

# Approximation numbers of composition operators on the Hardy space of the infinite polydisk

*Daniel Li, Hervé Queffélec, L. Rodríguez-Piazza*

March 14, 2017

**Abstract.** We study the composition operators of the Hardy space on  $\mathbb{D}^\infty \cap \ell_1$ , the  $\ell_1$  part of the infinite polydisk, and the behavior of their approximation numbers.

## 1 Introduction

Recently, in [2], we investigated approximation numbers  $a_n(C_\varphi)$ ,  $n \geq 1$ , of composition operators  $C_\varphi$ ,  $C_\varphi(f) = f \circ \varphi$ , on the Hardy or Bergman spaces  $H^2(\Omega)$ ,  $B^2(\Omega)$  over a bounded symmetric domain  $\Omega \subseteq \mathbb{C}^d$ . Assuming that  $\varphi(\Omega)$  has non-empty interior, one of the main results of this study was the following theorem.

**Theorem 1.1** ([2]). *Let  $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$  be compact. Then:*

- 1) *we always have  $a_n(C_\varphi) \geq c e^{-C n^{1/d}}$  where  $c, C$  are positive constants;*
- 2) *if  $\Omega$  is a product of balls and if  $\varphi(\Omega) \subseteq r \Omega$  for some  $r < 1$ , then:*

$$a_n(C_\varphi) \leq C e^{-c n^{1/d}}.$$

As a result, the minimal decay of approximation numbers is slower and slower as the dimension  $d$  increases, which might lead one to think that, in infinite-dimension, no compact composition operators can exist, since their approximation numbers will not tend to 0. After all, this is the case for the Hardy space of a half-plane, which supports no compact composition operator ([12], Theorem 3.1; in [9], it is moreover proved that  $\|C_\varphi\|_e = \|C_\varphi\|$  as soon as  $C_\varphi$  is bounded; see also [15] for a necessary and sufficient condition for  $H^2(\Omega)$  has compact composition operators, where  $\Omega$  is a domain of  $\mathbb{C}$ ). We will see that this is not quite the case here, even though the decay will be severely limited. In particular, we will never have a decay of the form  $C e^{-c n^\delta}$  for some  $c, C, \delta > 0$ .

## 2 Framework and reminders

### 2.1 Hardy spaces on $\mathbb{D}^\infty$

Let  $\mathbb{T} = \partial\mathbb{D}$  be the unit circle of the set of complex numbers. We consider  $\mathbb{T}^\infty$  and equip it with its Haar measure  $m$ . This is a compact Abelian group with dual  $\mathbb{Z}^{(\infty)}$ , the set of eventually zero sequences  $\alpha = (\alpha_j)_{j \geq 1}$  of integers. We denote  $L^2_{\mathbb{N}^{(\infty)}}(\mathbb{T}^\infty)$  the Hilbert subspace of  $L^2(\mathbb{T}^\infty)$  formed by the functions  $f$  whose Fourier spectrum is contained in  $\mathbb{N}^{(\infty)}$ :

$$\widehat{f}(\alpha) := \int_{\mathbb{T}^\infty} f(z) \bar{z}^\alpha dm(z) = 0 \quad \text{if } \alpha \notin \mathbb{N}^{(\infty)}.$$

The set  $E := \mathbb{N}^{(\infty)}$  is called the *narrow cone of Helson*, and we also denote  $L^2_{\mathbb{N}^{(\infty)}}(\mathbb{T}^\infty) = L^2_E(\mathbb{T}^\infty)$ . Any element of that subspace can be formally written as:

$$f = \sum_{\alpha \geq 0} c_\alpha e_\alpha \quad \text{with } c_\alpha = \widehat{f}(\alpha) \quad \text{and} \quad \sum_{\alpha \geq 0} |c_\alpha|^2 < \infty.$$

Here,  $(e_\alpha)_{\alpha \in \mathbb{Z}^{(\infty)}}$  is the canonical basis of  $L^2(\mathbb{T}^\infty)$  formed by characters, and accordingly  $(e_\alpha)_{\alpha \in \mathbb{N}^{(\infty)}}$  is the canonical basis of  $L^2_E(\mathbb{T}^\infty)$ .

Now we consider  $\Omega_2 = \mathbb{D}^\infty \cap \ell_2$ .

Any  $f \sim \sum_{\alpha \geq 0} c_\alpha e_\alpha \in L^2_E(\mathbb{T}^\infty)$  defines an analytic function on the infinite-dimensional Reinhardt domain  $\Omega_2$  by the formula:

$$(2.1) \quad f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha$$

where the series is absolutely convergent for each  $z = (z_j)_{j \geq 1} \in \Omega_2$ , as the pointwise product of two square-summable sequences. Indeed, using an Euler type formula, we get for  $z \in \Omega_2$ :

$$\sum_{\alpha \geq 0} |z^\alpha|^2 = \prod_{j=1}^{\infty} (1 - |z_j|^2)^{-1} < \infty,$$

and hence, by the Cauchy-Schwarz inequality:

$$\sum_{\alpha \geq 0} |c_\alpha z^\alpha| \leq \left( \sum_{\alpha \geq 0} |c_\alpha|^2 \right)^{1/2} \left( \sum_{\alpha \geq 0} |z^\alpha|^2 \right)^{1/2} < \infty.$$

If  $\alpha \in E$  and  $z \in \Omega_2$ , we have set, as usual,  $z^\alpha = \prod_{j \geq 1} z_j^{\alpha_j}$ .

This shows that  $L^2_E(\mathbb{T}^\infty)$  can be identified with  $H^2(\Omega_2)$ , the Hardy-Hilbert space of analytic functions  $f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha$  on  $\Omega_2$  with

$$\|f\|^2 := \sum_{\alpha \geq 0} |c_\alpha|^2 < \infty.$$

This setting is customary in connection with Dirichlet series (see [7]).

In this paper, for a technical reason appearing below in the proof of Proposition 2.5, we will consider, instead of  $\Omega_2 = \mathbb{D}^\infty \cap \ell_2$ , the sub-domain:

$$\Omega = \mathbb{D}^\infty \cap \ell_1,$$

i.e. the *open* subset of  $\ell^1$  formed by the sequences:

$$z = (z_n)_{n \geq 1} \quad \text{such that} \quad |z_n| < 1, \forall n \geq 1, \quad \text{and} \quad \sum_{n=1}^{\infty} |z_n| < \infty,$$

and the restrictions to  $\Omega$  of the functions  $f \in H^2(\Omega_2)$ . We denote  $H^2(\Omega)$  the space of such restrictions.

Hence  $f \in H^2(\Omega)$  if and only if:

$$f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha \quad \text{with } z \in \Omega,$$

and  $\|f\|^2 := \sum_{\alpha \geq 0} |c_\alpha|^2 < \infty$ .

We now identify the space  $L_E^2(\mathbb{T}^\infty)$  with the space  $H^2(\Omega)$ .

We more generally define Hardy spaces  $H^p(\Omega)$ , for  $1 \leq p < \infty$ , in the usual way:

$$H^p = H^p(\Omega) = \{f: \Omega \rightarrow \mathbb{C}; \|f\|_p < \infty\},$$

where  $f$  is analytic in  $\Omega$  and  $\|f\|_p = \sup_{0 < r < 1} M_p(r, f) = \lim_{r \rightarrow 1^-} M_p(r, f)$  with:

$$M_p(r, f) = \left( \int_{\mathbb{T}^\infty} |f(rz)|^p dm(z) \right)^{1/p}, \quad 0 < r < 1.$$

We have  $\|f\| = \|f\|_2$ . Moreover,  $H^q$  contractively embeds into  $H^p$  for  $p < q$ .

## 2.2 Singular numbers

We begin with a reminder of operator-theoretic facts. We recall that the approximation numbers  $a_n(T) = a_n$  of an operator  $T: H \rightarrow H$  (with  $H$  a Hilbert space) are defined by:

$$a_n = \inf_{\text{rank } R < n} \|T - R\|.$$

According to a 1957's result of Allahverdiev (see [3], page 155), we have  $a_n = s_n$ , the  $n$ -th singular number of  $T$ . We also recall a basic result due to H. Weyl and one obvious consequence:

**Theorem 2.1.** *Let  $T: H \rightarrow H$  be a compact operator with eigenvalues  $(\lambda_n)$  rearranged in decreasing order and singular numbers  $(a_n)$ . Then:*

$$\prod_{j=1}^n |\lambda_j| \leq \prod_{j=1}^n a_j \quad \text{for all } n \geq 1.$$

As a consequence:

$$|\lambda_{2n}|^2 \leq a_1 a_n.$$

### 2.3 Spectra of projective tensor products

The following operator-theoretic result will play a basic role in the sequel. Let  $E_1, \dots, E_n$  be Banach spaces and let  $E = \otimes_{i=1}^n E_i$  their *projective* tensor product (the only tensor product we shall use). If  $T_i \in \mathcal{L}(E_i)$ , we define as usual their projective tensor product  $T = \otimes_{i=1}^n T_i \in \mathcal{L}(E)$  by its action on the atoms of  $E$ , namely:

$$T(\otimes_{i=1}^n x_i) = \otimes_{i=1}^n T_i(x_i).$$

Denote in general  $\sigma(x)$  the spectrum of  $x \in \mathcal{A}$  where  $\mathcal{A}$  is a unital Banach algebra. We recall ([13], chap.11, Theorem 11.23) the following result.

**Lemma 2.2.** *Let  $\mathcal{A}$  be a unital Banach algebra, and  $x_1, \dots, x_n$  be pairwise commuting elements of  $\mathcal{A}$ . Then:*

$$\sigma(x_1 \cdots x_n) \subseteq \prod_{i=1}^n \sigma(x_i).$$

Here,  $\prod_{i=1}^n \sigma(x_i)$  is the product in the Minkowski sense, namely:

$$\prod_{i=1}^n \sigma(x_i) = \left\{ \prod_{i=1}^n \lambda_i : \lambda_i \in \sigma(x_i) \right\}.$$

As a consequence, we then have the following lemma due to Schechter, which we prove under a weakened form, sufficient here, and which is indeed already in [1] (we just add a few details because this is a central point in our estimates).

**Lemma 2.3.** *Let  $F$  be a Banach space,  $T_1, \dots, T_n \in \mathcal{L}(F)$  and  $T = \otimes_{i=1}^n T_i$ . Then  $\sigma(T) \subset \prod_{i=1}^n \sigma(T_i)$ .*

*Proof.* To save notation, we assume  $n = 2$ . Let  $x_1 = T_1 \otimes I_2$  and  $x_2 = I_1 \otimes T_2$  where  $I_i$  is the identity of  $E_i$ . Clearly,

$$x_1 x_2 = x_2 x_1 = T_1 \otimes T_2 = T \quad \text{and} \quad \sigma(x_i) = \sigma(T_i)$$

where the spectrum of  $x_i$  is in the Banach algebra  $\mathcal{L}(E)$  and that of  $T_i$  in  $\mathcal{L}(E_i)$ . Lemma 2.2 now gives:

$$\sigma(T) = \sigma(x_1 x_2) \subseteq \sigma(x_1) \sigma(x_2) = \sigma(T_1) \sigma(T_2),$$

hence the result. □

### 2.4 Schur maps and composition operators

We now pass to some general facts on composition operators  $C_\varphi$ , defined by  $C_\varphi(f) = f \circ \varphi$ , associated with a Schur map, namely a *non-constant* analytic self-map  $\varphi$  of  $\Omega$ . We say that  $\varphi$  is a *symbol* for  $H^2(\Omega)$  if  $C_\varphi$  is a bounded linear operator from  $H^2(\Omega)$  into itself.

The differential  $\varphi'(a)$  of  $\varphi$  at some point  $a \in \Omega$  is a bounded linear map  $\varphi'(a): \ell^1 \rightarrow \ell^1$ .

**Definition 2.4.** The symbol  $\varphi$  is said to be truly infinite-dimensional if the differential  $\varphi'(a)$  is an injective linear map from  $\ell^1$  into itself for at least one point  $a \in \Omega$ .

In finite dimension, this amounts to saying that  $\varphi(\Omega)$  has non-void interior.

We have the following general result.

**Proposition 2.5.** Let  $(\varphi_j)_{j \geq 1}$  be a sequence of analytic self-maps of  $\mathbb{D}$  such that  $\sum_{j \geq 1} |\varphi_j(0)| < \infty$ . Then, the mapping  $\varphi: \Omega \rightarrow \mathbb{C}^\infty$  defined by the formula  $\varphi(z) = (\varphi_j(z_j))_{j \geq 1}$  maps  $\Omega$  to itself and is a symbol for  $H^2(\Omega)$ .

*Proof.* First, the Schwarz inequality:

$$|\varphi_j(z_j) - \varphi_j(0)| \leq 2|z_j|$$

shows that  $\varphi(z) \in \Omega$  when  $z \in \Omega$ . To see that  $\varphi$  is moreover a symbol for  $H^2(\Omega)$ , we use the fact ([8]) that:

$$(2.2) \quad \|C_{\varphi_j}\| \leq \sqrt{\frac{1 + |\varphi_j(0)|}{1 - |\varphi_j(0)|}}.$$

Now, by the separation of variables and Fubini's theorem, we easily get:

$$(2.3) \quad \|C_\varphi\| \leq \prod_{j=1}^{\infty} \|C_{\varphi_j}\| < \infty.$$

As  $\sum_{j \geq 1} |\varphi_j(0)| < \infty$ , by hypothesis, the infinite product

$$\prod_{j \geq 1} \sqrt{\frac{1 + |\varphi_j(0)|}{1 - |\varphi_j(0)|}}$$

converges and, in view of (2.2) and (2.3),  $C_\varphi$  is bounded.  $\square$

We also have the following useful fact.

**Lemma 2.6.** The automorphisms of  $\Omega$  act transitively on  $\Omega$  and define bounded composition operators on  $H^2(\Omega)$ .

*Proof.* Let  $a = (a_j)_j \in \Omega$  and let  $\Psi_a: \Omega \rightarrow \mathbb{C}^\infty$  be defined by:

$$\Psi_a(z) = (\Phi_{a_j}(z_j))_{j \geq 1}$$

where in general  $\Phi_u: \mathbb{D} \rightarrow \mathbb{D}$  is defined by  $\Phi_u(z) = (z - u)/(1 - \bar{u}z)$ . The Schwarz lemma gives  $|\Phi_{a_j}(z_j) + a_j| \leq 2|z_j|$ , and shows that  $\Psi_a$  maps  $\Omega$  to itself. Clearly,  $\Psi_a$  is an automorphism of  $\Omega$  with inverse  $\Psi_{-a}$  and  $\Psi_a(a) = 0$ . The fact that the composition operator  $C_{\Psi_a}$  is bounded on  $H^2(\Omega)$  is a consequence of Proposition 2.5.  $\square$

### 3 Spectrum of compact composition operators

We begin with the following definition, following [10].

**Definition 3.1.** Let  $\varphi: \Omega \rightarrow \Omega$  be a truly infinite-dimensional symbol. We say that  $\varphi$  is compact if  $\overline{\varphi(\Omega)}$  is a compact subset of  $\Omega$ .

We then have the following result.

**Lemma 3.2.** If  $\varphi: \Omega \rightarrow \Omega$  is a compact mapping, then:

- 1)  $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$  is bounded and moreover compact.
- 2) If  $a \in \Omega$  a fixed point of  $\varphi$ ,  $\varphi'(a) \in \mathcal{L}(\ell^1)$  is a compact operator.

*Proof.* 1) follows from a H. Schwarz type criterion via an Ascoli-Montel type theorem: every sequence  $(f_n)$  of  $H^2(\Omega)$  bounded in norm contains a subsequence which converges uniformly on compact subsets of  $\Omega$ . Indeed, we have the following ([4], chap. 17, p. 274): if  $A$  is a locally bounded set of holomorphic functions on  $\Omega$ , then  $A$  is locally equi-Lipschitz, namely every point  $a \in \Omega$  has a neighbourhood  $U \subset \Omega$  such that:

$$z, w \in U \quad \text{and} \quad f \in A \quad \implies \quad |f(z) - f(w)| \leq C_{A,U} \|z - w\|.$$

The Ascoli-Montel theorem easily follows from this. Then, if  $f_n \in H^2(\Omega)$  converges weakly to 0, it converges uniformly to 0 on compact subsets of  $\Omega$ ; in particular on  $\overline{\varphi(\Omega)}$ . This means that  $\|C_\varphi(f_n)\|_\infty = \|f_n \circ \varphi\|_\infty \rightarrow 0$ , implying  $\|f_n \circ \varphi\|_2 \rightarrow 0$  and the compactness of  $C_\varphi$ .

Actually,  $C_\varphi$  is compact on every Hardy space  $H^p(\Omega)$ ,  $1 \leq p \leq \infty$ . This observation will be useful later on.

For 2), we may indeed dispense ourselves with the invariance of  $a$ , and force  $a = 0$  to be a fixed point of  $\varphi$ . Indeed, we can replace  $\varphi$  by  $\psi = \Psi_b \circ \varphi \circ \Psi_a$  where  $b = \varphi(a)$  is arbitrary, and use Lemma 2.6 as well as the ideal property of compact linear operators. We set  $A = \varphi'(0)$ . Expanding each coordinate  $\varphi_j$  of  $\varphi$  in a series of homogeneous polynomials, we may write (since  $\varphi(0) = 0$ ):

$$\varphi(z) = \sum_{|\alpha|=1} c_\alpha z^\alpha + \sum_{s=2}^{\infty} \left( \sum_{|\alpha|=s} c_\alpha z^\alpha \right) = A(z) + \sum_{s=2}^{\infty} \left( \sum_{|\alpha|=s} c_\alpha z^\alpha \right),$$

where  $c_\alpha = (c_{\alpha,j})_{j \geq 1} \in \mathbb{C}^\infty$ . We clearly have (looking at the Fourier series of  $\varphi(z e^{i\theta})$ ):

$$(3.1) \quad \|z\|_1 < 1 \quad \implies \quad z \in \Omega \quad \implies \quad A(z) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(z e^{i\theta}) e^{-i\theta} d\theta.$$

Since  $\varphi$  is compact, this clearly implies, with  $B$  the open unit ball of  $\ell^1$ , that  $A(B)$  is totally bounded, proving the compactness of  $A$ .  $\square$

The following extension of results of [11], then [1] and [6], which themselves extend a theorem of G. Königs ([14], p. 93) will play an essential role for lower bounds of approximation numbers.

**Theorem 3.3.** *Let  $\varphi: \Omega \rightarrow \Omega$  be a compact symbol. Assume there is  $a \in \Omega$  such that  $\varphi(a) = a$  and that  $\varphi'(a) \in \mathcal{L}(\ell^1)$  is injective. Then, the spectrum of  $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$  is exactly formed by the numbers  $\lambda^\alpha$ ,  $\alpha \in \mathbb{N}^{(\infty)}$ , and  $0, 1$ , where  $(\lambda_j)_{j \geq 1}$  denote the eigenvalues of  $A := \varphi'(a)$  and:*

$$\lambda^\alpha = \prod_{j \geq 1} \lambda_j^{\alpha_j} \quad \text{if } \alpha = (\alpha_j)_{j \geq 1} \in \mathbb{N}^{(\infty)}.$$

*Proof.* This is proved in [1] for the unit ball  $B_E$  of an arbitrary Banach space  $E$  and for the space  $H^\infty(B_E)$ , in four steps which are the following:

1. If  $\varphi(B_E)$  lies strictly inside  $B_E$  (namely if  $\varphi(B_E) \subseteq rB_E$  for some  $r < 1$ ), in particular when  $\varphi$  is compact,  $\varphi$  has a unique fixed point  $a \in B_E$ , according to a theorem of Earle and Hamilton.
2. The spectrum of  $C_\varphi$  contains the numbers  $\lambda$  where  $\lambda$  is an eigenvalue of  $\varphi'(a)$  or  $\lambda = 0, 1$ .
3. It is then proved that the spectrum of  $C_\varphi$  contains the numbers  $\lambda^\alpha$  and  $0, 1$ .
4. It is finally proved that spectrum of  $C_\varphi$  is contained in the numbers  $\lambda^\alpha$  and  $0, 1$ .

Here, handling with the domain  $\Omega$ , we see that:

1. True or not for  $\Omega$ , the Earle-Hamilton theorem is not needed since we will force, by a change of the compact symbol  $\varphi$  in another compact symbol  $\psi = \Psi_b \circ \varphi \circ \Psi_a$ , the point  $0$  to be a fixed point. Moreover  $A = \psi'(0)$  is injective if  $\varphi'(a)$  is, since  $\Psi'_a$  and  $\Psi'_b$  are invertible.
2. The second step (non-surjectivity) is valid for any domain and for  $H^2(\Omega)$ , or  $H^p(\Omega)$ , in exactly the same way.
3. The third step consists of proving  $\{\lambda^\alpha\} \subseteq \sigma(C_\varphi)$ .  
For that purpose, assume that  $\lambda^\alpha = \prod_{l=1}^m \lambda_l \neq 0$  with  $\lambda_l$  an eigenvalue of  $\varphi'(0)$  and with repetitions allowed. As we already mentioned, under the assumption of compactness of  $\varphi$ ,  $C_\varphi$  is compact on  $H^p(\Omega)$  as well, for any  $p \geq 1$ . We take here  $p = 2m$ . Step 2 provides us with non-zero functions  $f_i \in H^p(\Omega)$  such that  $f_i \circ \varphi = \lambda_i f_i$ ,  $1 \leq i \leq m$ , since for the compact operator  $C_\varphi: H^p \rightarrow H^p$ , non-surjectivity implies non-injectivity. Let  $f = \prod_{1 \leq i \leq m} f_i$ . Then, using the integral representation of the norm and the Hölder inequality, we see that  $f \in H^2(\Omega)$ ,  $f \neq 0$  and  $f \circ \varphi = \lambda^\alpha f$ , proving our claim.
4. The fourth step is valid as well, with a slight simplification: we have to show that, if  $\mu \neq 1$  is not of the form  $\lambda^\alpha$ , then  $C_\varphi - \mu I$  is injective. Let  $f \in H^2(\Omega)$  satisfying  $f \circ \varphi = \mu f$  and let:

$$f(z) = \sum_{m=0}^{\infty} \frac{d^m f(0)}{m!} (z^m)$$

be the Taylor expansion of  $f$  about  $z = 0$  (observe that  $\Omega$  is a Reinhardt domain). As usual,  $d^m f(0) =: L_m$  is an  $m$ -linear symmetric form on  $F = \ell^1$  and the notation  $L_m(z^m)$  means  $L_m(z, z, \dots, z)$ .

Observe that  $L_m$  can be isometrically identified with an element (denoted  $\overline{L_m}$ ) of  $\mathcal{L}(F^{\otimes n})$  defined by the formula:

$$\overline{L_m}(x_1 \otimes \cdots \otimes x_n) = L_m(x_1, \dots, x_m).$$

We will prove by induction that  $L_n = 0$  for each  $n$ . For this, we can avoid the appeal to transposes of [1] as follows: if the result holds for  $L_m$  with  $m < n$ , one gets (comparing the terms in  $z^n$  in both members of  $f \circ \varphi = \mu f$ ):

$$(3.2) \quad \mu A = A \circ B \quad \text{where} \quad A = \overline{L_n} \quad \text{and} \quad B = \varphi'(0)^{\otimes n}.$$

That is  $A(B - \mu I) = 0$  where  $I$  is the identity map of  $F^{\otimes n}$ . Now,  $B - \mu I$  is invertible in  $\mathcal{L}(F)$  by Lemma 3.3, so that  $A = A(B - \mu I)(B - \mu I)^{-1} = 0$ .

The proof is complete.  $\square$

The following theorem summarizes and exploits the preceding theorem. Possibly, some restrictions can be removed, and we could only assume the compactness of  $C_\varphi$ , not of  $\varphi$  itself. After all, in dimension one, there are symbols  $\varphi$  with  $\|\varphi\|_\infty = 1$  for which  $C_\varphi: H^2 \rightarrow H^2$  is compact.

**Theorem 3.4.** *Let  $\varphi: \Omega \rightarrow \Omega$  be a truly infinite-dimensional compact mapping of  $\Omega$ . Then:*

- 1)  $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$  is bounded and even compact.
- 2)  $A = \varphi'(0)$  is compact.
- 3) No  $\delta > 0$  can exist such that  $a_n(C_\varphi) \leq C e^{-c n^\delta}$  for all  $n \geq 1$ . More precisely, the numbers  $a_n$  satisfy:

$$(3.3) \quad \sum_{n \geq 1} \frac{1}{\log^p(1/a_n)} = \infty \quad \text{for all } p < \infty.$$

*Proof.* The proof is based on the previous Theorem 3.3. Without loss of generality, we can assume that  $\varphi(0) = 0$  and  $\varphi'(0)$  is injective, by using a point  $a$  at which  $\varphi'(a)$  is injective, and then the fact that automorphisms of  $\Omega$  act transitively on  $\Omega$ , act boundedly on  $H^2(\Omega)$ , and the ideal property of approximation numbers. More precisely, we pass to  $\Psi = \Psi_b \circ \varphi \circ \Psi_a$  with  $b = \varphi(a)$  and get:

$$\Psi(0) = 0 \quad \text{and} \quad \Psi'(b) = \Psi'_b(b) \varphi'(a) \Psi'_a(0)$$

injective, since  $\Psi'_b(b)$  and  $\Psi'_a(0)$  are, and  $\Psi_a$  and  $\Psi_b$  are automorphisms of  $\Omega$ .

We now have the following simple but crucial lemma.

**Lemma 3.5.** *Whatever the choice of the numbers  $\lambda_j$  with  $0 < |\lambda_j| < 1$ , denoting by  $(\delta_n)_{n \geq 1}$  the non-increasing rearrangement of the numbers  $\lambda^\alpha$ , one has:*

$$\sum_{n \geq 1} \frac{1}{\log^p(1/\delta_n)} = \infty \quad \text{for all } p < \infty.$$



*Proof of the Lemma.* For any positive integer  $p$ , we set:

$$q = 2p, \quad \log 1/|\lambda_j| = A_j,$$

and we use that:

$$\sum_{1 \leq j \leq q} \alpha_j A_j \leq \left( \sum_{1 \leq j \leq q} \alpha_j^2 \right) \left( \sum_{1 \leq j \leq q} A_j^2 \right) =: C_q \left( \sum_{1 \leq j \leq q} \alpha_j^2 \right) = C_q \|\alpha\|^2,$$

where  $\|\cdot\|$  stands for the euclidean norm in  $\mathbb{R}^q$ . We then get:

$$\begin{aligned} \sum_{n \geq 1} \frac{1}{\log^p(1/\delta_n)} &= \sum_{\alpha > 0} \frac{1}{\log^p(1/|\lambda^\alpha|)} \\ &\geq \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{\log^p(1/|\lambda_1^{\alpha_1}| \cdots 1/|\lambda_q^{\alpha_q}|)} \\ &= \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{(\alpha_1 A_1 + \cdots + \alpha_q A_q)^p} \\ &\geq C_q^{-p} \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{(\alpha_1^2 + \cdots + \alpha_q^2)^p} \\ &= C_q^{-p} \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{\|\alpha\|^q} = \infty, \end{aligned}$$

because:

$$\int_{x \in \mathbb{R}^q, \|x\| \geq 1} \frac{1}{\|x\|^q} dx = c_q \int_1^\infty \frac{r^{q-1}}{r^q} dr = \infty.$$

This proves the lemma.  $\square$

This can be transferred to the approximation numbers  $a_n = a_n(C_\varphi)$  to end the proof of Theorem 3.4. Indeed, we know from Lemma 3.5 that the non-increasing rearrangement  $(\delta_n)$  of the eigenvalues  $\lambda^\alpha$  of  $C_\varphi$  satisfies

$$\sum_{n \geq 1} \frac{1}{\log^p(1/\delta_n)} = \infty.$$

Since a divergent series of non-negative and non-increasing numbers  $u_n$  satisfies  $\sum u_{2n} = \infty$ , we further see that:

$$\sum_{n \geq 1} \frac{1}{\log^p(1/\delta_{2n})} = \infty \quad \text{for all } p < \infty.$$

Moreover, by Theorem 2.1 we have:

$$(3.4) \quad \left( \frac{1}{2 \log 1/\delta_{2n}} \right)^p \leq \left( \frac{1}{\log 1/(a_1 a_n)} \right)^p.$$

Since  $1/(\log 1/a_1 a_n) \sim 1/(\log 1/a_n)$ , Lemma 3.5 then gives the result. This clearly prevents an inequality of the form  $a_n \leq C e^{-c n^\delta}$  for some positive numbers  $c, C, \delta$  and all  $n \geq 1$ . Indeed, this would imply:

$$\sum_{n \geq 1} \frac{1}{\log^p(1/a_n)} < \infty \quad \text{for } p > 1/\delta,$$

contradicting (3.3). □

**Remarks.** Let us briefly comment on the assumptions in Theorem 3.4.

1) We do not need the Earle-Hamilton theorem under our assumptions. The Schauder-Tychonoff theorem gives the existence (if not the uniqueness) of a fixed point for  $\varphi$  in  $\Omega$  (bounded and convex).

2) The Earle-Hamilton theorem is in some sense more general (for analytic maps) since it remains valid when  $\overline{\varphi(\Omega)}$  is only assumed to lie strictly inside  $\Omega$ , i.e. when  $\varphi(\Omega) \subseteq r\Omega$  for some  $r < 1$ . But this assumption does not ensure the compactness of  $C_\varphi$  as indicated by the simple example  $\varphi(z) = rz$ ,  $0 < r < 1$ . The coordinate functions  $z \mapsto z_n$  converge weakly to 0, while  $\|C_\varphi(z_n)\|_{H^2(\Omega)} = r$ .

3) The mere assumption that  $\overline{\varphi(\Omega)}$  is compact is not sufficient either. Just take:

$$\varphi(z) = \left( \frac{1+z_1}{2}, 0, \dots, 0, \dots \right).$$

Since the composition operator  $C_{\varphi_1}$  associated with  $\varphi_1(z) = \frac{1+z}{2}$  is notoriously non-compact on  $H^2(\mathbb{D})$ , neither is  $C_\varphi$  on  $H^2(\Omega)$ . Yet,  $\overline{\varphi(\Omega)}$  is obviously compact in  $\ell^1$ .

## 4 Possible upper bounds

Recall that  $\Omega = \mathbb{D}^\infty \cap \ell^1$ .

### 4.1 A general example

**Theorem 4.1.** *Let  $\varphi((z_j)_j) = (\lambda_j z_j)_j$  with  $|\lambda_j| < 1$  for all  $j$ , so that  $\varphi(\Omega) \subseteq \Omega$  and  $\varphi'(0)$  is the diagonal operator with eigenvalues  $\lambda_j$ ,  $j \geq 1$ , on the canonical basis of  $\ell^1$ . Let  $p > 0$ . Then:*

$$(\lambda_j)_j \in \ell^p \quad \implies \quad C_\varphi \in S_p.$$

*In particular, there exist truly infinite-dimensional symbols on  $\Omega$  such that the composition operator  $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$  is in all Schatten classes  $S_p$ ,  $p > 0$ .*

*Proof.* Since  $C_\varphi$  is diagonal on the orthonormal basis  $(z^\alpha)_\alpha$  of the Hilbert space  $H^2(\Omega)$ , with  $C_\varphi(z^\alpha) = \varphi^\alpha$ , its approximation numbers are the non-increasing rearrangement of the moduli of eigenvalues  $\lambda^\alpha$ , so that an Euler product-type computation gives:

$$\sum_{n=1}^{\infty} a_n^p = \sum_{\alpha \in E} |\lambda^\alpha|^p = \sum_{\alpha_j \in \mathbb{N}} \prod_{j \geq 1} |\lambda_j|^{p\alpha_j} = \prod_{j=1}^{\infty} (1 - |\lambda_j|^p)^{-1} < \infty.$$

To obtain  $C_\varphi \in \bigcap_{p>0} S_p$ , just take  $\lambda_n = e^{-n}$ . This ends the proof.  $\square$

## 4.2 A sharper upper bound

By making a more quantitative study, we can prove the following result.

**Theorem 4.2.** *For any  $0 < \delta < 1$ , there exists a compact composition operator on  $H^2(\Omega)$ , with a truly infinite-dimensional symbol, such that, for some positive constants  $c, C, b$ , we have:*

$$a_n(C_\varphi) \leq C \exp\left(-c e^{b(\log n)^\delta}\right).$$

*Proof.* Take the same operator  $C_\varphi$  as in Theorem 4.1, with  $\lambda_n = e^{-A_n}$  where the positive numbers  $A_n$  have to be adjusted. Its approximation numbers  $a_N$  are then the non-increasing rearrangement of the sequence of numbers  $(\varepsilon_n)_n := (\lambda^\alpha)_\alpha$ . This suggests using a generating function argument, namely considering  $\sum \varepsilon_n x^n$ , but the rearrangement perturbs the picture. Accordingly, we follow a slightly different route. Fix an integer  $N \geq 1$  and a real number  $r > 0$ . Observe that, following the proof of Theorem 4.1:

$$N a_N^r \leq \sum_{n=1}^N a_n^r \leq \sum_{n=1}^{\infty} a_n^r = \prod_{n=1}^{\infty} (1 - e^{-rA_n})^{-1}.$$

First, consider the simple example  $A_n = n$ . We get:

$$N a_N^r \leq \eta(e^{-r})$$

where  $\eta$  is the Dedekind eta function (see [5]) given by:

$$\eta(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1} = \sum_{n=0}^{\infty} p(n) x^n, \quad |x| < 1,$$

where  $p(n)$  is the number of partitions of the integer  $n$ . It is well-known ([5], Ch. 7, p. 169) that  $\eta(e^{-r}) \leq e^{D/r}$  with  $D = \pi^2/6$ , so that:

$$a_N \leq \exp\left(\frac{D}{r^2} - \frac{\log N}{r}\right).$$

Optimizing with  $r = 2D/\log N$ , we get:

$$a_N \leq \exp(-c \log^2 N),$$

with  $c = 1/4D$ . This is more precise than Theorem 4.1.

We now show that if  $A_n$  increases faster, we can achieve the decay of Theorem 4.2. As before, we get in general:

$$(4.1) \quad a_N \leq \inf_{x>1} (\exp [x(\log F(x^{-1}) - \log N)]),$$

where

$$F(r) = \prod_{n=1}^{\infty} (1 - e^{-rA_n})^{-1}.$$

We have:

$$\log F(r) = \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{e^{-rmA_n}}{m} \right) = \sum_{m=1}^{\infty} \frac{1}{m} \left( \sum_{n=1}^{\infty} e^{-rmA_n} \right).$$

Now, take  $A_n = e^{n^\alpha}$  where  $\alpha > 0$  is to be chosen. We have:

$$\sum_{n=1}^{\infty} e^{-rm e^{n^\alpha}} \leq \int_0^{\infty} e^{-rm e^{t^\alpha}} dt =: I_m(r).$$

Standard estimates now give, for  $r < 1$ :

$$\begin{aligned} I_m(r) &= \int_1^{\infty} e^{-rmx} \frac{1}{\alpha} (\log x)^{\frac{1}{\alpha}-1} \frac{dx}{x} = \int_{rm}^{\infty} e^{-y} \frac{1}{\alpha} \left( \log \frac{y}{rm} \right)^{\frac{1}{\alpha}-1} \frac{dy}{y} \\ &\lesssim \left( \log \frac{1}{r} \right)^{\frac{1}{\alpha}-1} \int_{rm}^{\infty} e^{-y} \frac{dy}{y} \lesssim e^{-rm} \left( \log \frac{1}{r} \right)^{\frac{1}{\alpha}}, \end{aligned}$$

so that:

$$\log F(r) \lesssim (\log 1/r)^{\frac{1}{\alpha}} \sum_{m=1}^{\infty} m^{-1} e^{-rm} \lesssim (\log 1/r)^{\frac{1}{\alpha}+1}.$$

Going back to (4.1), we get, for some constant  $C > 0$ , and for  $x = 1/r > 1$ :

$$a_N \leq C \exp [C x ((\log x)^{\frac{1}{\alpha}+1} - \log N)].$$

Adjusting  $x = x_N > 1$  so as to have  $(\log x)^{\frac{1}{\alpha}+1} = \log N - 1$ , that is:

$$x_N = \exp [(\log(N/e))^{\frac{\alpha}{\alpha+1}}],$$

we get  $a_N \leq C e^{-c x_N}$ , which is the claimed result with  $\delta = \alpha/(\alpha + 1)$ .

This  $\delta$  can be taken arbitrarily in  $(0, 1)$  by choosing  $\alpha$  suitable, and we are done.  $\square$

**Remark.** Of course,  $\delta = 1$  is forbidden, because this would give  $a_n \leq C e^{-c n^b}$ , implying:

$$\sum_{n=1}^{\infty} \frac{1}{(\log 1/a_n)^p} \lesssim \sum_{n=1}^{\infty} n^{-bp} < \infty,$$

for large  $p$ , and contradicting Theorem 3.4.

## References

- [1] R. Aron, P. Galindo and M. Lindström, *Compact homomorphisms between algebras of analytic functions*, *Studia Math.* **123**(3) (1997), 235–247.
- [2] F. Bayart, D. Li, H. Queffélec and L. Rodríguez-Piazza, *Approximation numbers of composition operators on the Hardy and Bergman spaces of the ball and of the polydisk*, *Math. Proc. of the Cambridge Philos. Soc.*, to appear.
- [3] B. Carl and I. Stephani. *Entropy, compactness and the approximation of operators*, Cambridge Tracts in Mathematics 98, Cambridge University Press, Cambridge (1990).
- [4] S. Chae, *Holomorphy and Calculus in Normed Spaces*, Monographs and Textbooks in Pure and Applied Mathematics 92, Marcel Dekker (1985).
- [5] K. Chandrasekharan, *Arithmetical Functions*, Grundlehren Math. Wiss., Band 167, Berlin, Springer (1970).
- [6] D. Clahane, *Spectra of compact composition operators over bounded symmetric domains*, *Integral Equations Operator Theory* **51** (2005), 41–56.
- [7] B. Cole and T. Gamelin, *Representing measures and Hardy spaces for the infinite polydisk algebra*, *Proc. Lond. Math. Society* **53** (1986), 112–142.
- [8] C. Cowen, B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, C.R.C. Press (1994).
- [9] S. E. Elliott and M. T. Jury, *Composition operators on Hardy spaces of a half plane*, *Bull. London Math. Soc.* **44**, no. 3 (2012), 489–495.
- [10] P. Lefèvre, *Generalized Essential Norm of Weighted Composition Operators on some Uniform Algebras of Analytic Functions*, *Integral Equations Operator Theory* **63** (2009), 557–569.
- [11] B. MacCluer, *Spectra of compact composition operators on  $H^p(B_N)$* , *Analysis* **4** (1984), 87–103.
- [12] V. Matache, *Composition operators on Hardy spaces on a half-plane*, *Proc. Amer. Math. Soc.* **127**(5) (1999), 1483–1491.
- [13] W. Rudin, *Functional Analysis*, Second edition, Internat. Ser. Pure Appl. Math., McGraw-Hill (1991).
- [14] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Universitext, Tracts in Mathematics, Springer-Verlag, New York (1993).
- [15] J. H. Shapiro and W. Smith, *Hardy spaces that support no compact composition operators*, *J. Funct. Anal.* **205** no. 1 (2003), 62–89.

Daniel Li  
Univ. Artois, Laboratoire de Mathématiques de Lens (LML) EA 2462, &  
Fédération CNRS Nord-Pas-de-Calais FR 2956  
Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18  
F-62 300 LENS, FRANCE  
daniel.li@euler.univ-artois.fr

Hervé Queffélec  
Univ. Lille Nord de France, USTL  
Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération CNRS Nord-Pas-  
de-Calais FR 2956  
F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE  
Herve.Queffelec@univ-lille1.fr

Luis Rodríguez-Piazza  
Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis  
Matemático & IMUS  
Apartado de Correos 1160  
41 080 SEVILLA, SPAIN  
piazza@us.es