Invariant means and thin sets in harmonic analysis. Applications with prime numbers.

P. Lefèvre, L. Rodríguez-Piazza*

Abstract

We first prove a localization principle characterising Lust-Piquard sets. We obtain that the union of two Lust-Piquard sets is a Lust-Piquard set, provided that one of these two sets is closed for the Bohr topology. We also show that the closure of the set of prime numbers is a Lust-Piquard set, generalizing results of Lust-Piquard and Meyer, and even that the set of integers whose expansion uses less than r factors is a Lust-Piquard set. On the other hand, we use random methods to prove that there are some sets which are UC, $\Lambda(q)$ for every $q > 2$ and p-Sidon for every $p > 1$, but which are not Lust-Piquard sets. This is a consequence of the fact that a uniformly distributed set cannot be a Lust-Piquard set.

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0 Preliminary

Introduction

In this paper, we are interested in constructing some new Lust-Piquard sets (LP sets in short) and some new non LP sets. One of the main application involves the set of prime numbers.

Let G be compact abelian group. The Haar measure m defines an invariant mean on $L^\infty(G)$, and Rudin ([17]) showed that, if G is infinite, there always exist other invariant means on $L^\infty(G)$. A function $f \in L^\infty(G)$ has a unique invariant mean if $M(f) = \int f\,dm$ for every invariant mean M on $L^\infty(G)$. Every continuous function has a unique invariant mean. The same is true for every Riemann-integrable function; but they are not the only ones. Indeed there are non Riemann-integrable functions f such that fg has an unique invariant mean, for every continuous function $g \in C(G)$ (see [18]).

Using the terminology of Harmonic Analysis, Lust-Piquard sets is a class of, what are called "thin" sets. That is, subsets Λ of the dual group $\Gamma = \widehat{G}$, where the functions whose Fourier transform is supported by Λ have a particular behaviour. Lust-Piquard sets Λ are defined by the property that, fg has a unique invariant mean, for every $g \in C(G)$, and every $f \in L^\infty(G)$ with $\widehat{f}$ supported by Λ.

Mainly we will be working in the case where $G = \mathbb{T}$ is the torus, and then its dual group is identified with the set of integers $\mathbb{Z}$. In this setting, it was proved in [15] that $\mathbb{N}$ is not a LP set. On the other hand, a (non trivial) example of LP set is given in [12] with

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the set $\mathcal{P} \cap (2+5\mathbb{Z})$, where $\mathcal{P}$ are the prime numbers. This example is non trivial in the sense that the set is not a Rosenthal set (see the definition below). Another example of a LP set, which is not Rosenthal, is given in [5]: there, it is given a set which is discrete for the Bohr topology of $\mathbb{Z}$ and also a Hilbert set. It seems that the main references on the subject before the 90's are [11], [12], [15] and [17]. In [8], Li proved that LP sets are Riesz sets (see the def. below). This result was recently extended by the authors in [6], where they prove that the union of a Riesz set and a LP set is a Riesz set. This allows to recover that $\mathbb{N}$ is not a LP set.

In the first section of this paper, we prove a general criterion for LP sets. It is an extension of the classical localization principle due to Lust-Piquard in [12]. We are led to introduce the notion of strong LP set. It has to be linked with the notion of strong Riesz set, introduced by Meyer [13] to prove that the union of a strong Riesz set and a Riesz set is a Riesz set. Note that the union of two Riesz set is not a Riesz set in general: $\mathbb{Z} = \mathbb{Z}^- \cup \mathbb{N}$. It is unknown whether the union of two LP sets is still a LP set, but we can (positively) conclude when one is a strong LP set. There are some LP sets, which are not strong LP: for instance a Rosenthal set, dense for the Bohr topology, is constructed in [6]. Actually, the (so far) known examples of LP sets are either Rosenthal sets or discrete sets, for the Bohr topology. We obtain some new examples of LP sets which are neither Rosenthal sets nor discrete. For instance, it turns out that the set of prime numbers is a strong LP set and it gives us an example of LP set of different nature, compared to the previous known examples. We extend this result, showing that the set of integers whose expansion with prime numbers uses less than r factors is a strong LP set, where $r \geq 1$ is fixed but arbitrary.

In the second section, we construct some new non LP sets. The random selector method gives some sets which are uniformly distributed (hence dense for the Bohr topology) and thin in the sense that they are $\Lambda(s)$ for every $s > 1$ and p-Sidon for every $p > 1$. We show that a uniformly distributed set cannot be a LP set. In this way, we obtain thin sets in the previous sense, which are not LP sets. We also investigate the structure of the space $C_\Lambda(\mathbb{T})$ for such a Λ.

Notations

Let us recall some notations and definitions of the classes of thin sets involved in the paper. Throughout this paper, G will be a compact abelian group (with additive law), and $\Gamma = \hat{G}$ will be its (discrete) dual group. The Haar measure of G will be denoted by m, and integration with respect to m by dt or dx. We will denote by $\mathcal{M}(G)$ the space of complex Radon measures on G. For $f \in L^1(G)$, (or even for $\mu \in \mathcal{M}(G)$) the Fourier coefficient of f (μ) at $\gamma \in \Gamma$ is $\hat{f}(\gamma) = \int_G f(t)\,\overline{\gamma(t)}\,dt$ ($\hat{\mu}(\gamma) = \int_G \overline{\gamma}\,d\mu$).

If X is a linear subspace of $\mathcal{M}(G)$, X_Λ will be the subspace of those $f \in X$ whose Fourier coefficients vanish outside of Λ. If $x \in G$, and f is a function defined on G, we denote by f_x the translate of f by x, that is the function defined by $f_x(t) = f(t-x)$, for every $t \in G$.

An *invariant mean* M on $L^\infty(G)$ is a continuous linear functional on $L^\infty(G)$ such that $M(\mathbb{1}) = \|M\| = 1$ and $M(f_x) = M(f)$, for every $f \in L^\infty(G)$ and every $x \in G$. Clearly every invariant mean is a positive functional on $L^\infty(G)$.

A subset Λ of $\Gamma = \hat{G}$ is called a *Lust-Piquard set (LP set)* if γf has a unique invariant mean for every $f \in L^\infty_\Lambda(G)$ and every $\gamma \in \Gamma$. This is equivalent to say that fg has a unique invariant mean for every $f \in L^\infty_\Lambda(G)$ and every $g \in C(G)$. In [11], Lust-Piquard called these sets *totally ergodic sets*. In [5], the new terminology is introduced and we explain there why we do so. We introduce now the concept of strong LP set.

Definitions.

We will say that a LP set is a *strong LP set* if its closure in Γ for the Bohr topology is still a LP set.

We finish this introduction recalling the definitions of Riesz sets and Rosenthal sets.

A subset Λ of Γ is called a *Riesz set* if every measure whose Fourier transform vanishes out of Λ is absolutely continuous with respect to the Haar measure. That is, if we have $\mathcal{M}_\Lambda(G) = L^1_\Lambda(G)$.

A subset Λ of Γ is called a *Rosenthal set* if every $f \in L^\infty(G)$ whose Fourier transform vanishes out of Λ is equal (a.e.) to a continuous function. That is, if we have $L^\infty_\Lambda(G) = C_\Lambda(G)$.

The classical theorem of the Riesz brothers states that the set of positive integers is a Riesz set. We have the following implications

$$\Lambda \text{ is a Rosenthal set} \implies \Lambda \text{ is a LP set} \implies \Lambda \text{ is a Riesz set.}$$

As already signalled, the second one is due to Li [8], and the first one is clear.

1 Some new LP sets

Let us recall that the Bohr topology of a discrete abelian group Γ is the topology of pointwise convergence, when Γ is seen as a subset of $C(G)$; it is also the natural topology on Γ as a subset of the dual group of G_d, the group G with the discrete topology. See [16] for more informations.

We are going to prove the following criterion, which can be viewed as a restricted localization property for the class of LP sets (see the remark below).

Theorem 1.

Let Λ be a subset of Γ. We assume that there exists a subset E of Λ, such that E is a LP set and for every $\gamma \notin E$, there exists a LP set L_γ such that $\gamma \notin \overline{\Lambda \setminus L_\gamma}$. Then Λ is a LP set.

Remark: note that the condition on $\Gamma \setminus E$ actually means that for every $\gamma \notin E$, there exists a neighborhood V_γ of γ (for the Bohr topology), such that $L_\gamma = V_\gamma \cap \Lambda$ is a LP set. The converse of Theorem 1 is obviously true (take $E = \emptyset$ and $L_\gamma = \Lambda$).

This is a generalization of the following localization criterion, implicitely contained in [12] (see [5] too): if for every $\gamma \in \Gamma$, there exists a neighborhood V_γ of γ such that $V_\gamma \cap \Lambda$ is a LP set, then Λ is a LP set. In particular, every discrete subset of Γ is a LP set.

For the proof of Theorem 1, we need the following lemma.

Lemma 1. *Let $f \in L^\infty(G)$ and M be an invariant mean.*

Then there exists some $F \in L^\infty(G)$ such that for every $\gamma \in \Gamma$, $\hat{F}(\gamma) = M(\bar{\gamma} f)$ and $\|F\|_\infty \leq \|f\|_\infty$.

Proof. Let us first suppose that G is metrizable. Fix a function $f \in L^\infty(G)$ and M an invariant mean. Let $(K_n)_{n\geq 1}$ be a polynomial approximation of the unit in $L^1(G)$:

$\|K_n\|_1 = 1$ and for every $\gamma \in \Gamma$, $\lim_n \widehat{K_n}(\gamma) = 1$. Moreover, K_n is a positive trigonometric polynomial. We shall use the notation $\check{K}_n(t) = K_n(-t)$.

Now, for every $x \in G$ and every $n \geq 1$, we have (note that the sums are actually finite) using the properties of M,

$$\Big|\sum_\Gamma \widehat{K_n}(\gamma) M(\bar{\gamma} f)\gamma(x)\Big| = \Big|M\Big(\sum_\Gamma f \widehat{K_n}(\gamma)\gamma(x)\bar{\gamma}\Big)\Big| \leq M\Big(|f|\Big|\sum_\Gamma \widehat{K_n}(\gamma)\gamma(x)\bar{\gamma}\Big|\Big).$$

It means

$$\Big|\sum_\Gamma \widehat{K_n}(\gamma) M(\bar{\gamma} f)\gamma(x)\Big| \leq M(|f|.(\check{K}_n)_{-x}).$$

Almost everywhere, we have $|f|.|(\check{K}_n)_{-x}| \leq \|f\|_\infty |(\check{K}_n)_{-x}|$. Therefore, as K_n is continuous,

$$\Big|\sum_\Gamma \widehat{K_n}(\gamma) M(\bar{\gamma} f)\gamma(x)\Big| \leq \|f\|_\infty M((\check{K}_n)_{-x}) = \|f\|_\infty M(\check{K}_n) = \|f\|_\infty \widehat{\check{K}_n}(0) \leq \|f\|_\infty.$$

Now, we have a sequence $\Big(\sum_\Gamma \widehat{K_n}(\gamma) M(\bar{\gamma} f)\gamma\Big)_{n\in\mathbb{N}}$ in $L^\infty(G)$, bounded by $\|f\|_\infty$. By weak-star compactness, there exists some $F \in L^\infty(G)$, with $\|F\|_\infty \leq \|f\|_\infty$ and such that $\hat{F}(\gamma) = \lim_n \widehat{K_n}(\gamma) M(\bar{\gamma} f) = M(\bar{\gamma} f)$.

If G is not metrizable we can use, instead of the sequence $(K_n)_{n\geq 1}$, a net $(K_\delta)_{\delta\in D}$ (D could be for instance the directed set of neighbourhoods of 0) and the proof still works. ∎

Observe that one can show that F has its spectrum contained in Λ when f has its spectrum contained in Λ and Λ is closed for the Bohr topology (see the proof below). Nevertheless, we are not going to use this.

Proof of Theorem 1.

We fix $f \in L^\infty_\Lambda(G)$ and an invariant mean M. By lemma 1, a function $F \in L^\infty(G)$ is associated.

Let $\gamma \notin E$. By hypothesis, we are given a LP set L_γ and a discrete measure σ such that $\hat{\sigma} = 0$ on $\Lambda \setminus L_\gamma$ and $\hat{\sigma}(\gamma) = 1$. Denoting the Dirac measure in x by δ_x, we may write $\sigma = \sum_n s_n \delta_{x_n}$, for some $x_n \in G$, with $\sum_n |s_n|$ finite. Note that $1 = \hat{\sigma}(\gamma) = \sum_n s_n \bar{\gamma}(x_n)$.

We have $\hat{F}(\gamma) = M(\bar{\gamma} f) = \sum_n s_n \bar{\gamma}(x_n) M((\bar{\gamma} f)_{x_n})$, because M is an invariant mean. By continuity of M, this leads to

$$\hat{F}(\gamma) = M\Big(\sum_n s_n \bar{\gamma} f_{x_n}\Big) = M(\bar{\gamma}(f * \sigma)).$$

But $f * \sigma$ lies in $L^\infty_{L_\gamma}(G)$ and L_γ is a LP set, so

$$\hat{F}(\gamma) = \widehat{f * \sigma}(\gamma) = \hat{f}(\gamma).$$

It implies that $F \in L^\infty_\Lambda(G)$. Thus, for every $\gamma \in \Lambda \setminus E$, $\hat{F}(\gamma) = \hat{f}(\gamma)$.

Now, we write $F_0 = f$. By the preceding, there exists $F_1 \in L^\infty_\Lambda(G)$ such that for every $\gamma \in \Lambda \setminus E$, $\widehat{F_1}(\gamma) = \widehat{F_0}(\gamma)$, and for every $\gamma \in E$, $\widehat{F_1}(\gamma) = M(\bar{\gamma} F_0)$. Moreover, we have $\|F_1\|_\infty \leq \|F_0\|_\infty$. By induction, we construct a sequence $(F_n)_{n\in\mathbb{N}}$ in $L^\infty_\Lambda(G)$, with

- $\|F_{n+1}\|_\infty \leq \|F_n\|_\infty$.
- For every $\gamma \in E$, $\widehat{F_{n+1}}(\gamma) = M(\bar{\gamma} F_n)$.
- For every $\gamma \in \Lambda \setminus E$, $\widehat{F_{n+1}}(\gamma) = \widehat{F_n}(\gamma)$.

It is worth pointing out that $F_{n+1} - F_n$ lies in $L^\infty_E(G)$. So for every $\gamma \in \Gamma$,

$$M(\bar{\gamma}F_{n+1}) - M(\bar{\gamma}F_n) = M(\bar{\gamma}(F_{n+1} - F_n)) = (\widehat{F_{n+1} - F_n})(\gamma) = M(\bar{\gamma}F_n) - M(\bar{\gamma}F_{n-1})$$

because E is a LP set.

By induction, $M(\bar{\gamma}F_{n+1}) - M(\bar{\gamma}F_n) = (\widehat{F_1 - F_0})(\gamma) = M(\bar{\gamma}f) - \hat{f}(\gamma)$.

Thus, fixing $\gamma \in \Gamma$, the sequence $\Big(M(\bar{\gamma}F_n)\Big)_{n\in\mathbb{N}}$ is an arithmetical sequence, which is bounded by $\|F_n\|_\infty \leq \|f\|_\infty$. Then, it is a constant sequence. This implies that $M(\bar{\gamma}f) - \hat{f}(\gamma) = 0$. ∎

We apply this theorem in two "opposite" ways.

First, we are going to choose a uniform "big" L_γ. This gives a positive answer to the union problem for LP sets, provided one is closed for the Bohr topology.

Theorem 2. *Let E' be a LP set and E be a strong LP set. Then $\Lambda = E \cup E'$ is a LP set.*

Note that there are some LP sets which are not strong LP. Indeed, in [6], we construct some Rosenthal sets (a fortiori LP set), which are dense for the Bohr topology.

Proof. As a subset of a LP set is a LP set, we may (and do) suppose that E and E' are disjoint and that E is closed for the Bohr topology.

It suffices to apply Theorem 1, where $L_\gamma = E'$. Then for every $\gamma \notin E$, γ does not belong to $E = \bar{E} = \overline{\Lambda \setminus E'}$. ∎

In the following corollary of Theorem 1, the set L_γ is as thin as possible.

Proposition 1. *Let Λ be a closed subset of Γ. We assume that $\Lambda = E \cup E'$, where E is a LP set and for every $\gamma \in E'$, we have $\gamma \notin \overline{\Lambda \setminus \{\gamma\}}$.*
Then
Λ is a strong LP set.

Remark: the condition on E' actually means that E' is a discrete subset of Λ, equipped with the induced topology: for every $\gamma \in E'$, there exists a neighborhood V_γ of γ, such that $V_\gamma \cap \Lambda = \{\gamma\}$.

Proof. First for every $\gamma \notin \Lambda$, $\gamma \notin \overline{\Lambda}$. Thus, a singleton is trivially a LP set, Theorem 1 implies that Λ is a LP set. As Λ is closed, it is a strong LP set. ∎

Now, we are going to apply Theorem 1 to the set of prime numbers (actually, this is only a first step and we shall strengthen later the following theorem). Recall that in the framework of integers, a family of (clopen) neighborhood of $n_0 \in \mathbb{Z}$ is given by $n_0 + q\mathbb{Z}$, where $q \geq 1$. Concerning the set $\mathcal{P}$ of prime numbers, it is well known, and easy to check, that $\Lambda = \mathcal{P} \cup \{-1, +1\}$ is closed for the Bohr topology and that, more generally, every subset of Λ containing $\{-1, +1\}$ is closed for the Bohr topology (see [3] for instance). Actually, we cannot avoid adding -1 and $+1$: these are cluster points for $\mathcal{P}$ (this is not trivial and relies on a generalization, due to Schinzel, of a result of Vinogradov: see [4]). Point out that the reason why Lust-Piquard considered $\mathcal{P} \cap (2 + 5\mathbb{Z})$ in her work was to avoid the difficulty due to -1 and $+1$ (and this set was sufficient to construct her striking example): $\mathcal{P} \cap (2 + 5\mathbb{Z})$ is discrete for the Bohr topology. Here, thanks to Theorem 1, we are able to surround this problem because $\{-1, +1\}$ is finite. Nevertheless, in this paper, we never need to know the fact that -1 and $+1$ are cluster points for $\mathcal{P}$.

Theorem 3. *The set $\mathcal{P}$ of prime numbers is a strong LP set.*

Proof. It suffices to apply the preceding proposition with $E = \{-1, +1\}$ and $E' = \mathcal{P}$ to conclude that Λ is a strong LP set. ■

Remark. Actually, applying Theorem 2, we obtain that the union of the prime numbers with any LP set is still a LP set. Let us point out too that Theorem 3 generalizes the fact that the prime numbers is a strong Riesz set (see [13]). In the sequel, we are going to strongly generalize this.

We need to precise some notations: we shall write, as usual, that $k \,|\, n$ when $n = kq$ for some q, where n, k, q are integers. We shall write that $k \not| \, n$ when it does not hold. At last, for $n \in \mathbb{Z}$, with $n \neq 0$, and $p \in \mathcal{P}$: $\alpha_p(n) = \max\{a \in \mathbb{N};\ p^a \,|\, n\}$, so that

$$n = \varepsilon \prod_{p \in \mathcal{P}} p^{\alpha_p(n)} \qquad \text{where } \varepsilon \in \{-1, 1\}.$$

Theorem 4. *Let $r \geq 1$, the set $F_r = \{n \in \mathbb{Z} \setminus \{0\} |\ card\{p \in \mathcal{P} |\, \alpha_p(n) \neq 0\} \leq r\}$ is a closed LP set.*

Roughly speaking, F_r is the set of integers whose decomposition with prime numbers uses at most r factors. The proof uses several lemmas. The two following ones are inspired from [2].

Lemma 2. *The set $\mathcal{A} = \{n \in \mathbb{Z} \setminus \{0\} |\ p \in \mathcal{P},\ |k| \geq 2,\ p \,|\, n$ and $k \,|\, n \Rightarrow p \not| \, k + 1\}$ is a closed LP set.*

We have $0 \notin \mathcal{A}$ and $-1, 1 \in \mathcal{A}$.

Proof. First $0 \notin \overline{\mathcal{A}}$. Indeed, $6\mathbb{Z} \subset \mathbb{Z} \setminus \mathcal{A}$ because when $n \in 6\mathbb{Z}$, we have $2 \,|\, n$ and $3 \,|\, n$ so $n \notin \mathcal{A}$. Now, let $n \notin \mathcal{A}$ with $|n| \geq 2$. There exists some $p \in \mathcal{P}$ and $k \in \mathbb{Z}$, where $|k| \geq 2$, with $p \,|\, n$ and $k \,|\, n$ and $p \,|\, k + 1$. It is then easy to check that $n + pk\mathbb{Z} \subset \mathbb{Z} \setminus \mathcal{A}$. So $\mathcal{A}$ is closed for the Bohr topology.

We are going to use Proposition 1. The set $\{-1, 1\}$ is trivially a LP set. Now, fix $a \in \mathcal{A} \setminus \{-1, 1\}$. We claim that $(a + 3a^2\mathbb{Z}) \cap (\mathcal{A} \setminus \{a\}) = \emptyset$. Indeed, if $s \in (a + 3a^2\mathbb{Z}) \cap (\mathcal{A} \setminus \{a\})$, then $s = a(1 + 3az)$ where $z \in \mathbb{Z} \setminus \{0\}$. As $|a| \geq 2$, there exists some $p \in \mathcal{P}$ such that $p \,|\, a$, so $p \,|\, s$. As $|1 + 3az| \geq 2$ and $s \in \mathcal{A}$, we have $p \not| \, (-1 - 3az) + 1 = 3az$. This is a contradiction. ■

Lemma 3. *The set $F_1 = \{\pm p^\alpha |\ p \in \mathcal{P},\ \alpha \in \mathbb{N}\}$ is a strong LP set.*

Proof. It is sufficient to notice that $F_1 \subset \mathcal{A}$: suppose that, for some $q \in \mathcal{P}$ and some $k \in \mathbb{Z}$ with $|k| \geq 2$, we have $q \,|\, p^\alpha$ and $k \,|\, p^\alpha$, where $p \in \mathcal{P}$ and $\alpha \in \mathbb{N}$. Then $q = p$ and $k = \varepsilon p^\beta$ with $1 \leq \beta \leq \alpha$ and $\varepsilon \in \{-1, 1\}$. So we cannot have $q \,|\, k + 1$. ■

Lemma 4. *Let $\lambda \in \mathbb{Z}$.*
i) If $\Lambda \subset \mathbb{Z}$ is a LP set, then $\lambda.\Lambda$ is a LP set.
ii) If $\Lambda \subset \mathbb{Z}$ is closed, then $\lambda.\Lambda$ is closed.

Proof. The case $\lambda = 0$ is trivial. We assume that $|\lambda| \geq 1$.

i) For $g \in L^\infty(\mathbb{T})$, we define $g_\lambda \in L^\infty(\mathbb{T})$ by $g_\lambda(x) = g(\lambda x)$. When M is an invariant mean on $L^\infty(\mathbb{T})$, the functional $\tilde{M}(g) = M(g_\lambda)$ is an invariant mean as well. Now, for every $f \in L^\infty_{\lambda\Lambda}(\mathbb{T})$, we have $f = g_\lambda$ for some $g \in L^\infty_\Lambda(\mathbb{T})$. When $n \in \lambda\mathbb{Z}$ i.e. $n = \lambda k$, this implies that

$$M(\overline{e_n} f) = \tilde{M}(\overline{e_k} g) = \hat{g}(k) = \hat{f}(n)$$

because Λ is a LP set.

When $n \notin \lambda\mathbb{Z}$, $M(\overline{e_n} f) = 0 = \hat{f}(n)$ because $\lambda\mathbb{Z}$ is closed.

ii) It is a straightforward argument and left to the reader. ■

Lemma 5. *The set F_r is closed.*

Proof. $0 \notin \overline{F_r}$. Indeed take $p_1, \ldots, p_{r+1}$ the first $r+1$ prime numbers. Then $p_1 \cdots p_{r+1}\mathbb{Z} \cap F_r = \emptyset$.

Let $N \notin F_r$ with $|N| \geq 2$. The decomposition of N uses at least $r+1$ prime factors. So $N\mathbb{Z} \cap F_r = \emptyset$. ■

Let us point out that the points -1 and 1 are cluster points for F_r, because they are already cluster points for $\mathcal{P}$.

Proof of Theorem 4. The closeness was proved in the preceding lemma. We prove by induction that F_r is a LP set. The case $r = 1$ was proved in Lemma 3.

Suppose now that F_r is a LP set. We already know that F_{r+1} is closed. Fix $\gamma \in F_{r+1} \setminus F_r$. We can write $\gamma = \varepsilon q_1^{\alpha_1} \cdots q_{r+1}^{\alpha_{r+1}}$, where $q_1, \ldots, q_{r+1}$ are distincts prime numbers, $\varepsilon = \pm 1$ and $\alpha_j \geq 1$. We introduce the set: $L_\gamma = q_1^{\alpha_1}.F_r$. By Lemma 4., it is a LP set.

We claim that $\gamma \notin \overline{F_{r+1} \setminus L_\gamma}$. We are going to check that $(\gamma + \gamma^2\mathbb{Z}) \cap (F_{r+1} \setminus L_\gamma) = \emptyset$. Indeed, suppose that $\gamma' \in (\gamma + \gamma^2\mathbb{Z}) \cap (F_{r+1} \setminus L_\gamma)$, then $\gamma' = \gamma(1 + \gamma z)$ where $z \in \mathbb{Z}$. As $\gamma \,|\, \gamma'$ and $\gamma' \in F_{r+1}$, we have $\gamma' = \varepsilon' q_1^{\alpha'_1} \cdots q_{r+1}^{\alpha'_{r+1}}$ with $\varepsilon' = \pm 1$ and $\alpha_j \leq \alpha'_j$ for each $j \in \{1, \ldots, r+1\}$. But if $\alpha'_1 > \alpha_1$, then $q_1 \,|\, \frac{\gamma'}{\gamma} = 1 + \gamma z$. As $q_1 \,|\, \gamma$, we would have a contradiction. This yields $\alpha'_1 = \alpha_1$. We conclude that $\gamma' \in L_\gamma$, which is a contradiction and the claim is proved.

Applying Theorem 1, F_{r+1} is a LP set. ■

Remarks.

i) Actually, applying Theorem 2, we obtain that the union of F_r with any LP set is still a LP set. We would like to mention that this implies that the union of the preceeding set with any Riesz set is a Riesz set, thanks to the main result of [6]. It seems that even this application to Riesz sets was unknown.

ii) On the other hand, as previously claimed in the introduction, we obtain some new kinds of LP sets: neither Rosenthal sets, neither discrete. We precise now several examples. A first example is given by the set of prime numbers itself: using the fact (proved in [4]) that -1 and 1 are cluster points, this is not a discrete set, Lust-Piquard [12] proved that $\mathcal{P}$ is not a Rosenthal set. Another example is given by $E_1 \cup E_2$, where E_1 is the Rosenthal set, dense for the Bohr topology, constructed in [6] and E_2 is the discrete Hilbert set (hence not Rosenthal set) constructed in [5]. The set E_2 is a strong LP sets and Theorem 2 applies.

2 Some new non-LP sets

In this section, we investigate the links between the class of LP sets and other classical classes of thin sets in harmonic analysis. Let us recall some definitions. As we are going to focus our work on the Torus, we only precise the definitions in this framework (even if, obviously, the notions extend). We shall denote by $\mathcal{T}$ the space of trigonometric polynomials.

Definition. Let $1 \leq p < 2$ and Λ a subset of $\mathbb{Z}$. Λ is a *p-Sidon set* if there exists a constant $C > 0$ such that for any f in $\mathcal{T}_\Lambda$,

$$(\sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^p)^{1/p} \leq C \|f\|_\infty$$

For $p = 1$, this is in fact the notion of Sidon set.

Definition. A subset Λ of $\mathbb{Z}$ is a *set of uniform convergence* (*UC* set in short) if for all $f \in C_\Lambda(\mathbb{T})$, $(S_N f)_{N \geq 0}$ converges to f in $C_\Lambda(\mathbb{T})$, where $S_N f = \sum_{|n| \leq N} \hat{f}(n) e_n$.

Definition. Let $1 \leq p < \infty$ and let A be a subset of $\mathbb{Z}$. A is a $\Lambda(p)$ *set* if there exists $q \in]0, p[$ such that the norms $\|.\|_p$ and $\|.\|_q$ are equivalent on $\mathcal{T}_A$.

We recall that, if A is a $\Lambda(1)$ set, there is some $p > 1$ such that A is a $\Lambda(p)$ set. Thus, A is a $\Lambda(1)$ set if and only if $L^1_A(G)$ is a reflexive space.

The original example of Rosenthal [14] of a non trivial Rosenthal set (hence LP set) contains arbitrarily large arithmetical progressions. Then, it is neither a UC set, nor a p-Sidon, nor a $\Lambda(r)$ set, for any $p \in [1, 2[$ and $r \geq 1$. We are going to construct some non-LP sets which are UC, p-Sidon (for every $p > 1$), $\Lambda(r)$ set (for every $r \geq 1$). Of course, we cannot have a similar result with $p = 1$, because every Sidon set is a Rosenthal set, hence a LP set.

We shall need the following lemma. There, the torus is viewed as $(-1/2, 1/2)$.

Lemma 6. *Let E be a subset of $\mathbb{Z}$. We assume that for every $\varepsilon \in (0, 1/2)$, there exist $f_\varepsilon \in C_E(\mathbb{T})$ and $h_\varepsilon \in C(\mathbb{T})$ with*

. $h_\varepsilon(0) = 1$.

. $h_\varepsilon(\mathbb{T}) \subset [0, 1]$.

. $h_\varepsilon(\mathbb{T} \setminus (-\varepsilon, \varepsilon)) \subset [0, \varepsilon]$.

. $\|h_\varepsilon - f_\varepsilon\|_\infty \leq \varepsilon$.

Then

E is not a LP set.

It would be easy to show that the set E has the property that $C_E(\mathbb{T})$ contains a subspace isomorphic to c_0. Indeed, it is easy to construct some α_n and some ε_n such that the series $\sum f_{\varepsilon_n}(\cdot - \alpha_n)$ is weakly unconditionally Cauchy and the sequence $\{f_{\varepsilon_n}\}$ is not null. The classical Bessaga-Pełczyński Theorem gives the conclusion. Actually, the trick here is that this peak functions are very tightened.

Proof. Let $(\theta_n)_\mathbb{N}$ be a sequence of positive numbers such that $c = \sum_{n \in \mathbb{N}} \theta_n < 1/4$. Let $D = \{d_n\}$ be a dense countable subset of $\mathbb{T}$.

We first construct an increasing sequence of integers $(j_n)_\mathbb{N}$, a decreasing sequence of positive numbers $(\varepsilon_n)_\mathbb{N}$ and a sequence of open subset ω_n of $\mathbb{T}$, such that, for every $n \in \mathbb{N}$:

i) $\varepsilon_n \leq \theta_n$.

ii) $\omega_n = \{x \in \mathbb{T} |\, S_n(x) > 1 - c\}$.

iii) $0 \leq S_n \leq 2 + \sum\limits_{k=1}^{n} \theta_k$.

iv) $S_n(d_j) > 1 - c$, for every $j \in \{1, \ldots, j_n\}$.

where $S_n(x) = \sum\limits_{k=0}^{n} h_{\varepsilon_k}(x - d_{j_k})$.

The first step is clear (we take $j_1 = 1$).

Suppose that the construction is made up to the step m. Obviously S_m is continuous. ω_m is then well defined.

Note that $\|S_m\|_1 \leq \sum\limits_{n \in \mathbb{N}} \|h_{\varepsilon_n}\|_1 \leq 3 \sum\limits_{n \in \mathbb{N}} \varepsilon_n \leq 3 \sum\limits_{n \in \mathbb{N}} \theta_n = 3c$. There exists some integer $n \geq 1$ such that $S_m(d_n) \leq 1 - c$. Indeed, if not, with the density of D, this would imply that $S_m \geq 1 - c$ on the torus, so that $1 - c \leq \|S_m\|_1 \leq 3c$, which is impossible.

Therefore, we may define j_{m+1} as the least integer $n \geq 1$ such that $S_m(d_n) \leq 1 - c$. By (iv), we have $j_{m+1} > j_m$.

There exists $\varepsilon_{m+1} \leq \theta_{m+1}$ such that the open neighborhood $\tilde{\omega}_{m+1} = d_{j_{m+1}} + (-\varepsilon_{m+1}, \varepsilon_{m+1})$, of $d_{j_{m+1}}$ verifies $\tilde{\omega}_{m+1} \subset \{x \in \mathbb{T} |\, S_m(x) < 1\}$. Now, we have $S_{m+1} = S_m + h_{\varepsilon_{m+1}}(\cdot - d_{j_{m+1}})$.

If $x \in \tilde{\omega}_{m+1}$, then $S_{m+1}(x) \leq 2 \leq 2 + \sum\limits_{k=1}^{m+1} \theta_k$.

If $x \notin \tilde{\omega}_{m+1}$, then $S_{m+1}(x) \leq S_m(x) + \varepsilon_{m+1} \leq 2 + \sum\limits_{k=1}^{m+1} \theta_k$, thanks to the hypothesis on the h_ε's.

By induction, the construction is done.

Now, we define $\Omega = \bigcup\limits_{n \in \mathbb{N}} \omega_n$. It is a dense open subset of $\mathbb{T}$, because it contains D. The main result of [17] provides us with an invariant mean M such that $M(\mathbb{1}_\Omega) = 1$.

It is worth pointing out that the sequence (S_n) is non-decreasing, uniformly bounded by $2 + c$ and then converges pointwise to some $S \in L^\infty$. Observe that S is non-negative and that for every $n \in \mathbb{N}$, $S_{|\omega_n} \geq S_{n|\omega_n} \geq 1 - c$, hence $S_{|\Omega} \geq 1 - c$. We obtain that

$$M(S) \geq M(S\mathbb{1}_\Omega) \geq (1 - c)M(\mathbb{1}_\Omega) = 1 - c.$$

On the other hand,

$$\widehat{S}(0) \leq \|S\|_1 \leq \sum_{n \in \mathbb{N}} \|h_{\varepsilon_n}\|_1 \leq 3c.$$

Therefore

$$M(S) \neq \widehat{S}(0).$$

Now, for $x \in \mathbb{T}$ and $n \in \mathbb{N}$, we define

$$F_n(x) = \sum_{k=0}^{n} f_{\varepsilon_k}(x - d_{j_k}).$$

We have $F_n \in C_E(\mathbb{T})$ and the sequence is bounded. Actually, $(F_n - S_n)$ converges to a continuous function because the series $\sum\limits_{n \in \mathbb{N}} \|f_{\varepsilon_n} - h_{\varepsilon_n}\|_\infty$ is convergent.

Hence the sequence $(F_n)_{n \in \mathbb{N}}$ is pointwise convergent to some function F in $L^\infty_E(\mathbb{T})$. Since $F - S \in C(\mathbb{T})$,

$$M(F) - M(S) = M(F - S) = \widehat{F}(0) - \widehat{S}(0).$$

This shows that $M(F) \neq \hat{F}(0)$ and the lemma is proved. ■

Lemma 7. *Let X be a (closed) subspace of $C(\mathbb{T})$. The following assertions are equivalent:*

i) For every $\varepsilon > 0$, there exist $f_\varepsilon \in X$ and $h_\varepsilon \in C(\mathbb{T})$ with

. $h_\varepsilon(0) = 1$.

. $h_\varepsilon(\mathbb{T}) \subset [0,1]$.

. $h_\varepsilon(\mathbb{T} \setminus (-\varepsilon, \varepsilon)) \subset [0, \varepsilon]$.

. $\|h_\varepsilon - f_\varepsilon\|_\infty \leq \varepsilon$.

ii) There exists a sequence $(q_n)_n$ in the unit ball of X such that $q_n \to \mathbb{1}_{\{0\}}$ everywhere.

Proof. If (i) is true, then (ii) is clearly true.

Now suppose that (ii) is true and fix $\varepsilon > 0$. We introduce the real part ρ_n and the imaginary part I_n of q_n. The sequences are in the unit ball of $C(\mathbb{T})$. Obviously, ρ_n converges everywhere to $\mathbb{1}_{\{0\}}$ and I_n converges everywhere to 0. It means that I_n is weakly convergent to 0 in the space $C(\mathbb{T})$. By the Banach-Mazur Theorem, there exists some disjoints finite sets of integers $(E_n)_n$ and some non-negative reals $(a_j)_{j \in E_n}$, with $\sum\limits_{E_n} a_j = 1$, such that

$$\sum_{j \in E_n} a_j I_j \longrightarrow 0 \text{ uniformly on } \mathbb{T} \quad \text{and} \quad \min E_n \longrightarrow \infty.$$

We define $R_n = \sum\limits_{j \in E_n} a_j \rho_j$ and $g_n = \sum\limits_{j \in E_n} a_j q_j \in X$, we obviously have $\|g_n - R_n\|_\infty \longrightarrow 0$.

Now, consider $R_n^+ = \max(R_n, 0)$ and $R_n^- = \max(-R_n, 0)$. Of course, $R_n = R_n^+ - R_n^-$. As R_n^- is pointwise convergent to 0 on $\mathbb{T}$ and lies in the unit ball of $C(\mathbb{T})$, it is weakly convergent to 0 in the space $C(\mathbb{T})$. As previously, we can find some convex combination uniformly convergent to 0 on $\mathbb{T}$. Let F_n (resp. H_n) be the corresponding convex combination of the g_k (resp. R_k^+), with disjoint support tending to infinity. F_n lies in the unit ball of X and we have $\|F_n - H_n\|_\infty \longrightarrow 0$, $H_n(0) = 1$ and for every $t \neq 0$: $H_n(t) \longrightarrow 0$. Moreover, H_n is a non-negative function and $H_n(\mathbb{T}) \subset [0,1]$.

The (bounded) sequence H_n is weakly convergent to 0 in the space $C(\mathbb{T} \setminus (-\varepsilon, \varepsilon))$, so there are some convex combination uniformly convergent to 0 on $\mathbb{T} \setminus (-\varepsilon, \varepsilon)$. Let h_n (resp. f_n) be the corresponding convex combination of the H_k (resp. F_k). Choosing n large enough to have both $\|f_n - h_n\|_\infty \leq \varepsilon$ and $\sup\limits_{\mathbb{T} \setminus (-\varepsilon, \varepsilon)} h_n \leq \varepsilon$, the conclusion follows. ■

Definition. $\Lambda \subset \mathbb{N}$ is said to be uniformly distributed if

$$\frac{1}{N+1} \sum_{j=0}^{N} e_{\lambda_j} \longrightarrow \mathbb{1}_{\{0\}} \qquad \text{everywhere,}$$

where $\Lambda = \{\lambda_j | j \in \mathbb{N}\}$ and $\lambda_0 < \lambda_1 < \cdots$

As an immediate corollary of Lemmas 6 and 7, we obtain

Theorem 4. *Let $\Lambda \subset \mathbb{N}$. If Λ is uniformly distributed, then it is not a LP set.*

Recall that a classical result of Vinogradov asserts that for every $t \notin \mathbb{Q}$, we have

$$\frac{1}{N+1} \sum_{j=0}^{N} e_{p_j}(t) \longrightarrow 0$$

where $\mathcal{P} = \{p_j\}$ is the set a prime numbers. Nevertheless $\mathcal{P}$ is not uniformly distributed: first, it is not dense for the Bohr topology and moreover, we proved in the first part that it is a LP set. This shows that the assumption in the preceding lemmas is sharp:

for instance "everywhere" cannot be replaced by "almost everywhere" or "except on a countable set".

In [10], the authors proved that, almost surely, there are some subsets of $\mathbb{N}$ being a *UC* set, a p-Sidon set for every $p > 1$, a $\Lambda(p)$ set for every $p \geq 1$ and uniformly distributed. Let us precise the probability framework. The random method uses selectors (see Chap. 12 [9] to know more on this topic). We are going to use the same one than in [10]:

Let $(\varphi_n)_{n\geq 10}$ be a sequence of independent random variables (on some probability spave $(\Omega, \mathbb{P})$ taking values in $\{0,1\}$, with expectation $\delta_n = \dfrac{\ln(\ln(n))}{n}$.

Then, for $\omega \in \Omega$, we define the set

$$\Lambda_\omega = \{n \geq 10 |\, \varphi_n(\omega) = 1\} \subset \mathbb{N}.$$

Combining the main result of [10] and Theorem 4, we obtain the following result

Theorem 5. *Almost surely,*

i) Λ_ω *is a UC set.*

ii) Λ_ω *is a* p*-Sidon set for every* $p > 1$.

iii) Λ_ω *is a* $\Lambda(p)$ *set for every* $p \geq 1$.

iv) Λ_ω *is uniformly distributed.*

v) Λ_ω *is not a LP set.*

Thanks to the property (iv), the set Λ is dense for the Bohr topology. Moreover $C_\Lambda(\mathbb{T})$ contains a subspace isomorphic to c_0 (see [10]). A natural question is: when Λ is uniformly distributed, does the space $C_\Lambda(\mathbb{T})$ contain a lot of spaces c_0? More precisely, does this space have the Pełczyński property (V)?

Recall that a Banach space X has the property (V) of Pełczyński if, for every non relatively weakly compact bounded set $K \subset X^*$, there exists a weakly unconditionally Cauchy series $\sum x_n$ in X such that $\inf_n \sup\{|k(x_n)|;\ k \in K\} > 0$.

We are going to prove that this is false in general and that the previous sets Λ_ω do not have this property, almost surely.

In the sequel, we shall denote by $\mathcal{G}$ the following gap property for a set $\Lambda \subset \mathbb{N}$: for every infinite subset Λ' of Λ, there exists $p_0 \geq 1$ such that for infinitely many p, we may choose $\lambda_p \in \Lambda' \subset \Lambda$ with $\Lambda \cap ([\lambda_p - (p_0 + p), \lambda_p - p_0] \cup [\lambda_p + p_0, \lambda_p + p_0 + p]) = \emptyset$.

Obviously, up to an extraction, we may suppose that the sequence $(\lambda_p)_p$ is Hadamard.

We first have an easy criterion to check that a set has property $\mathcal{G}$.

Lemma 8. *A finite union of subsets of* $\mathbb{N}$ *whose pace tends to infinity has property* $\mathcal{G}$.

Recall that "the pace E tends to infinity" means that $\lambda_{n+1} - \lambda_n \to +\infty$, where $E = \{\lambda_n\}$.

Proof. Suppose that $\Lambda' \subset \Lambda = \cup_{1\leq j\leq d}\Lambda_j$, where Λ' is infinite and the pace of the sets Λ_j tends to infinity. We denote by δ_λ the distance from λ to its complementary in Λ. If $\sup_{\lambda\in\Lambda'} \delta_\lambda = +\infty$, then the conclusion follows with $p_0 = 1$. If not, $\sup_{\lambda\in\Lambda'} \delta_\lambda = M_1 \geq 1$ and we consider $\delta_\lambda^{(1)}$ the distance from λ to $\Lambda \setminus [\lambda - M_1, \lambda + M_1]$.

Once again, either $\sup_{\lambda\in\Lambda'} \delta_\lambda^{(1)} = +\infty$, in which case we choose $p_0 = M_1 + 1$, or $\sup_{\lambda\in\Lambda'} \delta_\lambda^{(1)} = M_2 > M_1$. And we continue this process.

We claim that the process stops and we have the p_0, as promised. Indeed, if we could continue until having M_d, we would have a contradiction: by hypothesis, for $\lambda \neq \lambda'$ big enough (upper than some N), belonging to the same Λ_j (for some j), we have $|\lambda - \lambda'| > 2M_d$. Choose $\lambda_1 \in \Lambda'$ with $\lambda_1 > N + M_d$. Actually $\lambda_1 \in \Lambda_{j_1}$ for some j_1. By definition of M_1, we can find $\lambda_2 \in \Lambda_{j_2}$ (where j_2 must be different from j_1) with $|\lambda_1 - \lambda_2| \leq M_1$. In the same way, we can find $\lambda_3 \in \Lambda_{j_3}$ (where $j_3 \notin \{j_1, j_2\}$) with $|\lambda_1 - \lambda_3| \leq M_2$ and $\lambda_3 \notin [\lambda_1 - M_1, \lambda_1 + M_1]$. This can be done until the step $d+1$. But we would construct $j_1, \ldots, j_{d+1}$ distincts points with values in $\{1, \ldots, d\}$. ■

Proposition 2. *Let $\Lambda \subset \mathbb{N}$ with property $\mathcal{G}$.*

Then every infinite subset $\Lambda' \subset \Lambda$ contains a Hadamard subset H such that $C_H(\mathbb{T})$ is complemented in $C_\Lambda(\mathbb{T})$.

We shall denote by $[A]$ the mesh spanned by the finite set A:

$$[A] = \Big\{ \sum_{a\in A} \varepsilon_a a |\ \varepsilon_a \in \{-1, 0, 1\} \Big\}.$$

Proof. Let p_0 be given by property $\mathcal{G}$. Suppose that $p_0 < h_1 < \cdots < h_p$ are constructed in $\Lambda' \subset \Lambda$ with $h_{j+1} \geq 3h_j$; $[\{h_1, \ldots, h_p\}] \cap \Lambda = \{h_1, \ldots, h_p\}$ and $[\{h_1, \ldots, h_p\}] \cap [-p_0, p_0] = \{0\}$. We write $L = h_1 + \ldots + h_p$. We can find $h_{p+1} \in \Lambda'$ with $[h_{p+1} - L, h_{p+1} + L] \cap (\Lambda \backslash]h_{p+1} - p_0, h_{p+1} + p_0[) = \emptyset$ and $h_{p+1} > 3h_p$. Now, the sequence (h_n) is Hadamard and $[\{h_1, \ldots, h_{p+1}\}] \cap \Lambda = \{h_1, \ldots, h_{p+1}\}$. We introduce the classical associated Riesz products

$$R_N(x) = 2 \prod_{n=1}^{N} [1 + \cos(2\pi h_n x)].$$

A standard weak compactness argument in $\mathcal{M}(\mathbb{T})$ provides us with a measure μ bounded by 2, whose Fourier coefficients take values 1 on H and cancel on $\Lambda \setminus H$. The convolution operator by μ is then a translation invariant projection from $C_\Lambda(\mathbb{T})$ onto $C_H(\mathbb{T})$. ■

We add the following remark: Lemma 8 is not an equivalence. Indeed, when A and B are infinite, $\Lambda = \{a + b |\ a \in A,\ b \in B, a \neq b\}$ cannot be a finite union of subsets of $\mathbb{N}$ whose pace tends to infinity. For instance, this is the case for $\{3^n + 3^m | n > m\}$. Nevertheless, this set has property $\mathcal{G}$ and we can construct some complemented copy of ℓ^1: it suffices to choose $p_0 = 1$ and the λ_j's among the $3^j + 3^{j-1}$.

As a corollary, we obtain

Theorem 6. *Let $\Lambda \subset \mathbb{N}$ be a finite union of subsets of $\mathbb{N}$ whose pace tends to infinity.*

Then

i) $C_\Lambda(\mathbb{T})$ contains a complemented copy of ℓ^1.

ii) $L^1_{\mathbb{Z}-\cup\Lambda}(\mathbb{T})$ contains a complemented copy of ℓ^2.

iii) $C_\Lambda(\mathbb{T})$ does not have property (V) of Pełczyński.

A special example. Observe that this applies to the case of polynomial sets $P(\mathbb{N})$ (where P is a polynomial with degree upper than 2) and this gives an example of a space $C_{P(\mathbb{N})}(\mathbb{T})$ which does not have property (V) of Pełczyński but contains a copy of c_0. The fact that $C_{P(\mathbb{N})}(\mathbb{T})$ contains some subspace isomorphic to c_0 was proved by Lust-Piquard [12] in the case $P(X) = X^s$, where $s \geq 2$. Nevertheless, the same argument works as well in full generality.

We did not find examples of translation invariant spaces with such properties in the litterature.

We also point out the following consequences of (i): $C_\Lambda(\mathbb{T})$ cannot verify the Theorem of Grothendieck (dual version). In the same spirit, (ii) implies that $L^1_{\mathbb{Z}^- \cup \Lambda}(\mathbb{T})$ cannot verify the Theorem of Grothendieck and does not have the Dunford-Pettis property.

Proof. (iii) immediatly follows from (i).

(i) and (ii) follow from the preceding proposition: the Hadamard set H is a Sidon set: $C_H(\mathbb{T})$ is isomorphic to ℓ^1. The space $L^1_H(\mathbb{T})$ is isomorphic to ℓ^2 (H being Sidon hence $\Lambda(2)$). The complementation in $C_\Lambda(\mathbb{T})$ being translation invariant, it is equivalent to the existence of a measure μ whose Fourier coefficients take value 1 on H and 0 on $\Lambda \setminus H$ (actually μ is given in the proof of the proposition). The convolution operator by μ is a bounded complementation from $L^1_\Lambda(\mathbb{T})$ to $L^1_H(\mathbb{T})$. At last, we can "add" the negative integers as in Lemma 1.5 [7]. ∎

Theorem 7. *Let $1 \leq p < 4/3$. If Λ is a p-Sidon set, then $C_\Lambda(\mathbb{T})$ does not have property (V) of Pełczyński.*

Proof. This immediatly follows from Theorem 6 and the fact that p-Sidon sets are finite union of sets whose pace tends to infinity: Déchamps-Gondim proved this result in the case $p = 1$ in [1] (see lemmes 6.1, 6.2 et corollaire). The same proof works as well in the case $p < 4/3$. In fact the proof works for sets which do not contain sets of the form $A + B$ with A and B finite sets but arbitrarily large. The details are left to the reader. ∎

We wish to finish the paper with some open problems

1) Is the union of two LP sets a LP set?

1') Let $\Lambda \subset \mathbb{N}$ and $\Lambda' \subset \mathbb{Z}^-$ be LP sets, $\Lambda \cup \Lambda'$ is a LP set?

1") Let $\Lambda \subset \mathbb{N}$ be a LP set, $-\Lambda \cup \Lambda$ is a LP set?

2) Let $E \subset \mathbb{N}$ be closed in $\mathbb{Z}$ for the Bohr topology. Is E a LP set?

2') Is the set of squares a LP set ?

3) Let $\Lambda \subset \mathbb{Z}$ such that $c_0 \not\subset C_\Lambda(\mathbb{T})$. Do we have that Λ is a LP set?

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Université d'Artois
Faculté Jean Perrin
rue Jean Souvraz S.P. 18.
62307 Lens cedex
France
e-mail : lefevre@euler.univ-artois.fr

Universidad de Sevilla
Faculdad de Matematica
Apdo 1160
41080 Sevilla
Spain
e-mail : piazza@us.es