L^p-VALUED MEASURES WITHOUT FINITE X-SEMIVARIATION FOR $2 < p < \infty$

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Abstract. We show that for $1 \le p < \infty$, the property that every L^p-valued vector measure has finite X-semivariation in $L^p(\mu, X)$ is equivalent to the property that every continuous linear map from ℓ^1 to X is p-summing. For $2 < p < \infty$, we explicitly construct an $L^p([0,1])$-valued measure without finite L^p-semivariation.

1. Introduction

Given a Banach space X, a number $1 \le p < \infty$ and a σ-finite measure space $(\Omega, \mathcal{S}, \mu)$, equip the the tensor product $X \otimes L^p(\mu)$ with the induced norm topology Δ_p from the Bochner space $\big(L^p(\mu, X), \|\cdot\|_{L^p(\mu,X)}\big)$ (see [4, p. 97]). It turns out that this induced norm is a reasonable crossnorm, [4, Definition VIII.1.1]. Moreover, the completion $X \widehat{\otimes}_{\Delta_p} L^p(\mu)$ of the normed tensor product $X \otimes_{\Delta_p} L^p(\mu)$ equals $L^p(\mu, X)$ because $X \otimes L^p(\mu)$ is dense in $L^p(\mu, X)$.

Now consider a vector measure $m : \mathcal{E} \to L^p(\mu)$ defined over a measurable space $(\Sigma, \mathcal{E})$. The X-*semivariation* of m in the completion $X \widehat{\otimes}_{\Delta_p} L^p(\mu) = L^p(\mu, X)$ of the normed tensor product $X \otimes_{\Delta_p} L^p(\mu)$ is the set function $\beta_X(m) : \mathcal{E} \to [0, \infty]$ defined by

$$\beta_X(m)(E) := \sup \left\{ \left\| \sum_{j=1}^{k} x_j \otimes m(E_j) \right\|_{L^p(\mu,X)} \right\} \tag{1.1}$$

for every $E \in \mathcal{E}$; the supremum is taken over all pairwise disjoint sets $E_1, \dots, E_k$ from $\mathcal{E} \cap E$ and vectors $x_1, \dots, x_k$ from X, such that $\|x_j\|_X \le 1$ for all $j = 1, \dots, k$ and $k = 1, 2, \dots$. If it happens that X is one-dimensional, that is, $X = \mathbb{C}$, then $\beta_{\mathbb{C}}$ coincides with the usual seminvariation $\|m\|$ of the vector measure m (see [4, Definition I.1.4 and Proposition I.1.11].

The condition that $\beta_X(m)(\Sigma) < \infty$ is related to the m-integrability of uniformly bounded, strongly measurable X-valued functions; see [9, Theorem 2.6] as motivated from the earlier work [6, *-Theorem] and [15, Theorem 6]. The problem of finding conditions for the finiteness of X-semivariation arose from the theory of random evolutions [7] and is relevant to stochastic integration. For example, an $L^p(P)$-valued gaussian random measure has finite $L^p(\mu)$-semivariation in $L^p(\mu \otimes P)$ if and only if $p \ge 2$, [14, Proposition 6.1].

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For the situation in which ν is a σ-finite measures and $X = L^p(\nu)$, we have the following natural identifications

$$L^p(\mu \otimes \nu) = L^p\big(\mu, L^p(\nu)\big) = L^p(\mu)\widehat{\otimes}_{\Delta_p} L^p(\nu).$$

In the case when $1 \leq p < 2$, we have explicitly constructed an $L^p(\mu)$-valued measure whose $L^p(\nu)$-semivariation in $L^p(\mu\otimes\nu)$ is infinite (see [9, Example 2.3] and Example 2.3(ii) below). For $p = 2$, the statement that *every* L^2-valued measure has finite L^2-semivariation is equivalent to Grothendieck's inequality; see [7, Proposition 4.5.3] or [9, Proposition 2.1].

In [9, Theorem 3.2], it was shown that, for every $2 < p < \infty$, there is *some* vector measure whose $L^p([0,1])$-semivariation in $L^p([0,1]^2)$ is infinite. In Theorem 2.1 below, by modifying the arguments of [9], we show that for any Banach space X and any $1 \leq p < \infty$, the condition that every vector measure $m : \mathcal{E} \to L^p([0,1])$ has finite X-semivariation in $L^p([0,1], X)$ is actually *equivalent* to the statement that every continuous linear map from ℓ^1 into X is p-summing.

For $2 < p < \infty$ and $X = L^p([0,1])$, the proof of the existence of a vector measure $m : \mathcal{E} \to L^p([0,1])$ without finite X-semivariation in $L^p([0,1], X)$ in [9, Theorem 3.2] appealed to a result of S. Kwapień [10, Theorem 7, 2^0] to show that not every continuous linear map from ℓ^1 into X is p-summing. However, we did not actually provide an explicit example of a measure with this property. In Section 3, we rectify the situation by exhibiting such a measure—this amounts to constructing a continuous linear map u from ℓ^1 into ℓ^p that is not p-summing and a sequence $\{x_n\}_{n=1}^\infty$ in ℓ^1 such that $\sum_{n=1}^\infty |\langle x_n, \xi\rangle|^p < \infty$ for each $\xi \in \ell^\infty$, but $\sum_{n=1}^\infty \|u(x_n)\|_{\ell^p}^p = \infty$. That this task is not straightforward is illustrated by the observation that any such map u is automatically q-summing for any $q > p \geq 2$; see [2, Corollary 24.6].

2. X-semivariation in L^p-spaces

Let X and Y be Banach spaces. The space of all continuous linear maps from X into Y is denoted by $\mathcal{L}(X,Y)$. Let $1 \leq p < \infty$. An operator $u \in \mathcal{L}(X,Y)$ is called *absolutely p-summing* (briefly p-summing) if there exists a constant $C > 0$ such that

$$\left(\sum_{j=1}^{k} \big\|u(x_j)\big\|_Y^p\right)^{1/p} \leq C \sup_{\|x'\|_{X'}\leq 1} \left(\sum_{j=1}^{k} \big|\langle x_j, x'\rangle\big|^p\right)^{1/p} \tag{2.1}$$

for all $x_j \in X$, $j = 1, \ldots, k$ and $k = 1, 2, \ldots$. The infimum of such numbers C is denoted by $\pi_p(u)$. The vector space of all absolutely p-summing maps from X into Y equipped with the norm π_p is denoted by $\Pi_p(X,Y)$. An absolutely summing map (for $p = 1$) is characterised by the fact that it maps unconditionally summable sequences to absolutely summable sequences, [4, Proposition VI.3.2]. For further details we refer to [5].

Let $\|m\| : \mathcal{E} \to [0,\infty)$ denote the usual semivariation of vector measure m, [4, Definition I.1.4] and let $\mathbb{P}$ denote Lebesgue measure on the Borel σ-algebra $\mathcal{B}([0,1])$, and $\mathbb{E}$ the associated expectation.

Theorem 2.1. *Let X be a nonzero Banach space, $1 \leq p < \infty$ and $(\Omega, \mathcal{S}, \mu)$ a σ-finite measure space containing infinitely many, pairwise disjoint non-μ-null sets, so that $L^p(\mu)$ has infinite dimension. The following conditions are equivalent.*

(i) $\mathcal{L}\big(L^1([0,1]), X\big) = \Pi_p\big(L^1([0,1]), X\big)$.

(ii) $\mathcal{L}(\ell^1,\, X) = \Pi_p(\ell^1,\, X)$.

(iii) *For every measurable space* $(\Sigma, \mathcal{E})$*, every vector measure* $m : \mathcal{E} \to L^p(\mu)$ *has finite* X*-semivariation in* $L^p(\mu, X)$*.*

If any of conditions (i)–(iii) *holds, then there exists a constant* $C > 0$ *such that*

$$\|m\|(\Sigma) \le \beta_X(m)(\Sigma) \le C\|m\|(\Sigma), \tag{2.2}$$

for every measurable space $(\Sigma, \mathcal{E})$ *and every vector measure* $m : \mathcal{E} \to L^p(\mu)$*.*

To prove this theorem we shall use the following result.

Lemma 2.2. *Let the assumption be as in Theorem 2.1. Suppose that* $g_j \in L^1([0,1])$*,* $j = 1, 2, \ldots,$ *are functions satisfying* $\sum_{j=1}^\infty |\langle g_j, f\rangle|^p < \infty$ *for every* $f \in L^\infty([0,1])$*. Then there exists a vector measure* $m : \mathcal{B}([0,1]) \to L^p(\mu)$ *such that*

$$\beta_X(m)(A) \ge \left(\sum_{j=1}^\infty \left\|u(g_j\chi_A)\right\|_X^p\right)^{1/p}, \qquad A \in \mathcal{B}([0,1]),$$

for all $u \in \mathcal{L}(L^1([0,1]), X)$ *with operator norm* $\|u\| \le 1$*.*

Proof. Let E_j, $j = 1, 2, \ldots,$ be pairwise disjoint sets belonging to the σ-algebra $\mathcal{S}$ with finite, nonzero μ-measure. Define a function $F : \Omega \to L^1([0,1])$ by

$$F(\omega) = \sum_{j=1}^\infty g_j \cdot \chi_{E_j}(\omega)/\mu(E_j)^{1/p}. \tag{2.3}$$

Then

$$\int_0^1 \left|\langle F(\omega), f\rangle\right|^p d\mu(\omega) = \sum_{j=1}^\infty \left|\langle g_j, f\rangle\right|^p < \infty,$$

that is, $\langle F(\cdot), f\rangle \in L^p(\mu)$ for all $f \in L^\infty([0,1])$.

Let $m : \mathcal{B}([0,1]) \to L^p(\mu)$ be the vector measure defined by

$$m(A)(\omega) = \langle F(\omega), \chi_A\rangle, \qquad A \in \mathcal{B}([0,1]),\ \omega \in \Omega. \tag{2.4}$$

That m is actually an $L^p(\mu)$-valued measure is easily seen by writing it as the composition of the embedding $\phi \longmapsto \sum_{j=1}^\infty \phi(j) \cdot \chi_{E_j}/\mu(E_j)^{1/p}$ of ℓ^p into $L^p(\mu)$ with the ℓ^p-valued measure $A \longmapsto \{\int_A g_j(t)\, dt\}_{j=1}^\infty$, $A \in \mathcal{B}([0,1])$.

Fix a set $A \in \mathcal{B}([0,1])$ and let $F_A(\omega) := F(\omega)\, \chi_A$, so that $m(A \cap B) = \langle F_A(\omega), \chi_B\rangle$ for all $B \in \mathcal{B}([0,1])$ and $\omega \in \Omega$. Let n be a positive integer and let $I_{n,k} = [(k-1)/2^n,\ k/2^n)$, $k = 1, \ldots, 2^n$, be the partition of $[0,1)$ into 2^n intervals of equal length. Let $P_n : L^1([0,1]) \to L^1([0,1])$ denote the associated conditional expectation operator with respect to the algebra of finite unions of the intervals $I_{n,k}$, $k = 1, \dots, 2^n$. Then for each $\omega \in \Omega$ we have

$$\begin{aligned} P_n \circ F_A(\omega) &= \sum_{j=1}^{\infty} P_n(g_j \chi_A) \cdot \chi_{E_j}(\omega)/\mu(E_j)^{1/p} \\ &= 2^n \sum_{j=1}^{\infty} \left(\sum_{k=1}^{2^n} \mathbb{E}(\chi_{I_{n,k} \cap A}\, g_j) \cdot \chi_{I_{n,k}} \right) \cdot \chi_{E_j}(\omega)/\mu(E_j)^{1/p} \\ &= 2^n \sum_{k=1}^{2^n} \left(\sum_{j=1}^{\infty} \mathbb{E}(\chi_{I_{n,k} \cap A}\, g_j) \cdot \chi_{E_j}(\omega)/\mu(E_j)^{1/p} \right) \chi_{I_{n,k}} \\ &= \sum_{k=1}^{2^n} (m(I_{n,k} \cap A))(\omega) \cdot 2^n \chi_{I_{n,k}}. \end{aligned}$$

Let $u \in \mathcal{L}(L^1([0,1]), X)$ have norm $\|u\| \le 1$. Then,

$$u(P_n \circ F_A(\omega)) = \sum_{k=1}^{2^n} \big(m(I_{n,k} \cap A)\big)(\omega) \cdot u(2^n \chi_{I_{n,k}}).$$

Each vector $x_{n,k} = u(2^n \chi_{I_{n,k}})$, $k = 1, \dots, 2^n$, belongs to the closed unit ball of X because $\|u\| \le 1$. Using the vectors $x_{n,k}$ to estimate the X-semivariation of m, we have

$$\begin{aligned} \left\| \sum_{k=1}^{2^n} x_{n,k} \otimes m(I_{n,k} \cap A) \right\|_{L^p(\mu,X)} &= \left(\int_\Omega \left\| \sum_{k=1}^{2^n} x_{n,k} \cdot \big(m(I_{n,k} \cap A)\big)(\omega) \right\|_X^p d\mu(\omega) \right)^{1/p} \\ &= \left(\int_\Omega \left\| u(P_n \circ F_A(\omega)) \right\|_X^p d\mu(\omega) \right)^{1/p}. \end{aligned}$$

Because $P_n(F_A(\omega)) \to F_A(\omega)$ for each $\omega \in \Omega$ as $n \to \infty$ and

$$\int_\Omega \left\| u(F_A(\omega)) \right\|_X^p d\mu(\omega) = \int_\Omega \sum_{j=1}^{\infty} \|u(g_j \chi_A)\|_X^p \, \chi_{E_j}(\omega)/\mu(E_j)\, d\mu(\omega),$$

it follows from Fatou's Lemma that

$$\begin{aligned} \liminf_{n \to \infty} \left\| \sum_{k=1}^{2^n} x_{n,k} \otimes m(I_{n,k} \cap A) \right\|_{L^p(\mu,X)} &\ge \left(\int_\Omega \left\| u(F_A(\omega)) \right\|_X^p d\mu(\omega) \right)^{1/p} \\ &= \left(\sum_{j=1}^{\infty} \|u(g_j \chi_A)\|_X^p \right)^{1/p}. \end{aligned}$$

Therefore, the lemma holds. □

Proof of Theorem 2.1. Suppose that condition (i) holds. To deduce part (ii), fix $T \in \mathcal{L}(\ell^1, X)$. Let B_j, $j = 1, 2, \dots$, be non-null, pairwise disjoint Borel subsets of $[0,1]$. If $J : \ell^1 \to L^1([0,1])$ denotes the isometry

$$\phi \longmapsto \sum_{j=1}^{\infty} \chi_{B_j} \cdot \phi(j)/\mathbb{P}(B_j), \quad \phi \in \ell^1,$$

then $Q \circ J$ is the identity map on ℓ^1 if $Q : L^1([0,1]) \to \ell^1$ denotes the continuous linear map $f \longmapsto \{\mathbb{E}(f \chi_{B_j})\}_{j=1}^{\infty}$, $f \in L^1([0,1])$. By condition (i), the operator $T \circ Q$ is p-summing. Because $T = (T \circ Q) \circ J$, it follows that $T \in \Pi_p(\ell^1, X)$ and part (ii) holds.

Now assume that condition (ii) is valid and $m : \mathcal{E} \to L^p(\mu)$ is a vector measure. Let n be a positive integer, let $A_j \in \mathcal{E}$, $j = 1, \ldots, n$, be pairwise disjoint sets and let $x_j \in X$, $j = 1, \ldots, n$, be vectors belonging to the closed unit ball of X. We establish a uniform bound for $\sum_{j=1}^n x_j \otimes m(A_j)$ in the norm of $L^p(\mu, X)$.

Let $u : \ell^1 \to X$ be a linear map with uniform norm bounded by one such that $u(e_j) = x_j$ for the standard basis vectors e_j of ℓ^1 and $j = 1, \ldots, n$. Then

$$\begin{aligned}\left\|\sum_{j=1}^n x_j \otimes m(A_j)\right\|_{L^p(\mu,X)} &= \left(\int_\Omega \left\|\sum_{j=1}^n x_j \cdot m(A_j)(\omega)\right\|_X^p d\mu(\omega)\right)^{1/p} \\ &= \left(\int_\Omega \left\|\sum_{j=1}^n u(e_j) \cdot m(A_j)(\omega)\right\|_X^p d\mu(\omega)\right)^{1/p} \\ &= \left(\int_\Omega \left\|u\left(\sum_{j=1}^n e_j \cdot m(A_j)(\omega)\right)\right\|_X^p d\mu(\omega)\right)^{1/p}.\end{aligned}$$

Since u is p-summing by condition (ii), it follows that

$$\begin{aligned}&\left(\int_\Omega \left\|u\left(\sum_{j=1}^n e_j \cdot m(A_j)(\omega)\right)\right\|_X^p d\mu(\omega)\right)^{1/p} \\ &\le \pi_p(u) \sup_{\|\xi\|_{\ell^\infty}\le 1}\left(\int_\Omega \left|\left\langle \sum_{j=1}^n e_j \cdot m(A_j)(\omega),\ \xi\right\rangle\right|^p d\mu(\omega)\right)^{1/p} \\ &= \pi_p(u)\left(\sup_{\|\xi\|_{\ell^\infty}\le 1}\left\|\sum_{j=1}^n \xi(j) m(A_j)\right\|_{L^p(\mu)}^p\right)^{1/p} \\ &\le \pi_p(u)\,\|m\|(\Sigma).\end{aligned}$$

Indeed, the first inequality follows from [12, Proposition 1.2] while the last inequality from [4, Proposition I.1.11]. Hence, we have

$$\left\|\sum_{j=1}^n x_j \otimes m(A_j)\right\|_{L^p(\mu,X)} \le \pi_p(u)\,\|m\|(\Sigma).$$

By condition (ii) and the Open Mapping Theorem, there exists a constant $C > 0$ such that $\pi_p(T) \le C\|T\|$ for every $T \in \mathcal{L}(X)$, which implies that $\beta_X(m)(\Sigma) \le C\|m\|(\Sigma)$. So, condition (iii) is satisfied. Moreover, the bound $\|m\|(\Sigma) \le \beta_X(m)(\Sigma)$ follows by taking $x_j = c_j x$, $j = 1, \ldots, n$, for a fixed unit vector $x \in X$ and $c_j \in \mathbb{C}$ with $|c_j| \le 1$, $j = 1, \ldots, n$. Consequently, (2.2) is established.

To prove that condition (iii) implies condition (i), we prove the contrapositive statement: suppose that $u \in \mathcal{L}(L^1([0,1]), X)$ has norm $\|u\| \le 1$ but is not p-summing, that is, there exist functions $g_j \in L^1([0,1])$, $j = 1, 2, \ldots$, such that $\sum_{j=1}^\infty |\langle g_j, f\rangle|^p < \infty$ for every $f \in L^\infty([0,1])$ and $\sum_{j=1}^\infty \|u(g_j)\|_X^p = \infty$. Take a vector measure $m : \mathcal{B}([0,1]) \to L^p(\mu)$ satisfying the conclusion of Lemma 2.2. Then,

$$\beta_X(m)(\Omega) \ge \left(\sum_{j=1}^\infty \|u(g_j)\|_X^p\right)^{1/p} = \infty.$$

So condition (iii) implies (i). □

Example 2.3. (i) Let $1 \le p < 2$. An example of an $L^p(\mu)$-valued measure without finite $L^p(\nu)$-semivariation in $L^p(\mu\otimes\nu)$ is given in [9, Example 2.3], so not every map from ℓ^1 to ℓ^p is p-summing. In fact, the embedding J of ℓ^1 into ℓ^p is not p-summing [5, p. 209]. We can see this more directly as follows. If the inclusion map $J : \ell^1 \to \ell^p$ were p-summing, then J would factor through ℓ^2 via Pietsch's Domination Theorem [5, Inclusion Theorem 2.8 and Corollary 2.16]. Since $1 \le p < 2$, every continuous linear map from ℓ^2 into ℓ^p is compact by Pitt's Theorem [11, Theorem 2.c.3], so it would follow that J is compact. But this is false because $\{J(e_k) : k = 1, 2, \dots\}$ is not relatively compact in ℓ^p.

(ii) Let $1 \le p < 2$. A concrete example of an $L^p(\mu)$-valued measure without finite $L^p(\nu)$-semivariation in $L^p(\mu \otimes \nu)$ on *any* set of positive measure is provided by a gaussian random measure $W : \mathcal{B}([0, 1]) \to L^p(\mu)$ with μ a probability measure (see [14, p. 184]). The gaussian random variable $W(B)$ has mean zero and variance $|B|$, the Lebesgue measure of $B \in \mathcal{B}([0, 1])$. Then there exists $C_p > 0$ such that $\|W(B)\|_{L^p(\mu)} = C_p\,|B|^{1/2}$ for every $B \in \mathcal{B}([0, 1])$. Consequently, the p-variation

$$\sup_{\pi}\left(\sum_{B\in\pi}\|W(B\cap A)\|^p_{L^p(\mu)}\right)^{1/p}$$

of W is infinite on any Borel set $A \subseteq [0, 1]$ with positive measure. Here the supremum is over all finite Borel partitions. An appeal to [9, Proposition 2.2] shows that $\beta_X(W)(A) = \infty$ with $X = L^p(\nu)$ for any scalar measure ν such that X is infinite-dimensional.

(iii) Let $2 < r < p < \infty$. By [2, Corollary 24.6], every continuous linear map from ℓ^1 to ℓ^r is p-summing, so every $L^p(\mu)$-valued vector measure has finite ℓ^r-semivariation in $L^p(\mu, \ell^r)$. More generally, $\Pi_p(Z, X) = \mathcal{L}(Z, X)$ if Z is an $\mathcal{L}^1$-space and X is an $\mathcal{L}^r$-space, see [5, p. 60] for the definition of $\mathcal{L}^q$-spaces. Further results on semivariation in tensor products of L^p-spaces are obtained in [1].

3. The measure

Let $2 < p < \infty$ and let q be the conjugate index satisfying $1/p + 1/q = 1$. We construct an L^p-valued measure m defined on the Borel σ-algebra $\mathcal{B}([0, 1])$ of the unit interval $[0, 1]$ via a family $\{g_j\}_{j=1}^{\infty}$ of independent, identically distributed, standard q-stable random variables with respect to Lebesgue measure $\mathbb{P}$ on $[0, 1]$. Here a $\mathcal{B}([0, 1])$-measurable function $f : [0, 1] \to \mathbb{R}$ is called a *standard q-stable random variable* if

$$\int_0^1 e^{isf(t)}\, d\mathbb{P}(t) = e^{-|s|^q}, \qquad s \in \mathbb{R}.$$

A discussion of q-stable random variables appears in [16, V.5.6]. In particular, by [16, Lemma V.5.4, p. 338], each standard q-stable random variable on $[0, 1]$ belongs to $L^r([0, 1])$ for every $1 \le r < q$ and the equality

$$\left\|\sum_{j=1}^{n} c_j g_j\right\|_{L^1([0,1])} = \left(\sum_{j=1}^{n}|c_j|^q\right)^{1/q}\cdot\|g_1\|_{L^1([0,1])}, \tag{3.1}$$

holds for all numbers $c_j \in \mathbb{C}$, $j = 1, \dots, n$, and $n = 1, 2, \dots$. The equality (3.1) determines an isometric embedding of ℓ^q into $L^1([0, 1])$.

Lemma 3.1. *The sequence* $\{g_j\}_{j=1}^\infty$ *is weakly p-summable in* $L^1([0,1])$*, that is,*

$$\sup_{\|f\|_{L^\infty([0,1])}\le 1} \sum_{j=1}^\infty |\langle g_j, f\rangle|^p < \infty.$$

Proof. Let $f \in L^\infty([0,1])$. Then, for all $n = 1, 2, \dots$ and all scalars $c_1, \dots, c_n$, we have

$$\begin{aligned}\left|\sum_{j=1}^n c_j \langle g_j, f\rangle\right| = \left|\sum_{j=1}^n c_j \,\mathbb{E}\big(f g_j\big)\right| &= \left|\mathbb{E}\Big(f \sum_{j=1}^n c_j g_j\Big)\right| \\ &\le \Big\|\sum_{j=1}^n c_j g_j\Big\|_{L^1([0,1])} \cdot \|f\|_{L^\infty([0,1])} \\ &= \left(\sum_{j=1}^n |c_j|^q\right)^{1/q} \cdot \|g_1\|_{L^1([0,1])} \cdot \|f\|_{L^\infty([0,1])}.\end{aligned}$$

Hence, $\sup_{\|f\|_{L^\infty([0,1])}\le 1} \sum_{j=1}^\infty |\langle g_j, f\rangle|^p$ is finite. □

Let $m : \mathcal{B}([0,1]) \to L^p([0,1])$ be the vector measure defined by formula (2.4) in the case that μ is Lebesgue measure $\mathbb{P}$ on $[0,1]$. Our goal is to prove the following result.

Theorem 3.2. *The $L^p([0,1])$-valued measure m has infinite $L^p([0,1])$-semivariation in the space $L^p(\mathbb{P}\otimes\mathbb{P}) = L^p([0,1]^2)$ on every Borel set of positive measure.*

In order to prove this, we find a continuous linear map $u : L^1([0,1]) \to \ell^p$ for which the sequence $\{g_j\}_{j=1}^\infty$ in $L^1([0,1])$ has the property that $\sum_{j=1}^\infty \left\|u(g_j\chi_A)\right\|_{\ell^p}^p = \infty$ for every Borel set $A \subseteq [0,1]$ of positive measure and then we appeal to Lemma 2.2.

4. A non-p-summing map

Let the notation be as in Section 3. Suppose that $\{g_j\}_{j=1}^\infty$ is the family of standard q-stable independent identically distributed random variables with respect to Lebesgue measure $\mathbb{P}$ on $[0,1]$ at the beginning of in Section 3 above. Next, we choose $\{c_j\}_{j=1}^\infty$ such that $\sum_{j=1}^\infty |c_j|^q \le 1$ and $\sum_{j=1}^\infty |c_j|^q|g_j|^q = \infty$ ($\mathbb{P}$-a.e.). This is possible according to [13, pp. 356–358]. In fact, choose such scalars c_j, $j = 1, 2, \dots$, satisfying $\sum_{j=1}^\infty |c_j|^q \ln(1/|c_j|) = \infty$.

To proceed, we need the following construction.

Lemma 4.1. *Let $\{f_j\}_{j=1}^\infty$ be a sequence in $L^1([0,1])$ such that $\sum_{j=1}^\infty |f_j(t)|^q = \infty$ for $\mathbb{P}$-almost all $t \in [0,1]$. Then there exist Borel measurable functions $h_1, h_2, \dots$ on $[0,1]$ such that*

1. $\sum_{j=1}^\infty |h_j(t)|^p \le 1$ *for all* $t \in [0,1]$,
2. $h_j(t)f_j(t) \ge 0$ *for all* $t \in [0,1]$ *and* $j = 1, 2, \dots,$ *and*
3. $\sum_{j=1}^\infty h_j(t)f_j(t) = \infty$ *for $\mathbb{P}$-almost all* $t \in [0,1]$.

Proof. For each $n = 1, 2, \dots$ and for $\mathbb{P}$-almost every $t \in [0,1]$, there exist numbers $h_j^{(n)}(t)$, $j = 1, \dots, n$, such that $\sum_{j=1}^n |h_j^{(n)}(t)|^p \le 1$ and $\sum_{j=1}^n h_j^{(n)}(t)f_j(t) = \sum_{j=1}^n |f_j(t)|^q \to \infty$ as $n \to \infty$. However, we need to choose h_j independently of n.

By applying the assumption that $\sum_{j=1}^\infty |f_j|^q = \infty$ ($\mathbb{P}$-a.e.), for any strictly increasing sequence $\alpha = \{\alpha_k\}_{k=1}^\infty$ of positive integers, there exists a strictly increasing sequence $\{N_k\}_{k=1}^\infty$

of positive integers such that the measure $\mathbb{P}(A_k)$ of the set

$$A_k = \left\{ t \in [0,1] : \sum_{n=1}^{N_k} |f_n(t)|^q > \alpha_k \right\} \tag{4.1}$$

is greater than $1 - (1/k)$. Then $\limsup_{k\to\infty} A_k = \cap_{j=1}^\infty \cup_{k=j}^\infty A_k$ is a set of full measure, so almost every $t \in [0,1]$ belongs to infinitely many sets A_k, $k = 1, 2, \ldots$. The sequence α will be chosen later.

For each $k = 1, 2, \ldots$ and $t \in [0,1]$, define

$$h_{j,k}(t) = \begin{cases} 0 & \text{if } j > N_k, \\ \dfrac{|f_j(t)|^q \cdot \chi_{A_k}(t)}{2^k f_j(t) \left(\sum_{n=1}^{N_k} |f_n(t)|^q\right)^{1/p}} & \text{if } j = 1, \ldots, N_k. \end{cases}$$

Here we set $0/0 = 0$.

For each $j, K = 1, 2, \ldots$, let $h_j^{(K)} = \sum_{k=1}^K |h_{j,k}|$ be the K'th partial sum of $|h_{j,k}|$, $k = 1, 2, \ldots$. Fix $t \in [0,1]$. Given $K = 1, 2, \ldots$, Minkowski's inequality yields that

$$\left(\sum_{j=1}^\infty \left| h_j^{(K)}(t) \right|^p \right)^{1/p} = \left(\sum_{j=1}^\infty \left(\sum_{k=1}^K |h_{j,k}(t)| \right)^p \right)^{1/p} \le \sum_{k=1}^K \left(\sum_{j=1}^\infty |h_{j,k}(t)|^p \right)^{1/p}. \tag{4.2}$$

Moreover, since $p(q-1) = q$, we have, for every $k = 1, \ldots, K$, that

$$\sum_{j=1}^\infty |h_{j,k}(t)|^p = 2^{-kp} \sum_{j=1}^{N_k} |f_j(t)|^{p(q-1)} \cdot \left(\sum_{n=1}^{N_k} |f_n(t)|^q \right)^{-1} \cdot \chi_{A_k}(t) \le 2^{-kp}.$$

So (4.2) implies that $\sum_{j=1}^\infty \left(h_j^{(K)}(t)\right)^p \le 1$ for all $K = 1, 2, \ldots$. In particular,

$$\sum_{k=1}^\infty |h_{j,k}(t)| = \lim_{K\to\infty} \sum_{k=1}^K |h_{j,k}(t)| = \lim_{K\to\infty} h_j^{(K)}(t) \le 1$$

for every $j = 1, 2, \ldots$, which enables us to define a Borel measurable function h_j on $[0,1]$ by $h_j(t) := \sum_{k=1}^\infty h_{j,k}(t)$ for all $t \in [0,1]$. Appealing to the Monotone Convergence Theorem ensures that

$$\begin{aligned} \sum_{j=1}^\infty |h_j(t)|^p &= \sum_{j=1}^\infty \left| \sum_{k=1}^\infty h_{j,k}(t) \right|^p \le \sum_{j=1}^\infty \left(\sum_{k=1}^\infty |h_{j,k}(t)| \right)^p \\ &= \sum_{j=1}^\infty \left(\lim_{K\to\infty} h_j^{(K)}(t) \right)^p = \lim_{K\to\infty} \sum_{j=1}^\infty \left(h_j^{(K)}(t) \right)^p \le 1. \end{aligned}$$

Therefore, property (1) holds and because $f_j(t)h_j(t) \ge 0$ for all $j = 1, 2, \ldots$ and $t \in [0,1]$, property (2) also holds.

To check property (3), let $j = 1, 2, \dots$ and $t \in [0, 1]$. Then

$$
\begin{aligned}
h_j(t) f_j(t) &= f_j(t) \sum_{k=1}^{\infty} h_{j,k}(t) \\
&= f_j(t) \left(\sum_{\{k : N_k < j\}} h_{j,k}(t) \; + \sum_{\{k : N_k \geq j\}} h_{j,k}(t) \right) \\
&= \sum_{\{k : N_k \geq j\}} 2^{-k} \cdot \chi_{A_k}(t) \cdot |f_j(t)|^q \left(\sum_{n=1}^{N_k} |f_n(t)|^q \right)^{-1/p} . \qquad (4.3)
\end{aligned}
$$

Then, given $k = 1, 2, \dots$ and $t \in A_k$, it follows from equation (4.3) that

$$
h_j(t) f_j(t) \geq 2^{-k} |f_j(t)|^q \left(\sum_{n=1}^{N_k} |f_n(t)|^q \right)^{-1/p}
$$

for all $j = 1, \dots, N_k$, and hence,

$$
\sum_{j=1}^{N_k} h_j(t) f_j(t) \geq 2^{-k} \left(\sum_{n=1}^{N_k} |f_n(t)|^q \right)^{1/q} > 2^{-k} \alpha_k^{1/q}.
$$

As noted above, $\mathbb{P}$-almost every $t \in [0, 1]$ belongs to infinitely many sets A_k, $k = 1, 2, \dots$, so choosing $\alpha_k := k 2^{kq}$ for each $k = 1, 2, \dots$ ensures that $\sum_{j=1}^{\infty} h_j(t) f_j(t) = \infty$ for $\mathbb{P}$-almost every $t \in [0, 1]$. □

Let $\{c_j\}_{j=1}^{\infty}$ be the sequence mentioned at the beginning of this section, $f_j = c_j g_j$ for $j = 1, 2, \dots$ and suppose that h_j, $j = 1, 2, \dots$, are any measurable functions satisfying properties (1), (2) and (3) of Lemma 4.1.

Lemma 4.2. *The mapping* $u : f \longmapsto \{\mathbb{E}(f h_j)\}_{j=1}^{\infty}$, $\quad f \in L^1([0, 1])$, *is a continuous linear map from* $L^1([0, 1])$ *into* ℓ^p *such that* $\sum_{k=1}^{\infty} \|u(g_k \chi_A)\|_{\ell^p}^p = \infty$ *whenever* A *is a Borel subset of* $[0, 1]$ *of positive measure.*

Proof. Let $f \in L^1([0, 1])$. To check that the sequence $\{\mathbb{E}(f h_j)\}_{j=1}^{\infty}$ belongs to ℓ^p, suppose that $\xi \in \ell^q$. Then, given $n = 1, 2, \dots$, we have $\sum_{j=1}^{n} \xi(j) \, \mathbb{E}(f h_j) = \mathbb{E}(f \sum_{j=1}^{n} \xi(j) \, h_j)$ and

$$
\begin{aligned}
\Big| \sum_{j=1}^{n} \xi(j) \, h_j(t) \Big| &\leq \Big(\sum_{j=1}^{n} |\xi(j)|^q \Big)^{1/q} \Big(\sum_{j=1}^{n} |h_j(t)|^p \Big)^{1/p} \\
&\leq \|\xi\|_{\ell^q}.
\end{aligned}
$$

for every $t \in [0, 1]$ by property (1) of Lemma 4.1. Therefore, $u(f) \in \ell^p$ and

$$
\|u(f)\|_{\ell^p} \leq \|f\|_{L^1([0,1])} \qquad \text{for every } f \in L^1([0, 1]).
$$

Appealing the Monotone Convergence Theorem and the fact that $c_j h_j g_j \geq 0$ for each $j = 1, 2, \ldots$, we have, for every non-null Borel set $A \subseteq [0,1]$, that

$$\begin{aligned}\sum_{k=1}^{\infty} \left\| u(g_k \chi_A) \right\|_{\ell^p}^p &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| \mathbb{E}(h_j g_k \chi_A) \right|^p \\ &\geq \sum_{j=1}^{\infty} \left| \mathbb{E}(h_j g_j \chi_A) \right|^p \\ &\geq \Big| \sum_{j=1}^{\infty} c_j \, \mathbb{E}(h_j g_j \chi_A) \Big|^p \\ &= \left| \mathbb{E}\Big(\sum_{j=1}^{\infty} c_j h_j g_j \chi_A \Big) \right|^p = \infty,\end{aligned}$$

because $\sum_{j=1}^{\infty} |c_j|^q \leq 1$ and because property (3) of Lemma 4.1 gives $\sum_{j=1}^{\infty} c_j h_j g_j \chi_A = \infty$ ($\mathbb{P}$-a.e. on A) . □

Proof of Theorem 3.2. Let $2 < p < \infty$. Then the continuous linear map $u : L^1([0,1]) \to \ell^p$ constructed above is not p-summing and by Lemmas 3.1 and 4.2, the sequence $\{g_j\}_{j=1}^{\infty}$ in $L^1([0,1])$ has the property that $\sum_{j=1}^{\infty} \left|\langle g_j, f\rangle\right|^p < \infty$, for every $f \in L^\infty[0,1]$, but $\sum_{j=1}^{\infty} \left\|u(g_j \chi_A)\right\|_{\ell^p}^p = \infty$ for every Borel set $A \subseteq [0,1]$ of positive measure. Now it follows from Lemma 2.2 that $\beta_X(m)(A) \geq \sum_{j=1}^{\infty} \left\|u(g_j \chi_A)\right\|_{\ell^p}^p = \infty$. □

Remark 4.3. The continuous linear map $u : L^1([0,1]) \to \ell^p$ is not p-summing because $\sup_{\|f\|_{L^\infty([0,1])} \leq 1} \sum_{k=1}^{\infty} |\langle g_k, f\rangle|^p < \infty$ by Lemma 3.1, but $\sum_{k=1}^{\infty} \|u(g_k)\|_{\ell^p}^p = \infty$ by Lemma 4.2. For $2 < p < \infty$, there are many examples of non-p-summing continuous linear maps from $L^1([0,1])$ to ℓ^p. Indeed, if X is any Banach space and $w : X \to \ell^p$ is a surjective continuous linear map, then the lifting property of ℓ^1 ensures that the following diagram is commutative:

$$\begin{array}{ccc} & \tilde{T} & X \\ & \nearrow & \downarrow \quad w \\ \ell^1 & \to & \ell^p \\ & T & \end{array}$$

If we choose T to be a non-p-summing continuous linear map [9, Lemma 4.1], then w cannot be p-summing, that is, no *surjective* continuous linear map $w : X \to \ell^p$ is absolutely p-summing. However, for the purpose of proving Theorem 3.2, we also need an explicit sequence busting the absolutely p-summing property.

References

[1] O. Blasco, *Remarks on the semivariation of vector measures with respect to Banach spaces*, preprint.

[2] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland, Amsterdam, 1993.

[3] J. Diestel, *Sequences and Series in Banach Spaces*, Springer-Verlag, New York, 1984.

[4] J. Diestel and J.J. Uhl Jr., *Vector Measures*, Math Surveys No.**15**, Amer. Math. Soc., Providence, 1977.

[5] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge University Press, Cambridge, 1995.

[6] I. Dobrakov, *On integration in Banach spaces I*, Czech. Math. J. **20** (1970), 511-536.

[7] B. Jefferies, *Evolution Processes and the Feynman-Kac Formula*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1996.

[8] B. Jefferies and S. Okada, *Bilinear integration in tensor products*, Rocky Mountain J. Math. **28** (1998), 517-545.

[9] ______, *Semivariation in L^p-spaces*, Comment. Math. Univ. Carolinae **46** (2005), 425–436.

[10] S. Kwapień, *On a theorem of L. Schwartz and its application to absolutely summing operators*, Studia Math, **38** (1970), 193–201.

[11] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence Spaces*, Springer-Verlag, Berlin, New York, 1977.

[12] G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, CBMS Regional Conference Series in Mathematics, 60, Amer. Math Soc., Providence, RI, 1986.

[13] H.P. Rosenthal, *On subspaces of L^p*, Ann. Math. **97** (1973), 344-373.

[14] J. Rosiński and Z. Suchanecki, On the space of vector-valued functions integrable with respect to the white noise, *Colloq. Math.* **43** (1980), 183–201.

[15] C. Swartz, *Integrating bounded functions for the Dobrakov integral*, Math. Slovaca **33** (1983), 141-144.

[16] N.N. Vakhania, V.I. Tarieladze and S.A. Chobanyan, *Probability Distributions on Banach Spaces* (English translation), D. Reidel Academic Publishing Co., Dordrecht, 1987.

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