer ℓ, denoted by $\zeta_\ell(x,y)$, is the maximum number of internally disjoint xy-paths in G whose lengths are at most ℓ. The *Menger number* of G with respect to ℓ is defined as

$$\zeta_\ell(G) = \min\{\zeta_\ell(x,y) : x, y \in V(G)\}.$$

Similarly to the average connectivity, if n is the order of a connected graph G, the *average Menger number* of G with respect to ℓ is defined as

$$\overline{\zeta}_\ell(G) = \frac{1}{\binom{n}{2}} \sum_{x,y \in V(G)} \zeta_\ell(x,y),$$

where the pair of vertices are also taken non ordered and $Z_\ell(G) = \sum\limits_{x,y \in V(G)} \zeta_\ell(x,y)$ is called the *total Menger number* of G.

Consider a subset of vertices $S = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$, of a connected graph G. A tree T in G is called an S-tree in G (or a $\{x_1, x_2, \ldots, x_k\}$-tree) if $S \subseteq V(T)$. Trees $T_1, T_2, \ldots, T_r$ are r internally disjoint S-trees in G when $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for any pair of integers i and j, with $1 \leq i < j \leq r$. For instance, in Figure 1.9 we depict a Tutte's wheel on seven vertices, denoted by $W_{1,6}$. If we choose the set of vertices $S = \{x, y, z\}$, then we can obtain at most three internally disjoint S-trees.

Denoting by $\kappa(S)$ the greatest number of internally disjoint S-trees in G, for an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* of G, $\kappa_k(G)$, is defined as

$$\kappa_k(G) = \min\{\kappa(S) : S \subseteq V(G) \text{ and } |S| = k\}.$$

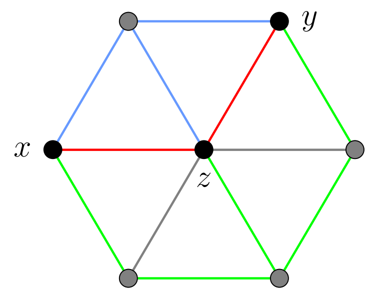

Figure 1.9: Three $\{x, y, z\}$-trees in the Tutte's wheel $W_{1,6}$.

For any connected graph G, the *Wiener index* of G, denoted by $W(G)$, is defined as

$$W(G) = \frac{1}{2} \sum_{x,y \in V(G)} d_G(x, y),$$

where the sum is taken through all ordered pairs of vertices of G.

The *average distance* of a connected graph G is defined as the mean on the distances between all the ordered pairs of vertices of G, that is,

$$\mu(G) = \frac{1}{2\binom{n}{2}} \sum_{x,y \in V(G)} d_G(x, y) = \frac{1}{n(n-1)} \sum_{x,y \in V(G)} d_G(x, y).$$

Finally, the *hyper-Wiener index* of G is defined as

$$WW(G) = \frac{1}{4} \sum_{x,y \in V(G)} \left(d_G(x, y) + d_G^2(x, y)\right),$$

where the sum is also taken through all ordered pairs of vertices of G.

1.2 Graph products

The construction of new graphs from two given ones is not unusual at all. Basically, the method consists on joining together several *copies* of one graph according to the structure of another one, the latter is usually called the *main graph* of the construction and both of them are called the *generator graphs* of the product.

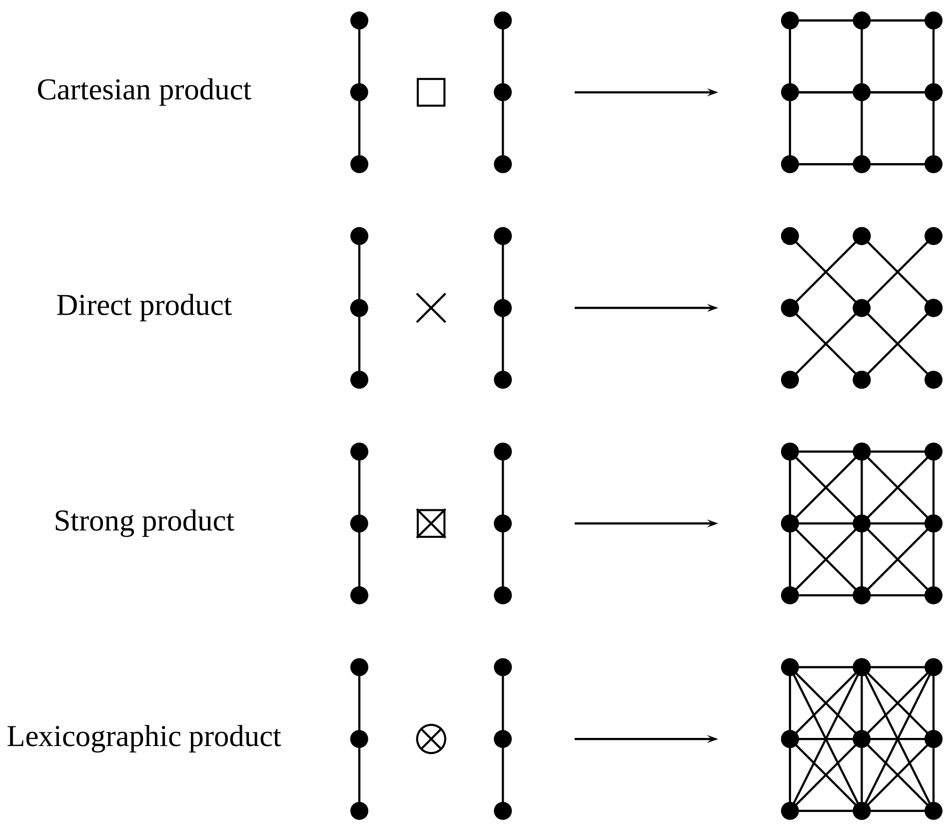

Figure 1.10: Main graphs products.

Bermond *et al.* introduced in [18] a *compound graph* $G[\Gamma]$ on the graphs Γ and G which is obtained by replacing each vertex of Γ by a copy of G plus one edge between the copies G^x, G^y (corresponding to vertices $x, y \in V(\Gamma)$) whenever x, y are adjacent vertices in Γ. Other similar types of compound graphs have been proposed (see for example [45, 60]), the difference between them coming from the number of edges between the copies graphs.

One of these types of compound graphs is the *strong product* of two given graphs, which is subject of study in this thesis. The strong product graph is one of the four standard graph products, together with the *cartesian product*, the *direct product* and the *lexicographic product* of two graphs (see Figure 1.10).

The *cartesian product* of two connected graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, denoted by $G_1 \square G_2$, has $V(G_1) \times V(G_2)$ as vertex set, such that two distinct vertices (x_1, x_2) and (y_1, y_2) of $G_1 \square G_2$ are adjacent if $x_1 = y_1$ and $x_2 y_2 \in E(G_2)$ or $x_1 y_1 \in E(G_1)$ and $x_2 = y_2$. This family of graphs has been extensively studied [25, 65, 67, 69, 71, 77, 90, 99, 100].

It is easy to deduce from the definition of the cartesian product some basic properties, as we show in next remark. The first one is on the degree of a vertex of $G_1 \square G_2$ and the second one is about the distance between two vertices.

Remark 1.2.1. *Let $(x_1, x_2), (y_1, y_2)$ be two distinct vertices in $V(G_1 \square G_2)$.*

(i) $d_{G_1 \square G_2}((x_1, x_2)) = d_{G_1}(x_1) + d_{G_2}(x_2)$. *Then* $\delta(G_1 \square G_2) = \delta(G_1) + \delta(G_2)$ *and* $\Delta(G_1 \square G_2) = \Delta(G_1) + \Delta(G_2)$.

(ii) $d_{G_1 \square G_2}((x_1, x_2), (y_1, y_2)) = d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2)$. *Then, it follows that* $D(G_1 \square G_2) = D(G_1) + D(G_2)$.

For instance, in the cartesian product $\mathcal{P}_3 \square \mathcal{P}_5$, where $\mathcal{P}_3$ and $\mathcal{P}_5$ are paths of length 3 and 5, respectively, it is clear that
$\delta(\mathcal{P}_3 \square \mathcal{P}_5) = 2$ (see the green vertex in Figure 1.11),
$\Delta(\mathcal{P}_3 \square \mathcal{P}_5) = 4$ (see the red vertex in Figure 1.11) and
$D(\mathcal{P}_3 \square \mathcal{P}_5) = 8$ (see the pair of blue vertices (x_1, x_2) and (y_1, y_2) in Figure 1.11).

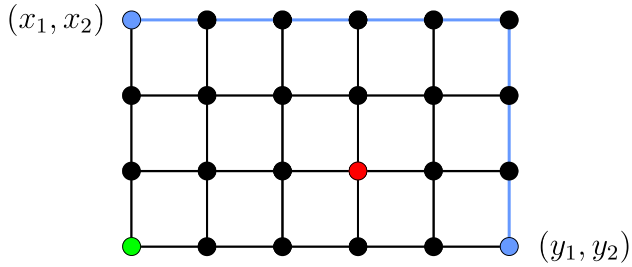

Figure 1.11: The cartesian product of two paths of length 3 and 5.

In last years many researches have been interested in the *strong product* of graphs. There are several works where different invariants and properties of this family are treated [23, 26, 66, 91, 92]. The *strong product of two connected graphs*, or simply, the *strong product graph*, was introduced in 1960 by Sabidussi [87] as follows.

Definition 1.2.1. ([87]) *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two connected graphs. The strong product $G_1 \boxtimes G_2$ of G_1 and G_2 has as vertex set $V(G_1) \times V(G_2)$, so that two distinct vertices (x_1, x_2) and (y_1, y_2) of $G_1 \boxtimes G_2$ are adjacent if $x_1 = y_1$ and $x_2y_2 \in E(G_2)$, or $x_1y_1 \in E(G_1)$ and $x_2 = y_2$, or $x_1y_1 \in E(G_1)$ and $x_2y_2 \in E(G_2)$.*

In this family of graphs, two kinds of edges are distinguished: the *copy edges*, which are the internal edges of every copy of G_2, and the *intercopy edges*, which are the external edges between the copies of G_2. Notice that for every edge $x_1y_1 \in E(G_1)$ and every vertex $x_2 \in V(G_2)$, the vertex (x_1, x_2) is adjacent to (y_1, x_2) in $G_1 \boxtimes G_2$ (see the red edge in Figure 1.12), and also to each vertex of $\bigcup_{v \in N_{G_2}(x_2)} (y_1, v)$, and reciprocally, the vertex (y_1, x_2) is adjacent to each vertex of $\bigcup_{v \in N_{G_2}(x_2)} (x_1, v)$ (see the blue edges in Figure 1.12). The former intercopy edges are called the *cartesian edges* (red edge) and the other intercopy edges are called the *non cartesian edges* (blue edges).

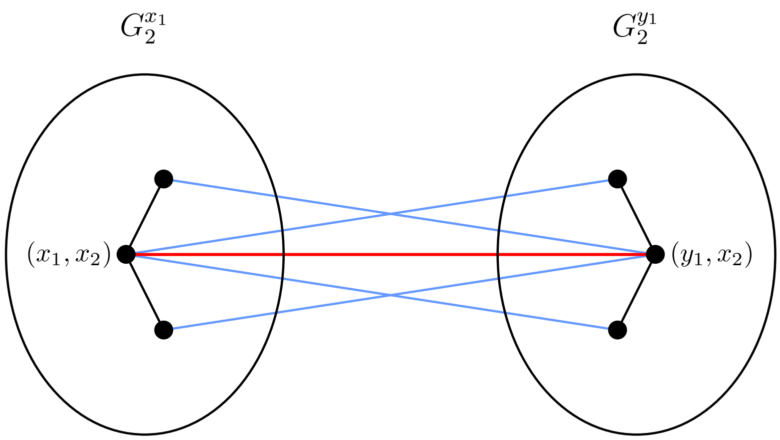

Figure 1.12: Two kinds of intercopy edges in the strong product graph.

Notice that the cartesian product graph is a subgraph of the strong product graph. In addition, from Definition 1.2.1, it clearly follows that the strong product of two graphs is commutative (see Figure 1.13).

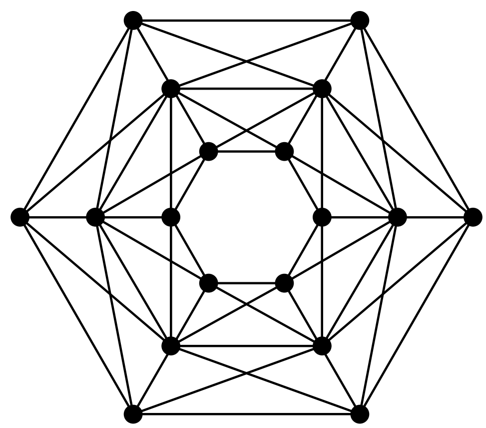

Figure 1.13: The strong product of a path and a cycle of order 3 and 6, respectively.

Observe that for every $x_2 \in V(G_2)$, the subgraph of $G_1 \boxtimes G_2$ induced by the set $\{(u, x_2) : u \in V(G_1)\}$ is isomorphic to G_1. This subgraph will be denoted by $G_1^{x_2}$. Analogously, for each $x_1 \in V(G_1)$, the set $\{(x_1, v) : v \in V(G_2)\}$ induces a subgraph isomorphic to G_2, which will be denoted by $G_2^{x_1}$. Thus, $G_1 \boxtimes G_2$ can be constructed by considering $|V(G_1)|$ copies of G_2, $G_2^{x_1}, \ldots, G_2^{x_n}$, corresponding to the set of vertices $V(G_1) = \{x_1, \ldots, x_n\}$, which are interconnected according to the definition.

Some relationships between the minimum degree and the maximum degree of $G_1 \boxtimes G_2$ in terms of the corresponding parameters of G_1 and G_2 can be found in [60]. We present some of them in the following remark, which directly comes from the definition.

Remark 1.2.2. ([60]) *Let G_1 and G_2 be two connected graphs. Let $\Delta(G_i)$ and $\delta(G_i)$ be the maximum and the minimum degree of G_i, for $i = 1, 2$. Let (x_1, x_2) be any vertex of $G_1 \boxtimes G_2$. Then*

$$d_{G_1 \boxtimes G_2}((x_1, x_2)) = d_{G_1}(x_1) d_{G_2}(x_2) + d_{G_1}(x_1) + d_{G_2}(x_2).$$

As a consequence, it follows that $\delta(G_1 \boxtimes G_2) = \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2)$ and $\Delta(G_1 \boxtimes G_2) = \Delta(G_1)\Delta(G_2) + \Delta(G_1) + \Delta(G_2)$.

It is important to know the distances between the vertices in $G_1 \boxtimes G_2$ in terms of the distances in the generator graphs G_1 and G_2. The following result was also proved in [60] and will be used in this thesis.

Lemma 1.2.1. ([60]) *For any pair of vertices (x_1, x_2) and (y_1, y_2) of $G_1 \boxtimes G_2$,*

$$d_{G_1 \boxtimes G_2}((x_1, x_2), (y_1, y_2)) = \max\{d_{G_1}(x_1, y_1), d_{G_2}(x_2, y_2)\}.$$

As a consequence of the previous lemma, we deduce that the diameter of the strong product graph is $D(G_1 \boxtimes G_2) = \max\{D(G_1), D(G_2)\}$.

For instance, in the strong product $\mathcal{P}_3 \boxtimes \mathcal{P}_5$, where $\mathcal{P}_3$ and $\mathcal{P}_5$ are the paths of length 3 and 5, respectively, we can easily observe that
$\delta(\mathcal{P}_3 \boxtimes \mathcal{P}_5) = 3$ (see the green vertex in Figure 1.14),
$\Delta(\mathcal{P}_3 \boxtimes \mathcal{P}_5) = 8$ (see the red vertex in Figure 1.14) and

$D(\mathcal{P}_3 \boxtimes \mathcal{P}_5) = \max\{3, 5\} = 5$ (see the blue vertices $(x_1, x_2), (y_1, y_2)$ in Figure 1.14).

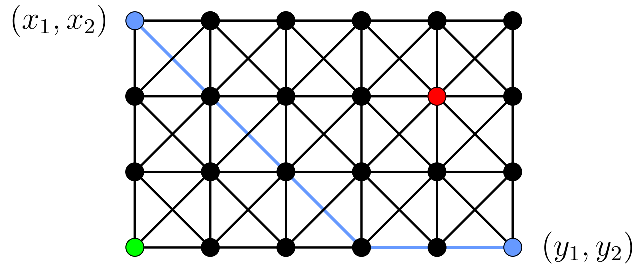

Figure 1.14: The strong product of two paths of length 3 and 5.

Chapter 2

Superconnectivity

We first give a lower bound for the connectivity of the strong product of two connected graphs, which is an improvement of a previous one. In addition, we prove that the strong product of two maximally connected graphs with minimum degree at least two and girth at least five is superconnected.

2.1 Introduction

The *connectivity* of a graph G, denoted by $\kappa(G)$, is one of the best studied measures of the vulnerability in graphs. It represents the minimum number of vertices whose deletion from G separates the graph in two or more components or produces an isolated vertex.

Recall that a *cut set* of a connected graph G is a set S of vertices such that $G - S$ is not connected or is an isolated vertex. Then, the parameter $\kappa(G)$ is defined as the minimum cardinality of a cut set of G.

As we mentioned in Introduction, the connectivity parameter has been studied in several products of graphs. For instance, the connectivity of the direct product $G_1 \times G_2$ of two connected graphs, G_1 and G_2, was upper bounded in 2008 by Brešar and Špacapan [24], who proved that

$$\kappa(G_1 \times G_2) \leq \min\{\kappa_b(G_1)|V(G_2)|, |V(G_1)|\kappa_b(G_2), \kappa_b(G_1)\kappa_b(G_2)\},$$

where $\kappa_b(G_i)$ is the smallest size of a set $S_i \subseteq V(G_i)$ such that $G_i - S_i$ is a bipartite graph, for $i = 1, 2$.

The lexicographic product graph, denoted by $G_1 \otimes G_2$, has been recently studied in 2013 by Yang and Xuin [101], who prove that $\kappa(G_1 \otimes G_2) = \kappa(G_1)|V(G_2)|$, whenever G_1 is a non complete graph. The connectivity of the cartesian product $G_1 \square G_2$ of two connected graphs, G_1 and G_2, was studied in 2008 by Špacapan in [90]. The author proved that

$$\kappa(G_1 \square G_2) = \min\{\kappa(G_1)|V(G_2)|, |V(G_1)|\kappa(G_2), \delta(G_1 \square G_2)\}.$$

Focus on the strong product graph, in 2010 Špacapan [91] proved the following lower bound

$$\kappa(G_1 \boxtimes G_2) \geq \min\{\kappa(G_1)(1 + \delta(G_2)), \kappa(G_2)(1 + \delta(G_1))\}, \tag{2.1}$$

which will be improved in this chapter. Before that, we present the description of certain vertex sets which was introduced in [91] and gives us an idea on the structure of the minimal cut sets in $G_1 \boxtimes G_2$.

Definition 2.1.1. ([91]) *Let G_1 and G_2 be two connected graphs. Let S_1 and S_2 be a cut set of the generator graphs G_1 and G_2, respectively. Then*

$$S_1 \times V(G_2) \quad \textit{and} \quad V(G_1) \times S_2$$

are cut sets of $G_1 \boxtimes G_2$ and each of them is called an I-set *(see Figure 2.1).*

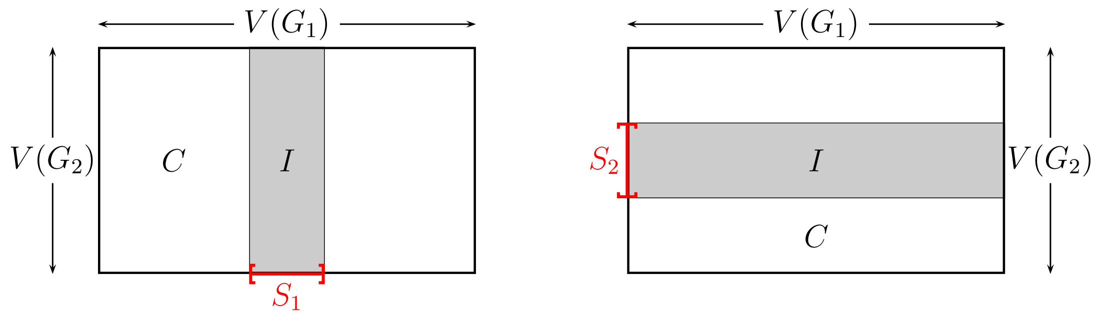

Figure 2.1: An I-set in $G_1 \boxtimes G_2$. C denotes a component of $(G_1 \boxtimes G_2) - I$.

Definition 2.1.2. ([91]) *Let G_1 and G_2 be two connected graphs. Let S_1 be a cut set of G_1 and set $A_1, \dots, A_k$ the components of $G_1 - S_1$. Similarly, let S_2 be a cut set of G_2 and set $B_1, \dots, B_\ell$ the components of $G_2 - S_2$. Then for every $i \in \{1, \dots, k\}$ and every $j \in \{1, \dots, \ell\}$, the set*

$$(S_1 \times V(B_j)) \cup (S_1 \times S_2) \cup (V(A_i) \times S_2)$$

is also a cut set of $G_1 \boxtimes G_2$ called an L-set *(see Figure 2.2).*

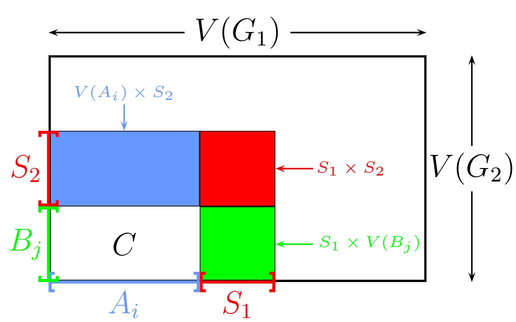

Figure 2.2: An L-set in $G_1 \boxtimes G_2$. C denotes a component of $(G_1 \boxtimes G_2) - L$.

Let us see an example for clarifying the definitions of I-sets and L-sets. Let us consider the strong product $\mathcal{P}_3 \boxtimes \mathcal{C}_4$ of a path and a cycle of order 4. Let us denote the path by $\mathcal{P}_3 = u_0u_1u_2u_3$, being its vertex set $V(\mathcal{P}_3) = \{u_0, u_1, u_2, u_3\}$. Let $V(\mathcal{C}_4) = \{v_0, v_1, v_2, v_3\}$ be the vertex set of $\mathcal{C}_4$.

Consider the cut set $S_1 = \{u_1\}$ of $\mathcal{P}_3$ whose removal produces two components, the isolated vertex u_0, which is denoted by A_1 and the edge u_2u_3, which is denoted by A_2 (see Figure 2.3).

Figure 2.3: A cut set $S_1 = \{u_1\}$ of $\mathcal{P}_3$ and the components of $\mathcal{P}_3 - S_1$.

Similarly, let $S_2 = \{v_0, v_2\}$ be a cut set of $\mathcal{C}_4$ whose removal produces also two components, the isolated vertices v_1 and v_3, which are denoted by B_1 and B_2, respectively (see Figure 2.4).

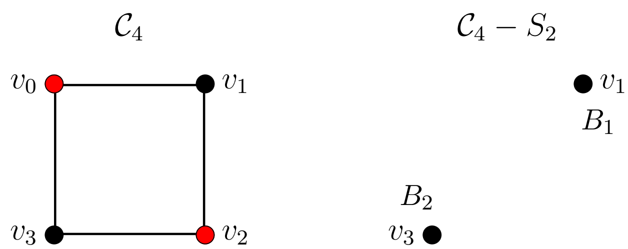

Figure 2.4: A cut set $S_2 = \{v_0, v_2\}$ of $\mathcal{C}_4$ and the components of $\mathcal{C}_4 - S_2$.

Notice that the set of red vertices in Figure 2.5 is the I-set $S_1 \times V(G_2)$ and also that the set of red vertices in Figure 2.6 is the I-set $V(G_1) \times S_2$ in $\mathcal{P}_3 \boxtimes \mathcal{C}_4$.

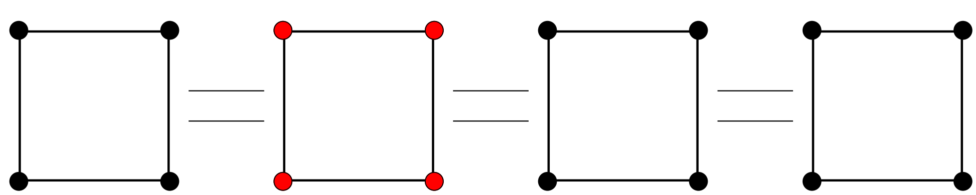

Figure 2.5: The I-set $S_1 \times V(G_2)$ in $\mathcal{P}_3 \boxtimes \mathcal{C}_4$.

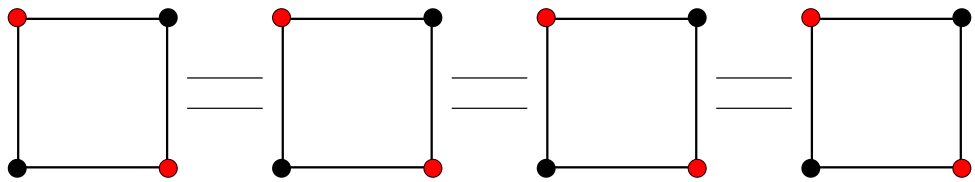

Figure 2.6: The I-set $V(G_1) \times S_2$ in $\mathcal{P}_3 \boxtimes \mathcal{C}_4$.

Observe in Figure 2.7 that the blue vertices correspond to the set $V(A_1) \times S_2$, the red vertices correspond to the set $S_1 \times S_2$ and the green vertex corresponds to the set $S_1 \times V(B_1)$. Furthermore,

$$(V(A_1) \times S_2) \cup (S_1 \times S_2) \cup (S_1 \times V(B_1))$$

is an L-set of $\mathcal{P}_3 \boxtimes \mathcal{C}_4$. Indeed, it separates the isolated vertex $A_1 \times B_1$ from the rest of the resultant graph by removing the L-set.

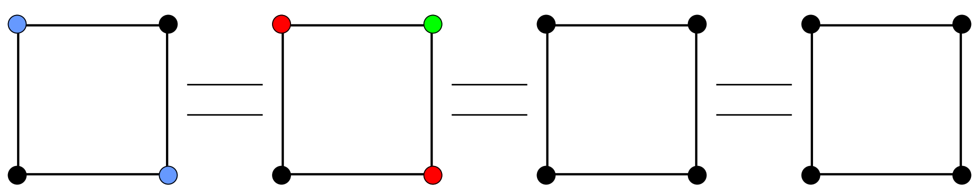

Figure 2.7: An L-set of $\mathcal{P}_3 \boxtimes \mathcal{C}_4$.

The following theorem proves that every minimum cut set in $G_1 \boxtimes G_2$ is induced by cut sets of its generator graphs.

Theorem 2.1.1. ([91]) *Let G_1 and G_2 be two connected graphs. Then every cut set in $G_1 \boxtimes G_2$ of minimum cardinality is either an I-set or an L-set.*

The cardinality of an I-set in $G_1 \boxtimes G_2$ can be easily computed. Indeed, if S is a cut set of $G_1 \boxtimes G_2$ of minimum cardinality and S is also an I-set, then

$$\kappa(G_1 \boxtimes G_2) = |S| = \min\{\kappa(G_1)|V(G_2)|, |V(G_1)|\kappa(G_2)\}.$$

Nevertheless, the cut sets S_1 and S_2 which define an L-set of minimum cardinality need not to be κ-sets in G_1 and G_2 respectively. We only can affirm that if an L-set

$$(V(A_i) \times S_2) \cup (S_1 \times S_2) \cup (S_1 \times V(B_j))$$

is a κ-set of $G_1 \boxtimes G_2$, then B_j is a smallest component of $G_2 - S_2$ and A_i is a smallest component of $G_1 - S_1$. Hence, as we know neither the size of the smallest component A_i of $G_1 - S_1$ nor the smallest component B_j of $G_2 - S_2$, we cannot compute the cardinality of such L-set.

Our aim in the next section is to get a more refined lower bound for $\kappa(G_1 \boxtimes G_2)$. We will complete this study by showing that the strong product of two non necessarily superconnected graphs may be superconnected.

2.2 News results on connectivity and superconnectivity

We first focus on the connectivity. The following definition will be used in this chapter.

Definition 2.2.1. *Let G_1 and G_2 be two connected graphs and $G = G_1 \boxtimes G_2$. Let $S \subset V(G)$ be a cut set of G. A copy $G_2^{x_i}$ of G_2 in G corresponding to a vertex $x_i \in V(G_1)$ is split by S if there exist at least two distinct components C and C' of $G - S$ such that*

$$V(G_2^{x_i}) \cap V(C) \neq \emptyset \quad \textit{and} \quad V(G_2^{x_i}) \cap V(C') \neq \emptyset.$$

The following remark comes from the definition of strong product of two graphs.

Remark 2.2.1. *Let* G_1 *and* G_2 *be two connected graphs and* $G = G_1 \boxtimes G_2$*. Given any vertex* $x_i \in V(G_1)$ *and any subset* $W \subseteq V(G_2^{x_i})$*, for every* $x_j \in N_{G_1}(x_i)$*, it follows that*

$$|N_{G_2^{x_i}}[W]| = |N_G(W) \cap V(G_2^{x_j})|.$$

Our first result gives sharp bounds on the connectivity parameter of the strong product $G_1 \boxtimes G_2$ of two connected graphs G_1 and G_2.

Theorem 2.2.1. *Let* G_1 *and* G_2 *be two connected graphs and* $G = G_1 \boxtimes G_2$*. If* G_1 *has girth at least* 4*, then*

$$\min\{|V(G_1)|\kappa(G_2), \kappa(G_1)|V(G_2)|, \delta(G_1)\kappa(G_2) + \delta(G_1) + \kappa(G_2)\} \leq \kappa\,(G) \leq \delta(G).$$

Proof. It is well known that $\kappa(G) \leq \delta(G)$ holds. Then we must prove the another inequality. Let $S \subset V(G)$ be a κ-set of G, that is $|S| = \kappa(G)$.

First, suppose that there is no split copy by S in G. Then, by applying Theorem 2.1.1, the cut set S must be an I-set, yielding that

$$|S| \geq \min\{|V(G_1)|\kappa(G_2), \kappa(G_1)|V(G_2)|\}.$$

Second, assume that there is some split copy by S in G. Set $V(G_1) = \{x_1, \ldots, x_n\}$. For each vertex $x_i \in V(G_1)$, denote by

$$S_{x_i} = V(G_2^{x_i}) \cap S.$$

Let $U \subseteq V(G_1)$ be the subset of vertices x_i for which the copy $G_2^{x_i}$ is split by S in G. Without loss of generality, assume that $U = \{x_1, \ldots, x_\ell\}$, with $1 \leq \ell \leq |V(G_1)|$.

Notice that

$$|S_{x_i}| \geq \kappa(G_2), \text{ for every } x_i \in U. \tag{2.2}$$

If $\ell = |V(G_1)|$, we have $U = V(G_1)$ and by (2.2),

$$|S| \geq \sum_{j=1}^{\ell} |S_{x_j}| \geq \kappa(G_2)|V(G_1)|.$$

Otherwise, assume that $1 \leq \ell \leq |V(G_1)| - 1$. For every $i = 1, \ldots, \ell$, let us denote by

$k_i = \min\{|V(G_2^{x_i}) \cap V(C)| : C$ is a component of $G - S$ and $V(G_2^{x_i}) \cap V(C) \neq \emptyset\}$.

Clearly, $k_i \geq 1$ for all $i = 1, \ldots, \ell$.

Given a vertex $x_q \in V(G_1)$ such that $x_q \notin U$, since the copy $G_2^{x_q}$ is not split by S, there exists a component $\widehat{C}$ of $G - S$ such that $V(G_2^{x_q}) \subset V(\widehat{C}) \cup S$.

Assume that $x_i x_q \in E(G_1)$ for some $i \in \{1, \ldots, \ell\}$. Then, for every component C of $G - S$ different from $\widehat{C}$, observe that

$$N_G\left(V(G_2^{x_i}) \cap V(C)\right) \cap V(G_2^{x_q}) \subseteq S_{x_q}.$$

Thus, by Remark 2.2.1, we have

$$|S_{x_q}| \geq |N_G\left(V(G_2^{x_i}) \cap V(C)\right) \cap V(G_2^{x_q})| = |N_{G_2^{x_i}}[V(G_2^{x_i}) \cap V(C)]|. \tag{2.3}$$

Since $N_{G_2^{x_i}}\left(V(G_2^{x_i}) \cap V(C)\right) \subseteq S_{x_i}$, for each component $C \neq \widehat{C}$, if it occurs that $V(G_2^{x_i}) \cap V(C) \neq \emptyset$ and $N_{G_2^{x_i}}\left(V(G_2^{x_i}) \cap V(C)\right) \neq S_{x_i}$ then $N_{G_2^{x_i}}\left(V(G_2^{x_i}) \cap V(C)\right)$ is a cut set of $G_2^{x_i}$, which means by (2.2) that

$$|N_{G_2^{x_i}}\left(V(G_2^{x_i}) \cap V(C)\right)| \geq \min\{|S_{x_i}|, \kappa(G_2)\} = \kappa(G_2).$$

Hence, from (2.3) it follows that

$$|S_{x_q}| \geq |N_{G_2^{x_i}}[V(G_2^{x_i}) \cap V(C)]| \geq k_i + \kappa(G_2). \tag{2.4}$$

Let $r = \min\{|N_{G_1}(x_i) \cap U| : i = 1, \ldots, \ell\}$. Clearly, $0 \leq r \leq \ell - 1$. Two cases need to be distinguished:

Case 1. Suppose that $r = 0$. That is, there exists $i \in \{1, \dots, \ell\}$ such that $N_{G_1}(x_i) \cap U = \emptyset$. Then, from (2.2) and (2.4) it follows that

$$|S| \geq \sum_{j=1}^{\ell} |S_{x_j}| + \sum_{x_q \in N_{G_1}(x_i)} |S_{x_q}| \geq \ell\kappa(G_2) + \delta(G_1)\left(k_i + \kappa(G_2)\right).$$

Since $\ell \geq 1$ and $k_i \geq 1$, we obtain

$$|S| \geq \kappa(G_2) + \delta(G_1)\left(1 + \kappa(G_2)\right) = \delta(G_1)\kappa(G_2) + \delta(G_1) + \kappa(G_2).$$

Case 2. Assume that $r \geq 1$. Since $U \neq V(G_1)$, there exist vertices $x_i, x_j \in U$ such that $x_i x_j \in E(G_1)$ and $|\left(N_{G_1}(x_i) \cup N_{G_1}(x_j)\right) \setminus U| \geq 1$ because G_1 is connected. Moreover, we know that $N_{G_1}(x_i) \cap N_{G_1}(x_j) = \emptyset$, due to $g(G_1) \geq 4$.

Hence, from inequalities (2.2) and (2.4) we have

$$\begin{aligned}
|S| &\geq \sum_{x \in N_{G_1}(x_i) \cap U} |S_x| + \sum_{x \in N_{G_1}(x_j) \cap U} |S_x| + \sum_{x \in N_{G_1}(x_i) \setminus U} |S_x| + \sum_{x \in N_{G_1}(x_j) \setminus U} |S_x| \\
&\geq |N_{G_1}(x_i) \cap U|\,\kappa(G_2) + |N_{G_1}(x_j) \cap U|\,\kappa(G_2) \\
&\quad + |N_{G_1}(x_i) \setminus U|\left(k_i + \kappa(G_2)\right) + |N_{G_1}(x_j) \setminus U|\left(k_j + \kappa(G_2)\right).
\end{aligned}$$

Since $k_i \geq 1$ and $k_j \geq 1$, we deduce that

$$|S| \geq (d_{G_1}(x_i) + d_{G_1}(x_j))\kappa(G_2) + |N_{G_1}(x_i) \setminus U| + |N_{G_1}(x_j) \setminus U| \geq 2\delta(G_1)\kappa(G_2) + 1.$$

As $\delta(G_1) \geq 1$ and $\kappa(G_2) \geq 1$, using that $ab + 1 \geq a + b$, for all integers $a, b \geq 1$, we finally have

$$|S| \geq \delta(G_1)\kappa(G_2) + \delta(G_1)\kappa(G_2) + 1 \geq \delta(G_1)\kappa(G_2) + \delta(G_1) + \kappa(G_2),$$

which finishes the proof. □

In order to present next result, let us denote by n_i, κ_i and δ_i the order, the connectivity and the minimum degree of G_i, for $i = 1, 2$. From Theorem 2.2.1 and the commutativity of the strong product graph, it follows this consequence whose proof is straightforward.

Corollary 2.2.1. *Let G_1 and G_2 be two connected graphs of girth at least 4 and $G = G_1 \boxtimes G_2$. Then*

$$\min\{n_1\kappa_2, \kappa_1 n_2, \max\{\delta_1\kappa_2 + \delta_1 + \kappa_2, \kappa_1\delta_2 + \kappa_1 + \delta_2\}\} \leq \kappa(G) \leq \delta(G).$$

Let us continue with a result which gives sufficient conditions for the strong product of two maximally connected graphs to be maximally connected. These conditions are addressed in terms of the minimum degree and the girth of the generator graphs.

To do that we use the well-known Moore bound (see [21] p. 105) which says that every graph with girth $g \geq 3$ and minimum degree $\delta \geq 2$ has at least $n_0(\delta, g)$ vertices, where

$$n_0(\delta, g) = \begin{cases} 1 + \delta \displaystyle\sum_{i=0}^{(g-3)/2} (\delta - 1)^i, & \text{if } g \text{ is odd} \\ 2 \displaystyle\sum_{i=0}^{g/2-1} (\delta - 1)^i, & \text{if } g \text{ is even} \end{cases} \tag{2.5}$$

Theorem 2.2.2. *Let G_1 and G_2 be two connected graphs with at least 3 vertices and girth at least 4. Then $G_1 \boxtimes G_2$ is maximally connected if both G_1 and G_2 are maximally connected and one of the following assertions holds:*

(i) One graph has minimum degree 1 and the other one has girth at least 5.

(ii) Both G_1 and G_2 have minimum degree at least 2.

Proof. Let $G = G_1 \boxtimes G_2$. To prove both points, we will apply Theorem 2.2.1, that is,

$$\min\{|V(G_1)|\kappa(G_2), \kappa(G_1)|V(G_2)|, \delta(G_1)\kappa(G_2) + \delta(G_1) + \kappa(G_2)\} \leq \kappa(G) \leq \delta(G).$$

Since G_1 and G_2 are maximally connected graphs, then we have, in fact, that

$$\min\{|V(G_1)|\delta(G_2), \delta(G_1)|V(G_2)|, \delta(G)\} \leq \kappa\,(G) \leq \delta(G).$$

(i) Without loss of generality, we may assume that G_1 has minimum degree 1 and G_2 has girth at least 5. The another case is analogous.

First, suppose that $\delta(G_2) = 1$. Since $\delta(G_1) = 1$, we have $\delta(G_1 \boxtimes G_2) = 3$. Hence,

$$\min\{|V(G_1)|, |V(G_2)|, 3\} \leq \kappa\,(G) \leq 3.$$

Also, $|V(G_1)| \geq 3 = \delta(G)$ and $|V(G_2)| \geq 3 = \delta(G)$, because both G_1 and G_2 have order at least 3. Therefore, $\kappa(G) = 3 = \delta(G)$ and we are done.

Second, suppose that $\delta(G_2) \geq 2$. As G_1 has at least 3 vertices and $\delta(G_1) = 1$, we have

$$|V(G_1)|\delta(G_2) \geq 3\delta(G_2) > \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2) = \delta\,(G)\,.$$

Since $g(G_2) \geq 5$, from the Moore bound (2.5) it follows that $|V(G_2)| \geq 1+\delta(G_2)^2$ and also, using that $a^2 \geq 2a$ for all integer $a \geq 2$,

$$\begin{aligned} \delta(G_1)|V(G_2)| &= |V(G_2)| \geq 1 + \delta(G_2)^2 \geq 1 + 2\delta(G_2) \\ &= \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2) = \delta\,(G)\,. \end{aligned}$$

Therefore, $\kappa\,(G) = \delta\,(G)$, that is, $G_1 \boxtimes G_2$ is maximally connected.

(ii) Assume that $\delta(G_1) \geq 2$ and $\delta(G_2) \geq 2$. Due to $g(G_1) \geq 4$ and $g(G_2) \geq 4$, from the Moore bound (2.5) it follows that

$$|V(G_1)| \geq 2\delta(G_1) \;\text{ and }\; |V(G_2)| \geq 2\delta(G_2).$$

Using that $ab \geq a + b$ for all integers $a, b \geq 2$, we have

$$\begin{aligned} |V(G_1)|\delta(G_2) &\geq 2\delta(G_1)\delta(G_2) = \delta(G_1)\delta(G_2) + \delta(G_1)\delta(G_2) \\ &\geq \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2) = \delta(G). \end{aligned}$$

Similarly, $\delta(G_1)|V(G_2)| \geq \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2) = \delta(G)$. Hence,

$$\kappa\,(G) = \delta(G),$$

and the result follows, that is, $G_1 \boxtimes G_2$ is maximally connected. $\square$

Theorem 2.2.2 is best possible in the sense that the hypothesis cannot be relaxed. On the one hand, observe that the hypothesis of girth at least 4 in both graphs is necessary. For instance, consider the strong product of a cycle $\mathcal{C}_g$ of length g, with $g \geq 4$, and any complete graph K_n with order $n \geq 3$, which has girth 3. We can disconnect $\mathcal{C}_g \boxtimes K_n$ by removing two copies of K_n corresponding to two nonadjacent vertices of $\mathcal{C}_g$ (see the red copies in Figure 2.8). Hence, $\kappa(\mathcal{C}_g \boxtimes K_n) \leq 2n < 2 + n - 1 + 2(n-1) = \delta(\mathcal{C}_g \boxtimes K_n)$.

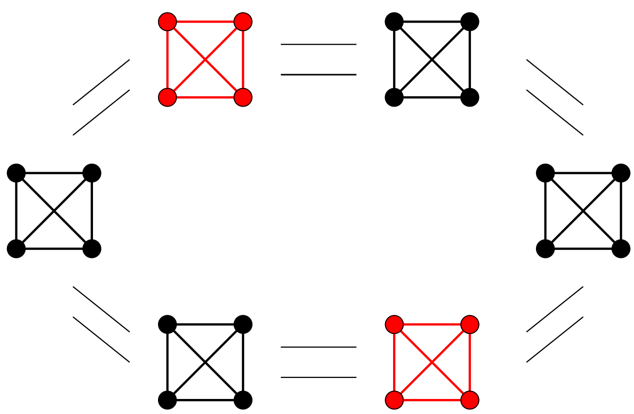

Figure 2.8: A cut set of the strong product $\mathcal{C}_6 \boxtimes K_4$.

On the other hand, we also check that the hypothesis of point (i) and (ii) of Theorem 2.2.2 cannot be relaxed. It suffices to consider the strong product of a path $\mathcal{P}_r$ of length r, with $r \geq 2$, and a cycle $\mathcal{C}_g$ of length g, with $g \leq 4$. Both $\mathcal{P}_r$ and $\mathcal{C}_g$ are maximally connected graphs. Observe that by removing one copy of $\mathcal{C}_g$ corresponding to whichever vertex of degree 2 in $\mathcal{P}_r$ (see the red copy in Figure 2.9), the resultant graph is disconnected. However, $\mathcal{P}_r \boxtimes \mathcal{C}_g$ is not maximally connected, due to $\kappa(\mathcal{P}_r \boxtimes \mathcal{C}_g) \leq g \leq 4 < 5 = \delta(\mathcal{P}_r \boxtimes \mathcal{C}_g)$.

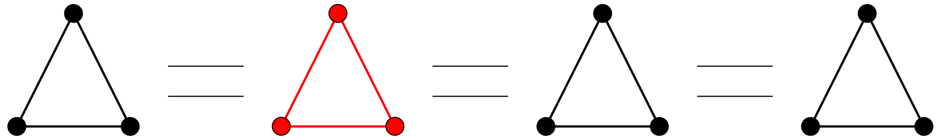

Figure 2.9: A cut set of the strong product $\mathcal{P}_3 \boxtimes \mathcal{C}_3$.

Finally, we can go one step further. We prove a sufficient condition on the generator graphs which permits us to know the structure of any κ-set. Namely, the next theorem shows that the strong product $G_1 \boxtimes G_2$ of two maximally connected graphs is superconnected if both G_1 and G_2 have girth at least 5 and minimum degree at least 2.

Theorem 2.2.3. *Let G_1 and G_2 be two maximally connected graphs of girth at least 5 and minimum degree at least 2. Then $G_1 \boxtimes G_2$ is superconnected.*

Proof. Let us set $G = G_1 \boxtimes G_2$. Due to G_1 and G_2 are maximally connected graphs and $\delta(G_1), \delta(G_2) \geq 2$, from Theorem 2.2.2, we know that G is maximally connected, that is, $\kappa(G) = \delta(G)$. Then, we must just prove that every minimum cut set isolates some vertex of G.

We reason by contradiction supposing that there exists a cut set S in G with $|S| = \delta(G)$ such that every component C of $G - S$ has order $|V(C)| \geq 2$.

We proceed in a similar way as in the proof of Theorem 2.2.1. For this goal, set $V(G_1) = \{x_1, \ldots, x_n\}$, and for each vertex $x_i \in V(G_1)$, let us denote by $S_{x_i} = V(G_2^{x_i}) \cap S$.

If there is no split copy by S in G, then by applying Theorem 2.1.1, the cut set S must be an I-set, yielding that

$$|S| \geq \min\{|V(G_1)|\kappa(G_2), \kappa(G_1)|V(G_2)|\} = \min\{|V(G_1)|\delta(G_2), \delta(G_1)|V(G_2)|\}.$$

By using the Moore bound (2.5) on the order of a graph with girth at least 5 and minimum degree at least 2, and also the inequality $ab \geq a+b+2$ for all $a \geq 4$, $b \geq 2$, we have

$$\begin{aligned}
|V(G_1)|\delta(G_2) &\geq (1+\delta(G_1)^2)\delta(G_2)\\
&= \delta(G_2)+\delta(G_1)\delta(G_2)\delta(G_1)\\
&\geq \delta(G_1)\delta(G_2)+\delta(G_1)+\delta(G_2)+2\\
&> \delta(G_1 \boxtimes G_2).
\end{aligned}$$

Analogously, we obtain that $\delta(G_1)|V(G_2)| > \delta(G_1 \boxtimes G_2)$.

We arrive at a contradiction in both cases, and therefore, we deduce that there must exist some split copy by S in G.

Let $U \subseteq V(G_1)$ be the subset of vertices x_i for which the copy $G_2^{x_i}$ is split by S in G. Without loss of generality, assume that $U = \{x_1, \ldots, x_\ell\}$, with $1 \leq \ell \leq n$. For every $i = 1, \ldots, \ell$, let us also denote by

$$k_i = \min\{|V(G_2^{x_i}) \cap V(C)| : C \text{ is a component of } G-S \text{ and } V(G_2^{x_i}) \cap V(C) \neq \emptyset\}.$$

Notice that, $k_i \geq 1$ for all $i = 1, \ldots, \ell$.

Let $r = \min\{|N_{G_1}(x_i) \cap U| : i = 1, \ldots, \ell\}$. Clearly, $0 \leq r \leq \ell - 1$.

Suppose that $r \geq 1$. By repeating the reasoning of Case 2 in the proof of Theorem 2.2.1, we get $|S| \geq 2\delta(G_1)\kappa(G_2)+1$. Since G_2 is maximally connected and $\delta(G_1), \delta(G_2) \geq 2$, using the inequality $ab \geq a+b$ for all $a \geq 2$, $b \geq 2$, we arrive at the following contradiction

$$|S| \geq 2\delta(G_1)\delta(G_2)+1 \geq \delta(G_1)\delta(G_2)+\delta(G_1)+\delta(G_2)+1 > \delta(G_1 \boxtimes G_2).$$

Thus, necessarily $r = 0$, which means that there exists $i \in \{1, \ldots, \ell\}$ such that $N_{G_1}(x_i) \cap U = \emptyset$. Then similarly to the Case 1 of the proof of Theorem 2.2.1, we have

$$\begin{aligned}|S| &\geq \sum_{j=1}^{\ell}|S_{x_j}| + \sum_{x_q \in N_{G_1}(x_i)} |S_{x_q}| \\ &\geq \ell\delta(G_2) + \delta(G_1)\left(k_i + \delta(G_2)\right) \\ &\geq \delta(G_2) + \delta(G_1)\left(1 + \delta(G_2)\right) \\ &= \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2),\end{aligned}$$

yielding that all the previous inequalities become equalities, due to G_2 is maximally connected. Therefore, we get $\ell = 1$, that is, $i = 1$ and $U = \{x_1\}$. Moreover,

$$|S_{x_1}| = \delta(G_2) \text{ and } k_1 = 1.$$

Thus, there exists a component C' of $G - S$ such that $V(G_2^{x_1}) \cap V(C') = \{(x_1, z)\}$, for some $z \in V(G_2)$. Furthermore, $N_{G_2^{x_1}}((x_1, z)) = S_{x_1}$, because $|S_{x_1}| = \delta(G_2)$ and G is maximally connected.

Since $G_2^{x_1}$ is a split copy by S in G, there is a component $C \neq C'$ such that $V(G_2^{x_1}) \cap V(C) \neq \emptyset$. Indeed, we can assure that $|V(G_2^{x_1}) \cap V(C)| \geq 2$, because otherwise, that is, if $V(G_2^{x_1}) \cap V(C) = \{(x_1, z^*)\}$, then $N_{G_2^{x_1}}((x_1, z^*)) = S_{x_1}$ and, this fact together with the hypothesis that $\delta(G_2) \geq 2$ yields that G_2 contains a cycle of length 4, being a contradiction with the assumption of $g(G_2) \geq 5$. Hence, $|V(G_2^{x_1}) \cap V(C)| \geq 2$.

As we suppose by contradiction that G is not superconnected, there exists at least one vertex $(x_p, z^*) \in N_{G_2^{x_1}}((x_1, z)) \cap V(C')$. Clearly, $x_1 \neq x_p$, because $V(G_2^{x_1}) \cap V(C') = \{(x_1, z)\}$, which means that $x_1x_p \in E(G_1)$. Since $r = 0$, $x_p \notin U$ and therefore the copy $G_2^{x_p}$ is not split by S in G, yielding that

$$V(G_2^{x_p}) = (V(G_2^{x_p}) \cap V(C')) \cup S_{x_p}.$$

Hence, $N_G(V(G_2^{x_1}) \cap V(C)) \cap V(G_2^{x_p}) \subseteq S_{x_p}$, since $x_1x_p \in E(G_1)$. From Remark 2.2.1, it follows that

$$\begin{aligned}
|S_{x_p}| &\geq |N_G(V(G_2^{x_1}) \cap V(C)) \cap V(G_2^{x_p})| \\
&= |N_{G_2^{x_1}}[V(G_2^{x_1}) \cap V(C)]| \qquad (2.6)\\
&= |V(G_2^{x_1}) \cap V(C)| + |N_{G_2^{x_1}}(V(G_2^{x_1}) \cap V(C))|.
\end{aligned}$$

Due to $N_{G_2^{x_1}}(V(G_2^{x_1}) \cap V(C)) \subseteq S_{x_1}$, if it occurs $N_{G_2^{x_1}}(V(G_2^{x_1}) \cap V(C)) \neq S_{x_1}$ then $N_{G_2^{x_1}}(V(G_2^{x_1}) \cap V(C))$ is a cut set of $G_2^{x_1}$ of cardinality less than $|S_{x_1}|$. This contradicts the fact that $|S_{x_1}| = \delta(G_2)$ and G_2 is maximally connected.

Therefore, $N_{G_2^{x_1}}(V(G_2^{x_1}) \cap V(C)) = S_{x_1}$ and from (2.6), we deduce that

$$\begin{aligned}
|S_{x_p}| &\geq |N_{G_2^{x_1}}[V(G_2^{x_1}) \cap V(C)]| \\
&= |V(G_2^{x_1}) \cap V(C)| + |S_{x_1}| \qquad (2.7)\\
&= |V(G_2^{x_1}) \cap V(C)| + \delta(G_2).
\end{aligned}$$

Then, from (2.7) we finally obtain that

$$\begin{aligned}
|S| &\geq |S_{x_1}| + |S_{x_p}| + \sum_{x_q \in N_{G_1}(x_1)\setminus\{x_p\}} |S_{x_q}| \\
&\geq \delta(G_2) + |V(G_2^{x_1}) \cap V(C)| + \delta(G_2) + (\delta(G_1) - 1)(1 + \delta(G_2)) \\
&\geq 2\delta(G_2) + 2 + (\delta(G_1) - 1)(1 + \delta(G_2)) \\
&= \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2) + 1 > \delta(G),
\end{aligned}$$

which is a contradiction. Hence, G is superconnected and this finishes the proof.

□

Theorem 2.2.3 is best possible in the sense that the hypothesis cannot be relaxed. First, the minimum degree at least 2 for each generator graph must be assumed. Otherwise, the strong product of a path $\mathcal{P}_r$ of length r, with $r \geq 2$, and a cycle $\mathcal{C}_g$ of length $g = 5$ would be a counterexample. In this case, both $\mathcal{P}_r$ and $\mathcal{C}_g$ are maximally connected graphs and $\kappa(\mathcal{P}_r \boxtimes \mathcal{C}_g) = 5 = \delta(\mathcal{P}_r \boxtimes \mathcal{C}_g)$.

However, the deletion of one copy of $\mathcal{C}_g$ corresponding to any vertex of degree 2 in $\mathcal{P}_r$ (see the red copy in Figure 2.10), produces a disconnected graph and every component is not an isolated vertex. Therefore, $\mathcal{P}_r \boxtimes \mathcal{C}_g$ is not superconnected.

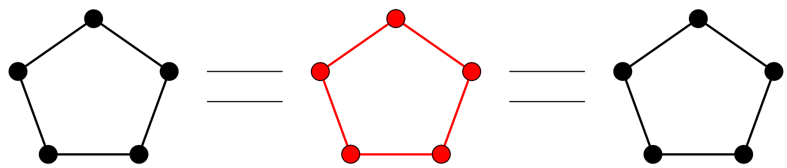

Figure 2.10: A cut set of the strong product $\mathcal{P}_2 \boxtimes \mathcal{C}_5$.

Second, the hypothesis of girth at least 5 for the generator graphs must be also assumed. Otherwise, the strong product of a cycle $\mathcal{C}_g$ of length g, with $g \geq 5$ and a cycle $\mathcal{C}_4$ of length 4 is a counterexample. Observe that the connectivity is $\kappa(\mathcal{C}_g \boxtimes \mathcal{C}_4) = 8 = \delta(\mathcal{C}_g \boxtimes \mathcal{C}_4)$ and we can disconnect $\mathcal{C}_g \boxtimes \mathcal{C}_4$ by removing two copies of $\mathcal{C}_4$ corresponding to two nonadjacent vertices of $\mathcal{C}_g$ (see the red copies in Figure 2.11).

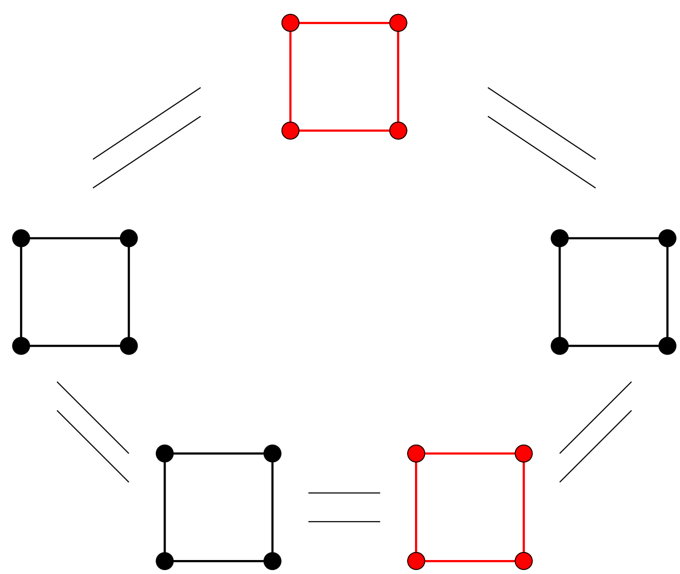

Figure 2.11: A cut set of the strong product $\mathcal{C}_5 \boxtimes \mathcal{C}_4$.

Chapter 3

Connectivity and Distances

We focus on three new indices of reliability in the strong product of two graphs: the average connectivity, the Menger number and the average Menger number. The average connectivity analyzes the reliability of a graph not focusing on the worst case, but providing a measure of the expected number of vertices that must fail to disrupt a graph. The Menger number and the average Menger number give us an information on the number of routes of bounded length. Sharp lower bounds on these parameters are obtained.

3.1 Introduction

As we mentioned in Chapter 1, from the relationship between the connectivity of a graph and the existence of internally disjoint paths given by Whitney [95], it makes sense to measure the connectivity between two distinct vertices x and y of G, denoted by $\kappa_G(x,y)$, as the maximum number of pairwise internally disjoint

xy-paths in G. By this way, the connectivity of a graph can be seen as

$$\kappa(G) = \min\{\kappa_G(x, y) : x, y \in V(G)\}.$$

The classical connectivity is a measure which focuses on the worst case and it cannot provide a complete information about the vulnerability of the graph. For example, Figure 3.1 shows two graphs with connectivity 1. Nevertheless, the former graph appears much more connected than the last one. Then it is interesting to consider different vulnerability parameters in graphs which give us a suitable information about their reliability.

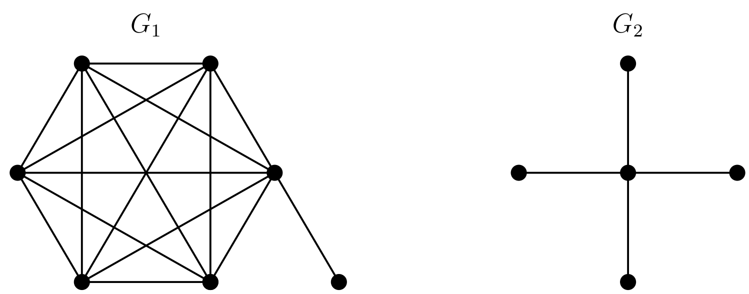

Figure 3.1: Two graphs with equal connectivity and distinct vulnerability.

In this chapter we pay attention to three of them. The *average connectivity*, defined by Beineke, Oellermann and Pippert in [16], represents the expected number of vertices that must been removed to disconnect a graph. By the theorem of Menger [79], it is related to the number of internally disjoint paths that exist between any two vertices. Another index in which we are interested is the *Menger number*, introduced by Lovász, Neumann-Lara and Plummer in [74]. This parameter takes into account, not only the pairwise disjoint paths between two vertices, but also their lengths. Finally, we also study the *average Menger number*, which reflects a mean on the number of pairwise disjoint paths with a bounded length between two vertices. We develop the study of these three parameters in Section 3.3, Section 3.4 and Section 3.5 of this chapter.

3.2 Previous Results

In this section we present some technical results which are necessary to approach the *average connectivity* and the *Menger number* of the strong product of two connected graphs. On the one hand, we are interested in guaranteeing a minimum number of internally disjoint paths between any two vertices in the strong product graph. On the other hand, we want to fix a maximum length for such paths. To study these indices, we need to introduce some previous notation.

Let G be a connected graph and consider two distinct vertices $x, y \in V(G)$. Let ℓ be a positive integer. Let us denote by $\zeta_\ell(x, y)$ the maximum number of internally disjoint xy-paths of length at most ℓ in G, and by

$$\zeta_\ell(G) = \min\{\zeta_\ell(x, y) : x, y \in V(G)\}.$$

For instance, consider a cycle $\mathcal{C}_6$ of length 6 as in Figure 3.2. Observe that $\zeta_3(x, y) = 2$, meanwhile $\zeta_3(u, v) = 1$. It is easy to check that the maximum number of internally disjoint paths of length at most 3 in $\mathcal{C}_6$ is always 1 or 2, for any pair of vertices in $\mathcal{C}_6$. Hence, in this case, we conclude that $\zeta_3(\mathcal{C}_6) = 1$.

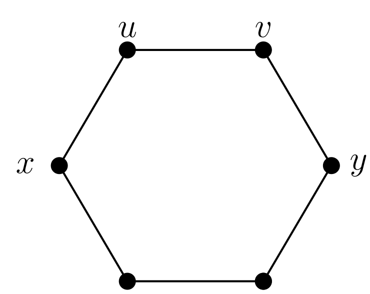

Figure 3.2: A cycle $\mathcal{C}_6$ verifies that $\zeta_3(\mathcal{C}_6) = 1$.

Let G_1 and G_2 be two connected graphs and let ℓ be a positive integer. Denote by $\zeta_i = \zeta_\ell(G_i)$, for $i = 1, 2$. Given vertices $(x_1, x_2), (y_1, y_2) \in V(G_1 \boxtimes G_2)$, let $P_1, \ldots, P_{\zeta_1}$ be ζ_1 internally disjoint x_1y_1-paths in G_1 and let $Q_1, \ldots, Q_{\zeta_2}$ be ζ_2 internally disjoint x_2y_2-paths in G_2, all of them of length at most ℓ.

Without loss of generality assume that $l(P_1) = \min\{l(P_i) : i = 1, \ldots, \zeta_1\}$ and $l(Q_1) = \min\{l(Q_j) : j = 1, \ldots, \zeta_2\}$. Also, for every x_1y_1-path P_i in G_1 and x_2y_2-path Q_j in G_2 of length at least 2, we denote by $\ddot{P}_i$ and $\ddot{Q}_j$, respectively, the new path obtained from P_i and Q_j by removing their end vertices (see Figure 3.3).

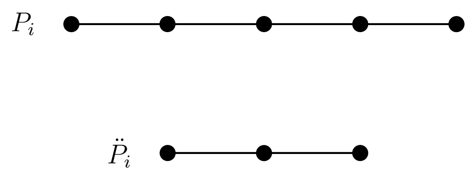

Figure 3.3: A path P_i and its corresponding $\ddot{P}_i$.

As we mentioned in Chapter1, for every $x_2 \in V(G_2)$, the subgraph of $G_1 \boxtimes G_2$ induced by the set $\{(u, x_2) : u \in V(G_1)\}$ is isomorphic to G_1 and it is denoted by $G_1^{x_2}$. Analogously, for each $x_1 \in V(G_1)$, the set $\{(x_1, v) : v \in V(G_2)\}$ induces a subgraph isomorphic to G_2 and it is denoted by $G_2^{x_1}$.

Thus, each x_1y_1-path P_i in G_1 induces an $(x_1, x_2)(y_1, x_2)$-path in $G_1^{x_2}$, which will be denoted by $P_i^{x_2}$, with vertex set $V(P_i^{x_2}) = \{(u, x_2) : u \in V(P_i)\}$ and edge set $E(P_i^{x_2}) = \{(u_1, x_2)(u_2, x_2) : u_1u_2 \in E(P_i)\}$. Similarly, each x_2y_2-path Q_j in the graph G_2 also induces an $(x_1, x_2)(x_1, y_2)$-path in $G_2^{x_1}$, which will be denoted by $Q_j^{x_1}$, with vertex set $V(Q_j^{x_1}) = \{(x_1, v) : v \in V(Q_j)\}$ and edge set $E(Q_j^{x_1}) = \{(x_1, v_1)(x_1, v_2) : v_1v_2 \in E(Q_j)\}$ (see Figure 3.4).

For computing the total number of internally disjoint paths that exist between any two distinct vertices in the strong product graph $G_1 \boxtimes G_2$, the position of such vertices is necessary to be considered.

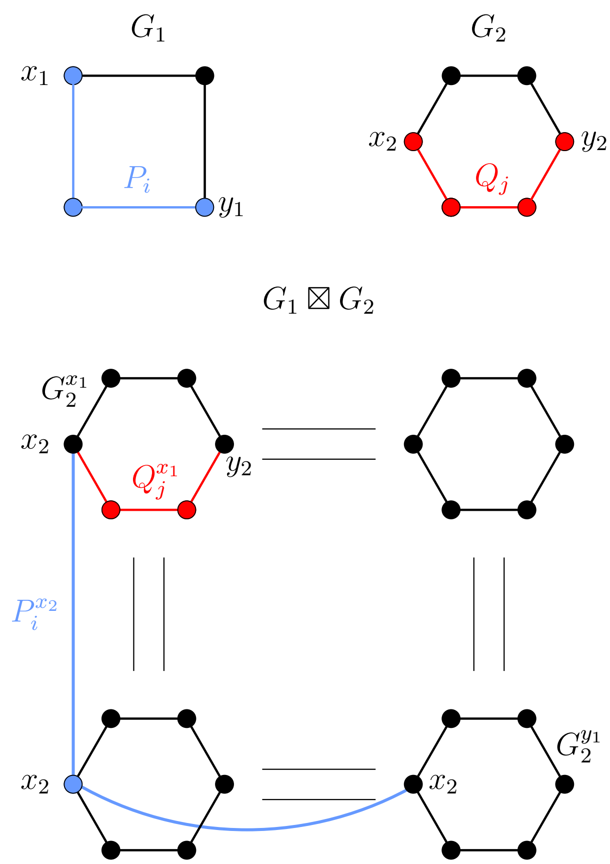

Figure 3.4: The induced paths $P_i^{x_2}$ and $Q_j^{x_1}$ in $\mathcal{C}_4 \boxtimes \mathcal{C}_6$.

Two vertices may belong to the same copy of G_2 and two different copies of G_1 (see the pair of green vertices in Figure 3.5). Also, two vertices may belong to the same copy of G_1 and different copies of G_2 (see the pair of blue vertices in Figure 3.5). These two situations will be analyzed in Lemma 3.2.1.

Finally, two vertices may be in different copies of G_1 and different copies of G_2 (see the pair of red vertices in Figure 3.5). This situation will be studied in Lemma 3.2.2 and Lemma 3.2.3.

First result gives a lower bound on the number of internally disjoint paths that exist between two distinct vertices $(x_1, x_2), (y_1, y_2)$ in $G_1 \boxtimes G_2$ such that either $x_1 = y_1$ or $x_2 = y_2$, that is, two vertices of $G_1 \boxtimes G_2$ which come from a single vertex of G_1 or a single vertex in G_2.

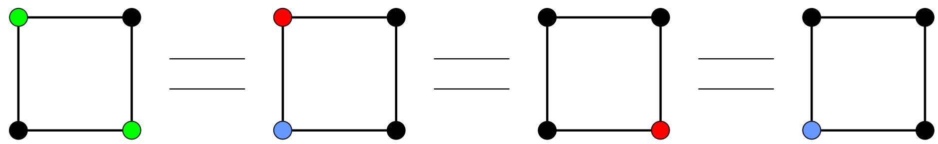

Figure 3.5: Possible positions of pairs of vertices in $P_3 \boxtimes C_4$.

Lemma 3.2.1. *Let G_1 and G_2 be two connected graphs with at least 3 vertices. Let $\ell \geq \max\{D(G_1), D(G_2), 2\}$ be an integer and $x_i, y_i \in V(G_i)$ be two distinct vertices, for $i = 1, 2$. Then the following assertions hold:*

(i) *There exist at least $(\delta(G_1)+1)\zeta_\ell(G_2)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$. Furthermore, if G_1 has girth at least 5, then there exist at least $\delta(G_1)$ additional internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most $\ell + 2$.*

(ii) *There exist at least $(\delta(G_2)+1)\zeta_\ell(G_1)$ internally disjoint $(x_1, x_2)(y_1, x_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$. Moreover, if G_2 has girth at least 5, then there exist at least $\delta(G_2)$ additional internally disjoint $(x_1, x_2)(y_1, x_2)$-paths of length at most $\ell + 2$.*

Proof. By the commutativity of the strong product of two graphs, it suffices to prove (i). Let $\zeta_2 = \zeta_\ell(G_2)$. Consider any vertex $x_1 \in V(G_1)$ and two distinct vertices $x_2, y_2 \in V(G_2)$. Then, by hypothesis, there are at least ζ_2 internally disjoint x_2y_2-paths, $Q_1, \ldots, Q_{\zeta_2}$, of length at most ℓ in G_2.

First, we introduce some general constructions of $(x_1, x_2)(x_1, y_2)$-paths in $G_1 \boxtimes G_2$. Consider $u \in N_{G_1}(x_1)$ and any Q_j, for $j \in \{1, \ldots, \zeta_2\}$. Notice that if $l(Q_j) \geq 2$, then vertices (x_1, x_2) and (x_1, y_2) are adjacent to the first and to the last internal vertex of Q_j^u, respectively. Hence, it makes sense to consider the following path in $G_1 \boxtimes G_2$ of length at most ℓ (see Figure 3.6),

$$R_{uj} : (x_1, x_2)\ddot{Q}_j^u(x_1, y_2).$$

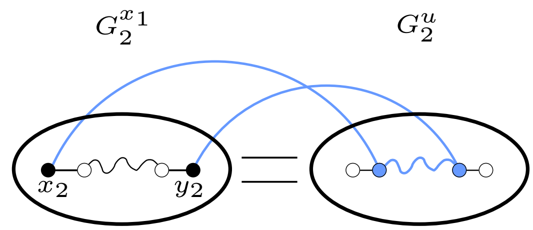

Figure 3.6: Construction of path R_{uj} in Lemma 3.2.1.

When there exists a vertex $w_u \in N_{G_1}(u) \setminus \{x_1\}$, by hypothesis there also exist ζ_2 internally disjoint x_2y_2-paths, $Q_1^{w_u}, \ldots, Q_{\zeta_2}^{w_u}$, of length at most ℓ in $G_2^{w_u}$. Then, only for one but whichever $Q_j^{w_u}$ verifying that $l(Q_j^{w_u}) \geq 2$, for $j \in \{1, \ldots, \zeta_2\}$, we can consider the $(x_1, x_2)(x_1, y_2)$-path

$$R_{w_u} : (x_1, x_2)(u, x_2)\ddot{Q}_j^{w_u}(u, y_2)(x_1, y_2)$$

of length at most $\ell + 2$ (see Figure 3.7).

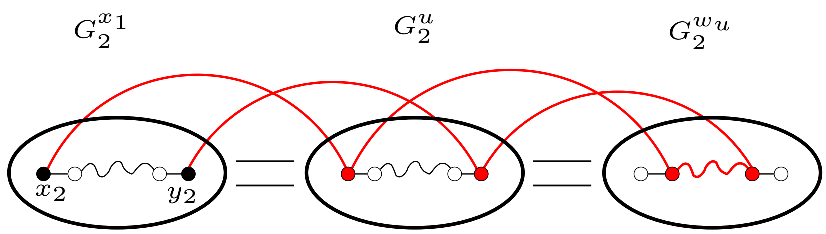

Figure 3.7: Construction of path R_{w_u} in Lemma 3.2.1.

Observe that R_{uj} and R_{w_u} are internally disjoint paths, for every $u \in N_{G_1}(x_1)$, every $w_u \in N_{G_1}(u) \setminus \{x_1\}$ and every $j \in \{1, \ldots, \zeta_2\}$ (see Figure 3.8).

Second, we obtain the $(\delta(G_1) + 1)\zeta_2$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most ℓ and the $\delta(G_1)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length

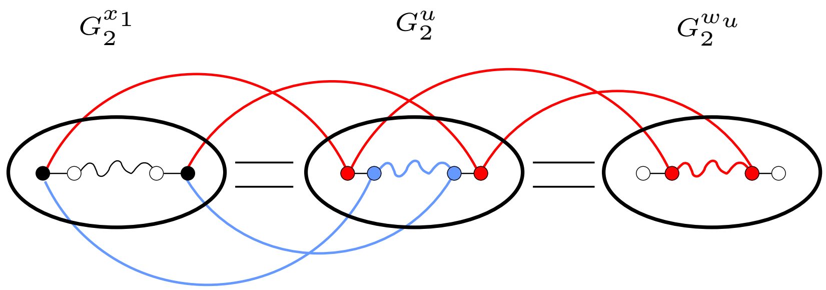

Figure 3.8: Both paths R_{uj} and R_{w_u} in Lemma 3.2.1.

at most $\ell + 2$ in $G_1 \boxtimes G_2$. Observe that vertices (x_1, x_2) and (x_1, y_2) belong to the same copy $G_2^{x_1}$ of $G_1 \boxtimes G_2$. Then, $Q_1^{x_1}, \ldots, Q_{\zeta_2}^{x_1}$ are ζ_2 internally disjoint $(x_1, x_2)(x_1, y_2)$-paths in $G_1 \boxtimes G_2$ of length at most ℓ. To construct the other $\delta(G_1)\zeta_2 + \delta(G_1)$ paths, we distinguish whether x_2y_2 is an edge of G_2 or not.

Assume that $x_2y_2 \in E(G_2)$, that is, $l(Q_1) = 1$. Let $u \in N_{G_1}(x_1)$. The paths

$$R'_u : (x_1, x_2)(u, x_2)(x_1, y_2) \quad \text{and} \quad R''_u : (x_1, x_2)(u, y_2)(x_1, y_2)$$

are contained in $G_1 \boxtimes G_2$ and they have length $2 \leq \ell$. Moreover, since G_2 is a simple graph, for $j = 2, \ldots, \zeta_2$, the paths Q_j have length at least 2 and it makes sense to consider the paths R_{uj}, for $j = 2, \ldots, \zeta_2$. Hence, we deduce that

$$Q_1^{x_1}, \ldots, Q_{\zeta_2}^{x_1}, R'_u, R''_u, R_{u2}, \ldots, R_{u\zeta_2},$$

for every $u \in N_{G_1}(x_1)$, are

$$\zeta_2 + 2d_{G_1}(x_1) + d_{G_1}(x_1)(\zeta_2 - 1) \geq \zeta_2 + 2\delta(G_1) + \delta(G_1)(\zeta_2 - 1) = (\delta(G_1) + 1)\zeta_2 + \delta(G_1)$$

internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$.

Now, suppose that $x_2y_2 \notin E(G_2)$. For $j = 1, \ldots, \zeta_2$ and $u \in N_{G_1}(x_1)$, we consider the paths $Q_j^{x_1}$ and R_{uj}. Thus, we have $(d_{G_1}(x_1) + 1)\zeta_2$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most ℓ.

Assume that $g(G_1) \geq 5$. If there exists a vertex $u \in N_{G_1}(x_1)$ such that $d_{G_1}(u) = 1$, then $d_{G_1}(x_1) \geq 2$ necessary, since G_1 has order at least 3. Hence

$$(d_{G_1}(x_1) + 1)\zeta_2 \geq 3\zeta_2 \geq 2\zeta_2 + 1 = (\delta(G_1) + 1)\zeta_2 + \delta(G_1).$$

Otherwise, there exists a vertex $w_u \in N_{G_1}(u) \setminus \{x_1\}$ for every $u \in N_{G_1}(x_1)$. As $g(G_1) \geq 5$, then $w_u \neq w_v$ for all $u, v \in N_{G_1}(x_1)$ with $u \neq v$. Therefore, the paths R_{w_u}, for each $u \in N_{G_1}(x_1)$, are at least $\delta(G_1)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$. Hence, in this case,

$$Q_1^{x_1}, \ldots, Q_{\zeta_2}^{x_1},\ R_{u1}, \ldots, R_{u\zeta_2},\ R_{w_u}$$

are at least $(\delta(G_1)+1)\zeta_2+\delta(G_1)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$, which finishes the proof. $\square$

Next two lemmas provide the number of internally disjoint paths between two vertices in $G_1 \boxtimes G_2$ which come from x_1, y_1, two distinct vertices of G_1, and x_2, y_2, two different vertices of G_2.

For this goal, we introduce the following description of the paths P_i and Q_j in G_1 and G_2, respectively. For $i \in \{1, \ldots, \zeta_1\}$ and $j \in \{1, \ldots, \zeta_2\}$, denote by $P_i : u_0^i u_1^i \ldots u_{r_i}^i$ and $Q_j : v_0^j v_1^j \ldots v_{s_j}^j$, where notice that $(u_0^i, v_0^j) = (x_1, x_2)$ and $(u_{r_i}^i, v_{s_j}^j) = (y_1, y_2)$. Thus, the length of each path P_i and Q_j is r_i and s_j, respectively.

Using paths of length at most ℓ in the generator graphs G_1 and G_2, next lemma shows constructions of paths in $G_1 \boxtimes G_2$ whose lengths are also at most ℓ.

Lemma 3.2.2. *Let G_1 and G_2 be two connected graphs with at least 3 vertices and $\ell \geq \max\{D(G_1), D(G_2)\}$ be an integer. For every two distinct vertices $x_1, y_1 \in V(G_1)$ and every two distinct vertices $x_2, y_2 \in V(G_2)$, there exist at least $\zeta_\ell(G_1)\zeta_\ell(G_2)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$ of length at most ℓ.*

Proof. The $\zeta_\ell(G_1)\zeta_\ell(G_2)$ paths in $G_1 \boxtimes G_2$ are constructed considering all the possible combinations of pairs of paths, one from G_1 and other from G_2. Hence, for each $i \in \{1, \dots, \zeta_1\}$ and each $j \in \{1, \dots, \zeta_2\}$, associated to the x_1y_1-path P_i in G_1 and to the x_2y_2-path Q_j in G_2, we consider the $(x_1, x_2)(y_1, y_2)$-path R_{ij} in $G_1 \boxtimes G_2$ as follows:

(i) If P_i is shorter than Q_j, that is, if $r_i < s_j$ then (see Figure 3.9)

$$R_{ij} : \begin{cases} (u_0^i, v_0^j)(u_1^i, v_1^j) \dots (u_1^i, v_{s_j}^j), & \text{if } r_i = 1 \\ (u_0^i, v_0^j) \dots (u_{r_i-1}^i, v_{r_i-1}^j) \dots (u_{r_i-1}^i, v_{s_j-1}^j)(u_{r_i}^i, v_{s_j}^j), & \text{if } r_i \geq 2. \end{cases}$$

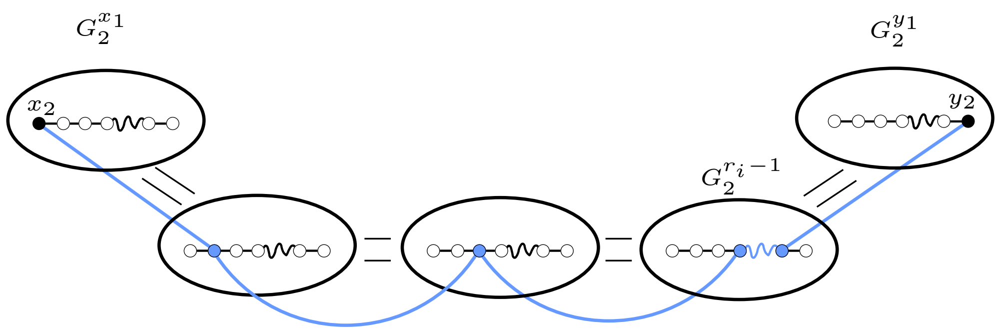

Figure 3.9: Construction of path R_{ij} when $r_i < s_j$ and $r_i \geq 2$ in Lemma 3.2.2.

(ii) If P_i is longer than Q_j, that is, if $r_i \geq s_j$ then (see Figure 3.10)

$$R_{ij} : \begin{cases} (u_0^i, v_0^j)(u_1^i, v_1^j) \dots (u_{r_i}^i, v_1^j), & \text{if } s_j = 1 \\ (u_0^i, v_0^j) \dots (u_{s_j-1}^i, v_{s_j-1}^j) \dots (u_{r_i-1}^i, v_{s_j-1}^j)(u_{r_i}^i, v_{s_j}^j), & \text{if } s_j \geq 2. \end{cases}$$

To complete the proof, notice that $l(R_{ij}) = \max\{r_i, s_j\} \leq \ell$ and all these paths are internally disjoint in $G_1 \boxtimes G_2$, since each path R_{ij} is associated to specific and different paths P_i in G_1 and Q_j in G_2. $\square$

Notice that Lemma 3.2.2 provides a tight bound and it is not difficult to find several families of graphs which achieve it. For example, the strong product

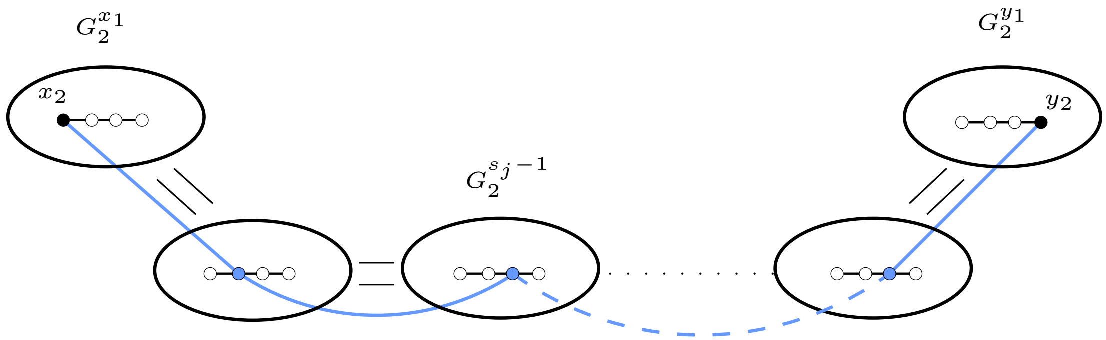

Figure 3.10: Construction of path R_{ij} when $r_i \geq s_j$ and $s_j \geq 2$ in Lemma 3.2.2.

of two paths, of two cycles or the strong product of a path and a cycle, are some of them.

We have just constructed $\zeta_\ell(G_1)\,\zeta_\ell(G_2)$ internally disjoint paths in $G_1 \boxtimes G_2$ of length at most ℓ which join two given vertices of $G_1 \boxtimes G_2$, using paths of length at most ℓ in the generator graphs G_1 and G_2. But if we allow the length of the paths in $G_1 \boxtimes G_2$ to be at most $\ell + 2$, it is possible to construct more paths as we prove in next result.

Lemma 3.2.3. *Let G_1 and G_2 be two connected graphs with at least 3 vertices and girth at least 5. Let $\ell \geq \max\{D(G_1), D(G_2)\}$ be an integer. For every two distinct vertices $x_1, y_1 \in V(G_1)$ and every two distinct vertices $x_2, y_2 \in V(G_2)$ there exist at least*

$$\zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$$

internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$.

Proof. Let us denote by $\zeta_1 = \zeta_\ell(G_1)$ and $\zeta_2 = \zeta_\ell(G_2)$. Let $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$ be two pairs of distinct vertices. Let $P_1, \ldots, P_{\zeta_1}$ be ζ_1 internally disjoint x_1y_1-paths of length at most ℓ in G_1 and let $Q_1, \ldots, Q_{\zeta_2}$ be ζ_2 internally disjoint x_2y_2-paths of length at most ℓ in G_2. We need to find $\zeta_1\zeta_2 + \zeta_1 + \zeta_2$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$ of length at most $\ell + 2$.

(I) First, by considering the x_1y_1-path P_1 in G_1 and the x_2y_2-path Q_1 in G_2, we construct three pairwise disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$ of length at most $\ell + 2$. These paths are denoted by R_{11}, R'_{11} and R^* and their construction is done according to the length of the paths P_1 and Q_1.

(a) Assume that $r_1 = 1$ and $s_1 = 1$. Thus $P_1 : x_1y_1 \in E(G_1)$, $Q_1 : x_2y_2 \in E(G_2)$. Then, the three internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$ are

$R_{11} : (x_1, x_2)(x_1, y_2)(y_1, y_2)$,

$R'_{11} : (x_1, x_2)(y_1, x_2)(y_1, y_2)$ and

$R^* : (x_1, x_2)(y_1, y_2)$.

Their lengths are $l(R_{11}) = l(R'_{11}) = 2$ and $l(R^*) = 1$.

(b) Assume that $r_1 = 1$ and $s_1 \geq 2$. Then, the first two $(x_1, x_2)(y_1, y_2)$-paths are

$R_{11} : (u_0^1, v_0^1)(u_1^1, v_1^1) \dots (u_1^1, v_{s_1}^1)$ and

$R'_{11} : (u_0^1, v_0^1) \dots (u_0^1, v_{s_1-1}^1)(u_1^1, v_{s_1}^1)$.

Notice that $l(R_{11}) = l(R'_{11}) = s_1 \leq \ell$. In this case, it is impossible to construct in $G_1 \boxtimes G_2$ the third path induced only by P_1 and Q_1. We solve this problem in two different ways depending on the value ζ_1.

First, suppose that $\zeta_1 = 1$. Since $x_1y_1 \in E(G_1)$ and G_1 is a connected graph with at least three vertices, there exists a vertex $u \in V(G_1)$ such that either $ux_1 \in E(G_1)$ or $uy_1 \in E(G_1)$.

Without loss of generality, we may assume that $ux_1 \in E(G_1)$. In such case, observe that the first and the last internal vertex of the path Q_1^u are adjacent in $G_1 \boxtimes G_2$ to (x_1, x_2) and (x_1, y_2), respectively. Then we obtain the third

$(x_1, x_2)(y_1, y_2)$-path as follows (see Figure 3.11):

$$R^* : (x_1, x_2)\ddot{Q}_1^u(x_1, y_2)(y_1, y_2),$$

which has length $1 + s_1 - 2 + 1 + 1 \leq \ell + 1$.

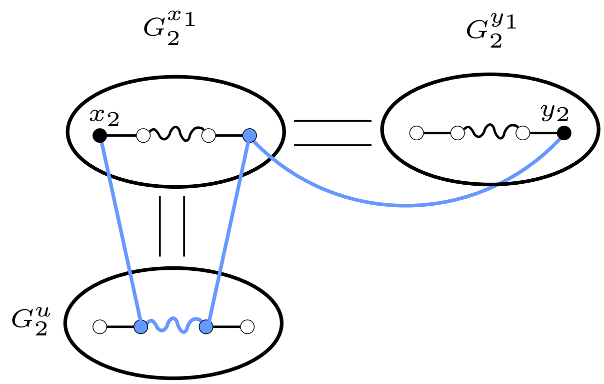

Figure 3.11: Path R^* when $r_1 = 1$, $s_1 \geq 2$ and $\zeta_1 = 1$ in Lemma 3.2.3.

Second, suppose that $\zeta_1 \geq 2$. Then there exists at least one path P_2 in G_1. Moreover, since $g(G_1) \geq 5$ and $r_1 = 1$, the path P_2 has length $r_2 \geq 4$. Then observe that $u_0^1 = u_0^2 = x_1$, $u_1^1 = u_{r_2}^2 = y_1$, $v_0^1 = x_2$ and $v_{s_1}^1 = y_2$. In this case, we construct the third path as follows:

If $s_1 = 2$, then (see Figure 3.12)

$$R^* : (u_0^1, v_0^1)(u_1^1, v_0^1)(u_{r_2-1}^2, v_0^1)(u_{r_2-2}^2, v_1^1) \ldots (u_2^2, v_1^1)(u_1^2, v_2^1)(u_0^1, v_{s_1}^1)(u_1^1, v_{s_1}^1).$$

If $r_2 > s_1$ and $s_1 \geq 3$, then (see Figure 3.13)

$$R^* : (u_0^1, v_0^1)(u_1^1, v_0^1)(u_{r_2-1}^2, v_0^1) \ldots (u_{r_2-s_1}^2, v_{s_1-1}^1) \ldots (u_1^2, v_{s_1-1}^1)(u_0^1, v_{s_1}^1)(u_1^1, v_{s_1}^1).$$

If $r_2 \leq s_1$ and $s_1 \geq 3$, then (see Figure 3.14)

$$R^* : (u_0^1, v_0^1)(u_1^1, v_0^1)(u_{r_2-1}^2, v_1^1) \ldots (u_{r_2-1}^2, v_{s_1-r_2+1}^1) \ldots (u_1^2, v_{s_1-1}^1)(u_0^1, v_{s_1}^1)(u_1^1, v_{s_1}^1).$$

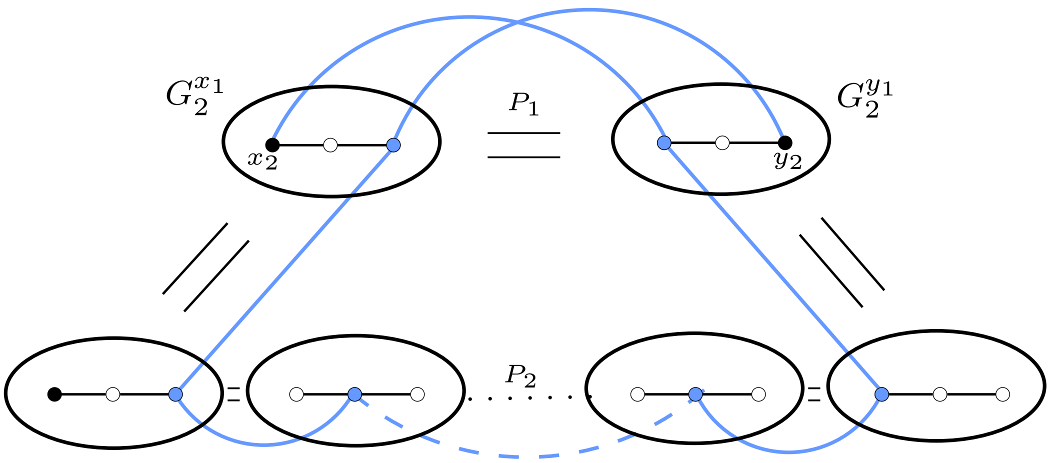

Figure 3.12: Path R^* when $\zeta_1 \geq 2$, $r_1 = 1$ and $s_1 = 2$ in Lemma 3.2.3.

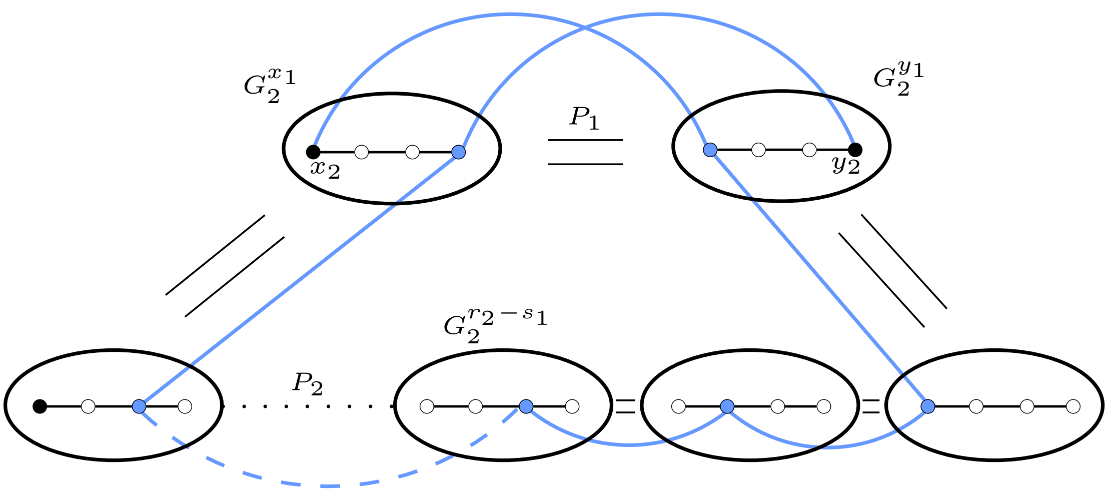

Figure 3.13: Path R^* when $\zeta_1 \geq 2$, $r_2 > s_1$ and $s_1 \geq 3$ in Lemma 3.2.3.

Notice that $l(R^*) = \max\{s_1, r_2\} + 2 \leq \ell + 2$, in either case. The design of R^* is very special and different with respect to the previous ones since it must be combined with the paths that will be described in **(III)**.

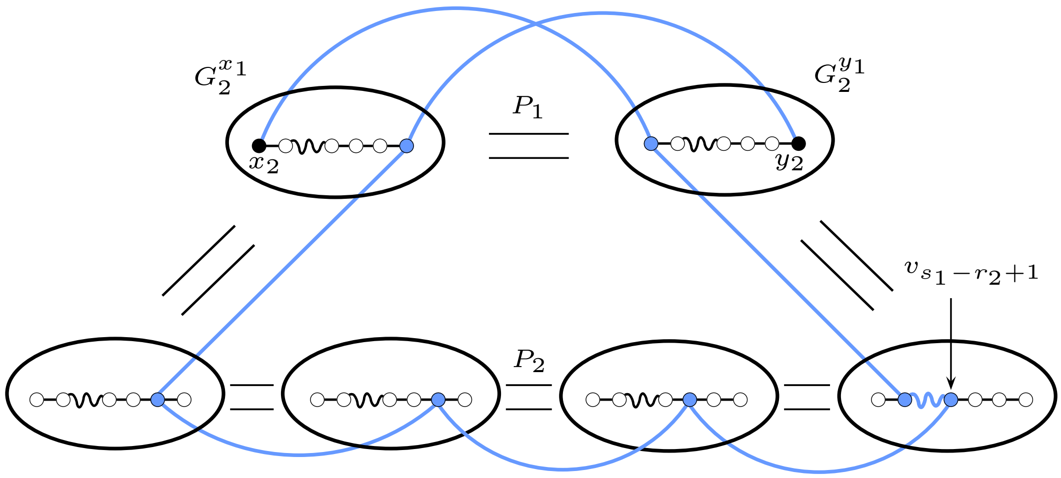

Figure 3.14: Path R^* when $\zeta_1 \geq 2$, $r_2 \leq s_1$ and $s_1 \geq 3$ in Lemma 3.2.3.

(c) The case $r_1 \geq 2$ and $s_1 = 1$ is symmetric to the previous one due to the commutativity of the strong product of graphs.

(d) Assume that $r_1 \geq 2$ and $s_1 \geq 2$. Then, the $(x_1, x_2)(y_1, y_2)$-paths are

$$R_{11}: \begin{cases} (u_0^1, v_0^1) \dots (u_0^1, v_{s_1-r_1+1}^1) \dots (u_{r_1-1}^1, v_{s_1}^1)(u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 \leq s_1 \\ (u_0^1, v_0^1)(u_0^1, v_1^1) \dots (u_{s_1-1}^1, v_{s_1}^1) \dots (u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 > s_1, \end{cases}$$

$$R'_{11}: \begin{cases} (u_0^1, v_0^1)(u_1^1, v_0^1) \dots (u_{r_1}^1, v_{r_1-1}^1) \dots (u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 \leq s_1 \\ (u_0^1, v_0^1) \dots (u_{r_1-s_1+1}^1, v_0^1) \dots (u_{r_1}^1, v_{s_1-1}^1)(u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 > s_1, \end{cases}$$

and

$$R^*: \begin{cases} (u_0^1, v_0^1) \dots (u_{r_1-1}^1, v_{r_1-1}^1) \dots (u_{r_1-1}^1, v_{s_1-1}^1)(u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 \leq s_1 \\ (u_0^1, v_0^1) \dots (u_{s_1-1}^1, v_{s_1-1}^1) \dots (u_{r_1-1}^1, v_{s_1-1}^1)(u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 > s_1. \end{cases}$$

In this case, observe that $l(R_{11}) = l(R'_{11}) = \max\{r_1, s_1\} + 1 \leq \ell + 1$, whereas $l(R^*) \leq \ell$. Hence, these three paths constructively prove the desired result when $\zeta_1 = \zeta_2 = 1$.

(II) If $\zeta_2 \geq 2$, then there exist the x_2y_2-paths $Q_2, \ldots, Q_{\zeta_2}$ of length at most ℓ in G_2 and $s_j \geq 3$, for $j \in \{2, \ldots, \zeta_2\}$, since $g(G_2) \geq 5$. Then associated to the only x_1y_1-path P_1 in G_1 and to the x_2y_2-paths $Q_2, \ldots, Q_{\zeta_2}$ in G_2 we construct two $(x_1, x_2)(y_1, y_2)$-paths R_{1j} and R'_{1j}, for $j \in \{2, \ldots, \zeta_2\}$ in $G_1 \boxtimes G_2$ of length at most $\ell + 2$ in $G_1 \boxtimes G_2$ as follows, distinguishing two cases.

$$R_{1j} : \begin{cases} (u_0^1, v_0^j) \ldots (u_0^1, v_{s_j-r_1}^j) \ldots (u_{r_1-1}^1, v_{s_j-1}^j)(u_{r_1}^1, v_{s_j}^j), & \text{if } r_1 < s_j \\ (u_0^1, v_0^j)(u_0^1, v_1^j) \ldots (u_{s_j-2}^1, v_{s_j-1}^j) \ldots (u_{r_1-1}^1, v_{s_j-1}^j)(u_{r_1}^1, v_{s_j}^j), & \text{if } r_1 \geq s_j, \end{cases}$$

$$R'_{1j} : \begin{cases} (u_0^1, v_0^j) \ldots (u_{r_1-1}^1, v_{r_1-1}^j)(u_{r_1}^1, v_{r_1}^j) \ldots (u_{r_1}^1, v_{s_j}^j), & \text{if } r_1 < s_j \\ (u_0^1, v_0^j)(u_1^1, v_1^j) \ldots (u_{r_1-s_j+2}^1, v_1^j) \ldots & \\ \ldots (u_{r_1-1}^1, v_{s_j-2}^j)(u_{r_1}^1, v_{s_j-1}^j)(u_{r_1}^1, v_{s_j}^j), & \text{if } r_1 \geq s_j. \end{cases}$$

The lengths of the paths R_{1j} and R'_{1j} are at most $\max\{r_1, s_j\} + 1 \leq \ell + 1$ and observe that all these paths are internally disjoint with the three paths described in **(I)** in either case, since they depend on each path Q_j for $j \in \{2, \ldots, \zeta_2\}$.

If $\zeta_1 = 1$ and $\zeta_2 \geq 2$, then **(I)** and **(II)** provide $3 + 2(\zeta_2 - 1) = 2\zeta_2 + 1$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$ and the proof is finished.

(III) If $\zeta_1 \geq 2$ then there exist the x_1y_1-paths $P_2, \ldots, P_{\zeta_1}$ of length at most ℓ in G_1 and $r_i \geq 3$, for $i \in \{2, \ldots, \zeta_1\}$, since $g(G_1) \geq 5$. Then associated to the only x_2y_2-path Q_1 in G_2 and to each x_1y_1-path P_i in G_1, for $i \in \{2, \ldots, \zeta_1\}$, we construct two $(x_1, x_2)(y_1, y_2)$-paths $R_{i,1}$ and $R'_{i,1}$ of length at most $\ell + 2$ in $G_1 \boxtimes G_2$. Observe that the difficulty to construct the paths R_{i1} and R'_{i1} takes root in the fact that they must be internally disjoint with the path R^* considered in **(I)**. For this reason, we need to distinguish different cases depending on the length of the path Q_1.

(a) If $s_1 = 1$ then

$R_{i1} : (u_0^i, v_0^1) \dots (u_{r_i-1}^i, v_0^1)(u_{r_i}^i, v_1^1)$,

$R'_{i1} : (u_0^i, v_0^1)(u_1^i, v_1^1) \dots (u_{r_i}^i, v_1^1)$.

(b) If $s_1 = 2$ then

$R_{i1} : (u_0^i, v_0^1) \dots (u_{r_i-2}^i, v_0^1)(u_{r_i-1}^i, v_1^1)(u_{r_i}^i, v_2^1)$,

$R'_{i1} : (u_0^i, v_0^1)(u_1^i, v_1^1)(u_2^i, v_2^1) \dots (u_{r_i}^i, v_2^1)$.

(c) If $r_i = 3$ and $s_1 \geq 3$, then

$R_{i1} : (u_0^i, v_0^1)(u_1^i, v_1^1) \dots (u_1^i, v_{s_1-1}^1)(u_2^i, v_{s_1}^1)(u_3^i, v_{s_1}^1)$,

$R'_{i1} : (u_0^i, v_0^1)(u_1^i, v_0^1)(u_2^i, v_1^1) \dots (u_2^i, v_{s_1-1}^1)(u_3^i, v_{s_1}^1)$.

(d) If $r_i > s_1 \geq 3$ then

$$R_{i1} : \begin{cases} (u_0^i, v_0^1)(u_1^i, v_1^1) \dots (u_{r_i-s_1}^i, v_1^1) \dots (u_{r_i-1}^i, v_{s_1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } s_1 \text{ is odd} \\ \\ (u_0^i, v_0^1)(u_1^i, v_1^1) \dots (u_{r_i-s_1+1}^i, v_1^1) \dots & \\ \dots (u_{r_i-2}^i, v_{s_1-2}^1)(u_{r_i-2}^i, v_{s_1-1}^1)(u_{r_i-1}^i, v_{s_1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } s_1 \text{ is even,} \end{cases}$$

$$R'_{i1} : \begin{cases} (u_0^i, v_0^1) \dots (u_{r_i-s_1+1}^i, v_0^1) \dots & \\ \dots (u_{r_i-1}^i, v_{s_1-2}^1)(u_{r_i-1}^i, v_{s_1-1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } s_1 \text{ is odd} \\ \\ (u_0^i, v_0^1) \dots (u_{r_i-s_1+2}^i, v_0^1) \dots & \\ \dots (u_{r_i-1}^i, v_{s_1-3}^1) \dots (u_{r_i-1}^i, v_{s_1-1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } s_1 \text{ is even,} \end{cases}$$

(e) If $s_1 \geq r_i > 3$ then

$$R_{i1} : \begin{cases} (u_0^i, v_0^1)(u_1^i, v_1^1) \dots (u_1^i, v_{s_1-r_i+3}^1) \dots (u_{r_i-2}^i, v_{s_1}^1) \dots (u_{r_i}^i, v_{s_1}^1), & \text{if } r_i \text{ is odd} \\ \\ (u_0^i, v_0^1)(u_1^i, v_1^1) \dots (u_1^i, v_{s_1-r_i+2}^1) \dots (u_{r_i-1}^i, v_{s_1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } r_i \text{ is even,} \end{cases}$$

$$R'_{i1} : \begin{cases} (u_0^i, v_0^1)(u_1^i, v_0^1)(u_2^i, v_1^1) \dots (u_2^i, v_{s_1-r_i+2}^1) \dots (u_{r_i}^i, v_{s_1}^1), & \text{if } r_i \text{ is odd} \\ \\ (u_0^i, v_0^1)(u_1^i, v_0^1)(u_2^i, v_1^1) \dots (u_2^i, v_{s_1-r_i+1}^1) \dots & \\ \dots (u_{r_i-1}^i, v_{s_1-2}^1)(u_{r_i-1}^i, v_{s_1-1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } r_i \text{ is even.} \end{cases}$$

The lengths of the paths R_{i1} and R'_{i1} are at most $\max\{r_i, s_1\}+2 \leq \ell+2$. Notice that they are internally disjoint with all the paths described in **(I)** and **(II)**.

If $\zeta_2 = 1$ and $\zeta_1 \geq 2$, then **(I)** and **(III)** provide $3+2(\zeta_1-1) = 2\zeta_1+1$ internally disjoint $(x_1,x_2)(y_1,y_2)$-paths of length at most $\ell+2$ in $G_1 \boxtimes G_2$, as we have desired.

(IV) If $\zeta_1 \geq 2$ and $\zeta_2 \geq 2$, there exist the x_1y_1-paths $P_2, \dots, P_{\zeta_1}$ and the x_2y_2-paths $Q_2, \dots, Q_{\zeta_2}$ of length at most ℓ in G_1 and G_2, respectively. Moreover, as $g(G_1) \geq 5$ and $g(G_2) \geq 5$, $r_i \geq 3$ and $s_j \geq 3$, for every $i \in \{2, \dots, \zeta_1\}$ and every $j \in \{2, \dots, \zeta_2\}$.

Then, for $i \in \{2, \dots, \zeta_1\}$ and $j \in \{2, \dots, \zeta_2\}$, associated to each x_1y_1-path P_i in G_1 and to each x_2y_2-path Q_j in G_2, we consider the path described in Lema 3.2.2 (see Figure 3.9 and Figure 3.10):

$$R_{ij} : \begin{cases} (u_0^i, v_0^j) \dots (u_{r_i-1}^i, v_{r_i-1}^j) \dots (u_{r_i-1}^i, v_{s_j-1}^j)(u_{r_i}^i, v_{s_j}^j), & \text{if } r_i < s_j \\ (u_0^i, v_0^j) \dots (u_{s_j-1}^i, v_{s_j-1}^j) \dots (u_{r_i-1}^i, v_{s_j-1}^j)(u_{r_i}^i, v_{s_j}^j), & \text{if } r_i \geq s_j. \end{cases}$$

It is easy to check that $l(R_{ij}) = \max\{r_i, s_j\} \leq \ell$ and that these $(\zeta_1-1)(\zeta_2-1)$ paths R_{ij} are internally disjoint with all the previous paths because they are associated to different paths in the generator graphs G_1 and G_2. If $\zeta_1 \geq 2$ and $\zeta_2 \geq 2$, then **(I)** to **(IV)** provide

$$3+2(\zeta_2-1)+2(\zeta_1-1)+(\zeta_1-1)(\zeta_2-1) = \zeta_1\zeta_2+\zeta_1+\zeta_2$$

pairwise internally disjoint $(x_1,x_2)(y_1,y_2)$-paths in $G_1 \boxtimes G_2$ of length at most $\ell+2$, which finishes the proof. $\square$

This section has been devoted to present the results which will be the key point for next sections of this chapter. We have studied the number of internally disjoint paths that exists between any two vertices in the strong product of two connected graphs as well as their lengths.

3.3 The Menger number

The *Menger number* is the first of the three parameters that we study in this chapter. Given two distinct vertices x, y of a connected graph G, the xy-Menger number with respect to a positive integer ℓ, denoted by $\zeta_\ell(x, y)$, is the maximum number of internally disjoint xy-paths in G whose lengths are at most ℓ. The Menger number of G with respect to ℓ is defined as

$$\zeta_\ell(G) = \min\{\zeta_\ell(x, y) : x, y \in V(G)\}.$$

Observe that $\zeta_\ell(G)$ is an increasing function on ℓ and $\zeta_\ell(G) \leq \kappa(G) \leq \delta(G)$ for every positive integer ℓ. Clearly, if $\ell < D(G)$ then $\zeta_\ell(G) = 0$ and also, for every integer $\ell \geq |V(G)| - 1$, the Menger number is $\zeta_\ell(G) = \kappa(G)$. Thus, the determination of $\zeta_\ell(G)$ when $D(G) \leq \ell \leq |V(G)| - 2$ is an open and interesting problem.

Ma, Xu, and Zhu in [77] studied the *Menger number* of the cartesian product of two connected graphs G_1 and G_2. Namely, for two integers $\ell_1 \geq 2$ and $\ell_2 \geq 2$, they proved that

$$\zeta_{\ell_1+\ell_2}(G_1 \Box G_2) \geq \zeta_{\ell_1}(G_1) + \zeta_{\ell_2}(G_2),$$

which is an equality when both G_1 and G_2 are paths and, therefore, $G_1 \Box G_2$ is a grid.

In this section we focus on $\zeta_\ell(G_1 \boxtimes G_2)$, the Menger number of the strong product of two connected graphs G_1 and G_2 with respect to a positive integer ℓ.

We have mentioned that for any connected graph G, the Menger number $\zeta_\ell(G) = 0$ for all $\ell < D(G)$. Hence, since the diameter of the strong product graph is $D(G_1 \boxtimes G_2) = \max\{D(G_1), D(G_2)\}$, from now on, we will consider only integers $\ell \geq \max\{D(G_1), D(G_2)\}$.

To estimate the Menger number $\zeta_\ell(G_1 \boxtimes G_2)$, we make use of the lower bounds on the number of internally disjoint paths of length at most ℓ that join any two arbitrary vertices in $G_1 \boxtimes G_2$, provided in Section 3.2.

Theorem 3.3.1. *Let G_1 and G_2 be two connected graphs with at least 3 vertices and $\ell \geq \max\{D(G_1), D(G_2)\}$ be an integer. The following assertions hold:*

(i) $\zeta_\ell(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2)$.

(ii) $\zeta_{\ell+2}(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$*, if* $g(G_i) \geq 5$ *for* $i = 1, 2$.

Proof. Let us consider vertices $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$.

(i) If $\ell = 1$, then G_1 and G_2 are complete graphs, yielding that $G_1 \boxtimes G_2$ is a complete graph and $\zeta_1(G_1 \boxtimes G_2) = \zeta_1(G_1) = \zeta_1(G_2) = 1$. Thus, point (i) directly holds.

Hence, assume that $\ell \geq 2$. If $x_1 = y_1$ and $x_2 \neq y_2$, then, by applying point (i) of Lemma 3.2.1, there exist at least $(\delta(G_1) + 1)\zeta_\ell(G_2) > \zeta_\ell(G_1)\zeta_\ell(G_2)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$. Analogously, if $x_1 \neq y_1$ and $x_2 = y_2$, by point (ii) of Lemma 3.2.1, there exist at least $(\delta(G_2)+1)\zeta_\ell(G_1) > \zeta_\ell(G_1)\zeta_\ell(G_2)$ internally disjoint $(x_1, x_2)(y_1, x_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$. If $x_1 \neq y_1$ and $x_2 \neq y_2$ then, by Lemma 3.2.2, there exist at least $\zeta_\ell(G_1)\zeta_\ell(G_2)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$. Therefore, $\zeta_\ell(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2)$.

(ii) Assume also that both G_1 and G_2 have girth at least 5. If $x_1 = y_1$ and $x_2 \neq y_2$, then by point (i) of Lemma 3.2.1, there exist at least

$$(\delta(G_1) + 1)\zeta_\ell(G_2) + \delta(G_1) \geq \zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_2) + \zeta_\ell(G_1)$$

internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$.

The same conclusion is obtained when $x_1 \neq y_1$ and $x_2 = y_2$, due to point (ii) of Lemma 3.2.1.

If $x_1 \neq y_1$ and $x_2 \neq y_2$ then, by Lemma 3.2.3, there exist at least

$$\zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$$

internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$.

Hence, $\zeta_{\ell+2}(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$, which finishes the proof.

□

Theorem 3.3.1 (i) provides a tight bound. There exist several examples for which equality $\zeta_\ell(G_1 \boxtimes G_2) = \zeta_\ell(G_1)\zeta_\ell(G_2)$ holds, for instance, when both G_1 and G_2 are isomorphic to the path $\mathcal{P}_\ell$ of length ℓ, when both G_1 and G_2 are isomorphic to the cycle $\mathcal{C}_{2\ell+1}$ of length $2\ell + 1$ and also when $G_1 = \mathcal{P}_\ell$ and $G_2 = \mathcal{C}_{2\ell+1}$.

For example, consider the case $G_1 = \mathcal{P}_2$ and $G_2 = \mathcal{C}_5$, and $\ell = 2$. It is clear that $\zeta_2(\mathcal{P}_2) = 1$ and $\zeta_2(\mathcal{C}_5) = 1$. Also, it is easy to check in this case that $\zeta_2(\mathcal{P}_2 \boxtimes \mathcal{C}_5) = 1$, because there are pairs of vertices in $\mathcal{P}_2 \boxtimes \mathcal{C}_5$ for which there exists only one path of length at most 2 (see Figure 3.15).

Theorem 3.3.1 (ii) is also best possible in the sense that both hypothesis cannot be relaxed. On the one hand, the bound in Theorem 3.3.1 (ii) may not be attained when at least one of the generator graphs has two vertices. For example, by considering the paths $\mathcal{P}_1$ and $\mathcal{P}_2$ whose lengths are 1 and 2, respectively, if $\ell \geq 2$, observe that

$$\zeta_\ell(\mathcal{P}_1 \boxtimes \mathcal{P}_2) \leq \kappa(\mathcal{P}_1 \boxtimes \mathcal{P}_2) = 2 < 3 = \zeta_\ell(\mathcal{P}_1)\zeta_\ell(\mathcal{P}_2) + \zeta_\ell(\mathcal{P}_1) + \zeta_\ell(\mathcal{P}_2).$$

On the other hand, the same bound may fail when the hypothesis of girth at least five is not fulfilled. For example, let $\mathcal{C}_4 \boxtimes \mathcal{C}_4$ be the strong product of

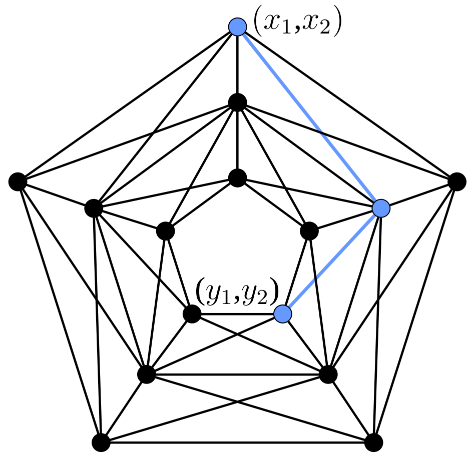

Figure 3.15: Unique $(x_1,x_2)(y_1,y_2)$-path of length 2 in $\mathcal{P}_2 \boxtimes \mathcal{C}_5$.

two cycles of length 4 and consider any integer $\ell > D(\mathcal{C}_4) = 2$, for instance, $\ell = 3$. Clearly $\zeta_3(\mathcal{C}_4) = 2$. From Theorem 3.3.1 (ii) there should exist at least 8 internally disjoint paths between whichever pair of vertices in $\mathcal{C}_4 \boxtimes \mathcal{C}_4$. However, if we choose two vertices, $(x_1,x_2),(y_1,y_2)$, in $\mathcal{C}_4 \boxtimes \mathcal{C}_4$ such that $d_{G_1}(x_1,y_1) = 1$ and $d_{G_2}(x_2,y_2) = 2$, then it is only possible to construct just 7 internally disjoint paths of length at most $\ell + 2 = 5$ between such vertices as we can see in Figure 3.16.

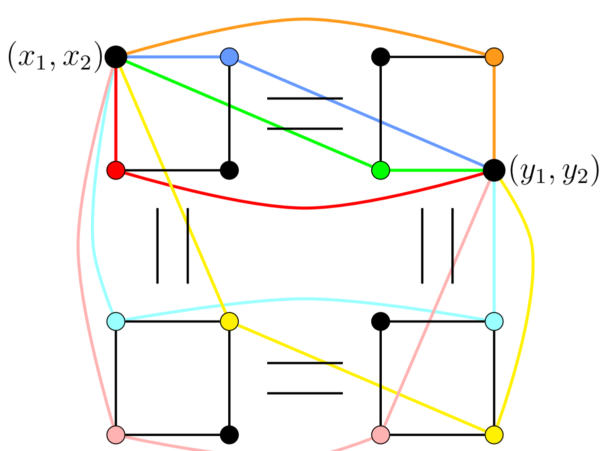

Figure 3.16: Pair of vertices in $\mathcal{C}_4 \boxtimes \mathcal{C}_4$ for which the lower bound (ii) of Theorem 3.3.1 is not attained after relaxing the hypothesis.

The first consequence of Theorem 3.3.1 is the following result. It gives a sufficient condition to guarantee the maximum possible value of $\zeta_{\ell+2}(G_1 \boxtimes G_2)$, for certain ℓ.

Corollary 3.3.1. *Let G_1 and G_2 be two maximally connected graphs with at least 3 vertices and girth at least 5. If ℓ is a positive integer such that $\zeta_\ell(G_1) = \kappa(G_1)$ and $\zeta_\ell(G_2) = \kappa(G_2)$, then*

$$\zeta_{\ell+2}(G_1 \boxtimes G_2) = \delta(G_1 \boxtimes G_2).$$

Proof. Inequality $\zeta_{\ell+2}(G_1 \boxtimes G_2) \leq \delta(G_1 \boxtimes G_2)$ clearly holds. We just need to prove the another one. From Theorem 3.3.1, we have

$$\zeta_{\ell+2}(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\,\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2).$$

Since $\zeta_\ell(G_1) = \kappa(G_1)$ and $\zeta_\ell(G_2) = \kappa(G_2)$,

$$\zeta_{\ell+2}(G_1 \boxtimes G_2) \geq \kappa(G_1)\,\kappa(G_2) + \kappa(G_1) + \kappa(G_2).$$

Also we know that both graphs are maximally connected graphs, hence

$$\zeta_{\ell+2}(G_1 \boxtimes G_2) \geq \delta(G_1)\,\delta(G_2) + \delta(G_1) + \delta(G_2) = \delta(G_1 \boxtimes G_2),$$

and the desired result is proved. □

As we mentioned above, $\zeta_\ell(G)$ is an increasing function on ℓ, therefore $\zeta_\ell(G) \leq \kappa(G) \leq \delta(G)$ for every positive integer ℓ. In fact, for every connected graph G there exists a positive integer $\ell \leq |V(G)| - 1$ for which equality $\zeta_\ell(G) = \kappa(G)$ holds. Hence, from this fact and Theorem 3.3.1, it follows the next consequence whose proof is straightforward.

Corollary 3.3.2. *Let G_1 and G_2 be two maximally connected graphs with at least 3 vertices and girth at least 5. Then $G_1 \boxtimes G_2$ is maximally connected.*

We finish this section determining the exact Menger number of the strong product of two graphs $G_1 \boxtimes G_2$ when both G_1 and G_2 are paths, when both of them are cycles and if one of them is a path and the another one is a cycle, for certain values of ℓ and their orders. First, observe that:

If both G_1 and G_2 are paths, $\mathcal{P}_{r_1}$ and $\mathcal{P}_{r_2}$, with lengths r_1 and r_2, respectively, being $r_1 \geq r_2$, then $\zeta_{r_1}(\mathcal{P}_{r_1}) = \zeta_{r_1}(\mathcal{P}_{r_2}) = 1$.

If both G_1 and G_2 are cycles, $\mathcal{C}_{r_1}$ and $\mathcal{C}_{r_2}$, with lengths r_1 and r_2, respectively, being $r_1 \geq r_2$, then $\zeta_{r_1-1}(\mathcal{C}_{r_1}) = \zeta_{r_1-1}(\mathcal{C}_{r_2}) = 2$.

Hence, by applying point (ii) of Theorem 3.3.1 and taking into account the previous observations, next result can be proved directly.

Corollary 3.3.3. *For integers $r_1 \geq r_2 \geq 2$, the following assertions hold.*

(i) If $\mathcal{P}_{r_1}$ and $\mathcal{P}_{r_2}$ are paths, with lengths r_1 and r_2, respectively, then

$$\zeta_{r_1+2}(\mathcal{P}_{r_1} \boxtimes \mathcal{P}_{r_2}) = 3.$$

(ii) If $\mathcal{P}_{r_1}$ is a path and $\mathcal{C}_{r_2}$ is a cycle, with lengths r_1 and r_2, respectively, such that $r_2 \geq 5$, then

$$\zeta_{r_1+2}(\mathcal{P}_{r_1} \boxtimes \mathcal{C}_{r_2}) = 5.$$

(iii) If $\mathcal{C}_{r_1}$ and $\mathcal{C}_{r_2}$ are cycles, with lengths r_1 and r_2, respectively, such that $r_2 \geq 5$, then

$$\zeta_{r_1+1}(\mathcal{C}_{r_1} \boxtimes \mathcal{C}_{r_2}) = 8.$$

3.4 The average connectivity

The average connectivity is the next vulnerability parameter which is studied in this chapter. For a connected graph G of order n, recall that the connectivity between two distinct vertices x and y in G, denoted by $\kappa_G(x,y)$, is the maximum number of pairwise internally disjoint xy-paths in G. Then, the *average connectivity*, denoted by $\overline{\kappa}(G)$, is defined as the mean of the connectivities between all the non ordered pairs of vertices in G, that is,

$$\overline{\kappa}(G) = \frac{1}{\binom{n}{2}} \sum_{x,y \in V(G)} \kappa_G(x,y).$$

Sometimes, in order to avoid fractions, we also consider the *total connectivity* of G, denoted by

$$K(G) = \sum_{x,y \in V(G)} \kappa_G(x,y).$$

The difference between the classical connectivity and the average connectivity is that, while the connectivity is the minimum number of vertices whose removal separates at least one connected pair of vertices, the average connectivity is a measure for the expected number of vertices that have to be removed to separate a randomly chosen pair of vertices. For instance, in Figure 3.17 there are two graphs with connectivity 1, but it is obvious the graph G_2 is more vulnerable than G_2. In fact, $\overline{\kappa}(G_1) = 2.2$ and $\overline{\kappa}(G_2) = 1$.

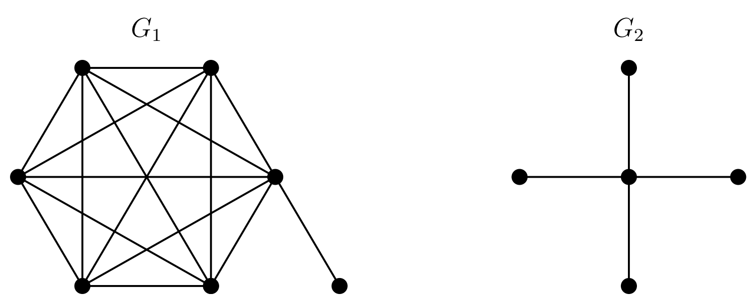

Figure 3.17: Two graphs with equal connectivity but $\overline{\kappa}(G_1) > \overline{\kappa}(G_2)$.

To estimate the average connectivity of the strong product of two connected graphs G_1 and G_2, we must compute the number of pairwise disjoint paths connecting two arbitrary vertices in $G_1 \boxtimes G_2$.

In Section 3.2 we have proved that if both G_1 and G_2 have at least three vertices and girth at least 5, for any two vertices $x_1, y_1 \in V(G_1)$, $x_2, y_2 \in V(G_2)$ and $\ell \geq \max\{D(G_1), D(G_2)\}$, there exist at least $\zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$.

When the length of the paths between vertices $(x_1, x_2), (y_1, y_2)$ of $G_1 \boxtimes G_2$ is not decisive and we are only interested in computing how many paths are internally disjoint, the $(x_1, x_2)(y_1, y_2)$-Menger number with respect to a large enough value of ℓ leads us to study the connectivity between (x_1, x_2) and (y_1, y_2) in $G_1 \boxtimes G_2$. Namely, for $\ell \geq |V(G_1 \boxtimes G_2)| - 1$, we have

$$\kappa_{G_1 \boxtimes G_2}((x_1, x_2), (y_1, y_2)) = \zeta_\ell((x_1, x_2), (y_1, y_2)),$$

which means that Lemma 3.2.1 and Lemma 3.2.3 can be applied as we see in the following remark.

Remark 3.4.1. *Let G_1 and G_2 be two connected graphs with at least 3 vertices and girth at least 5. Let $x_i, y_i \in V(G_i)$ be two distinct vertices, for $i = 1, 2$. The following assertions hold:*

(i) There exist at least $(\delta(G_1) + 1)\kappa_{G_2}(x_2, y_2) + \delta(G_1)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths in $G_1 \boxtimes G_2$.

(ii) There exist at least $\kappa_{G_1}(x_1, y_1)(\delta(G_2) + 1) + \delta(G_2)$ internally disjoint $(x_1, x_2)(y_1, x_2)$-paths in $G_1 \boxtimes G_2$.

(iii) There exist at least $\kappa_{G_1}(x_1, y_1)\kappa_{G_2}(x_2, y_2) + \kappa_{G_1}(x_1, y_1) + \kappa_{G_2}(x_2, y_2)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$.

Remark 3.4.1 leads us to get a lower bound for the average connectivity of the strong product of two connected graphs, as we will see below.

Let G_1 and G_2 be two connected graphs of order n_1 and n_2, size e_1 and e_2, average connectivity $\overline{\kappa}(G_1)$ and $\overline{\kappa}(G_2)$, and average degree $\overline{d}(G_1)$ and $\overline{d}(G_2)$, respectively. We will obtain such lower bound on $\overline{\kappa}\left(G_1 \boxtimes G_2\right)$ in terms of the aforementioned parameters of G_1 and G_2.

Theorem 3.4.1. *Let G_1 and G_2 be two connected graphs with order $n_1, n_2 \geq 3$, respectively, and girth at least 5. Then*

$$\begin{aligned} \overline{\kappa}\left(G_1 \boxtimes G_2\right) \geq & \frac{1}{n_1 n_2 - 1} \big[(n_1-1)(n_2+\overline{d}(G_2))\overline{\kappa}(G_1) + (n_2-1)(n_1+\overline{d}(G_1))\overline{\kappa}(G_2) \\ & +(n_1-1)(n_2-1)\overline{\kappa}(G_1)\overline{\kappa}(G_2) + (n_2-1)\overline{d}(G_1) + (n_1-1)\overline{d}(G_2) \big]. \end{aligned}$$

Proof. Let $G = G_1 \boxtimes G_2$. Let $x_i, y_i \in V(G_i)$ be two distinct vertices, for $i = 1, 2$, and denote by $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Let $\mathbb{P}(G_1 \boxtimes G_2)$ be the set of non ordered pairs of vertices of $G_1 \boxtimes G_2$. Then $\mathbb{P}(G_1 \boxtimes G_2)$ can be partitioned into the following sets:

$$\begin{aligned} A &= \bigcup_{x_2, y_2 \in V(G_2)} \left\{ \{(u, x_2), (u, y_2)\} : u \in V(G_1) \right\}, \\ B &= \bigcup_{x_1, y_1 \in V(G_1)} \left\{ \{(x_1, v), (y_1, v)\} : v \in V(G_2) \right\}, \\ C &= \bigcup_{x, y \in V(G_1 \boxtimes G_2)} \left\{ \{(x_1, x_2), (y_1, y_2)\} : x_1 \neq y_1 \text{ and } x_2 \neq y_2 \right\}. \end{aligned}$$

Moreover, their cardinalities are

$$|V(G_1 \boxtimes G_2)| = n_1 n_2,$$

$$|A| = n_1 \binom{n_2}{2},$$

$$|B| = n_2 \binom{n_1}{2} \text{ and}$$

$$|C| = 2 \binom{n_1}{2} \binom{n_2}{2}.$$

Hence,

$$\begin{aligned}
\overline{\kappa}\left(G_1 \boxtimes G_2\right) &= \frac{1}{\binom{n}{2}} \sum_{\mathbb{P}(G_1 \boxtimes G_2)} \kappa_G\left((x_1,x_2),(y_1,y_2)\right) \\
&= \frac{1}{\binom{n}{2}} \left[\sum_A \kappa_G\left((u,x_2),(u,y_2)\right) + \sum_B \kappa_G\left((x_1,v),(y_1,v)\right) \right. \qquad (3.1) \\
&\quad \left. + \sum_C \kappa_G\left((x_1,x_2),(y_1,y_2)\right) \right].
\end{aligned}$$

To get the desired lower bound of the average connectivity of the strong product graph, we start computing the three sums of equality (3.1).

Since the elements of A satisfy the hypothesis of Remark 3.4.1, it follows that

$$\begin{aligned}
\sum_A \kappa_G\left((u,x_2),(u,y_2)\right) &\geq \sum_A \left[(1+d_{G_1}(u))\kappa_{G_2}(x_2,y_2) + d_{G_1}(u)\right] \\
&= \sum_{x_2,y_2 \in V(G_2)} \kappa_{G_2}(x_2,y_2) \sum_{u \in V(G_1)} (1+d_{G_1}(u)) \\
&\quad + \binom{n_2}{2} \sum_{u \in V(G_1)} d_{G_1}(u) \\
&= \sum_{x_2,y_2 \in V(G_2)} \kappa_{G_2}(x_2,y_2)(n_1+2e_1) + 2e_1 \binom{n_2}{2} \\
&= (n_1+2e_1)K(G_2) + 2e_1 \binom{n_2}{2}.
\end{aligned}$$

Similarly, as the elements of B satisfy the hypothesis of Remark 3.4.1 and by the commutativity of the strong product of graphs, we also deduce that

$$\sum_B \kappa_G\left((x_1,v),(y_1,v)\right) \geq (n_2+2e_2)K(G_1) + 2e_2 \binom{n_1}{2}.$$

Since the elements of C satisfy the hypothesis of Remark 3.4.1, we have

$$\begin{aligned}
\sum_{C} \kappa_G\left((x_1,x_2),(y_1,y_2)\right) &\geq \sum_{C} \left[\kappa_{G_1}(x_1,y_1)\kappa_{G_2}(x_2,y_2) + \kappa_{G_1}(x_1,y_1) + \kappa_{G_2}(x_2,y_2)\right] \\
&= \sum_{C} \left[\left(\kappa_{G_1}(x_1,y_1)+1\right)\left(\kappa_{G_2}(x_2,y_2)+1\right) - 1\right] \\
&= 2 \sum_{x_1,y_1 \in V(G_1)} \left(\kappa_{G_1}(x_1,y_1)+1\right) \sum_{x_2,y_2 \in V(G_2)} \left(\kappa_{G_2}(x_2,y_2)+1\right) \\
&\quad -|C| \\
&= 2K(G_1)K(G_2) + 2\binom{n_2}{2}K(G_1) + 2\binom{n_1}{2}K(G_2).
\end{aligned}$$

Thus, from the partition of $\mathbb{P}(G_1 \boxtimes G_2)$ into the sets A, B, C, we deduce that

$$\begin{aligned}
K(G_1 \boxtimes G_2) &= \sum_{\mathbb{P}(G_1 \boxtimes G_2)} \kappa_G\left((x_1,x_2),(y_1,y_2)\right) = \sum_{A} \kappa_G\left((u,x_2),(u,y_2)\right) \\
&\quad + \sum_{B} \kappa_G\left((x_1,v),(y_1,v)\right) + \sum_{C} \kappa_G\left((x_1,x_2),(y_1,y_2)\right) \\
&\geq (n_1+2e_1)K(G_2) + 2e_1\binom{n_2}{2} + (n_2+2e_2)K(G_1) + 2e_2\binom{n_1}{2} \\
&\quad + 2K(G_1)K(G_2) + 2\binom{n_2}{2}K(G_1) + 2\binom{n_1}{2}K(G_2) \\
&= (n_2^2+2e_2)K(G_1) + (n_1^2+2e_1)K(G_2) + 2K(G_1)K(G_2) \\
&\quad + 2e_1\binom{n_2}{2} + 2e_2\binom{n_1}{2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\overline{\kappa}\left(G_1 \boxtimes G_2\right) &= \frac{2}{n_1n_2(n_1n_2-1)} K\left(G_1 \boxtimes G_2\right) \\
&\geq \frac{1}{n_1n_2-1}\Big[(n_1-1)(n_2+\overline{d}(G_2))\overline{\kappa}(G_1) \\
&\quad + (n_2-1)(n_1+\overline{d}(G_1))\overline{\kappa}(G_2) + (n_1-1)(n_2-1)\overline{\kappa}(G_1)\overline{\kappa}(G_2) \\
&\quad + (n_2-1)\overline{d}(G_1) + (n_1-1)\overline{d}(G_2)\Big].
\end{aligned}$$

$\square$

Theorem 3.4.1 is best possible in the sense that the hypothesis of girth at least 5 cannot be relaxed. Indeed, let G_1 be the graph formed by two cycles of length 5 which share a common vertex z, and let G_2 be a cycle of length 4. Clearly G_1 is 1-connected, since z is a cut vertex of G_1, and G_2 is 2-connected. Let us consider two distinct vertices $x_1, y_1 \in V(G_1) \setminus \{z\}$ such that any x_1y_1-path in G_1 pass through z. For any two vertices $x_2, y_2 \in V(G_2)$, it is impossible to find five internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$, because each of these paths must contain a vertex of the subgraph G_2^z. But this graph has only four vertices (see Figure 3.18).

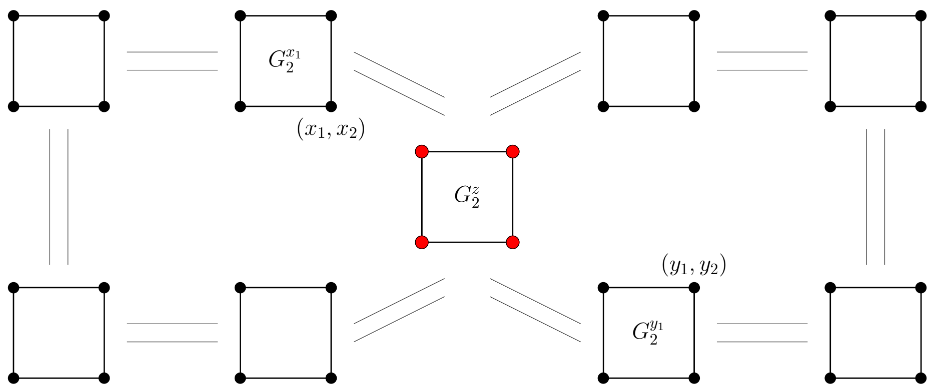

Figure 3.18: It is not possible to construct 5 internally disjoint paths between (x_1, x_2) and (y_1, y_2) in $G_1 \boxtimes G_2$.

We finish this section, giving an upper bound on the average connectivity of the strong product graph which es optimal under certain requirements. To do that, we use the following result, proved in [16].

Theorem 3.4.2. *([16]) Let G be a graph on n vertices and e edges with $e \geq n$, and let $r = 2e - n\lfloor 2e/n \rfloor$. Then*

$$\overline{\kappa}(G) \leq \overline{d}(G) - \frac{r(n-r)}{n(n-1)}.$$

The following result is a directly consequence from the Theorem 3.4.2.

Corollary 3.4.1. *Let G be a connected graph. Then $\overline{\kappa}(G) \leq \overline{d}(G)$.*

Proof. If $|E(G)| \geq |V(G)|$ then we are done, due to the Theorem 3.4.2.

Otherwise, G must be a tree, since it is connected. In such case, observe that $\kappa_G(x,y) = 1$, for every $x, y \in V(G)$, which means that $\overline{\kappa}(G) = 1$.

Furthermore, $d_G(x) \geq 1$, for every $x, y \in V(G)$, thus $\overline{d}(G) \geq 1$ and therefore, $\overline{\kappa}(G) = \overline{d}(G)$. □

Finally, from Theorem 3.4.1 and Corollary 3.4.1, we give a sufficient condition for the average connectivity of the strong product graph to be equal to its average degree.

Corollary 3.4.2. *Let G_1 and G_2 be two connected graphs with at least 3 vertices and girth at least 5. If $\overline{\kappa}(G_i) = \overline{d}(G_i)$, for $i = 1, 2$, then*

$$\overline{\kappa}\left(G_1 \boxtimes G_2\right) = \overline{d}\left(G_1 \boxtimes G_2\right).$$

Proof. We know that $\overline{\kappa}\left(G_1 \boxtimes G_2\right) \leq \overline{d}\left(G_1 \boxtimes G_2\right)$. Thus we must prove the another inequality. By applying Corollary 3.4.1 to the lower bound of Theorem 3.4.1, we deduce that

$$\begin{aligned}
\overline{\kappa}\left(G_1 \boxtimes G_2\right) &\geq \frac{1}{n_1 n_2 - 1}\big[(n_1-1)(n_2+\overline{d}(G_2))\overline{\kappa}(G_1) + (n_2-1)(n_1+\overline{d}(G_1))\overline{\kappa}(G_2) \\
&\quad +(n_1-1)(n_2-1)\overline{\kappa}(G_1)\overline{\kappa}(G_2) + (n_2-1)\overline{d}(G_1) + (n_1-1)\overline{d}(G_2)\big] \\
&\geq \frac{1}{n_1 n_2 - 1}\left[(n_1-1)(n_2+\overline{\kappa}(G_2))\overline{\kappa}(G_1) + (n_2-1)(n_1+\overline{\kappa}(G_1))\overline{\kappa}(G_2)\right. \\
&\quad \left.+(n_1-1)(n_2-1)\overline{\kappa}(G_1)\overline{\kappa}(G_2) + (n_2-1)\overline{\kappa}(G_1) + (n_1-1)\overline{\kappa}(G_2)\right] \\
&= \frac{1}{n_1 n_2 - 1}\left[(n_1 n_2 + (n_1-1)\overline{\kappa}(G_2) - 1)\overline{\kappa}(G_1)\right. \\
&\quad +(n_1 n_2 + (n_2-1)\overline{\kappa}(G_1) - 1)\overline{\kappa}(G_2) \\
&\quad \left.+(n_1 n_2 - n_1 - n_2 + 1)\overline{\kappa}(G_1)\overline{\kappa}(G_2)\right].
\end{aligned}$$

Hence,

$$\begin{aligned}\overline{\kappa}\left(G_1 \boxtimes G_2\right) &\geq \frac{1}{n_1n_2-1}\left[(n_1n_2-1)\overline{\kappa}(G_1)\overline{\kappa}(G_2)+(n_1n_2-1)\overline{\kappa}(G_1)\right.\\ &\quad \left.+(n_1n_2-1)\overline{\kappa}(G_2)\right]\\ &= \overline{\kappa}(G_1)\overline{\kappa}(G_2)+\overline{\kappa}(G_1)+\overline{\kappa}(G_2).\end{aligned}$$

By applying the hypothesis $\overline{\kappa}(G_1)=\overline{d}(G_1)$ and $\overline{\kappa}(G_2)=\overline{d}(G_2)$, we have

$$\overline{\kappa}\left(G_1 \boxtimes G_2\right) \geq \overline{d}(G_1)\overline{d}(G_2)+\overline{d}(G_1)+\overline{d}(G_2)=\overline{d}\left(G_1 \boxtimes G_2\right),$$

obtaining the desired result. □

3.5 The average Menger number

Similarly to the average connectivity, when the requirement of the lengths of the disjoint paths has to be considered, it may be interesting to study the Menger number not as a worst-case measure but also as a measure of the expected number of pairwise disjoint paths with an upper bounded length.

Let G be a connected graph on n vertices. Let ℓ be a positive integer and $x, y \in V(G)$ be two distinct vertices. We have denoted by $\zeta_\ell(x,y)$ the maximum number of internally disjoint xy-paths of length at most ℓ in G. The *average Menger number* of G with respect to ℓ is defined as

$$\overline{\zeta}_\ell(G)=\frac{1}{\binom{n}{2}}\sum_{x,y\in V(G)}\zeta_\ell(x,y),$$

where the pair of vertices are taken non ordered and being the *total Menger number* of G

$$Z_\ell(G)=\sum_{x,y\in V(G)}\zeta_\ell(x,y).$$

Lemmas proved in Section 3.3 lead us to get lower bounds on the average Menger number of the strong product of two connected graphs, similarly to the bounds on the average connectivity obtained in Section 3.4.

Let G_1 and G_2 be two connected graphs with order n_1 and n_2, size e_1 and e_2, average Menger number $\overline{\zeta}_\ell(G_1)$ and $\overline{\zeta}_\ell(G_2)$, and average degree $\overline{d}(G_1)$ and $\overline{d}(G_2)$, respectively. Next we present the analogous result to Theorem 3.4.1 for the average Menger number of the strong product graph, but now we distinguish two cases depending on the permitted length of the paths, as we have taken into account in Theorem 3.3.1.

Theorem 3.5.1. *Let G_1 and G_2 be two connected graphs with order n_1, n_2, respectively. Let ℓ be a positive integer. The following assertions hold:*
(i) If both G_1 and G_2 have order at least 3, then

$$\begin{aligned}\overline{\zeta}_\ell\,(G_1 \boxtimes G_2) &= \frac{1}{n_1n_2-1}\big[(n_1-1)(1+\overline{d}(G_2))\overline{\zeta}_\ell(G_1)\\ &+ (n_2-1)(1+\overline{d}(G_1))\overline{\zeta}_\ell(G_2) + (n_1-1)(n_2-1)\overline{\zeta}_\ell(G_1)\overline{\zeta}_\ell(G_2)\big]\,.\end{aligned}$$

(ii) If both G_1 and G_2 have order at least 3 and girth at least 5, then

$$\begin{aligned}\overline{\zeta}_\ell\,(G_1 \boxtimes G_2) &\geq \frac{1}{n_1n_2-1}\big[(n_1-1)(n_2+\overline{d}(G_2))\overline{\zeta}_\ell(G_1) + (n_2-1)(n_1+\overline{d}(G_1))\overline{\zeta}_\ell(G_2)\\ &+(n_1-1)(n_2-1)\overline{\zeta}_\ell(G_1)\overline{\zeta}_\ell(G_2) + (n_2-1)\overline{d}(G_1) + (n_1-1)\overline{d}(G_2)\big]\,.\end{aligned}$$

Proof. Let $x_i, y_i \in V(G_i)$ be two distinct vertices, for $i = 1, 2$, and denote by $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Let $\mathbb{P}(G_1 \boxtimes G_2)$ be the set of unordered pairs of vertices of $V(G_1 \boxtimes G_2)$, and consider the following partition of $\mathbb{P}(G_1 \boxtimes G_2)$:

$$\begin{aligned}A &= \bigcup_{x_2,y_2\in V(G_2)} \{\{(u,x_2),(u,y_2)\} : u \in V(G_1)\}\,,\\ B &= \bigcup_{x_1,y_1\in V(G_1)} \{\{(x_1,v),(y_1,v)\} : v \in V(G_2)\}\,,\\ C &= \bigcup_{x,y\in V(G_1\boxtimes G_2)} \{\{(x_1,x_2),(y_1,y_2)\} : x_1 \neq y_1 \text{ and } x_2 \neq y_2\}\,.\end{aligned}$$

Their cardinalities are $|V(G_1 \boxtimes G_2)| = n_1 n_2$, $|A| = n_1 \binom{n_2}{2}$, $|B| = n_2 \binom{n_1}{2}$ and $|C| = 2\binom{n_1}{2}\binom{n_2}{2}$.

Then the following equality holds:

$$\begin{aligned}
\overline{\zeta}_\ell\left(G_1 \boxtimes G_2\right) &= \frac{1}{\binom{n}{2}} \sum_{\mathbb{P}(G_1 \boxtimes G_2)} \zeta_\ell\left((x_1, x_2), (y_1, y_2)\right) \\
&= \frac{1}{\binom{n}{2}} \left[\sum_A \zeta_\ell\left((u, x_2), (u, y_2)\right) + \sum_B \zeta_\ell\left((x_1, v), (y_1, v)\right) \right. \qquad (3.2) \\
&\quad \left. + \sum_C \zeta_\ell\left((x_1, x_2), (y_1, y_2)\right) \right].
\end{aligned}$$

We firstly prove case (i). To do that, we begin computing the sums of equality (3.2).

By applying Lemma 3.2.1 to the elements of A, it follows that

$$\begin{aligned}
\sum_A \zeta_\ell\left((u, x_2), (u, y_2)\right) &\geq \sum_A \left[(1 + d_{G_1}(u))\zeta_\ell(x_2, y_2)\right] \\
&= \sum_{x_2, y_2 \in V(G_2)} \zeta_\ell(x_2, y_2) \sum_{u \in V(G_1)} (1 + d_{G_1}(u)) \\
&= (n_1 + 2e_1) \sum_{x_2, y_2 \in V(G_2)} \zeta_\ell(x_2, y_2) \\
&= (n_1 + 2e_1) Z_\ell(G_2).
\end{aligned}$$

By Lemma 3.2.1 and the commutativity of the strong product of two graphs, we also obtain that

$$\sum_B \zeta_\ell\left((x_1, v), (y_1, v)\right) \geq (n_2 + 2e_2) Z_\ell(G_1).$$

Since the elements of C satisfy the hypothesis of Lemma 3.2.2, we have

$$\begin{aligned}
\sum_C \zeta_\ell\left((x_1, x_2), (y_1, y_2)\right) &\geq \sum_C \left[\zeta_\ell(x_1, y_1)\zeta_\ell(x_2, y_2)\right] \\
&= 2 \sum_{x_1, y_1 \in V(G_1)} \zeta_\ell(x_1, y_1) \sum_{x_2, y_2 \in V(G_2)} \zeta_\ell(x_2, y_2) \\
&= 2 Z_\ell(G_1) Z_\ell(G_2).
\end{aligned}$$

From the partition of $\mathbb{P}(G_1 \boxtimes G_2)$ we deduce that

$$Z_\ell\left(G_1 \boxtimes G_2\right) \geq (n_1+2e_1)Z_\ell(G_2) + (n_2+2e_2)Z_\ell(G_1) + 2Z_\ell(G_1)Z_\ell(G_2).$$

Hence,

$$\begin{aligned}
\overline{\zeta}_\ell\left(G_1 \boxtimes G_2\right) &= \frac{2}{n_1n_2(n_1n_2-1)} Z_\ell\left(G_1 \boxtimes G_2\right) \\
&= \frac{1}{n_1n_2-1}\left[(n_1-1)(1+\overline{d}(G_2))\overline{\zeta}_\ell(G_1)\right. \\
&+ \left.(n_2-1)(1+\overline{d}(G_1))\overline{\zeta}_\ell(G_2) + (n_1-1)(n_2-1)\overline{\zeta}_\ell(G_1)\overline{\zeta}_\ell(G_2)\right].
\end{aligned}$$

We secondly prove case (ii). Assume that both G_1 and G_2 have girth at least 5. Again, from Lemma 3.2.1 and Lemma 3.2.3 applied to inequality (3.2), it follows that:

$$\begin{aligned}
\sum_A \zeta_\ell\left((u,x_2),(u,y_2)\right) &\geq \sum_A \left[(1+d_{G_1}(u))\zeta_\ell(x_2,y_2) + d_{G_1}(u)\right] \\
&= \sum_{x_2,y_2\in V(G_2)} \zeta_\ell(x_2,y_2) \sum_{u\in V(G_1)} (1+d_{G_1}(u)) \\
&\quad + \binom{n_2}{2} \sum_{u\in V(G_1)} d_{G_1}(u) \qquad (3.3) \\
&= (n_1+2e_1) \sum_{x_2,y_2\in V(G_2)} \zeta_\ell(x_2,y_2) + 2e_1\binom{n_2}{2} \\
&= (n_1+2e_1)Z_\ell(G_2) + 2e_1\binom{n_2}{2}.
\end{aligned}$$

$$\sum_B \zeta_\ell\left((x_1,v),(y_1,v)\right) \geq (n_2+2e_2)Z_\ell(G_1) + 2e_2\binom{n_1}{2}. \qquad (3.4)$$

$$
\begin{aligned}
\sum_C \zeta_\ell\left((x_1,x_2),(y_1,y_2)\right) &\geq \sum_C \left[\zeta_\ell(x_1,y_1)\zeta_\ell(x_2,y_2)+\zeta_\ell(x_1,y_1)+\zeta_\ell(x_2,y_2)\right] \\
&= \sum_C \left[\left(\zeta_\ell(x_1,y_1)+1\right)\left(\zeta_\ell(x_2,y_2)+1\right)-1\right] \\
&= 2\sum_{x_1,y_1\in V(G_1)}\left(\zeta_\ell(x_1,y_1)+1\right)\sum_{x_2,y_2\in V(G_2)}\left(\zeta_\ell(x_2,y_2)+1\right) \\
&\quad -|C| \\
&= 2Z_\ell(G_1)Z_\ell(G_2)+2\binom{n_2}{2}Z_\ell(G_1)+2\binom{n_1}{2}Z_\ell(G_2).
\end{aligned} \tag{3.5}
$$

From inequalities (3.3), (3.4), and (3.5) and taking into account that the sets A, B and C form a partition of $\mathbb{P}(G_1 \boxtimes G_2)$, it follows that

$$
\begin{aligned}
Z_\ell\left(G_1 \boxtimes G_2\right) &\geq (n_1+2e_1)Z_\ell(G_2)+2e_1\binom{n_2}{2} \\
&+ (n_2+2e_2)Z_\ell(G_1)+2e_2\binom{n_1}{2} \\
&+ 2Z_\ell(G_1)Z_\ell(G_2)+2\binom{n_2}{2}Z_\ell(G_1) \\
&+ 2\binom{n_1}{2}Z_\ell(G_2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\overline{\zeta}_\ell\left(G_1 \boxtimes G_2\right) &= \frac{2}{n_1n_2(n_1n_2-1)}Z_\ell\left(G_1 \boxtimes G_2\right) \\
&\geq \frac{1}{n_1n_2-1}\Big[(n_1-1)(n_2+\overline{d}(G_2))\overline{\zeta}_\ell(G_1) \\
&\quad +(n_2-1)(n_1+\overline{d}(G_1))\overline{\zeta}_\ell(G_2) \\
&\quad +(n_1-1)(n_2-1)\overline{\zeta}_\ell(G_1)\overline{\zeta}_\ell(G_2) \\
&\quad +(n_2-1)\overline{d}(G_1)+(n_1-1)\overline{d}(G_2)\Big].
\end{aligned}
$$

□

We finish this section with a consequence on the average Menger number of the strong product of two connected graphs. Under certain hypothesis, we guarantee that the average Menger number is as large as possible. From Theorem 3.4.2 inequalities $\overline{\zeta}_\ell(G) \leq \overline{\kappa}(G) \leq \overline{d}(G)$ clearly hold. Thus, the proof of the next corollary is straightforward.

Corollary 3.5.1. *Let G_1 and G_2 be two connected graphs with at least 3 vertices and girth at least 5. Let ℓ be a positive integer. If $\overline{\zeta}_\ell(G_i) = \overline{d}(G_i)$, for $i = 1, 2$, then*

$$\overline{\zeta}_\ell \left(G_1 \boxtimes G_2\right) = \overline{d} \left(G_1 \boxtimes G_2\right).$$

Chapter 4

Generalized 3-connectivity

In this chapter a natural extension of the connectivity parameter is treated, called the generalized k-connectivity. It is oriented to quantify how connected any set of k vertices in a graph is. We focus on studying the case of the generalized 3-connectivity of the strong product of two connected graphs. A lower bound is given, being best possible when the generalized 3-connectivity of a factor graph is one.

4.1 Introduction

The generalized k-connectivity was introduced by Chartrand, Okamoto and Zhang in [29]. Briefly, a graph is said generalized k-connected if and only if there exists at least k pairwise disjoint trees connecting any set of k vertices in such graph. Let us define it formally. Let G be a connected graph and $S = \{x_1, \ldots, x_k\} \subseteq V(G)$. A tree T is called an S-tree (or an $\{x_1, x_2, \ldots, x_k\} - tree$) if $S \subseteq V(T)$. A family of trees $T_1, T_2, \ldots, T_r$ are internally disjoint S-trees if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$, for any pair of integers i and j, with $1 \leq i < j \leq r$.

Denote by $\kappa(S)$ the greatest number of internally disjoint S-trees. For an integer k with $2 \le k \le n$, the generalized k-connectivity $\kappa_k(G)$ of G is defined as

$$\kappa_k(G) = \min\{\kappa(S) : S \subseteq V(G) \text{ and } |S| = k\}.$$

Clearly, when $|S| = 2$, we have $\kappa_2(G) = \kappa(G)$, the classical connectivity of G. If G has less than k vertices, $\kappa_k(G) = 1$ is adopted. In [72] was proved that $\kappa_3(G) \le \kappa(G)$, for any connected graph G. Thus, $\kappa(G) \ge 1$ if and only if $\kappa_3(G) \ge 1$.

Some papers and partial results have appeared in last years using this parameter. Regarding the graphs products, Li, Li and Sun in [71], studied the generalized 3-connectivity of the cartesian product graph. They gave sharp lower bounds in terms of the generalized 3-connectivity of the factor graphs. More precisely, they proved the following theorem.

Theorem 4.1.1 ([71]). *Let G and H be two connected graphs such that $\kappa_3(G) \ge \kappa_3(H)$. The following assertions hold:*
(i) If $\kappa(G) = \kappa_3(G)$ then $\kappa_3(G \Box H) \ge \kappa_3(G) + \kappa_3(H) - 1$. Moreover, the bound is sharp.
(ii) If $\kappa(G) > \kappa_3(G)$ then $\kappa_3(G \Box H) \ge \kappa_3(G) + \kappa_3(H)$. Moreover, the bound is sharp.

In this chapter, we study the generalized 3-connectivity of the strong product graph $G_1 \boxtimes G_2$ when $\kappa_3 = 1$, that is, when there exists only a tree joining some subset of three vertices in one of the factor graphs. We give sharp lower bounds of $\kappa_3(G_1 \boxtimes G_2)$ in terms of the connectivity and the generalized 3-connectivity of G_1 and G_2.

4.2 Specific notation and remark

Before proceeding with the main results of this chapter, we need to introduce some basic definitions, specific notation as well as a useful observation.

Let T be a tree and $x, y, z \in V(T)$. When x, y and z are end vertices of T, we say that T is an r-*rooted* tree, for certain particular vertex $r \in V(T)$ called the *root* of T. Moreover, given an $\{x, y, z\}$-tree T, simply deleting extra vertices, we can construct an $\{x, y, z\}$-tree $\widetilde{T} \subset T$ with the minimum number of vertices (see [71]). This tree $\widetilde{T}$ is called a *minimal* $\{x, y, z\}$-tree. The tree T is called an xyz-*path* when T is an xz-path with y as an internal vertex. When an specific order need not to be specified among the vertices x, y, z in such path T we call it an $\{x, y, z\}$-path.

Next, we introduce a kind of trees which will play an important role in this study.

Definition 4.2.1. *Let G be a connected graph and x, y, z three distinct vertices of G. An $\{x, y, z\}$-tree T of G is said to be* special *if either T is an r-rooted $\{x, y, z\}$-tree with edge set $E(T) = \{rx, ry, rz\}$ or T is a $\{x, y, z\}$-path such that $d_T(x, y) \leq 2$ or $d_T(y, z) \leq 2$ or $d_T(x, z) \leq 2$.*

Remark 4.2.1. *Let G be a connected graph and x, y, z three distinct vertices of G. If $g(G) \geq 5$ and $\kappa_3(G) \geq 2$, let us notice that:*

(i) If G contains an r-rooted $\{x, y, z\}$-tree T with edge set $E(T) = \{rx, ry, rz\}$, then any other $\{x, y, z\}$-tree T' of G is not special.

(ii) If there exist T_1 and T_2 special *xyz-paths in G such that $d_{T_1}(x, y) \leq 2$ and $d_{T_2}(y, z) \leq 2$, combining T_1 and T_2, we can find another pair of xyz-paths T_1' and T_2' such that only T_1' is* special *(see the first case in Figure 4.1).*

(iii) If there exist three special $\{x, y, z\}$*-paths* T_1*,* T_2 *and* T_3 *in* G *such that* $d_{T_1}(x, y) \leq 2$*,* $d_{T_2}(y, z) \leq 2$ *and* $d_{T_3}(x, z) \leq 2$*, combining these paths, we can consider another set of paths* T_1'*,* T_2' *and* T_3' *verifying that*

$$V(T_1' \cup T_2' \cup T_3') = V(T_1 \cup T_2 \cup T_3)$$

and such that at most two of these paths are special *(see the second case in Figure 4.1).*

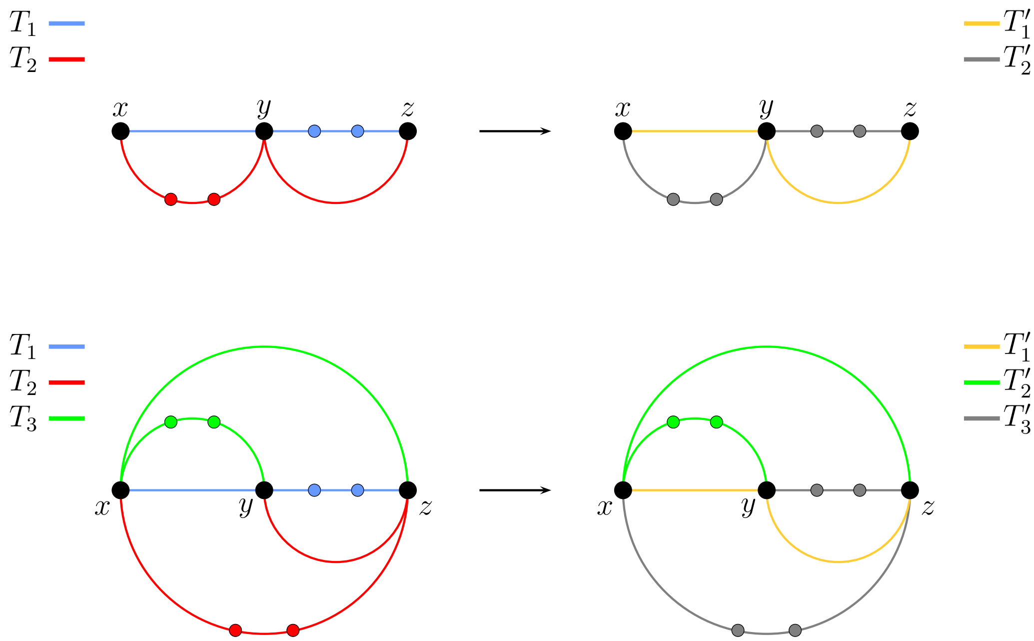

Figure 4.1: It can be considered that there exist at most two *special* $\{x, y, z\}$-trees in a connected graph with girth at least five.

To illustrate some constructions in Section 4.3 we will use the structure of Figure 4.2, which represents two *special* $\{x, y, z\}$-paths, when they need to be considered.

To fix notation, let us consider three vertices, $x_1, y_1, z_1 \in V(G_1)$ and $x_2, y_2, z_2 \in V(G_2)$ in each factor graph. Our goal is to study the maximum number of internally disjoint trees that connect vertices $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ in $V(G_1 \boxtimes G_2)$.

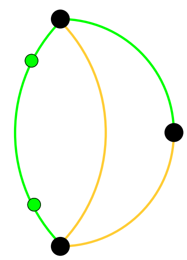

Figure 4.2: Representation of two *special* $\{x, y, z\}$-paths in a connected graph with girth at least five.

To do that, throughout the proofs of next results, $P_1, \ldots, P_{\ell_1}$ denote internally disjoint minimal $\{x_1, y_1, z_1\}$-trees in G_1 while $Q_1, \ldots, Q_{\ell_2}$ internally disjoint minimal $\{x_2, y_2, z_2\}$-trees in G_2. We always assume that $P_1, \ldots, P_{\ell_1}$ and $Q_1, \ldots, Q_{\ell_2}$ contain the minimum number of *special* trees. Without loss of generality, in cases (i) and (ii) of Remark 4.2.1 we denote by P_1 (or Q_1, respectively) the unique *special* tree, whereas in case (iii), we consider that Q_1 and Q_2 are the *special* trees of G_2, and the same consideration holds for P_1 and P_2. Otherwise, we consider that $|V(P_1)| = \min\{|V(P_i)| : i = 1, \ldots, \ell_1\}$ and $|V(Q_1)| = \min\{|V(Q_j)| : j = 1, \ldots, \ell_2\}$, when no tree is *special*.

Let us also give a general idea of the notation used to describe the trees. If P_i is an $x_1y_1z_1$-path, we denote it as $P_i : x_1\overline{x}_1^i \ldots \underline{y}_1^i y_1 \overline{y}_1^i \ldots \underline{z}_1^i z_1$ (see Figure 4.3).

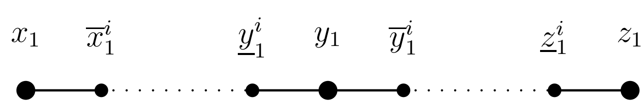

Figure 4.3: Description of an $x_1y_1z_1$-path P_i.

If P_i is an r^i-rooted $\{x_1, y_1, z_1\}$-tree formed by three paths, we write it as $P_i : r^i \ldots \underline{x}_1^i x_1 \cup r^i \ldots \underline{y}_1^i y_1 \cup r^i \ldots \underline{z}_1^i z_1$ (see Figure 4.4).

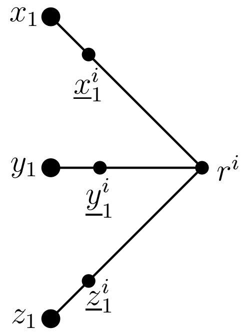

Figure 4.4: Description of an r^i-rooted $\{x_1, y_1, z_1\}$-tree P_i.

Similarly, when Q_j is an $x_2y_2z_2$-path, we denote it $Q_j : x_2\overline{x}_2^j \dots \underline{y}_2^j y_2 \overline{y}_2^j \dots \underline{z}_2^j z_2$ taking into account that $\overline{x}_2^j \neq \underline{y}_2^j$ and $\overline{y}_2^j \neq \underline{z}_2^j$ for $j \geq 3$, whenever $g(G_2) \geq 5$. If Q_j is an s^j-rooted tree formed by three paths, we write

$$Q_j : s^j \dots \underline{x}_2^j x_2 \cup s^j \dots \underline{y}_2^j y_2 \cup s^j \dots \underline{z}_2^j z_2$$

and, in this case, at least one element of the set $\{\underline{x}_2^j, \underline{y}_2^j, \underline{z}_2^j\}$ is not equal to s^j for $j \geq 2$. Moreover, we say that a *tree T_{ij} in $G_1 \boxtimes G_2$ is associated to trees P_i in G_1 and Q_j in G_2* when every vertex $(u, v) \in V(T_{ij})$ is such that $u \in P_i$ and $v \in Q_j$.

4.3 Lower bounds on $\kappa_3(G_1 \boxtimes G_2)$

To estimate $\kappa_3(G_1 \boxtimes G_2)$, we construct internally disjoint trees connecting any three distinct vertices $x, y, z \in V(G_1 \boxtimes G_2)$ based on trees of the generator graphs G_1 and G_2. To do that, for $u \in V(G_1)$, notice that every $\{x_2, y_2, z_2\}$-tree Q_j of G_2 induces an $\{(u, x_2), (u, y_2), (u, z_2)\}$-tree Q_j^u in the copy G_2^u taking into account that $V(Q_j^u) = \{(u, v) : v \in V(Q_j)\}$ and $E(Q_j^u) = \{(u, v_1)(u, v_2) : v_1v_2 \in E(Q_j)\}$ are the vertex and edge set, respectively.

Depending on the copies of G_1 and G_2 to which x, y, z belong, we distinguish four cases.

First, we assume that x, y, z belong to a unique copy $G_2^{x_1}$, and hence, $x_1 = y_1 = z_1$.

Lemma 4.3.1. *Let G_1 and G_2 be two connected graphs with at least 3 vertices. Consider distinct vertices $x = (x_1, x_2)$, $y = (x_1, y_2)$ and $z = (x_1, z_2)$ of $G_1 \boxtimes G_2$.*

(i) There exist at least $(\delta(G_1) + 1)\kappa_3(G_2)$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$.

(ii) If $g(G_1) \geq 5$, there exist at least $(\delta(G_1)+1)\kappa_3(G_2)+\delta(G_1)$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$.

Proof. Since the vertices x, y, z belong to a unique copy $G_2^{x_1}$ in $G_1 \boxtimes G_2$ and x_2, y_2, z_2 are connected at least by $\ell_2 = \kappa_3(G_2)$ internally disjoint trees $Q_1, \ldots, Q_{\ell_2}$ in G_2, then trees $Q_1^{x_1} \ldots, Q_{\ell_2}^{x_1}$ are ℓ_2 internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$.

To construct another $\delta(G_1)\kappa_3(G_2)$ trees, we define next an $\{x, y, z\}$-tree T_j^u for each $u \in N_{G_1}(x_1)$ and $j \in \{1, \ldots, \ell_2\}$. To do that, we distinguish if none, one or two trees of the family $Q_1, \ldots, Q_{\ell_2}$ contain direct edges between the vertices of the set $\{x_2, y_2, z_2\}$.

For each tree Q_j such that $x_2y_2 \notin E(Q_j)$, $y_2z_2 \notin E(Q_j)$ and $x_2z_2 \notin E(Q_j)$, let us denote

$$\ddot{Q}_j^u : \; Q_j^u - \{(u, x_2), (u, y_2), (u, z_2)\}.$$

If Q_1 is an $x_2y_2z_2$-path such that x_2y_2 and/or y_2z_2 belong to $E(Q_1)$, then

$$\ddot{Q}_1^u : \; Q_1^u - \{(u, x_2), (u, z_2)\}.$$

If Q_2 also contains a direct edge between two vertices of the set $\{x_2, y_2, z_2\}$, we assume that $x_2y_2 \in E(Q_1)$ and $x_2z_2 \in E(Q_2)$ and then

$$\ddot{Q}_2^u : \; Q_2^u - \{(u, x_2), (u, y_2)\}.$$

By the definition of the strong product of graphs, for $j \in \{1, \ldots, \ell_2\}$, each end vertex of $\ddot{Q}_j^u$ is adjacent to at least one vertex of the set $\{x, y, z\}$. We define T_j^u as a tree contained in $G_1 \boxtimes G_2$ such that (see Figure 4.5, Figure 4.6 and Figure 4.7)

$$V(T_j^u) = V(\ddot{Q}_j^u) \cup \{x, y, z\}.$$

Therefore $Q_1^{x_1}, \ldots, Q_{\ell_2}^{x_1}$, $T_1^u, \ldots, T_{\ell_2}^u$ are at least $\kappa_3(G_2) + \delta(G_1)\kappa_3(G_2)$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ and item (i) is proved.

If there exists $u \in N_{G_1}(x_1)$ such that $d_{G_1}(u) = 1$, then $d_{G_1}(x_1) \geq 2$, and the previous bound leads to

$$(1 + d_{G_1}(x_1))\kappa_3(G_2) \geq 3\kappa_3(G_2) \geq 2\kappa_3(G_2) + 1 = (1 + \delta(G_1))\kappa_3(G_2) + \delta(G_1),$$

which proves (ii).

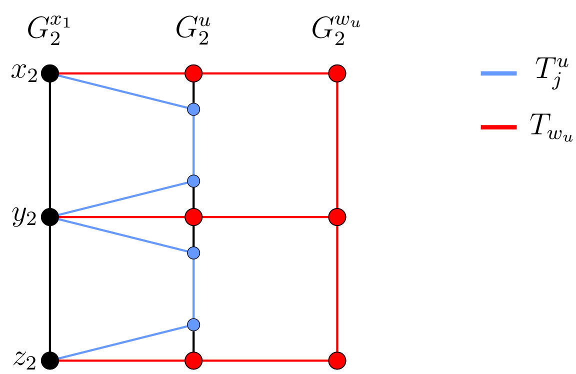

Figure 4.5: Trees T_j^u and T_{w_u} for every $x_2y_2z_2$-path Q_j such that $x_2y_2 \notin E(Q_j)$, $y_2z_2 \notin E(Q_j)$ and $x_2z_2 \notin E(Q_j)$.

Otherwise, let us assume that $g(G_1) \geq 5$. For each $u \in N_{G_1}(x_1)$, we consider $w_u \in N_{G_1}(u) \backslash \{x_1\}$. Clearly, $w_u \neq w_v$ for all $u, v \in N_{G_1}(x_1)$ with $u \neq v$, and this fact makes feasible the construction of another $\{x, y, z\}$-tree denoted by T_{w_u}, for each $u \in N_{G_1}(x_1)$, internally disjoint with all the previous ones and such that $T_{w_u} \cap T_{w_v} = \{x, y, z\}$.

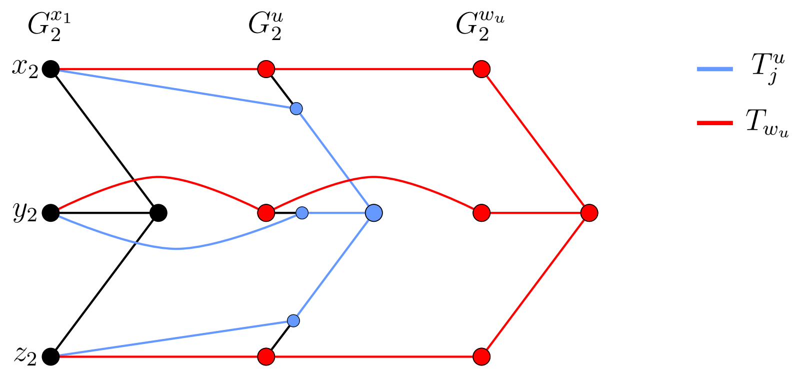

Figure 4.6: Trees T_j^u and T_{w_u} for every $x_2y_2z_2$-path Q_j such that $x_2y_2 \notin E(Q_j)$, $y_2z_2 \notin E(Q_j)$ and $x_2z_2 \notin E(Q_j)$.

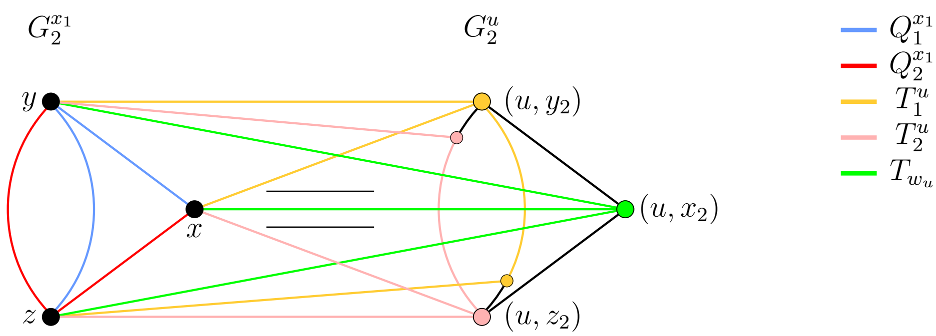

Figure 4.7: Five $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ when $x_2y_2 \in E(Q_1)$ and $x_2z_2 \in E(Q_2)$.

If $x_2y_2 \notin E(Q_1 \cup Q_2)$, $y_2z_2 \notin E(Q_1 \cup Q_2)$ and $x_2z_2 \notin E(Q_1 \cup Q_2)$, then (see Figure 4.5 and Figure 4.6)

$$T_{w_u} : Q_1^{w_u} \cup (x_1, x_2)(u, x_2)(w_u, x_2) \cup (x_1, y_2)(u, y_2)(w_u, y_2) \cup (x_1, z_2)(u, z_2)(w_u, z_2).$$

In case Q_1 is an $x_2y_2z_2$-path such that $x_2y_2 \in E(Q_1)$, then

$$T_{w_u} : Q_1^{w_u} \cup (x_1, x_2)(u, x_2)(w_u, x_2) \cup (x_1, y_2)(u, x_2) \cup (x_1, z_2)(u, z_2)(w_u, z_2).$$

A similar tree can be constructed in the symmetrical case $y_2z_2 \in E(Q_1)$.

If $x_2y_2 \in E(Q_1)$ and $x_2z_2 \in E(Q_2)$, then the vertex (u, x_2) is adjacent to the three vertices x, y, z, (see Figure 4.7), and therefore, we consider

$$T_{w_u} : (x_1, x_2)(u, x_2) \cup (x_1, y_2)(u, x_2) \cup (x_1, z_2)(u, x_2).$$

Hence, when $g(G_1) \geq 5$, the trees T_{w_u}, for $u \in N_{G_1}(x_1)$ are at least $\delta(G_1)$ additional $\{x, y, z\}$-trees internally disjoint with $Q_1^{x_1}, \ldots, Q_{\ell_2}^{x_1}$, $T_1^u, \ldots, T_{\ell_2}^u$. $\square$

Now, we consider that x, y, z belong to two copies of G_1 and to two copies of G_2.

Lemma 4.3.2. *Let G_1 and G_2 be two connected graphs with at least three vertices. For distinct vertices $x = (x_1, x_2)$, $y = (x_1, z_2)$ and $z = (z_1, z_2)$ of $G_1 \boxtimes G_2$, the following assertions hold:*

(i) If $g(G_2) \geq 5$, there exist at least $2\kappa(G_2)$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$.

(ii) If $g(G_1) \geq 5$, there exist at least $2\kappa(G_1)$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$.

(iii) If $g(G_1) \geq 5$ and $g(G_2) \geq 5$, there exist at least $\kappa(G_1) + \kappa(G_1)\kappa(G_2) + \kappa(G_1) - 1$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$.

Proof. Notice that x, y belong to the copy $G_2^{x_1}$ while $z \in G_2^{z_1}$. By the theorem of Menger (see [79]),there exist $k_1 = \kappa(G_1)$ internally disjoint x_1z_1-paths $P_1, ..., P_{k_1}$ in G_1 and $k_2 = \kappa(G_2)$ internally disjoint x_2z_2-paths $Q_1, \ldots, Q_{k_2}$ in G_2. Without loss of generality we may assume that $|V(P_1)| = \min\{|V(P_i)| : i = 1, \ldots, \ell_1\}$ and $|V(Q_1)| = \min\{|V(Q_j)| : j = 1, \ldots, \ell_2\}$. Then $\overline{x}_1^1 = z_1$, $\underline{x_2}^1 = z_2$ may occur.

(i) Associated to paths P_1 and Q_1, we construct trees (see Figure 4.8)
$T_{11} : Q_1^{x_1} \cup (x_1, z_2) \ldots (z_1, z_2)$ and

$$T'_{11} : \begin{cases} Q_1^{z_1} \cup (z_1, x_2) \ldots (x_1, x_2) \cup (\overline{x}_1, x_2)(x_1, z_2), & \text{if } Q_1 : x_2z_2 \\ Q_1^{z_1} \cup (z_1, x_2) \ldots (x_1, x_2) \cup (z_1, \underline{z}_2) \ldots (\overline{x}_1, \underline{z}_2)(x_1, z_2), & \text{if } x_2z_2 \notin E(Q_1). \end{cases}$$

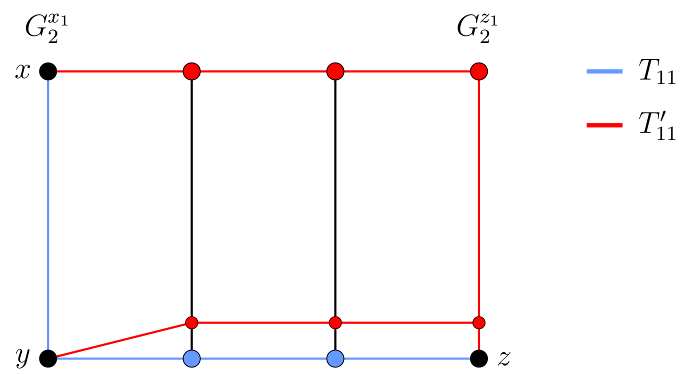

Figure 4.8: Trees T_{11}, T'_{11} associated to paths P_1 and Q_1 such that $x_2y_2 \notin E(Q_1)$.

If $k_2 \geq 2$, we consider that $g(G_2) \geq 5$ to guarantee that $\overline{x}_2^j \neq \underline{z}_2^j$ for $j \geq 2$. Associated to paths P_1 and Q_j we construct the following two $\{x, y, z\}$-trees T_{1j}, T'_{1j} in $G_1 \boxtimes G_2$, for each $j \in \{2, \ldots, k_2\}$.

If $x_1z_1 \in E(P_1)$, for $j = 2, \ldots, k_2$, then

$T_{1j} : Q_j^{x_1} \cup (x_1, \underline{z}_2^j)(z_1, z_2)$ and

$T'_{1j} : (x_1, x_2)(z_1, \overline{x}_2^j) \ldots (z_1, z_2) \cup (z_1, \underline{z}_2^j)(x_1, z_2)$.

If $x_1z_1 \notin E(P_1)$, for $j = 2, \ldots, k_2$, then (see Figure 4.9)

$T_{1j} : (x_1, x_2) \ldots (x_1, \underline{\underline{z}}_2^j)(\overline{x}_1^1, \underline{z}_2^j) \ldots (\underline{z}_1^1, \underline{z}_2^j)(z_1, z_2) \cup (\overline{x}_1^1, \underline{z}_2^j)(x_1, z_2)$ and

$T'_{1j} : (x_1, x_2)(\overline{x}_1^1, \overline{x}_2^j) \ldots (z_1, \overline{x}_2^j) \ldots (z_1, z_2) \cup (\overline{x}_1^1, \overline{x}_2^j) \ldots (\overline{x}_1^1, \underline{\underline{z}}_2^j)(x_1, \underline{z}_2^j)(x_1, z_2)$,

Therefore, trees $T_{11}, \ldots T_{1k_2}, T'_{11}, \ldots T'_{1k_2}$ are $2k_2$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ and (i) is proved.

(ii) It directly follows from item (i) due to the symmetrical position of vertices x, y, z in $V(G_1 \boxtimes G_2)$ and the commutativity of the strong product graph.

(iii) Let us assume that $g(G_1) \geq 5$ and $g(G_2) \geq 5$. If $k_1 = 1$ and/or $k_2 = 1$, the proof is finished.

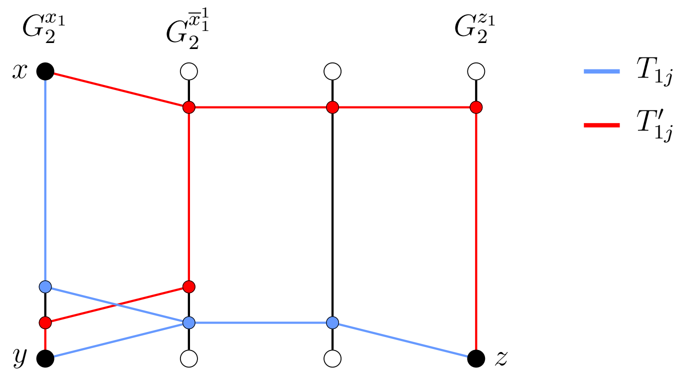

Figure 4.9: Trees T_{1j}, T'_{1j} associated to paths P_1 and Q_j such that $x_1z_1 \notin E(P_1)$, for $j \in \{2, \ldots, k_2\}$.

Otherwise, it remains to construct $(k_1 - 1)(k_2 - 1)$ additional trees. To do that, for every $i \in \{2, \ldots, k_1\}$ and every $j \in \{2, \ldots, k_2\}$, associated to paths P_i and Q_j, we consider the tree

$$T_{ij} : (x_1, x_2)(\overline{x}_1^i, \overline{x}_2^j) \ldots (\overline{x}_1^i, \underline{z}_2^j)(x_1, z_2) \cup (\overline{x}_1^i, \underline{z}_2^j) \ldots (\underline{z}_1^i, \underline{z}_2^j)(z_1, z_2). \qquad \square$$

Next, we consider that x, y, z belong to two copies of G_1 and to three copies of G_2.

Lemma 4.3.3. *Let G_1 and G_2 be connected graphs with at least three vertices. For distinct vertices $x = (x_1, x_2)$, $y = (x_1, y_2)$ and $z = (z_1, z_2)$ in $G_1 \boxtimes G_2$, the following assertions hold:*

(i) If $g(G_2) \geq 5$, there exist at least $2\kappa_3(G_2)$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$.

(ii) If $g(G_1) \geq 5$, there exist at least $2\kappa(G_1)$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$.

Proof. Notice that x, y belong to the copy $G_2^{x_1}$ while $z \in G_2^{z_1}$. Since G_1 and G_2 are connected graphs with at least three vertices, $\kappa_3(G_i) \geq 1$, $i = 1, 2$, clearly holds. Let us consider $k_1 = \kappa(G_1)$ internally disjoint x_1z_1-paths $P_1, \ldots, P_{k_1}$ in G_1, for which we assume that $|V(P_1)| = \min\{|V(P_i)| : i = 1, \ldots, k_1\}$. Let $\ell_2 = \kappa_3(G_2)$ be internally disjoint $\{x_2, y_2, z_2\}$-trees $Q_1, \ldots, Q_{\ell_2}$ in G_2, such that at most Q_1 and Q_2 are *special* trees.

(i) First, assume that at most Q_1 is an *special* tree. In this case, associated to P_1 and Q_1, we consider the trees (see Figure 4.10)

$T_{11} : Q_1^{x_1} \cup (x_1, z_2) \ldots (z_1, z_2)$ and

$T'_{11} : Q_1^{z_1} \cup (x_1, x_2) \ldots (z_1, x_2) \cup (x_1, y_2) \ldots (z_1, y_2)$.

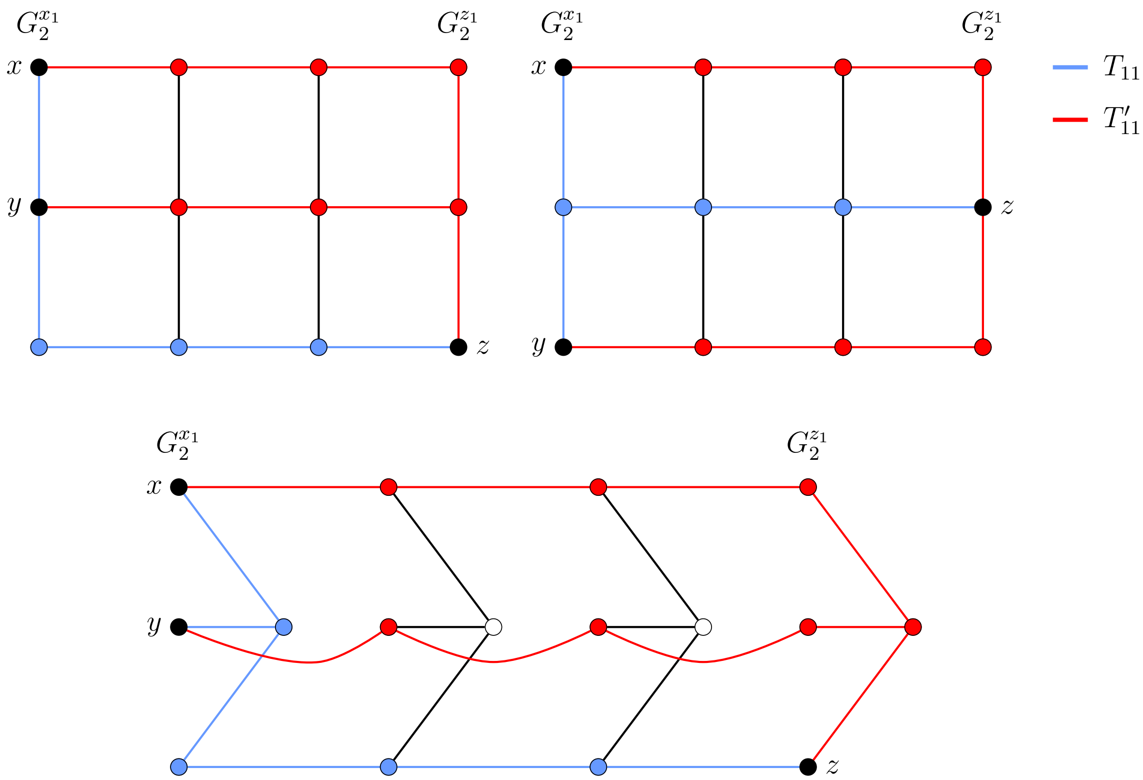

Figure 4.10: Trees T_{11}, T'_{11} associated to trees P_1, Q_1 when Q_1 is an special tree.

If $\kappa_3(G_2) = 1$, item (i) is proved. If $\kappa_3(G_2) \geq 2$, we assume that $g(G_2) \geq 5$. In case both Q_1 and Q_2 are *special* trees, we construct four trees associated to P_1, Q_1 and Q_2 as depict Figure 4.11, depending on whether both x, y have the same degree or not.

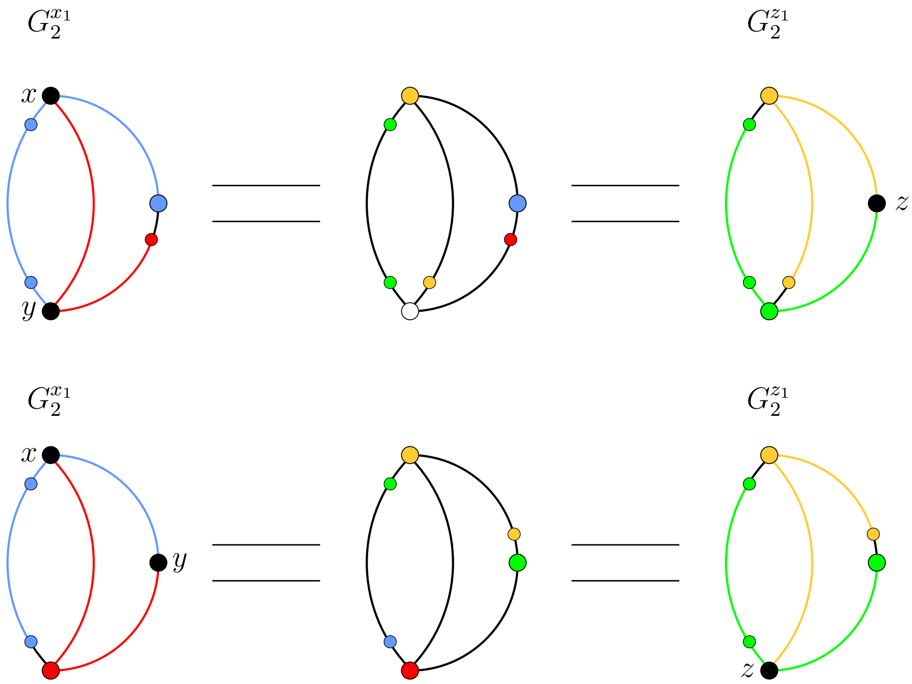

Figure 4.11: Four $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to paths P_1, Q_1 and Q_2, when both Q_1 and Q_2 are *special* trees.

For each $j \in \{2, \dots, \ell_2\}$ such that Q_j is not an *special* tree, we construct two $\{x, y, z\}$-trees T_{1j}, T'_{1j} in $G_1 \boxtimes G_2$ associated to P_1 and Q_j.

First, we focus on a particular case. Assume that $x_1 z_1 \notin E(P_1)$ and that Q_j is an s^j-rooted tree such that $d_{Q_j}(s^j, x_2) \geq 2$, $s^j y_2 \in E(Q_j)$, $s^j z_2 \in E(Q_j)$. It means that $Q_j : s^j \dots \underline{\underline{x}}_2^j \underline{x}_2^j x_2 \cup s^j y_2 \cup s^j z_2$ where $\underline{\underline{x}}_2^j$ may be equal to s^j. Then

$T_{1j} : (x_1, x_2)(\overline{x}_1^1, \underline{x}_2^j)(x_1, \underline{\underline{x}}_2^j) \dots (x_1, y_2) \cup (\overline{x}_1^1, \underline{x}_2^j) \dots (z_1, \underline{x}_2^j) \dots (z_1, z_2)$ and

$T'_{1j} : (x_1, x_2)(x_1, \underline{x}_2^j)(\overline{x}_1^1, \underline{\underline{x}}_2^j) \dots (\overline{x}_1^1, s^j)(x_1, y_2) \cup (\overline{x}_1^1, s^j) \dots (\underline{z}_1^1, s^j)(z_1, z_2)$.

Notice that a symmetrical construction holds when $s^j x_2 \in E(Q_j)$, $d_{Q_j}(s^j, y_2) \geq 2$ and $s^j z_2 \in E(Q_j)$.

In any other case, the tree T_{1j} is any tree contained in $G_1 \boxtimes G_2$ such that

$$\begin{aligned} V(T_{1j}) = \quad & \{x,y,z\} \cup V(Q_j^{x_1} - (x_1,z_2)) \cup \\ & \{(u,v) : u \in P_1 - \{x_1,z_1\}, v \in N_{Q_j}(z_2)\}. \end{aligned}$$

Similarly, T'_{1j} is such that (see Figure 4.12)

$$\begin{aligned} V(T'_{1j}) = \quad & \{x,y,z\} \cup V(\, Q_j^{z_1} - \{(z_1,x_2),(z_1,y_2)\, \}) \cup \\ & \{(u,v) : u \in P_1 - \{x_1,z_1\}, v \in N_{Q_j}(x_2) \cup N_{Q_j}(y_2)\}. \end{aligned}$$

Then, if $g(G_2) \geq 5$, we have constructed $2\kappa_3(G_2)$ internally disjoint $\{x,y,z\}$-trees in $G_1 \boxtimes G_2$.

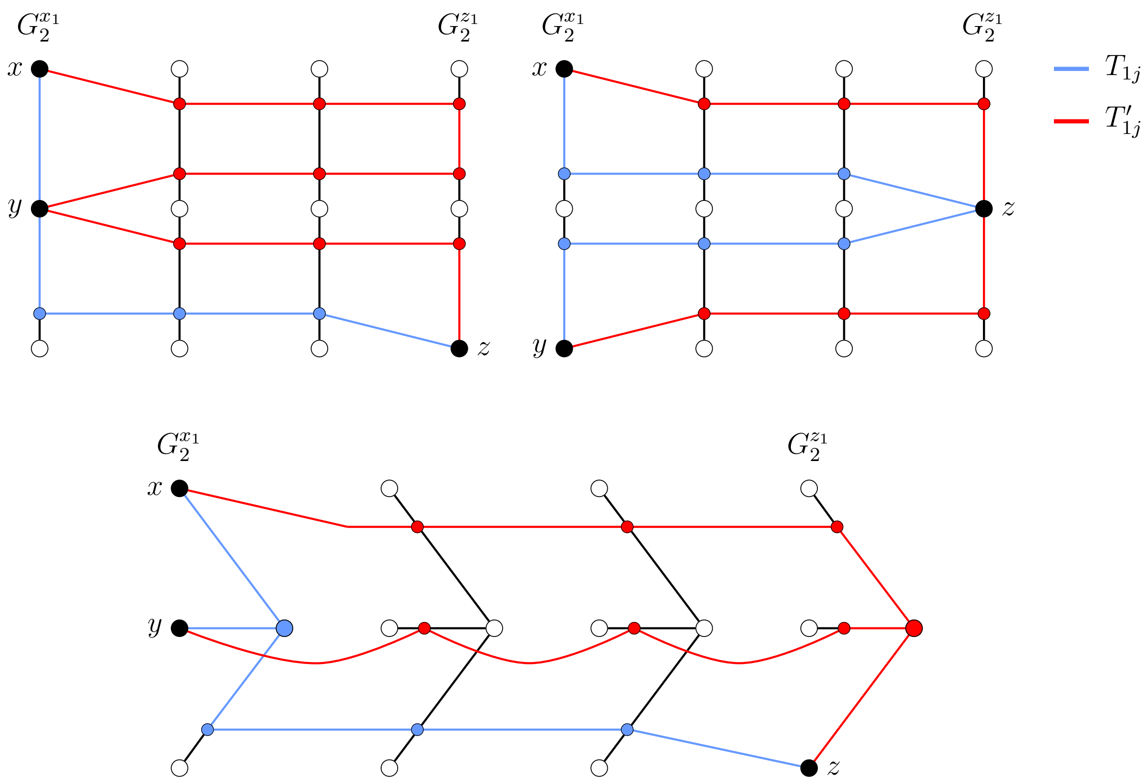

Figure 4.12: General construction of trees T_{1j}, T'_{1j} associated to P_1 and Q_j, for $j \in \{2,\ldots,\ell_2\}$.

(ii) Let us assume that $g(G_1) \geq 5$. We consider $P_i : x_1\overline{x}_1^i \ldots \underline{z}_1^i z_1$ an x_1z_1-path in G_1, for $i \in \{1,\ldots,k_1\}$. Notice that $x_1 \neq \overline{x}_1^i \neq \underline{z}_1^i \neq z_1$ for $i \geq 2$.

If $\kappa(G_1) = 1$, then trees T_{11} and T'_{11} provide the desired result. Otherwise, $\kappa(G_1) \geq 2$ and associated to P_i and Q_1, for $i \in \{2, \ldots, k_1\}$, we construct two $\{x, y, z\}$-trees T_{i1}, T'_{i1} in $G_1 \boxtimes G_2$.

If Q_1 is an $x_2y_2z_2$-path such that $x_2y_2 \in E(Q_1)$, then

$T_{i1} : (x_1, x_2)(\overline{x}_1^i, y_2)(x_1, y_2) \cup (\overline{x}_1^i, y_2) \ldots (\overline{x}_1^i, z_2) \ldots (z_1, z_2)$ and

$T'_{i1} : (x_1, x_2)(\overline{x}_1^i, x_2)(x_1, y_2) \cup (\overline{x}_1^i, x_2) \ldots (\underline{z}_1^i, x_2) \ldots (\underline{z}_1^i, \underline{z}_2^1)(z_1, z_2)$.

In any other case, we have

$$\begin{aligned} V(T_{i1}) = \; & \{x, y, z\} \cup V(\, Q_1^{\overline{x}_1^i} - \{(\overline{x}_1^i, x_2), (\overline{x}_1^i, y_2)\}\,) \cup \\ & \{(u, z_2) : u \in P_i - \{x_1, z_1\}\} \end{aligned}$$

and

$$\begin{aligned} V(T'_{i1}) = \; & \{x, y, z\} \cup V(\, Q_1^{\underline{z}_1^i} - (\underline{z}_1^i, z_2)\,) \cup \\ & \{(u, v) : u \in P_i - \{x_1, z_1\}, v \in \{x_2, y_2\}\} \end{aligned}$$

Trees $T_{11}, \ldots, T_{k_11}, T'_{11}, \ldots, T'_{k_11}$ prove item (ii). $\square$

The bounds of Lemmas 4.3.2 and 4.3.3 are sharp. To see that, it is enough to check out that $\kappa_3(\mathcal{P}_2 \boxtimes \mathcal{P}_2) = 2$, where $\mathcal{P}_2$ denotes a path of length two (see Figure 4.13.)

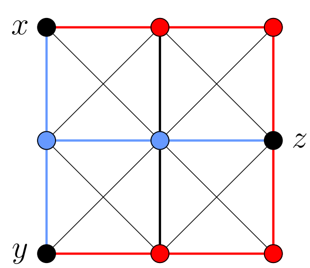

Figure 4.13: There exist 2 internally disjoint trees connecting vertices x, y, z in $\mathcal{P}_2 \boxtimes \mathcal{P}_2$.

Finally, we assume that x, y, z belong to three distinct copies of G_1 and to three distinct copies of G_2. Next lemma shows a lower bound of $\kappa_3(G_1 \boxtimes G_2)$ which is attained when $\kappa_3(G_1) = 1$.

Lemma 4.3.4. *Let G_1 and G_2 be two connected graphs with at least three vertices and such that $g(G_2) \geq 5$. For distinct vertices $x_1, y_1, z_1 \in V(G_1)$ and distinct vertices $x_2, y_2, z_2 \in V(G_2)$, there exist at least $2\kappa_3(G_2) + 1$ internally disjoint trees connecting vertices $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ in $G_1 \boxtimes G_2$.*

Proof. Notice that vertices x, y, z belong to different copies $G_2^{x_1}$, $G_2^{y_1}$, $G_2^{z_1}$, respectively. Associated to one $\{x_1, y_1, z_1\}$-tree P_1 in G_1 and to $\ell_2 = \kappa_3(G_2)$ internally disjoint $\{x_2, y_2, z_2\}$-trees $Q_1, \ldots, Q_{\ell_2}$ in G_2, we construct $2\ell_2 + 1$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$. Without loss of generality, when P_1 is a path, we assume that it is an $x_1y_1z_1$-path described as $P_1 : x_1\overline{x}_1 \ldots \underline{y}_1 y_1 \overline{y}_1 \ldots \underline{z}_1 z_1$ (where $\overline{x}_1 = y_1$, $\underline{y}_1 = x_1$, $\overline{y}_1 = z_1$ and/or $\underline{z}_1 = y_1$ are possible).

(I) Associated to trees P_1 and Q_1, we construct three trees T_{11}, T'_{11} and T^* in $G_1 \boxtimes G_2$.

a) If P_1 is a path and Q_1 is a tree with leaves x_2, y_2, z_2, (see Figure 4.14), denoting $\ddot{Q}_1^{y_1} = Q_1^{y_1} - \{(y_1, x_2), (y_1, z_2)\}$, we consider

$T_{11} : Q_1^{x_1} \cup (x_1, y_2) \ldots (y_1, y_2) \cup (x_1, z_2) \ldots (z_1, z_2)$,

$T'_{11} : Q_1^{z_1} \cup (x_1, x_2) \ldots (z_1, x_2) \cup (y_1, y_2) \ldots (z_1, y_2)$ and

$T^* : \ddot{Q}_1^{y_1} \cup (x_1, x_2)(\overline{x}_1, \underline{x}_2^1) \ldots (y_1, \underline{x}_2^1) \cup (y_1, \underline{z}_2^1) \ldots (\underline{z}_1, \underline{z}_2^1)(z_1, z_2)$.

b) If both P_1 and Q_1 are paths, we need to distinguish several cases depending on the position of the vertices in the grid defined by P_1 and Q_1.

$b1$) If Q_1 is an $x_2y_2z_2$-path described as $Q_1 : x_2\overline{x}_2^1 \ldots \underline{y}_2^1 y_2 \overline{y}_2^1 \ldots \underline{z}_2^1 z_2$ and

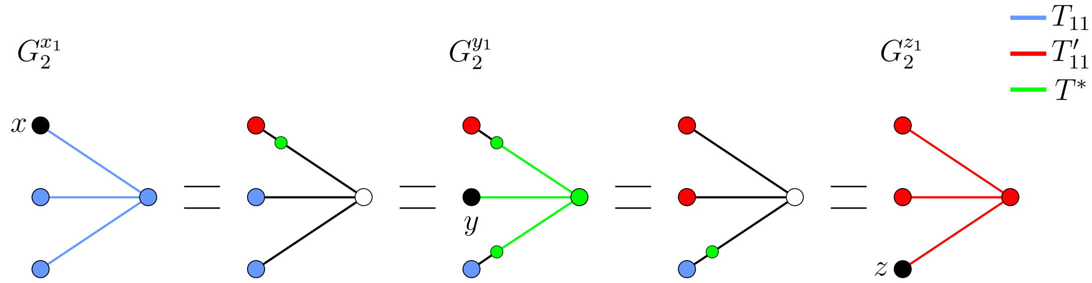

Figure 4.14: Three trees in $G_1 \boxtimes G_2$ associated to a path P_1 and a tree Q_1.

taking into account that it may occur that $\overline{x}_2^1 = y_2$, $\underline{y}_2^1 = x_2$, $\overline{y}_2^1 = z_2$ or $\underline{z}_2^1 = y_2$, (see Case $b1$ in Figure 4.15), we consider

$T_{11} : Q_1^{x_1} \cup (x_1, \overline{y}_2) \ldots (\underline{y}_1, \overline{y}_2)(y_1, y_2) \cup (x_1, z_2) \ldots (z_1, z_2)$,

$T'_{11} : Q_1^{z_1} \cup (x_1, x_2) \ldots (z_1, x_2) \cup (z_1, \underline{y}_2) \ldots (\overline{y}_1, \underline{y}_2)(y_1, y_2)$ and

$T^* : (x_1, x_2)(\overline{x}_1, \overline{x}_2^1) \ldots (y_1, \overline{x}_2^1) \ldots (y_1, \underline{z}_2^1) \ldots (\underline{z}_1, \underline{z}_2^1)(z_1, z_2)$.

$b2$) Now, we assume that Q_1 is an $x_2z_2y_2$-path. If $y_1z_1 \in E(P_1)$ and $y_2z_2 \in E(Q_1)$, (see Case $b2$ in Figure 4.15) and therefore, we consider

$T_{11} : Q_1^{x_1} \cup (x_1, y_2) \ldots (y_1, y_2)(z_1, z_2)$,

$T'_{11} : Q_1^{z_1} \cup (x_1, x_2) \ldots (z_1, x_2) \cup (y_1, y_2)(z_1, y_2)$ and

$T^* : (x_1, x_2)(\overline{x}_1, \overline{x}_2^1) \ldots (y_1, \overline{x}_2^1) \ldots (y_1, z_2)(y_1, y_2) \cup (y_1, z_2)(z_1, z_2)$.

$b3$) Otherwise, $d_{P_1}(y_1, z_1) \geq 2$ or $d_{Q_1}(z_2, y_2) \geq 2$. Both cases are symmetrical and hence we assume without loss of generality that $d_{Q_1}(z_2, y_2) \geq 2$. It means that $Q_1 : x_2\overline{x}_2^1 \ldots z_2\overline{z}_2^1 \ldots y_2$ with $z_2 \neq \overline{z}_2^1 \neq y_2$, (see Case $b3$ in Figure 4.15). In this case,

$T_{11} :\ (x_1, x_2) \ldots (x_1, z_2)(\overline{x}_1, \overline{z}_2^1) \ldots (\underline{z}_1, \overline{z}_2^1)(z_1, z_2) \cup (y_1, \overline{z}_2^1) \ldots (y_1, y_2)$,

$T'_{11} :\ Q_1^{z_1} \cup (x_1, x_2) \ldots (z_1, x_2) \cup (y_1, y_2) \ldots (z_1, y_2)$ and

$T^*: (x_1, x_2)(\overline{x}_1, \overline{x}_2^1)\dots(\overline{x}_1, z_2)(x_1, \overline{z}_2^1)\dots(x_1, y_2)\dots(y_1, y_2) \cup (\overline{x}_1, z_2)\dots(z_1, z_2)$.

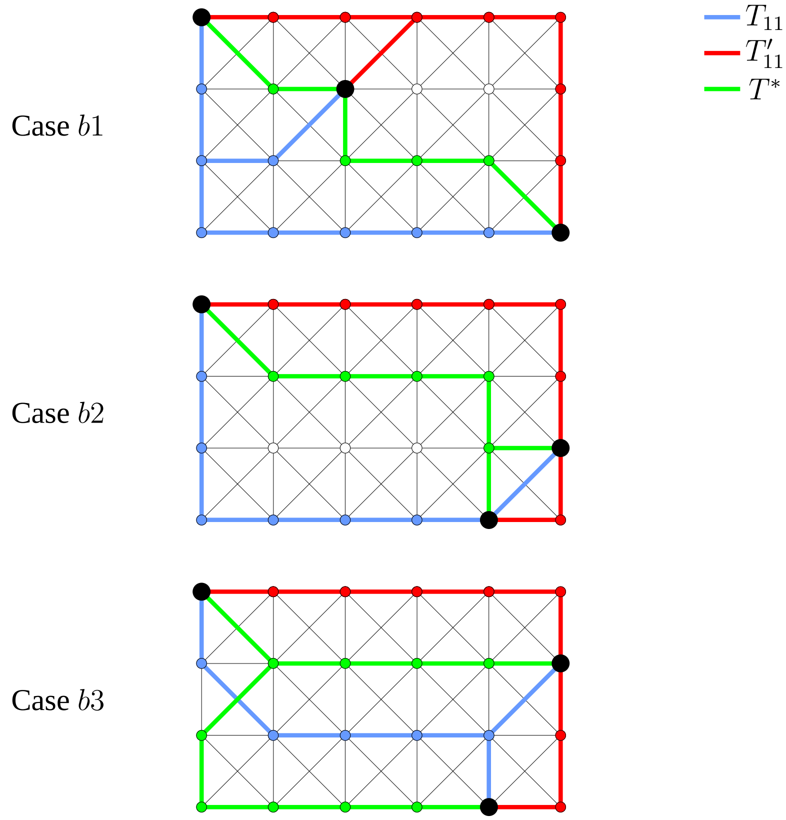

Figure 4.15: Three $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to P_1 and Q_1 paths.

c) Assume that P_1 is an r-rooted tree and that Q_1 is also a rooted tree. Denoting by $\ddot{Q}_1^r = Q_1^r - \{(r, x_2), (r, y_2), (r, z_2)\}$, (see Figure 4.16), we consider

$T_{11}: Q_1^{x_1} \cup (x_1, y_2)\dots(r, y_2)\dots(y_1, y_2) \cup (x_1, z_2)\dots(r, z_2)\dots(z_1, z_2)$,

$T'_{11}: (x_1, x_2)\dots(r, x_2)\dots(y_1, x_2)\dots(y_1, y_2) \cup (r, x_2)\dots(z_1, x_2)\dots(z_1, z_2)$ and

$T^*: \ddot{Q}_1^r \cup (r, \underline{x}_2^1)\dots(\underline{x}_1, \underline{x}_2^1)(x_1, x_2) \cup (r, \underline{y}_2^1)\dots(\underline{y}_1, \underline{y}_2^1)(y_1, y_2) \cup (r, \underline{z}_2^1)\dots(\underline{z}_1, \underline{z}_2^1)(z_1, z_2)$.

d) The case P_1 being a tree and Q_1 being a path is symmetrical to *a)* due to the commutativity of the strong product of graphs.

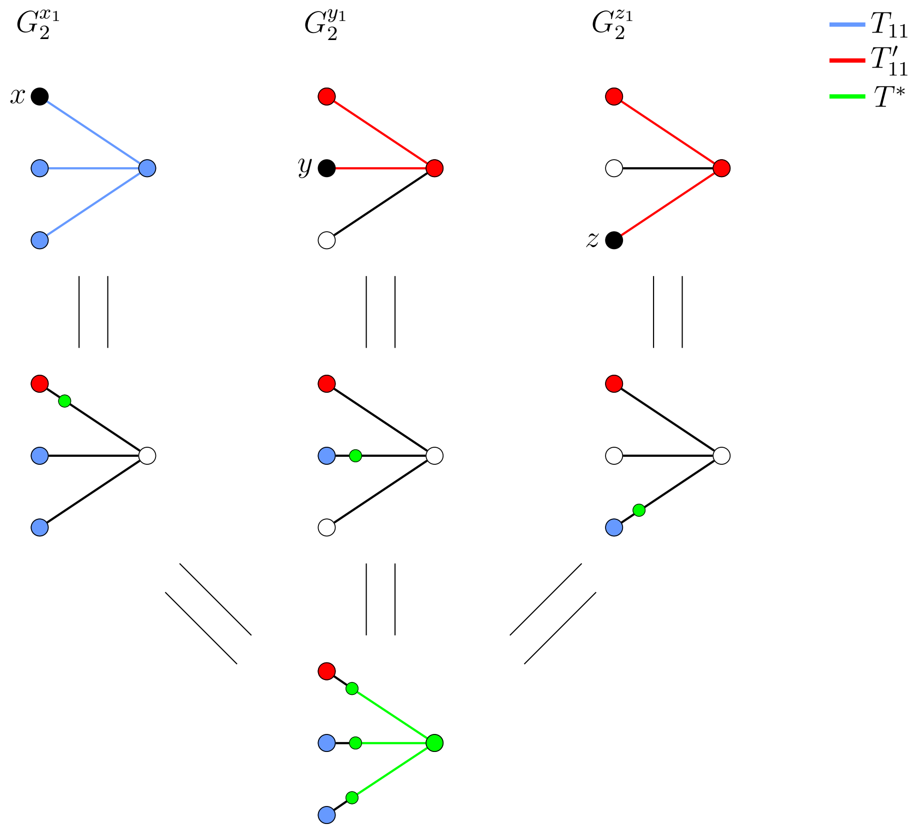

Figure 4.16: Three $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to trees P_1 and Q_1.

Notice that the desired result is proved when $\kappa_3(G_2) = 1$. From now on, we assume that $\ell_2 = \kappa_3(G_2) \geq 2$ and we must consider whether Q_2 is an *special* tree or not.

(II) Assume that both Q_1 and Q_2 are *special* trees. In this case, we construct five trees associated to P_1, Q_1 and Q_2. First, we assume that P_1 is a path. If $d_{Q_1}(x_2) = d_{Q_2}(x_2) = 1$, then the five trees can be constructed with the help of Figure 4.17. As a consequence of the symmetry of the path P_1, similar constructions hold when $d_{Q_1}(z_2) = d_{Q_2}(z_2) = 1$.

We consider the remaining case $d_{Q_1}(y_2) = d_{Q_2}(y_2) = 1$. If $x_2y_2 \in E(Q_1)$, then Figure 4.18 helps us to construct the five trees associated to P_1, Q_1 and Q_2.

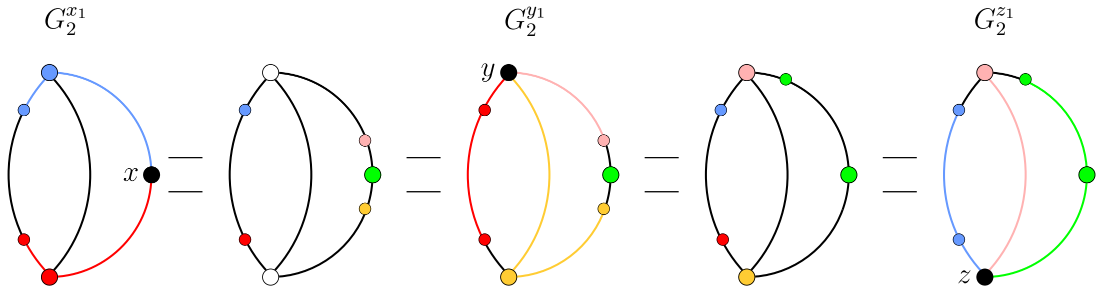

Figure 4.17: Five $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to paths P_1, Q_1 and Q_2 .

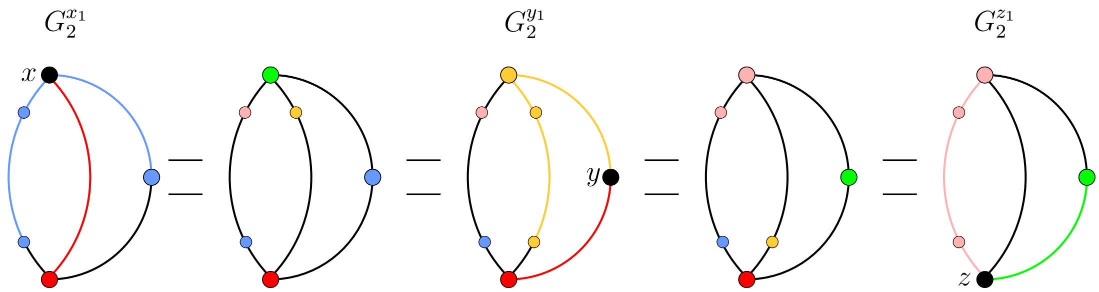

Figure 4.18: Five $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to paths P_1, Q_1 and Q_2, when $x_2y_2 \in E(Q_1)$.

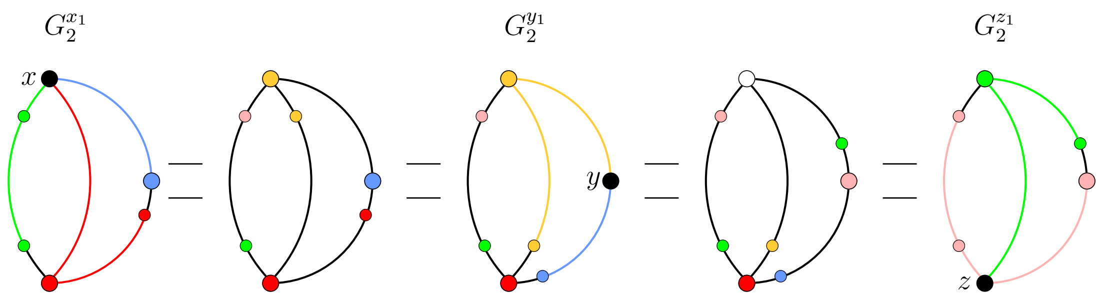

Figure 4.19: Five $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to paths P_1, Q_1 and Q_2, when $x_2y_2 \notin E(Q_1)$ and $y_2z_2 \notin E(Q_2)$.

Symmetrical solution holds when $y_2z_2 \in E(Q_2)$.

Otherwise, $x_2y_2 \notin E(Q_1)$, $y_2z_2 \notin E(Q_2)$ and Figure 4.19 shows the vertices contained in each of the desired five trees. Finally, when P_1 is a tree, see Figure 4.20.

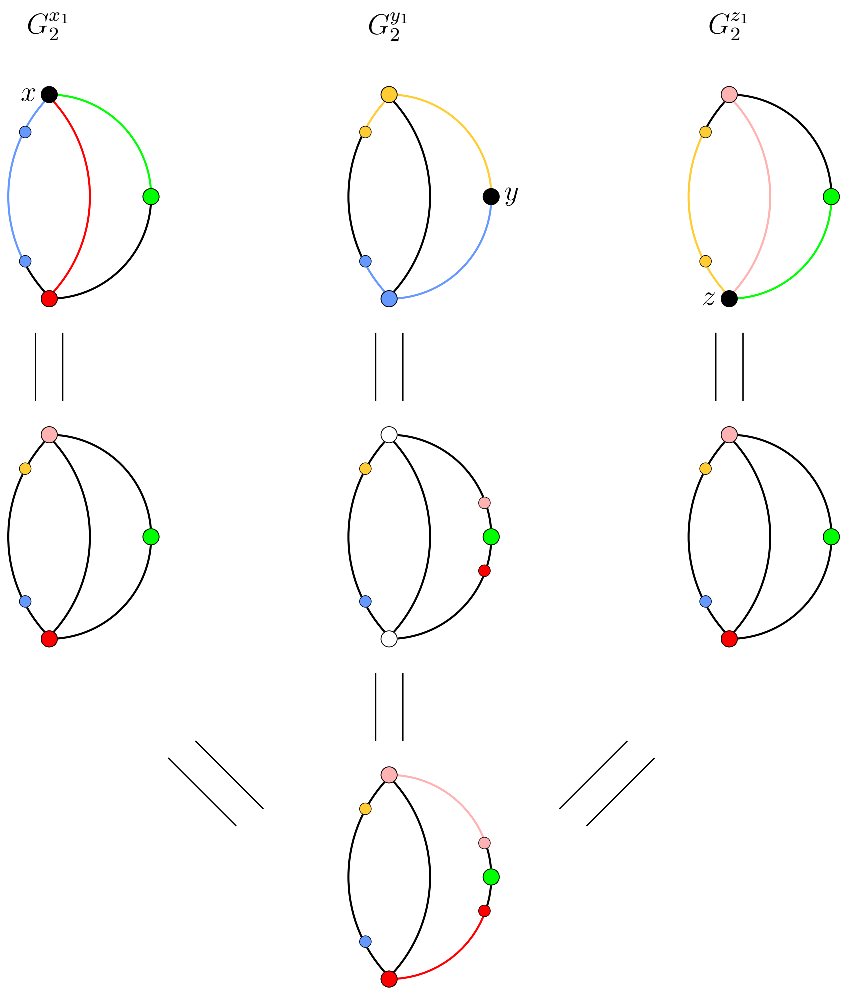

Figure 4.20: Five $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to a tree P_1 and paths Q_1 and Q_2.

(III) For each $j \in \{2, \ldots, \ell_2\}$ such that Q_j is not an *special* tree, we construct two trees T_{1j}, T'_{1j} in $G_1 \boxtimes G_2$ associated to trees P_1 and Q_j. We need to distinguish three cases:

a) Assume that P_1 is an $x_1y_1z_1$-path and Q_j is an $x_2z_2y_2$-path. Then

$T_{1j} : (x_1, x_2)(\overline{x}_1, \overline{x}_2^j) \dots (z_1, \overline{x}_2^j) \dots (z_1, z_2) \dots (z_1, \underline{y}_2^j) \dots (\overline{y}_1, \underline{y}_2^j)(y_1, y_2)$ and

$T'_{1j} : (x_1, x_2) \dots (x_1, \underline{z}_2^j) \dots (\underline{z}_1, \underline{z}_2^j)(z_1, z_2)(\underline{z}_1, \overline{z}_2^j) \dots (y_1, \overline{z}_2^j) \dots (y_1, y_2)$.

b) Assume that P_1 is an $x_1y_1z_1$-path and Q_j is an s^j-rooted tree. As usual, $Q_j : s^j \dots \underline{x}_2^j x_2 \cup s^j \dots \underline{y}_2^j y_2 \cup s^j \dots \underline{z}_2^j z_2$. Since $j \neq 1$, at least one element of $\{\underline{x}_2^j, \underline{y}_2^j, \underline{z}_2^j\}$ is different to s^j.

If $d_{P_1}(x_1, y_1) \geq 2$ and $\underline{x}_2^j = \underline{y}_2^j = s^j$, then

$T_{1j} : (x_1, x_2) \dots (x_1, \underline{z}_2^j) \dots (y_1, \underline{z}_2^j) \dots (y_1, y_2) \cup (y_1, \underline{z}_2^j) \dots (\underline{z}_1, \underline{z}_2^j)(z_1, z_2)$ and

$T'_{1j} : (x_1, x_2)(\overline{x}_1, s^j) \dots (\underline{y}_1, s^j)(y_1, y_2)(\overline{y}_1, s^j) \dots (z_1, s^j) \dots (z_1, z_2)$.

Similar constructions hold for $d_{P_1}(y_1, z_1) \geq 2$ and $\underline{z}_2^j = \underline{y}_2^j = s^j$.

c) In any other case, to unify the description of the trees T_{1j} and T'_{1j}, without loss of generality we provide an specific role to the vertex $y = (y_1, y_2)$. As usual, if P_1 is a path, then we consider that P_1 is an $x_1y_1z_1$-path. If P_1 is an r-rooted tree, we assume either that $d_{Q_j}(s^j, y_2) \geq 2$ or that Q_j is an $x_2y_2z_2$-path. Also, we consider that $\overline{y}_2^j = \underline{y}_2^j$ when Q_j is a tree. Under these assumptions, to construct the tree T_{1j} it is enough to consider that

$$\begin{aligned} V(T_{1j}) = \quad & Q_j^{y_1} - \{(y_1, x_2), (y_1, z_2)\} \cup \\ & \{(u, v) : u \in P_1 - \{x_1, z_1\}, v \in N_{Q_j}(x_2) \cup N_{Q_j}(z_2)\} \cup \\ & \{x, y, z\} \end{aligned}$$

Similarly, T'_{1j} is such that

$$\begin{aligned} V(T'_{1j}) = \quad & \{(x_1, x_2), \dots, (x_1, \underline{y}_2)\} \cup \\ & \{(z_1, z_2), \dots, (z_1, \overline{y}_2)\} \cup \\ & \{(u, v) : u \in P_1 - \{x_1, z_1\}, v \in N_{Q_j}(y_2)\} \cup \\ & \{x, y, z\} \end{aligned}$$

If $\ell_2 \geq 2$ and Q_1 is the unique *special* tree, **(I)** and **(III)** provide

$$3 + 2(\ell_2 - 1) = 2\ell_2 + 1$$

internally disjoint $\{x, y, z\}$-trees, as desired. Otherwise, **(II)** and **(III)** provide $5 + 2(\ell_2 - 2) = 2\ell_2 + 1$ such trees. $\square$

The bounds of Lemmas 4.3.1 and 4.3.4 are sharp. In fact, applied to P_1 and P_2 paths with at least three vertices, they provide three internally disjoint trees connecting any three vertices x, y, z of $G_1 \boxtimes G_2$ and this bound is sharp because $\kappa_3(P_1 \boxtimes P_2) \leq \delta(P_1 \boxtimes P_2) = 3$.

Due to the commutativity of the strong product of graphs, it is not necessary to study the remaining positions of three vertices x, y, z in $G_1 \boxtimes G_2$. In fact, as a consequence of Lemma 4.3.1, it follows the existence of at least $2\kappa_3(G_2) + 1$ internally disjoint trees joining vertices $(x_1, x_2), (y_1, x_2), (z_1, x_2)$ when $g(G_2) \geq 5$. Also, from Lemma 4.3.3, it follows the existence of at least $2\kappa_3(G_2)$ internally disjoint trees connecting vertices $(x_1, x_2), (y_1, x_2), (z_1, z_2)$ when $g(G_2) \geq 5$.

Now, we are ready to prove the main result of this chapter.

Theorem 4.3.1. *Let G_1 and G_2 be two connected graphs with at least 3 vertices and such that $g(G_2) \geq 5$. Then $\kappa_3(G_1 \boxtimes G_2) \geq 2\kappa_3(G_2)$. The bound is sharp.*

Proof. The bound $\kappa_3(G_1 \boxtimes G_2) \geq 2\kappa_3(G_2)$ is consequence of the inequality $\delta(G_2) \geq \kappa(G_2) \geq \kappa_3(G_2)$ and Lemmas 4.3.1, 4.3.2, 4.3.3 and 4.3.4. Notice that the given bound is only attained when we consider vertices $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$ in $G_1 \boxtimes G_2$ such that $x_1 = y_1 \neq z_1$ or $x_2 = y_2 \neq z_2$. $\square$

When the generator graphs G_1 and G_2 are such that $\kappa(G_1) = \delta(G_1) = 1$ and $\kappa_3(G_2) = \delta(G_2)$, the generalized 3-connectivity of the strong product of G_1 and G_2 is almost determined, as we show in next result.

Corollary 4.3.1. *Let* G_1 *and* G_2 *be two graphs with at least three vertices and such that* $\kappa(G_1) = \delta(G_1) = 1$ *and* $\kappa_3(G_2) = \delta(G_2)$*. Assume also that* $g(G_2) \geq 5$*. Then* $\delta(G_1 \boxtimes G_2) - 1 \leq \kappa_3(G_1 \boxtimes G_2) \leq \delta(G_1 \boxtimes G_2)$*.*

Proof. Since $\kappa_3(G) \leq \kappa(G) \leq \delta(G)$ for every connected graph G, it remains only to prove the given lower bound of $\kappa_3(G_1 \boxtimes G_2)$. Taking into account that the minimum degree of the strong product of G_1 and G_2 is

$$\delta(G_1 \boxtimes G_2) = \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2) = 2\kappa_3(G_2) + 1,$$

from Theorem 4.3.1, it follows that $\kappa_3(G_1 \boxtimes G_2) \geq 2\kappa_3(G_2) = \delta(G_1 \boxtimes G_2) - 1$. $\square$

Chapter 5

The Wiener index

This chapter deals with a parameter related to the control on the distances in a connected graph, called the Wiener index. Sharp lower and upper bounds on the Wiener index of the strong product of two connected graphs are given and the exact value for the strong product of some families of graphs is determined.

5.1 Introduction

Classical parameters on distance in graphs, as the diameter, the radius or the eccentricity, are very studied in Graph Theory. However, sometimes these indices cannot provide a suitable information about the graph description. Let us see an example. Consider the graph G obtained by a complete graph and a path $\mathcal{P}$ sharing a common vertex (see Figure 5.1). In this case, notice that there are $|V(G)| - (l(\mathcal{P}) + 1)$ pairs of vertices separated at diametral distance (for example, x and y in Figure 5.1), which is small if the length of $\mathcal{P}$ is large enough, but there are many pairs of vertices in the graph directly communicated.

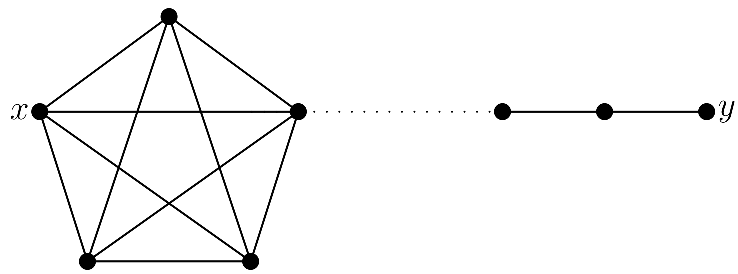

Figure 5.1: A graph G with a big difference between its diameter and its average distance.

Then, it is interesting to search another type of parameters to obtain a more comprehensive information about the distances in a graph. We focus on the *Wiener index* of the strong product of two connected graphs. As we saw in Introduction, the Wiener index was introduced in [96] and for any connected graph G, it is denoted by $W(G)$ and defined as

$$W(G) = \frac{1}{2} \sum_{x,y \in V(G)} d_G(x,y),$$

where the sum is taken through all the ordered pairs of vertices of G.

The Wiener index is related to the *average distance*, introduced by Doyle and Graver in [38] and denoted by $\mu(G)$. The average distance of a connected graph G is the expected distance between a randomly chosen pair of distinct vertices, namely, the mean on the distances between all the ordered pairs of vertices of G, that is,

$$\mu(G) = \frac{\sum\limits_{x,y \in V(G)} d_G(x,y)}{2\binom{n}{2}} = \frac{2W(G)}{n(n-1)}.$$

Therefore, studying the Wiener index of any connected graph, we could obtain directly the average distance of such graph. This is a reason for which we consider interesting the Wiener index.

One can compute directly the exact values of this kind of parameters in small graphs, see for instance the graph G of Figure 5.2. It is easy to check that $W(G) = 21$ and $\mu(G) = 1.4$.

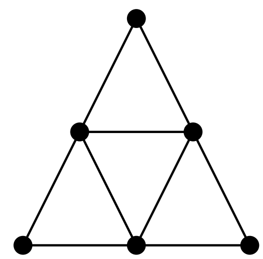

Figure 5.2: A graph G with $W(G) = 21$ and $\mu(G) = 1.4$.

Indeed, as we mentioned in Introduction, for complete graphs, paths or cycles, the Wiener index and the average distance were deduced, as well as certain general bounds [40] as the following ones.

$$\frac{n(n-1)}{2} \leq W(G) \leq \frac{n^3 - n}{6},$$

for every graph G of order n. This bound comes from inequality

$$W(K) \leq W(G) \leq W(\mathcal{P}),$$

respectively, where K and $\mathcal{P}$ are the complete graph and the path on n vertices.

Our aim is to find bounds of $W(G_1 \boxtimes G_2)$ for any two connected graphs G_1 and G_2. First, we need to introduce some general notations that we will use throughout this chapter.

Let G_1 and G_2 be two connected graphs with order n_1, n_2, size e_1, e_2, Wiener indices

$$W_1 = W(G_1) = \frac{1}{2} \sum_{x_1, y_1 \in V(G_1)} d_{G_1}(x_1, y_1)$$

and

$$W_2 = W(G_2) = \frac{1}{2} \sum_{x_2, y_2 \in V(G_2)} d_{G_2}(x_2, y_2),$$

where all the sums are taken through all the ordered pairs of vertices of G_1 and G_2, respectively.

First of all, we introduce a sets of ordered pairs of vertices of G_1 and G_2 which will play an important role in the development of this study. For every $i = 0, \dots, D(G_1)$ and every $j = 0, \dots, D(G_2)$, let us denote

$$R_i = \{x_1, y_1 \in V(G_1) : d_{G_1}(x_1, y_1) = i\}, \text{ with cardinality } r_i = |R_i|$$

and

$$S_j = \{x_2, y_2 \in V(G_2) : d_{G_2}(x_2, y_2) = j\}, \text{ with cardinality } s_j = |S_j|.$$

Notice that the sets R_i and S_j form a partition of the sets of ordered pairs of vertices of G_1 and G_2, respectively. That is

$$\bigcup_{i=0}^{D(G_1)} R_i = V(G_1) \times V(G_1) \text{ and } \bigcup_{j=0}^{D(G_2)} S_j = V(G_2) \times V(G_2).$$

Observe also that

$$\sum_{i=0}^{D(G_1)} i r_i = \sum_{x_1, y_1 \in V(G_1)} d_{G_1}(x_1, y_1),$$

and similarly,

$$\sum_{j=0}^{D(G_2)} j s_j = \sum_{x_2, y_2 \in V(G_2)} d_{G_2}(x_2, y_2).$$

Attending to this notation, we have the following equalities:

$$r_0 = n_1,\ r_1 = 2e_1 \text{ and } s_0 = n_2,\ s_1 = 2e_2 \tag{5.1}$$

$$\sum_{i=0}^{D(G_1)} r_i = n_1^2 \text{ and } \sum_{j=0}^{D(G_2)} s_j = n_2^2 \tag{5.2}$$

$$\frac{1}{2} \sum_{i=1}^{D(G_1)} i r_i = W_1 \text{ and } \frac{1}{2} \sum_{j=1}^{D(G_2)} j s_j = W_2. \tag{5.3}$$

Given any two vertices (x_1, x_2) and (y_1, y_2) of $G_1 \boxtimes G_2$, recall that

$$d_{G_1 \boxtimes G_2}((x_1, x_2), (y_1, y_2)) = \max\{d_{G_1}(x_1, y_1), d_{G_2}(x_2, y_2)\}. \tag{5.4}$$

Instead of working properly with the distance matrix of $V(G_1 \boxtimes G_2)$, we consider more useful to regroup the original distances in boxes depending on the distance between the ordered pair of vertices in the generator graphs G_1 and G_2. We will denote this new matrix as $D = [d_{ij}]$ (see Table 5.1).

	S_0	S_1	$\cdots$	S_j	$\cdots$	$S_{D(G_2)}$
R_0	d_{00}	d_{01}				
R_1	d_{10}	d_{11}				
$\vdots$						
R_i				d_{ij}		
$\vdots$						
$R_{D(G_1)}$						$d_{D(G_1)D(G_2)}$

Table 5.1: Matrix $D = [d_{ij}]$.

The entrances of this matrix D are the sets R_i and S_j described above, for $i = 0, \ldots, D(G_1)$ and $j = 0, \ldots, D(G_2)$. Each element d_{ij} of D represents the distance between any pair $\{(x_1, x_2), (y_1, y_2)\}$ and $\{(y_1, y_2), (x_1, x_2)\}$ of the set of ordered pairs of vertices of $G_1 \boxtimes G_2$ such that $x_1, y_1 \in R_i$ and $x_2, y_2 \in S_j$. In fact, the value of each cell is $d_{ij} = \max\{i, j\}$ and the number of ordered pairs of vertices which are at distance d_{ij} in $G_1 \boxtimes G_2$ is exactly $r_i s_j$.

Moreover, denoting by $x = (x_1, x_2)$ and $y = (y_1, y_2)$ two any vertices in $V(G_1 \boxtimes G_2)$, observe that

$$\sum_{x,y \in V(G_1 \boxtimes G_2)} d_{G_1 \boxtimes G_2}(x, y) = \sum_{j=0}^{D(G_2)} \sum_{i=0}^{D(G_1)} d_{ij} r_i s_j, \tag{5.5}$$

where the sum is taken through all the ordered pairs of vertices of $G_1 \boxtimes G_2$.

We want to clarify it through an example. Consider any two connected graphs G_1 and G_2 such that $D(G_1) \leq D(G_2)$. Table 5.2 is the matrix of distances D described above. This table represents the distances between any ordered pair of vertices in $G_1 \boxtimes G_2$ from the point of view of the distances between the vertices in each generator graph. For example, two distinct vertices $(x_1, x_2), (y_1, y_2) \in V(G_1 \boxtimes G_2)$ coming from $x_1, y_1 \in R_2$ and $x_2, y_2 \in S_5$, are at distance 5 in $G_1 \boxtimes G_2$, which is the value that exists in the cell $(2, 5)$ of Table 5.2.

	S_0	S_1	S_2	S_3	S_4	S_5	S_6
R_0	0	1	2	3	4	5	6
R_1	1	1	2	3	4	5	6
R_2	2	2	2	3	4	5	6
R_3	3	3	3	3	4	5	6

Table 5.2: Matrix D for $G_1 \boxtimes G_2$ with $D(G_1) = 3$ and $D(G_2) = 6$.

5.2 General equality of $W(G_1 \boxtimes G_2)$ and lower bounds

We start this section with a result which provides a general equality for the Wiener index of the strong product graph $W(G_1 \boxtimes G_2)$ in terms of known parameters of the generator graphs G_1 and G_2, as their orders, their Wiener indices and their cardinalities r_i and s_j, which were defined in Section 5.1 as follows:

$$r_i = |R_i| \text{ where } R_i = \{x_1, y_1 \in V(G_1) : d_{G_1}(x_1, y_1) = i\}$$

and

$$s_j = |S_j| \text{ where } S_j = \{x_2, y_2 \in V(G_2) : d_{G_2}(x_2, y_2) = j\}.$$

Theorem 5.2.1. *Let G_k be a connected graph with order n_k and Wiener index W_k, for $k = 1, 2$, such that $D(G_1) \leq D(G_2)$. Let r_i and s_j be defined in Section 5.1. The following assertions hold:*

(i) If $D(G_1) = 1$ then

$$W(G_1 \boxtimes G_2) = n_2 W_1 + n_1^2 W_2.$$

(ii) If $2 \leq D(G_1) \leq D(G_2)$ then

$$W(G_1 \boxtimes G_2) = n_2 W_1 + n_1^2 W_2 + \frac{1}{2} \sum_{j=1}^{D(G_1)-1} \sum_{i=j+1}^{D(G_1)} (i-j) r_i s_j.$$

Proof. Let $x = (x_1, x_2), y = (y_1, y_2)$ be two vertices in $V(G_1 \boxtimes G_2)$. By the definition of the Wiener index and applying equality (5.5), we deduce that

$$W(G_1 \boxtimes G_2) = \frac{1}{2} \sum_{x,y \in V(G_1 \boxtimes G_2)} d_{G_1 \boxtimes G_2}(x, y) = \frac{1}{2} \sum_{i=0}^{D(G_1)} \sum_{j=0}^{D(G_2)} d_{ij} r_i s_j,$$

where, r_i and s_j are the cardinalities of the sets R_i and S_j, as well as d_{ij} are the distances of the matrix D defined in Section 5.1, for $0 \leq i \leq D(G_1)$ and $0 \leq j \leq D(G_2)$.

Furthermore, since $d_{ij} = \max\{i, j\}$, we may describe

$$d_{ij} r_i s_j = \max\{i, j\} r_i s_j = \begin{cases} j r_i s_j, \text{ for } i \le j \\ \\ (i - j + j) r_i s_j, \text{ otherwise.} \end{cases} \tag{5.6}$$

Then, by applying the definition of the Wiener index and by (5.6), observe that

$$\begin{aligned} W(G_1 \boxtimes G_2) &= \frac{1}{2} \sum_{(x,y) \in V(G_1 \boxtimes G_2)} d_{G_1 \boxtimes G_2}(x, y) = \frac{1}{2} \sum_{j=0}^{D(G_2)} \sum_{i=0}^{D(G_1)} d_{ij} r_i s_j \\ &= \frac{1}{2} \sum_{j=0}^{D(G_1)-1} \sum_{i=0}^{D(G_1)} d_{ij} r_i s_j \; + \; \frac{1}{2} \sum_{j=D(G_1)}^{D(G_2)} \sum_{i=0}^{D(G_1)} d_{ij} r_i s_j. \end{aligned} \tag{5.7}$$

(i) Assume that $D(G_1) = 1$. Then, by (5.7), we have

$$W(G_1 \boxtimes G_2) = \frac{1}{2} s_0 \sum_{i=0}^{1} i r_i \; + \; \frac{1}{2} \sum_{j=1}^{D(G_2)} j s_j \sum_{i=0}^{1} r_i.$$

Taking into account equalities (5.1), (5.2) and (5.3), we know that

$$s_0 = n_2, \;\; \frac{1}{2} \sum_{i=0}^{D(G_1)} i r_i = W_1, \;\; \sum_{i=0}^{D(G_1)} r_i = n_1^2 \;\text{ and }\; \frac{1}{2} \sum_{j=1}^{D(G_2)} j s_j = W_2,$$

yielding that

$$W(G_1 \boxtimes G_2) = n_2 W_1 + n_1^2 W_2.$$

(ii) Assume that $D(G_1) \geq 2$. From equality (5.7), we deduce that

$$\begin{aligned}
W(G_1 \boxtimes G_2) &= \frac{1}{2}\sum_{j=0}^{D(G_1)-1}\left(\sum_{i=0}^{j} jr_is_j + \sum_{i=j+1}^{D(G_1)}(i-j+j)r_is_j\right) \\
&\quad + \frac{1}{2}\sum_{j=D(G_1)}^{D(G_2)}\sum_{i=0}^{D(G_1)} jr_is_j \\
&= \frac{1}{2}\sum_{j=0}^{D(G_1)-1}\left(\sum_{i=0}^{D(G_1)} jr_is_j + \sum_{i=j+1}^{D(G_1)}(i-j)r_is_j\right) \\
&\quad + \frac{1}{2}\sum_{j=D(G_1)}^{D(G_2)}\sum_{i=0}^{D(G_1)} jr_is_j \\
&= \frac{1}{2}\sum_{j=0}^{D(G_1)-1}\sum_{i=0}^{D(G_1)} jr_is_j + \frac{1}{2}\sum_{j=0}^{D(G_1)-1}\sum_{i=j+1}^{D(G_1)}(i-j)r_is_j \\
&\quad + \frac{1}{2}\sum_{j=D(G_1)}^{D(G_2)}\sum_{i=0}^{D(G_1)} jr_is_j \\
&= \frac{1}{2}\sum_{j=0}^{D(G_2)} js_j \sum_{i=0}^{D(G_1)} r_i + \frac{1}{2}\sum_{j=0}^{D(G_1)-1}\sum_{i=j+1}^{D(G_1)}(i-j)r_is_j \\
&= \frac{1}{2}\sum_{j=0}^{D(G_2)} js_j \sum_{i=0}^{D(G_1)} r_i + s_0\frac{1}{2}\sum_{i=1}^{D(G_1)} ir_i + \frac{1}{2}\sum_{j=1}^{D(G_1)-1}\sum_{i=j+1}^{D(G_1)}(i-j)r_is_j.
\end{aligned}$$

Again, from equalities (5.1), (5.2) and (5.3), it follows that

$$\frac{1}{2}\sum_{j=0}^{D(G_2)} js_j = W_2, \quad \sum_{i=0}^{D(G_1)} r_i = n_1^2, \quad s_0 = n_2 \quad \text{and} \quad \frac{1}{2}\sum_{i=0}^{D(G_1)} ir_i = W_1,$$

leading to

$$W(G_1 \boxtimes G_2) = n_2W_1 + n_1^2W_2 + \frac{1}{2}\sum_{j=1}^{D(G_1)-1}\sum_{i=j+1}^{D(G_1)}(i-j)r_is_j.$$

□

Equality (ii) of Theorem 5.2.1 leads us to deduce several lower bounds on $W(G_1 \boxtimes G_2)$ in terms of known invariants as the order, the size, the minimum degree and the Wiener index of the generator graphs, as the next theorem shows.

Theorem 5.2.2. *Let G_k be a connected graph with order n_k, size e_k, minimum degree δ_k and Wiener index W_k, for $k = 1, 2$, such that $2 \leq D(G_1) \leq D(G_2)$. The following assertions hold:*

(i) If $D(G_1) \geq 2$ then
$W(G_1 \boxtimes G_2) \geq n_2W_1 + n_1^2W_2 + e_2\left(2W_1 - n_1^2 + n_1\right)$,
with equality if and only if $D(G_1) = 2$.

(ii) If $D(G_1) \geq 3$ then
$W\left(G_1 \boxtimes G_2\right) \geq n_2W_1+n_1^2W_2+e_2\left(2W_1 - n_1^2 + n_1\right)+s_2\left(W_1 - n_1^2 + n_1 + e_1\right)$,
with equality if and only if $D(G_1) = 3$.

(iii) If $D(G_1) \geq 3$ and G_2 has minimum degree at least 2 and girth at least 5, then
$W\left(G_1 \boxtimes G_2\right) \geq n_2W_1+n_1^2W_2+2\delta_2e_2W_1+2e_1e_2(\delta_2-1)-e_2(2\delta_2-1)(n_1^2-n_1)$,
with equality if and only if $D(G_1) = 3$ and G_2 is regular.

Proof. (i) Assume that $D(G_1) \geq 2$. Then, by Theorem 5.2.1, we have

$$\begin{aligned}
W\left(G_1 \boxtimes G_2\right) &= n_2W_1 + n_1^2W_2 + \frac{1}{2}\sum_{j=1}^{D(G_1)-1}\sum_{i=j+1}^{D(G_1)}(i-j)r_is_j \\
&\geq n_2W_1 + n_1^2W_2 + s_1\frac{1}{2}\sum_{i=2}^{D(G_1)}(i-1)r_i \\
&= n_2W_1 + n_1^2W_2 + s_1\frac{1}{2}\left(\sum_{i=2}^{D(G_1)} ir_i - \sum_{i=2}^{D(G_1)} r_i\right) \\
&= n_2W_1 + n_1^2W_2 + 2e_2\left(\frac{1}{2}\sum_{i=2}^{D(G_1)} ir_i - \frac{1}{2}\sum_{i=2}^{D(G_1)} r_i\right).
\end{aligned}$$

From (5.2) and (5.3), it follows that

$$\frac{1}{2}\sum_{i=2}^{D(G_1)} ir_i = W_1 - \frac{1}{2}r_1 = W_1 - e_1$$

and

$$\sum_{i=2}^{D(G_1)} r_i = n_1^2 - r_0 - r_1 = n_1^2 - n_1 - 2e_1,$$

yielding that

$$\begin{aligned} W\left(G_1 \boxtimes G_2\right) &\geq n_2 W_1 + n_1^2 W_2 + 2e_2\left((W_1 - e_1) - \frac{1}{2}(n_1^2 - n_1 - 2e_1)\right) \\ &= n_2 W_1 + n_1^2 W_2 + e_2\left(2W_1 - n_1^2 + n_1\right). \end{aligned}$$

Further, observe that the previous equality becomes equality if and only if $r_i = 0$ for $i \geq 3$, that is, when $D(G_1) = 2$, since $r_0 = n_1$, $r_1 = 2e_1$ and

$$r_2 = n_1^2 - r_0 - r_1 = n_1^2 - n_1 - 2e_1.$$

(ii) Suppose that $D(G_1) \geq 3$. Similarly to the first case, by Theorem 5.2.1, we have

$$\begin{aligned} W\left(G_1 \boxtimes G_2\right) &= n_2 W_1 + n_1^2 W_2 + \frac{1}{2}\sum_{j=1}^{D(G_1)-1}\sum_{i=j+1}^{D(G_1)}(i-j) r_i s_j \\ &\geq n_2 W_1 + n_1^2 W_2 + s_1 \frac{1}{2}\sum_{i=2}^{D(G_1)}(i-1) r_i + s_2 \frac{1}{2}\sum_{i=3}^{D(G_1)}(i-2) r_i \\ &= n_2 W_1 + n_1^2 W_2 + s_1\left(\frac{1}{2}\sum_{i=2}^{D(G_1)} i r_i - \frac{1}{2}\sum_{i=2}^{D(G_1)} r_i\right) \\ &\quad + s_2\left(\frac{1}{2}\sum_{i=3}^{D(G_1)} i r_i - \sum_{i=3}^{D(G_1)} r_i\right). \end{aligned}$$

As $\sum_{i=0}^{D(G_1)} r_i = n_1^2$, it follows that

$$\frac{1}{2}\sum_{i=2}^{D(G_1)} r_i = \frac{1}{2}(n_1^2 - r_0 - r_1) = \frac{1}{2}(n_1^2 - n_1 - 2e_1) = \frac{n_1^2 - n_1}{2} - e_1$$

and

$$\sum_{i=3}^{D(G_1)} r_i = n_1^2 - r_0 - r_1 - r_2 = n_1^2 - n_1 - 2e_1 - r_2,$$

yielding that

$$\begin{aligned} W(G_1 \boxtimes G_2) \;\geq\; & n_2W_1 + n_1^2W_2 + s_1\left(\frac{1}{2}\sum_{i=2}^{D(G_1)} ir_i - \left(\frac{n_1^2-n_1}{2} - e_1\right)\right) \\ & + s_2\left(\frac{1}{2}\sum_{i=3}^{D(G_1)} ir_i - (n_1^2 - n_1 - 2e_1 - r_2)\right). \end{aligned}$$

As $\frac{1}{2}\sum_{i=1}^{D(G_1)} ir_i = W_1$, we have

$$\frac{1}{2}\sum_{i=2}^{D(G_1)} ir_i = W_1 - \frac{1}{2}r_1 = W_1 - e_1$$

and

$$\frac{1}{2}\sum_{i=3}^{D(G_1)} ir_i = W_1 - \frac{1}{2}r_1 - r_2 = W_1 - e_1 - r_2,$$

which means, together with $s_1 = 2e_2$, that

$$\begin{aligned} W(G_1 \boxtimes G_2) \;\geq\; & n_2W_1 + n_1^2W_2 + s_1\left((W_1 - e_1) - \left(\frac{n_1^2-n_1}{2} - e_1\right)\right) \\ & + s_2\left((W_1 - e_1 - r_2) - (n_1^2 - n_1 - 2e_1 - r_2)\right) \\ =\; & n_2W_1 + n_1^2W_2 + e_2\left(2W_1 - n_1^2 + n_1\right) + s_2\left(W_1 - n_1^2 + n_1 + e_1\right). \end{aligned} \tag{5.8}$$

Observe that the equality is attained if and only if $D(G_1) = 3$, due to

$$\frac{1}{2}\sum_{j=1}^{2}\left(\sum_{i=j+1}^{3}(i-j)r_i\right)s_j = \left(\frac{r_2}{2} + r_3\right)s_1 + \frac{r_3}{2}s_2$$

and r_0, r_1, r_2 and r_3 satisfy equations: $r_0 = n_1$, $r_1 = 2e_1$, $r_0 + r_1 + r_2 + r_3 = n_1^2$ and $\frac{1}{2}(r_1 + 2r_2 + 3r_3) = W_1$, that is,

$$\frac{r_2}{2} + r_3 = W_1 - \frac{n_1^2 - n_1}{2} \quad \text{and} \quad \frac{r_3}{2} = W_1 - n_1^2 + n_1 + e_1.$$

(iii) Assume that G_2 has minimum degree $\delta_2 \geq 2$ and girth at least 5. Then the neighborhood of each vertex $v \in V(G_2)$ is an independent set and there are $2\binom{d_{G_2}(v)}{2}$ ordered pairs of vertices at distance 2. Moreover, any two vertices in G_2 at distance 2 have exactly one common neighbor v. Therefore,

$$\begin{aligned} s_2 = & \sum_{v\in V(G_2)} 2\binom{d_{G_2}(v)}{2} = \sum_{v\in V(G_2)} \left(d_{G_2}(v)^2 - d_{G_2}(v)\right) \\ = & \sum_{v\in V(G_2)} d_{G_2}(v)^2 - \sum_{v\in V(G_2)} d_{G_2}(v) = \sum_{v\in V(G_2)} d_{G_2}(v)d_{G_2}(v) - 2e_2 \\ \geq & \ \delta_2 \sum_{v\in V(G_2)} d_{G_2}(v) - 2e_2 \geq \delta_2 2e_2 - 2e_2 = 2e_2(\delta_2 - 1), \end{aligned}$$

with equality if and only if $d_{G_2}(v) = \delta_2$ for every vertex $v \in V(G_2)$, that is, when the graph G_2 is regular. Then, we obtain another lower bound of $W(G_1 \boxtimes G_2)$ only in terms of known parameters of the generator graphs just replacing s_2 by $2e_2(\delta_2 - 1)$ in (5.8), as follows.

$$\begin{aligned} W(G_1 \boxtimes G_2) \ \geq \ & n_2W_1 + n_1^2W_2 + e_2\left(2W_1 - n_1^2 + n_1\right) \\ & +(2\delta_2e_2 - 2e_2)\left(W_1 - n_1^2 + n_1 + e_1\right). \end{aligned}$$

That is,

$$W(G_1 \boxtimes G_2) \geq n_2W_1 + n_1^2W_2 + 2\delta_2e_2W_1 + 2e_1e_2(\delta_2 - 1) - e_2(2\delta_2 - 1)(n_1^2 - n_1),$$

obtaining the desired lower bound, which is an equality if and only if G_2 is regular and $D(G_1) = 3$. □

Point (i) of Theorem 5.2.2 generalizes Theorem 1 of Pattabiraman and Paulraja in [83], as we see in the next remark.

Remark 5.2.1. *(Theorem 1 of [83]) Let $G_1 = K_{m_1,\dots,m_r}$ be a complete bipartite graph with $n_1 = \sum_{i=1}^{k} m_i$ vertices and e_1 edges. Let G_2 be a connected graph on n_2 vertices and e_2 edges. Then*

$$W(K_{m_1,\dots,m_k} \boxtimes G_2) = n_2(n_1^2 - n_1 - e_1) + n_1^2W_2 + e_2(n_1^2 - n_1 - 2e_1).$$

5.3 Wiener index of the strong products of paths and cycles

We provide the exact value of the Wiener index of the strong products of paths and cycles in terms of their orders. For this aim, we use equality (ii) of Theorem 5.2.1, due to the cardinalities r_i, $i = 1, \ldots, D(G_1)$ and s_j, $j = 1, \ldots, D(G_2)$ can be easily determined.

We start with the strong product of two any paths. Notice that a path $\mathcal{P}$ of order n has diameter $n-1$, Wiener index $\dfrac{n^3-n}{6}$ and admits $2n-2i$ ordered pairs of vertices at distance i, for $i = 1, \ldots, n-1$.

Corollary 5.3.1. *Let $\mathcal{P}_1, \mathcal{P}_2$ be two paths on n_1, n_2 vertices, respectively, such that $2 \le n_1 \le n_2$. Then*

$$W(\mathcal{P}_1 \boxtimes \mathcal{P}_2) = \frac{n_1^2 n_2}{2}\left(\frac{n_1^2}{6} + \frac{n_2^2}{3} - \frac{1}{2}\right) - \frac{n_1}{3}\left(\frac{n_1^4}{20} - \frac{n_1^2}{4} + \frac{1}{5}\right).$$

Proof. By the hypothesis, $D(\mathcal{P}_1) \le D(\mathcal{P}_2)$. We apply equality (ii) of Theorem 5.2.1 with $G_1 = \mathcal{P}_1$ and $G_2 = \mathcal{P}_2$. Observe that $r_i = 2n_1 - 2i$, for $i = 1, \ldots, n_1 - 1$ and $s_j = 2n_2 - 2j$, for $j = 1, \ldots, n_2 - 1$. Hence,

$$\begin{aligned}
W(\mathcal{P}_1 \boxtimes \mathcal{P}_2) &= n_2 W_1 + n_1^2 W_2 + \frac{1}{2}\sum_{j=1}^{n_1-2}\left((2n_2 - 2j)\sum_{i=j+1}^{n_1-1}(i-j)(2n_1-2i)\right)\\
&= \frac{n_1^3 - n_1}{6} n_2 + \frac{n_2^3 - n_2}{6} n_1^2\\
&\quad + \frac{n_1 n_2}{3}\left(\frac{n_1^3}{4} - \frac{n_1^2}{2} - \frac{n_1}{4} + \frac{1}{2}\right) - \frac{n_1^5}{60} + \frac{n_1^3}{12} - \frac{n_1}{15}\\
&= \frac{n_1^2 n_2}{2}\left(\frac{n_1^2}{6} + \frac{n_2^2}{3} - \frac{1}{2}\right) - \frac{n_1}{3}\left(\frac{n_1^4}{20} - \frac{n_1^2}{4} + \frac{1}{5}\right).
\end{aligned}$$

□

Next result shows the Wiener index of the strong product of any two cycles.

Observe that a cycle $\mathcal{C}$ of order n has diameter $D(\mathcal{C}) = \left\lfloor \dfrac{n}{2} \right\rfloor$ and Wiener index

$$W(\mathcal{C}) = \begin{cases} \dfrac{n^3}{8}, & \text{if } n \text{ is even.} \\ \dfrac{n^3 - n}{8}, & \text{if } n \text{ is odd.} \end{cases}$$

If n is odd, it is easy to check that $\mathcal{C}$ admits $2n$ ordered pairs of vertices at distance i, for $i = 1, \ldots, D(\mathcal{C})$. If n is even, then $\mathcal{C}$ admits $2n$ ordered pairs of vertices at distance i, for $i = 1, \ldots, D(\mathcal{C}) - 1$ and n ordered pairs of vertices at distance $D(\mathcal{C})$.

Corollary 5.3.2. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two cycles with n_1, n_2 vertices, respectively, such that $3 \leq n_1 \leq n_2$. Then*

$$W\left(\mathcal{C}_1 \boxtimes \mathcal{C}_2\right) = \begin{cases} \dfrac{n_1^2 n_2}{2}\left(\dfrac{n_1^2}{12} + \dfrac{n_2^2}{4} + \dfrac{1}{6}\right), & \textit{if } n_1 \textit{ and } n_2 \textit{ are even.} \\ \dfrac{n_1^2 n_2}{2}\left(\dfrac{n_1^2}{12} + \dfrac{n_2^2}{4} - \dfrac{1}{3}\right), & \textit{if } n_1 \textit{ and } n_2 \textit{ are odd.} \\ \dfrac{n_1^2 n_2}{2}\left(\dfrac{n_1^2}{12} + \dfrac{n_2^2}{4} - \dfrac{1}{12}\right), & \textit{otherwise.} \end{cases}$$

Proof. By the hypothesis, $D(\mathcal{C}_1) \leq D(\mathcal{C}_2)$. Then we use equality (ii) of Theorem 5.2.1 taking $G_1 = \mathcal{C}_1$ and $G_2 = \mathcal{C}_2$. First, observe that for graph $\mathcal{C}_1$, we have $r_i = 2n_1$, for $i = 1, \ldots, D(\mathcal{C}_1) - 1$ and

$$r_{D(\mathcal{C}_1)} = \begin{cases} n_1, & \text{if } n_1 \text{ is even.} \\ 2n_1, & \text{if } n_1 \text{ is odd.} \end{cases}$$

Analogously, for $\mathcal{C}_2$, we have $s_j = 2n_2$, for $j = 1, \ldots, D(\mathcal{C}_2) - 1$ and

$$s_{D(\mathcal{C}_2)} = \begin{cases} n_2, & \text{if } n_2 \text{ is even.} \\ 2n_2, & \text{if } n_2 \text{ is odd.} \end{cases}$$

We must distinguish two cases depending on the parity of the number of vertices of the shortest cycle $\mathcal{C}_1$.

Case 1. Suppose that n_1 is even. Then $D(\mathcal{C}_1) = \frac{n_1}{2}$, $W_1 = \frac{n_1^3}{8}$ and $r_{D(\mathcal{C}_1)} = n_1$. By applying equality (ii) of Theorem 5.2.1, we get

$$\begin{aligned} W\left(\mathcal{C}_1 \boxtimes \mathcal{C}_2\right) &= \frac{n_1^3}{8}n_2 + \left\lfloor \frac{n_2^2}{8} \right\rfloor n_1^2 n_2 + \frac{1}{2}\sum_{j=1}^{\frac{n_1}{2}-1} 2n_2 \left(\left(\frac{n_1}{2} - j\right) n_1 + \sum_{i=j+1}^{\frac{n_1}{2}-1} (i-j)2n_1 \right) \\ &= \frac{n_1^3}{8}n_2 + \left\lfloor \frac{n_2^2}{8} \right\rfloor n_1^2 n_2 + \frac{n_1^2 n_2}{4}\left(\frac{n_1^2}{6} - \frac{n_1}{2} + \frac{1}{3}\right) \\ &= \begin{cases} \dfrac{n_1^2 n_2}{2}\left(\dfrac{n_1^2}{12} + \dfrac{n_2^2}{4} + \dfrac{1}{6}\right), & \text{if } n_2 \text{ is even.} \\ \dfrac{n_1^2 n_2}{2}\left(\dfrac{n_1^2}{12} + \dfrac{n_2^2}{4} - \dfrac{1}{12}\right), & \text{if } n_2 \text{ is odd.} \end{cases} \end{aligned}$$

Case 2. Assume that n_1 is odd. Then $D(\mathcal{C}_1) = \frac{n_1 - 1}{2}$ and $W_1 = \frac{n_1^3 - n_1}{8}$. Similarly to the previous case, we obtain

$$\begin{aligned} W\left(\mathcal{C}_1 \boxtimes \mathcal{C}_2\right) &= \frac{n_1^3 - n_1}{8}n_2 + \left\lfloor \frac{n_2^2}{8} \right\rfloor n_1^2 n_2 + \frac{1}{2}\sum_{j=1}^{\frac{n_1-3}{2}} 2n_2 \left(\sum_{i=j+1}^{\frac{n_1-1}{2}} (i-j)2n_1 \right) \\ &= \frac{n_1^3 - n_1}{8}n_2 + \left\lfloor \frac{n_2^2}{8} \right\rfloor n_1^2 n_2 + \frac{n_1^2 n_2}{8}\left(\frac{n_1^3}{3} - n_1^2 - \frac{n_1}{3} + 1\right) \\ &= \begin{cases} \dfrac{n_1^2 n_2}{2}\left(\dfrac{n_1^2}{12} + \dfrac{n_2^2}{4} - \dfrac{1}{12}\right), & \text{if } n_2 \text{ is even.} \\ \dfrac{n_1^2 n_2}{2}\left(\dfrac{n_1^2}{12} + \dfrac{n_2^2}{4} - \dfrac{1}{3}\right), & \text{if } n_2 \text{ is odd.} \end{cases} \end{aligned}$$

□

Last, we finish with the Wiener index of the strong product of any path and any cycle.

Corollary 5.3.3. *Let $\mathcal{P}, \mathcal{C}$ be a path and a cycle with n_1 and n_2 vertices, respectively, such that $n_1, n_2 \geq 3$. The following assertions hold:*

(i) If $n_1 \leq \left\lfloor \dfrac{n_2}{2} \right\rfloor + 1$ *then*

$$W(\mathcal{P} \boxtimes \mathcal{C}) = \begin{cases} \dfrac{n_1^2 n_2}{4}\left(\dfrac{n_1^2}{3} + \dfrac{n_2^2}{2} - \dfrac{1}{3}\right), & \textit{if } n_2 \textit{ is even.} \\ \dfrac{n_1^2 n_2}{4}\left(\dfrac{n_1^2}{3} + \dfrac{n_2^2}{2} - \dfrac{5}{6}\right), & \textit{if } n_2 \textit{ is odd.} \end{cases}$$

(ii) If $n_1 > \left\lfloor \dfrac{n_2}{2} \right\rfloor + 1$ *then*

$$W(\mathcal{C} \boxtimes \mathcal{P}) = \begin{cases} \dfrac{n_2^2 n_1}{6}\left(n_1^2 + \dfrac{n_2^2}{4} - \dfrac{1}{2}\right) - \dfrac{n_2^3}{48}\left(\dfrac{n_2^2}{4} - 1\right), & \textit{if } n_2 \textit{ is even.} \\ \dfrac{n_2^2 n_1}{6}\left(n_1^2 + \dfrac{n_2^2}{4} - \dfrac{5}{4}\right) - \dfrac{n_2^3}{48}\left(\dfrac{n_2}{4} - \dfrac{5}{2}\right) - \dfrac{3n_2}{64}, & \textit{if } n_2 \textit{ is odd.} \end{cases}$$

Proof. As $D(\mathcal{P}) = n_1 - 1$ and $D(\mathcal{C}) = \left\lfloor \dfrac{n_2}{2} \right\rfloor$, it follows that $D(\mathcal{P}) \leq D(\mathcal{C})$ if and only if $n_1 \leq \left\lfloor \dfrac{n_2}{2} \right\rfloor + 1$. Thus, two cases need to be distinguished.

Case 1. Suppose that $n_1 \leq \left\lfloor \dfrac{n_2}{2} \right\rfloor + 1$. Then $D(\mathcal{P}) \leq D(\mathcal{C})$ and we apply equality (ii) of Theorem 5.2.1 with $G_1 = \mathcal{P}$ and $G_2 = \mathcal{C}$.

We know that $r_i = 2n_1 - 2i$ holds in $\mathcal{P}$, for $i = 1, \ldots, n_1 - 1$ and $s_j = 2n_2$ holds in $\mathcal{C}$, for $j = 1, \ldots, \left\lfloor \dfrac{n_2}{2} \right\rfloor - 1$. Then, we deduce that

$$\begin{aligned} W(\mathcal{P} \boxtimes \mathcal{C}) &= \frac{n_1^3 - n_1}{6} n_2 + \left\lfloor \frac{n_2^2}{8} \right\rfloor n_1^2 n_2 + \frac{1}{2} \sum_{j=1}^{n_1-2} 2n_2 \left(\sum_{i=j+1}^{n_1-1} (i-j)(2n_1 - 2i) \right) \\ &= \frac{n_1^3 - n_1}{6} n_2 + \left\lfloor \frac{n_2^2}{8} \right\rfloor n_1^2 n_2 + \frac{n_1 n_2}{6}\left(\frac{n_1^3}{2} - n_1^2 - \frac{n_1}{2} + 1 \right) \\ &= \begin{cases} \dfrac{n_1^2 n_2}{4}\left(\dfrac{n_1^2}{3} + \dfrac{n_2^2}{2} - \dfrac{1}{3}\right), & \text{if } n_2 \text{ is even.} \\ \dfrac{n_1^2 n_2}{4}\left(\dfrac{n_1^2}{3} + \dfrac{n_2^2}{2} - \dfrac{5}{6}\right), & \text{if } n_2 \text{ is odd.} \end{cases} \end{aligned}$$

obtaining the equality (i).

Case 2. Suppose that $n_1 > \left\lfloor \dfrac{n_2}{2} \right\rfloor + 1$. Then $D(\mathcal{C}) < D(\mathcal{P})$ and we use the equality (ii) of Theorem 5.2.1 with $G_1 = \mathcal{C}$ and $G_2 = \mathcal{P}$. Now, we have $s_j = 2n_1 - 2j$ in $\mathcal{P}$, for $j \in \{1, \ldots, n_1 - 1\}$ and we have $r_i = 2n_2$ in $\mathcal{C}$, for $i \in \{1, \ldots, D(\mathcal{C}) - 1\}$ and $r_{D(\mathcal{C})} = \begin{cases} n_2, & \text{if } n_2 \text{ even.} \\ 2n_2, & \text{if } n_2 \text{ odd.} \end{cases}$

If n_2 is even, then

$$\begin{aligned} W(\mathcal{C} \boxtimes \mathcal{P}) &= \frac{n_2^3}{8} n_1 + \frac{n_1^3 - n_1}{6} n_2^2 \\ &\quad + \frac{1}{2} \sum_{j=1}^{\frac{n_2-2}{2}} (2n_1 - 2j) \left(\left(\frac{n_2}{2} - j \right) n_2 + \sum_{i=j+1}^{\frac{n_2-2}{2}} (i-j) 2n_2 \right) \\ &= \frac{n_2^3}{8} n_1 + \frac{n_1^3 - n_1}{6} n_2^2 + \frac{n_1 n_2^2}{4} \left(\frac{n_2^2}{6} - \frac{n_2}{2} + \frac{1}{3} \right) - \frac{n_2^3}{16} \left(\frac{n_2^2}{12} - \frac{1}{3} \right) \\ &= \frac{n_2^2 n_1}{6} \left(n^2 + \frac{n_2^2}{4} - \frac{1}{2} \right) - \frac{n_2^3}{48} \left(\frac{n_2^2}{4} - 1 \right). \end{aligned}$$

If n_2 is odd, then

$$\begin{aligned} W(\mathcal{C} \boxtimes \mathcal{P}) &= \frac{n_2^3 - n_2}{8} n_1 + \frac{n_1^3 - n_1}{6} n_2^2 + \frac{1}{2} \sum_{j=1}^{\frac{n_2-3}{2}} (2n_1 - 2j) \left(\sum_{i=j+1}^{\frac{n_2-1}{2}} (i-j) 2n_2 \right) \\ &= \frac{n_2^3 - n_2}{8} n_1 + \frac{n_1^3 - n_1}{6} n_2^2 + \frac{n_1 n_2^4}{24} - \frac{n_1 n_2^3}{8} - \frac{n_1 n_2^2}{24} \\ &\quad + \frac{n_1 n_2}{8} - \frac{n_2^5}{192} + \frac{5 n_2^3}{96} - \frac{3 n_2}{64} \\ &= \frac{n_2^2 n_1}{6} \left(n_1^2 + \frac{n_2^2}{4} - \frac{5}{4} \right) - \frac{n_2^3}{48} \left(\frac{n_2}{4} - \frac{5}{2} \right) - \frac{3 n_2}{64}. \end{aligned}$$

This proves (ii) and the result. $\square$

5.4 An upper bound for $W(G_1 \boxtimes G_2)$

Entringer, Jackson and Snyder in [40] proved that $W(G) \leq \dfrac{n^3 - n}{6}$, for any connected graph G on n vertices, with equality if and only if G is a path on n vertices. This upper bound comes from inequality $W(G) \leq W(T)$, where T is a spanning tree of G, with equality if and only if $G = T$. In this subsection, this upper bound is extended to the strong product graph.

It is clear that if G is a connected graph and $H \subseteq G$ is a subgraph of G such that $|V(G)| = |V(H)|$, then $W(G) \leq W(H)$. Hence,

$$W(G_1 \boxtimes G_2) \leq W(G_1 \Box G_2) = |V(G_2)|^2 W(G_1) + |V(G_1)|^2 W(G_2).$$

In this section we provide a sharp upper bound. We first prove that $W(G_1 \boxtimes G_2) \leq W(\mathcal{P}_1 \boxtimes G_2)$, for any two connected graphs G_1 and G_2 on n_1 and n_2 vertices, respectively, and where $\mathcal{P}_1$ is the path with the same order as G_1. Then, as consequence, the upper bound $W(G_1 \boxtimes G_2) \leq W(\mathcal{P}_1 \boxtimes \mathcal{P}_2)$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the paths on n_1 and n_2 vertices, respectively, clearly holds.

Theorem 5.4.1. *Let G_k be a connected graph with order n_k, for $k = 1, 2$. Let $\mathcal{P}_1$ be the path on n_1 vertices. Then*

$$W(G_1 \boxtimes G_2) \leq W(\mathcal{P}_1 \boxtimes G_2),$$

with equality if and only if G_1 is the path $\mathcal{P}_1$.

Proof. Let G_1 and G_2 any two connected graphs with n_1 and n_2 vertices, respectively. Notice that it is sufficient to prove that $W(T_1 \boxtimes G_2) \leq W(\mathcal{P}_1 \boxtimes G_2)$, where T_1 is a spanning tree of G_1, due to $T_1 \boxtimes G_2 \subseteq G_1 \boxtimes G_2$ and hence, $W(G_1 \boxtimes G_2) \leq W(T_1 \boxtimes G_2)$.

Without loss of generality, we may suppose that $n_1 \geq 3$, since if $n_1 = 1$ or $n_1 = 2$, then $T_1 = K_1$ or $T_1 = K_2$ and the result directly holds.

We apply induction on n_1. If $n_1 = 3$, then T_1 is exactly a path on 3 vertices, yielding the upper bound. Assume that the result holds for $n_1 - 1$. That is, suppose that $W(T_1' \boxtimes G_2) \leq W(\mathcal{P}'_1 \boxtimes G_2)$, being T_1' and $\mathcal{P}'_1$ a tree and the path on $n_1 - 1$ vertices. We will prove it for n_1.

Let us denote by $x = (x_1, x_2)$ and $y = (y_1, y_2)$ any two vertices in $V(T_1 \boxtimes G_2)$. By (5.5), we have

$$W(T_1 \boxtimes G_2) = \frac{1}{2} \sum_{x,y \in V(T_1 \boxtimes G_2)} d_{T_1 \boxtimes G_2}(x, y) = \frac{1}{2} \sum_{i=0}^{D(T_1)} \sum_{j=0}^{D(G_2)} d_{ij} r_i s_j,$$

where the sum is taken through all ordered pairs of vertices of $T_1 \boxtimes G_2$, r_i and s_j are the cardinalities of the sets R_i and S_j and $d_{ij} = \max\{i, j\}$ are the distances of matrix D defined in Section 5.1, for $0 \leq i \leq D(T_1)$ and $0 \leq j \leq D(G_2)$.

Let u be a leaf of T_1 and let G_2^u the copy of G_2 in $T_1 \boxtimes G_2$ corresponding to vertex $u \in V(T_1)$. Let us consider a partition of the set of ordered pairs of vertices of $T_1 \boxtimes G_2$ into the following two sets:

$$A = \{x, y \in V(T_1 \boxtimes G_2) : x_1 \neq u \text{ and } y_1 \neq u\}$$

and

$$B = \{x, y \in V(T_1 \boxtimes G_2) : x_1 = u \text{ or } y_1 = u\}.$$

Namely, on the one hand, A is the set of ordered pairs of vertices of $T_1 \boxtimes G_2$ such that $x \notin G_2^u$ and $y \notin G_2^u$. On the other hand, B is the set of ordered pairs of vertices of $T_1 \boxtimes G_2$ such that $x \in G_2^u$ or $y \in G_2^u$ or $x, y \in G_2^u$.

Hence,

$$\begin{aligned} W(T_1 \boxtimes G_2) &= \frac{1}{2} \sum_{x,y \in V(T_1 \boxtimes G_2)} d_{T_1 \boxtimes G_2}(x, y) \\ &= \frac{1}{2} \sum_{x,y \in A} d_{T_1 \boxtimes G_2}(x, y) + \frac{1}{2} \sum_{x,y \in B} d_{T_1 \boxtimes G_2}(x, y). \end{aligned}$$

First, we bound $\frac{1}{2}\sum_{x,y\in A} d_{T_1\boxtimes G_2}(x,y)$. Let us denote by $T_1' = T_1 - \{u\}$. Since u is a leaf of T_1, observe that $d_{T_1}(x_1,y_1) = d_{T_1'}(x_1,y_1)$, for any two vertices x_1, y_1 of T_1'. Then for any two vertices $x, y \in V(T_1 \boxtimes G_2)$ we deduce that

$$\begin{aligned} d_{T_1\boxtimes G_2}(x,y) &= \max\{d_{T_1}(x_1,y_1), d_{G_2}(x_2,y_2)\} \\ &= \max\{d_{T_1'}(x_1,y_1), d_{G_2}(x_2,y_2)\} = d_{T_1'\boxtimes G_2}(x,y). \end{aligned}$$

Hence, applying the induction hypothesis, it directly follows that

$$\begin{aligned} \frac{1}{2}\sum_{x,y\in A} d_{T_1\boxtimes G_2}(x,y) &= \frac{1}{2}\sum_{x,y\in V(T_1'\boxtimes G_2)} d_{T_1'\boxtimes G_2}(x,y) \\ &\le \frac{1}{2}\sum_{x,y\in V(\mathcal{P}_1'\boxtimes G_2)} d_{\mathcal{P}_1'\boxtimes G_2}(x,y). \end{aligned} \tag{5.9}$$

Second, we estimate $\frac{1}{2}\sum_{x,y\in B} d_{T_1\boxtimes G_2}(x,y)$. Denote by $e_u = ecc_{T_1}(u)$ the eccentricity of vertex u in $V(T_1)$. Let us define the next subsets of ordered pairs of vertices of the sets R_i,

$$R_i^u = \{(x_1,y_1) \in R_i : x_1 = u \text{ or } y_1 = u\},$$

for every $i = 0,\ldots,e_u$, due to $R_i^u = \emptyset$, for $i = e_u+1,\ldots,D(T_1)$. The cardinalities of these subsets are denoted by $r_i^u = |R_i^u|$. Observe that $r_0^u = 1$ and also that $r_i^u = 2|N_{T_1}^i(u)|$, for every $i = 1,\ldots,e_u$. Moreover, $\sum_{i=1}^{e_u} r_i^u = 2(n_1-1)$.

Hence, by (5.5), we get

$$\frac{1}{2}\sum_{x,y\in B} d_{T_1\boxtimes G_2}(x,y) = \frac{1}{2}\sum_{i=0}^{e_u}\sum_{j=0}^{D(G_2)} d_{ij} r_i^u s_j.$$

Due to $d_{ij} = \max\{i,j\}$, it follows that

$$\begin{aligned}
\frac{1}{2}\sum_{x,y\in B} d_{T_1\boxtimes G_2}(x,y) &= \frac{1}{2}\sum_{i=0}^{e_u} r_i^u \left(\sum_{j=0}^{D(G_2)} d_{ij}s_j\right)\\
&= \frac{1}{2}\sum_{i=0}^{e_u} r_i^u \left(\sum_{j=0}^{i} is_j + \sum_{j=1}^{D(G_2)} js_j - \sum_{j=0}^{i} js_j\right)\\
&= \sum_{i=0}^{e_u} r_i^u \left(\frac{1}{2}\sum_{j=0}^{i}(i-j)s_j + W(G_2)\right),
\end{aligned}$$

where $f_i = \dfrac{1}{2}\displaystyle\sum_{j=0}^{i}(i-j)s_j + W(G_2)$ is a increasing function on i thanks to the cardinalities s_j are positive for all $j = 0,\ldots,D(G_2)$. Hence,

$$\frac{1}{2}\sum_{x,y\in B} d_{T_1\boxtimes G_2}(x,y) = \sum_{i=0}^{e_u} r_i^u f_i.$$

Now, we distinguish two cases depending on the eccentricity of u in T_1. Suppose that $e_u = n_1 - 1$. Then $r_i^u = 2$, for all $i = 0,\ldots,n_1-1$ and therefore, $T_1 = \mathcal{P}_1$, which finishes the proof.

Assume that $e_u < n_1 - 1$. Then $T_1 \neq \mathcal{P}_1$ and hence, there exists $i_0 \in \{0,\ldots,e_u\}$ such that $|N_{T_1}^{i_0}(u)| \geq 2$. Indeed, at least one vertex of $N_{T_1}^{i_0}(u)$ is a leaf. In such case, we make the following transformation on the tree T_1.

Between all the possible vertices of $N_{T_1}^{i_0}(u)$, we delete a leaf and add it to a vertex at distance e_u from u in T_1. For instance, in Figure 5.3 we illustrate a possible case, where for $i = 3$, there are three vertices, and two of them are leaves. Then we delete vertex v (the red vertex) and we add it to a vertex at distance e_u from u (the blue vertex).

By this way, if we represent the cardinalities $(r_i^u)_{i=1}^{e_u}$ in a list of length $n_1 - 1$, we have initially

$$(r_1^u, .., r_{i_0}^u, .., r_{e_u}^u, 0, \ldots, 0)$$

Figure 5.3: Transformation in T_1.

and after the transformation we get

$$(r_1^u, .., r_{i_0}^u - 2, .., r_{e_u}^u, 2, 0, \ldots, 0).$$

Notice that both lists have the same length $n_1 - 1$ and the same sum of elements, $2(n_1 - 1)$.

Moreover, due to the function f_i is an increasing function on i, we have

$$\sum_{i=1}^{e_u} r_i^u f_i < \sum_{i=1}^{i_0-1} r_i^u f_i + (r_{i_0}^u - 2) f_{i_0} + \sum_{i=i_0+1}^{e_u} r_i^u f_i + 2 f_{e_u+1}. \tag{5.10}$$

Using iteratively the previous transformation, from the sequence

$$(r_0^u, r_1^u, .., r_{i_0}^u, .., r_{e_u}^u, 0, \ldots, 0)$$

we obtain the sequence

$$(1, 2, \ldots, 2, 2, \ldots, 2)$$

which is, in fact, the sequence of neighborhoods of u in a path $\mathcal{P}_1$ of n_1 vertices, being u one of the two end vertices of $\mathcal{P}_1$.

Hence, by (5.10), we deduce that

$$\begin{aligned}
\frac{1}{2}\sum_{x,y\in B} d_{T_1\boxtimes G_2}(x,y) &= \sum_{i=0}^{e_u} r_i^u f_i \\
&= r_0^u f_0 + \sum_{i=1}^{e_u} r_i^u f_i \\
&= f_0 + \sum_{i=1}^{e_u} r_i^u f_i \qquad (5.11)\\
&< f_0 + \sum_{i=1}^{n_1-1} 2 f_i \\
&= \frac{1}{2}\sum_{(x_1=u)\vee(y_1=u)} d_{\mathcal{P}_1\boxtimes G_2}(x,y),
\end{aligned}$$

being u one of the two end vertices of $\mathcal{P}_1$.

Finally, from (5.9) and (5.11), it follows that

$$\begin{aligned}
W(T_1\boxtimes G_2) &= \frac{1}{2}\sum_{x,y\in V(T_1\boxtimes G_2)} d_{T_1\boxtimes G_2}(x,y) \\
&= \frac{1}{2}\sum_{x,y\in A} d_{T_1\boxtimes G_2}(x,y) + \frac{1}{2}\sum_{x,y\in B} d_{T_1\boxtimes G_2}(x,y) \\
&\leq \frac{1}{2}\sum_{x,y\in V(\mathcal{P}_1'\boxtimes G_2)} d_{\mathcal{P}_1'\boxtimes G_2}(x,y) + \frac{1}{2}\sum_{(x_1=u)\vee(y_1=u)} d_{\mathcal{P}_1\boxtimes G_2}(x,y) \\
&= \frac{1}{2}\sum_{x,y\in V(\mathcal{P}_1\boxtimes G_2)} d_{\mathcal{P}_1\boxtimes G_2}(x,y) \\
&= W(\mathcal{P}_1\boxtimes G_2).
\end{aligned}$$

with equality if and only if $T_1=\mathcal{P}_1$, which finishes the proof. □

To end this subsection, we present the following corollary which directly comes from Theorem 5.4.1, thanks to the commutativity of the strong product graph and whose proof is straightforward.

Corollary 5.4.1. *Let G_k be a connected graph with order n_k, for $k=\{1,2\}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two paths on n_1 and n_2 vertices, respectively. Then*

$$W(G_1\boxtimes G_2)\leq W(\mathcal{P}_1\boxtimes\mathcal{P}_2),$$

with equality if and only if $G_1 = \mathcal{P}_1$ *and* $G_2 = \mathcal{P}_2$.

Notice also that if we suppose that G_1 and G_2 are two connected graphs with order n_1 and n_2, respectively, such that $2 \leq n_1 \leq n_2$, then Corollary 5.3.1 and Corollary 5.4.1, lead us to get the upper bound

$$W(G_1 \boxtimes G_2) \leq \frac{n_1^2 n_2}{2}\left(\frac{n_1^2}{6} + \frac{n_2^2}{3} - \frac{1}{2}\right) - \frac{n_1}{3}\left(\frac{n_1^4}{20} - \frac{n_1^2}{4} + \frac{1}{5}\right).$$

5.5 Hyper-Wiener index of the strong product graph

This last section is devoted to present some results on the *hyper-Wiener index* of the strong product graph. It was introduced by Randié [85] in 1993 with the aim to search new optimal molecular descriptors or invariants of a relatively simple structural interpretation which can describe molecular properties of interest. For any connected graph G, it is denoted by $WW(G)$ and defined as

$$WW(G) = \frac{1}{4}\sum_{x,y\in V(G)} \left(d_G(x,y) + d_G^2(x,y)\right),$$

where the sum is taken through all the ordered pairs of vertices of G. Hence, it is clear that the Wiener and the hyper-Wiener indices are directly related by the equality

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{4}\sum_{x,y\in V(G)} d_G^2(x,y).$$

Similar bounds those obtained in [40] for the Wiener index have been deduced for the hyper-Wiener index. For any connected graph G on n vertices, Gutman et al. in [48] proved that

$$\frac{n(n-1)}{2} \leq WW(G) \leq \frac{n^4 + 2n^3 - n^2 - 2n}{24},$$

which comes from the fact that $WW(K) \leq WW(G) \leq WW(\mathcal{P})$, where K and $\mathcal{P}$ are the complete graph and the path on n vertices, respectively.

Since the hyper-Wiener index has a high correlation to the Wiener index, it is possible to establish analogous results for $WW(G_1 \boxtimes G_2)$ to the obtained ones for $W(G_1 \boxtimes G_2)$, as we show below. Thus, the proofs are omitted.

Continuing with similar notations, we use WW_1 and WW_2 to denote the hyper-Wiener index of G_1 and G_2, respectively. That is,

$$WW_1 = \frac{1}{4} \sum_{x_1,y_1 \in V(G_1)} \left(d_{G_1}(x_1, y_1) + d^2_{G_1}(x_1, y_1) \right) = \frac{1}{4} \sum_{i=1}^{D(G_1)} (i + i^2) r_i, \tag{5.12}$$

and

$$WW_2 = \frac{1}{4} \sum_{x_2,y_2 \in V(G_2)} \left(d_{G_2}(x_2, y_2) + d^2_{G_2}(x_2, y_2) \right) = \frac{1}{4} \sum_{j=1}^{D(G_2)} (j + j^2) s_j, \tag{5.13}$$

where both sums are taken through all the ordered pairs of vertices of G_1 and G_2, respectively.

Theorem 5.5.1. *Let G_k be a connected graph with order n_k, Wiener index and hyper-Wiener index W_k and WW_k, for $k = 1, 2$, such that $D(G_1) \leq D(G_2)$. Let r_i and s_j be defined as above. The following assertions hold:*

(i) If $D(G_1) = 1$ then

$$WW(G_1 \boxtimes G_2) = n_2 WW_1 + n_1^2 WW_2.$$

(ii) If $2 \leq D(G_1) \leq D(G_2)$ then

$$WW(G_1 \boxtimes G_2) = n_2 WW_1 + n_1^2 WW_2 + \frac{1}{4} \sum_{j=1}^{D(G_1)-1} \sum_{i=j+1}^{D(G_1)} (i - j + i^2 - j^2) r_i s_j.$$

Theorem 5.5.2. *Let G_k be a connected graph with order n_k, size e_k, minimum degree δ_k and hyper-Wiener index WW_k, for $k = 1, 2$. Let $2 \leq D(G_1) \leq D(G_2)$. The following assertions hold:*

(i) If $D(G_1) \geq 2$ *then*

$$WW(G_1 \boxtimes G_2) \geq n_2 WW_1 + n_1^2 WW_2 + e_2\left(2WW_1 - n_1^2 + n_1\right),$$

with equality if and only if $D(G_1) = 2$*.*

(ii) If $D(G_1) \geq 3$ *then*

$$\begin{aligned} WW\left(G_1 \boxtimes G_2\right) \ &\geq \ n_2 WW_1 + n_1^2 WW_2 + e_2\left(2WW_1 - n_1^2 + n_1\right) \\ &\quad + s_2\left(WW_1 - \frac{3}{2}(n_1^2 - n_1) + 2e_1\right), \end{aligned}$$

with equality if and only if $D(G_1) = 3$*.*

(iii) If $D(G_1) \geq 3$ *and* G_2 *has minimum degree at least* 2 *and girth at least* 5*, then*

$$\begin{aligned} WW\left(G_1 \boxtimes G_2\right) \ &\geq \ n_2 WW_1 + n_1^2 WW_2 + 2\delta_2 e_2 WW_1 \\ &\quad + 4e_1 e_2(\delta_2 - 1) - e_2(3\delta_2 - 2)(n_1^2 - n_1), \end{aligned}$$

with equality if and only if $D(G_1) = 3$ *and* G_2 *is regular.*

Finally, we present an upper bound for the hyper-Wiener index $WW(G_1 \boxtimes G_2)$ of the strong product of two connected graphs.

Theorem 5.5.3. *Let* G_k *be a connected graph with order* n_k*, for* $k = 1, 2$*. Let* $\mathcal{P}_1$ *be the path on* n_1 *vertices. Then*

$$WW(G_1 \boxtimes G_2) \leq WW(\mathcal{P}_1 \boxtimes G_2)$$

with equality if and only if G_1 *is the path* $\mathcal{P}_1$*.*

Thanks to the commutativity of the strong product graph we present the next corollary which directly comes from Theorem 5.5.3.

Corollary 5.5.1. *Let* G_k *be a connected graph with order* n_k*, for* $k = \{1, 2\}$*. Let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *be the paths on* n_1 *and* n_2 *vertices, respectively. Then*

$$WW(G_1 \boxtimes G_2) \leq WW(\mathcal{P}_1 \boxtimes \mathcal{P}_2)$$

with equality if and only if $G_1 = \mathcal{P}_1$ *and* $G_2 = \mathcal{P}_2$*.*

Chapter 6

Conclusions and Open problems

This chapter is devoted to summarize the main results of this thesis, as well as to propose some open problems whose solution would be an advance in the research of vulnerability parameters in graphs products.

6.1 Superconnectivity (Chapter 2)

In *Chapter* 2 we have focused on the connectivity and superconnectivity in the strong product of two connected graphs. We have obtained the bound

$$\min\{n_1\kappa_2, \kappa_1 n_2, \max\{\delta_1\kappa_2 + \delta_1 + \kappa_2, \kappa_1\delta_2 + \kappa_1 + \delta_2\}\} \le \kappa(G) \le \delta(G),$$

being $G = G_1 \boxtimes G_2$ and n_i, δ_i, κ_i the order, the minimum degree and the connectivity of G_i, for $i = 1, 2$.

Then sufficient conditions for this family to be maximally connected have been given in Theorem 2.2.2. Namely, we have proven that if G_1 and G_2 are two connected graphs with at least 3 vertices and such that at least one of them has girth at least 4, then $G_1 \boxtimes G_2$ is maximally connected if both G_1 and G_2 are maximally connected and either one graph has minimum degree 1 and the other one has girth at least 5 or the minimum degree is at least 2 in both graphs.

In addition, we have proven that the strong product of two maximally connected graphs with girth at least 5 and minimum degree at least 2 is super-connected.

After this brief summary, some possible lines of future work are:

- *On the restricted connectivity of the strong product graph.*

  The connectivity is a very extensively studied parameter, as we saw in Introduction, and it has given rise to many others connectivity-type indices last years. Two of them are the *restricted connectivity* and the *restricted edge-connectivity*, denoted by κ' and λ', respectively. There exist some works which deal with these parameters in graphs products [12, 13, 82]. Our proposal is to find sufficient conditions for the strong product of two connected graphs to be λ'-optimal, that is, $\lambda'(G_1 \boxtimes G_2) = \xi(G_1 \boxtimes G_2)$, where $\xi(G_1 \boxtimes G_2)$ is the minimum edge degree.

- *On the toughness of the strong product graph.*

  Another interesting parameter of the resilience in graphs is the socalled *toughness* which pays special attention to the relationship between the cardinality of a cut set in a graph and the number of components after deletion. For any connected graph G, this parameter is denoted by $\tau(G)$ and defined as

  $$\tau(G) = \min\left\{|S|/\omega(G-S) : S \subseteq J(G)\right\},$$

  where $J(G) = \{S \subset V(G) : S$ is a cutset of G or $G{-}S$ is an isolated vertex$\}$, and $\omega(G{-}S)$ denotes the number of components in the resultant graph $G{-}S$ by removing S. We have started the study of the toughness in some graphs products as the corona and the cartesian product of graphs. For this last family we have proved the following result:

Theorem 6.1.1. *Let G be a connected graph with minimum degree δ and independence number β. Then*

$$\min\left\{\tau(G), \frac{|V(G)|}{1+\beta}\right\} \leq \tau(K_2 \Box G) \leq \min\left\{2\tau(G), \frac{\delta+1}{2}\right\}.$$

We consider interesting to generalize this result to the cartesian product of two arbitrary connected graphs.

6.2 Connectivity and Distances (Chapter 3)

In *Chapter* 3 sharp lower bounds on the average connectivity, the Menger number and the average Menger number of the strong product family have been given.

For the average connectivity of the strong product graph, $\overline{\kappa}\,(G_1 \boxtimes G_2)$, in Theorem 3.4.1 we have shown that if G_1 and G_2 are two connected graphs, with orders $n_1, n_2 \geq 3$ and girth at least 5, then

$$\begin{aligned}\overline{\kappa}\,(G_1 \boxtimes G_2) \geq \quad & \frac{1}{n_1 n_2 - 1}\left[(n_1-1)(n_2+\overline{d}(G_2))\overline{\kappa}(G_1) + (n_2-1)(n_1+\overline{d}(G_1))\overline{\kappa}(G_2)\right. \\ & \left. +(n_1-1)(n_2-1)\overline{\kappa}(G_1)\overline{\kappa}(G_2) + (n_2-1)\overline{d}(G_1) + (n_1-1)\overline{d}(G_2)\right].\end{aligned}$$

Moreover, in Corollary 3.4.2 we gave sufficient conditions to guarantee that if G_1 and G_2 are two connected graphs with at least 3 vertices, girth at least 5 and $\overline{\kappa}(G_i) = \overline{d}(G_i)$, for $i = 1, 2$, then

$$\overline{\kappa}\,(G_1 \boxtimes G_2) = \overline{d}\,(G_1 \boxtimes G_2)\,.$$

In the case of the Menger number of this family of graphs, $\zeta_\ell(G_1 \boxtimes G_2)$, we found in Theorem 3.3.1 two different lower bounds depending on the permitted lengths in the paths. For any two connected graphs, G_1 and G_2, with at least 3 vertices and $\ell \geq \max\{D(G_1), D(G_2)\}$, it follows that

(i) $\zeta_\ell(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2)$.

(ii) $\zeta_{\ell+2}(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$, if $g(G_i) \geq 5$ for $i = 1, 2$.

Similarly to the average connectivity, in Corollary 3.3.1 we proved that if G_1 and G_2 are two maximally connected graphs with at least 3 vertices, girth at least 5 and ℓ is a positive integer such that $\zeta_\ell(G_1) = \kappa(G_1)$ and $\zeta_\ell(G_2) = \kappa(G_2)$, then

$$\zeta_{\ell+2}(G_1 \boxtimes G_2) = \delta(G_1 \boxtimes G_2).$$

Finally we also provided bounds on the expected number of pairwise disjoint paths with a maximum fixed length in the strong product graph. That is, the average Menger number, $\overline{\zeta}_\ell\,(G_1 \boxtimes G_2)$. For two connected graphs, G_1 and G_2, with n_1, n_2 vertices, respectively, in Theorem 3.5.1 we have proved that

(i) If both G_1 and G_2 have order at least 3, then

$$\begin{aligned}\overline{\zeta}_\ell\,(G_1 \boxtimes G_2) &= \frac{1}{n_1 n_2 - 1}\big[(n_1-1)(1+\overline{d}(G_2))\overline{\zeta}_\ell(G_1) \\ &+ (n_2-1)(1+\overline{d}(G_1))\overline{\zeta}_\ell(G_2) + (n_1-1)(n_2-1)\overline{\zeta}_\ell(G_1)\overline{\zeta}_\ell(G_2)\big]\,.\end{aligned}$$

(ii) If both G_1 and G_2 have order at least 3 and girth at least 5, then

$$\begin{aligned}\overline{\zeta}_\ell\,(G_1 \boxtimes G_2) &\geq \frac{1}{n_1 n_2 - 1}\big[(n_1-1)(n_2+\overline{d}(G_2))\overline{\zeta}_\ell(G_1) + (n_2-1)(n_1+\overline{d}(G_1))\overline{\zeta}_\ell(G_2) \\ &+(n_1-1)(n_2-1)\overline{\zeta}_\ell(G_1)\overline{\zeta}_\ell(G_2) + (n_2-1)\overline{d}(G_1) + (n_1-1)\overline{d}(G_2)\big]\,.\end{aligned}$$

Again, we have shown that

$$\overline{\zeta}_\ell\,(G_1 \boxtimes G_2) = \overline{d}\,(G_1 \boxtimes G_2)\,,$$

when G_1 and G_2 are two connected graphs with at least 3 vertices, girth at least 5 and $\overline{\zeta}_\ell(G_i) = \overline{d}(G_i)$, for $i = 1, 2$.

To continue this line of research, some other parameters are of interest.

- *On the persistence of the strong product graph.*

  The diameter is often taken as a measure of efficiency in a network. An interesting question is how many edges at most can be removed from a graph without increasing the diameter of the remaining graph. This question is called the *edge-deletion problem.* One of parameter related to the edge-deletion problems is the *edge-persistence*, introduced by Boesch *et al.* [19] and by Exoo [41]. The edge-persistence of a connected graph G, denoted by $D^+(G)$, is the minimum number of edges whose deletion from G increases the diameter. In 2001, Xu [98] generalized this notion to the socalled *bounded edge-connectivity.* For any connected graph G, a positive integer k and $x, y \in V(G)$, the *xy-bounded edge-connectivity*, denoted by $\lambda_k(G; x, y)$ with respect to k is the minimum number of edges whose deletion from G destroys all xy-paths of length at most k. The *bounded edge-connectivity* of G with respect to k is defined as

  $$\lambda_k(G) = \min\{\lambda_k(G; x, y) : x, y \in V(G)\}.$$

  A natural question is whether the persistence (edge-persistence) is equal to the Menger number. For any connected graph G, does $\zeta_{D(G)}(G) = D^+(G)$ holds? The answer is negative. Bondy and Hell gave in [22] several counterexamples. Recently, Lu, Xu, and Hou [76] deduced bounds on the *bounded edge-connectivity* and the *edge-persistence* for the cartesian product graph. More precisely, they showed that for any two connected graphs, G_1 and G_2, and two positive integers, $k_1, k_2 \geq 2$,

  $$\lambda_{k_1+k_2}(G_1 \times G_2) \geq \lambda_{k_1}(G_1) + \lambda_{k_2}(G_2).$$

  We propose to study the bounded edge-connectivity in the strong product graph, trying to give a sharp lower bound as well as to study the particular case of the edge-persistence.

- *On the average connectivity and the Menger number in other graphs products.*

  The average connectivity and the Menger number are parameters not very extensively studied in graphs products. Thus, it may be of interest to continue the findness of bounds for some families, as the lexicographic and the Kronecker products.

6.3 Generalized 3-connectivity (Chapter 4)

In *Chapter* 4 we have focused on the generalized 3-connectivity of the strong product graph, $\kappa_3(G_1 \boxtimes G_2)$. We gave some sharp lower bounds in terms of the connectivity and generalized 3-connectivity of the generator graphs, depending on the additional constraints over G_1 and G_2. Finally, in Theorem 4.3.1 we deduced that $\kappa_3(G_1 \boxtimes G_2) \geq 2\kappa_3(G_2)$, for G_1 and G_2 two connected graphs with at least 3 vertices and girth at least 5. Moreover, this bound is sharp when $\kappa_3(G_1) = 1$.

To complete this study on the generalized 3-connectivity of the strong product graph, we describe some of the problems which are still open:

- *To obtain a sharp lower bound on $\kappa_3(G_1 \boxtimes G_2)$, for any two connected graphs.*

  From *Chapter* 4, one can suppose that the number of cases to be studied to obtain a sharp lower bound on $\kappa_3(G_1 \boxtimes G_2)$, for any two connected graphs, is high and difficult to describe. We have the following conjecture anyway.

  Conjecture 6.3.1. *For any connected graphs G_1 and G_2 with at least 3 vertices and girth at least 5, it holds*

  $$\kappa_3(G_1 \boxtimes G_2) \geq \kappa_3(G_1)\kappa_3(G_2) + \kappa_3(G_1) + \kappa_3(G_2) - 1.$$

If Conjecture 6.3.1 is true, it would follow that the generalized 3-connectivity of the strong product graph is almost determined:

$$\delta(G_1 \boxtimes G_2) - 1 \leq \kappa_3(G_1 \boxtimes G_2) \leq \delta(G_1 \boxtimes G_2)$$

for graphs G_i, with at least 3 vertices, with girth $g(G_i) \geq 5$ and such that $\kappa_3(G_i) = \delta(G_i)$, for $i \in \{1, 2\}$.

- *To deduce conditions to assure that $\kappa_3(G_1 \boxtimes G_2)$ attains its maximum value.*

  As for connectivity, it is always an interesting open problem to determine additional conditions on the factor graphs G_1 and G_2 that guarantee the equality $\kappa_3(G_1 \boxtimes G_2) = \delta(G_1 \boxtimes G_2)$.

- *On the generalized k-connectivity.*

  As far as we know, there are a lot of lines of research still open for this parameter. For instance, to study $\kappa_k(G)$, for $k \geq 4$ and G a connected graph. In particular, we consider specially interesting the case $k = |V(G)|$, that is, how connected all the vertices of a graph are.

- *On the generalized k-connectivity in graphs products.*

  It would be also interesting to approach $\kappa_k(G_1 \boxtimes G_2)$, for $k \geq 4$, as well as the generalized k-connectivity, for $k \geq 3$, in other products of graphs for which this parameter has not been studied yet.

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