The Menger number of the strong product of graphs

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A B S T R A C T

The xy-Menger number with respect to a given integer ℓ, for every two vertices x, y in a connected graph G, denoted by $\zeta_\ell(x, y)$, is the maximum number of internally disjoint xy-paths whose lengths are at most ℓ in G. The Menger number of G with respect to ℓ is defined as $\zeta_\ell(G) = \min\{\zeta_\ell(x, y) : x, y \in V(G)\}$. In this paper we focus on the Menger number of the strong product $G_1 \boxtimes G_2$ of two connected graphs G_1 and G_2 with at least three vertices. We show that $\zeta_\ell(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2)$ and furthermore, that $\zeta_{\ell+2}(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$ if both G_1 and G_2 have girth at least 5. These bounds are best possible, and in particular, we prove that the last inequality is reached when G_1 and G_2 are maximally connected graphs.

1. Introduction

Throughout this paper, all the graphs are simple, that is, with neither loops nor multiple edges. Notations and terminology not explicitly given here can be found in the book by Chartrand and Lesniak [2].

Let G be a graph with a vertex set $V = V(G)$ and an edge set $E = E(G)$. Let x and y be two distinct vertices of G. A path from x to y, also called an xy-path in G, is a subgraph P with vertex set $V(P) = \{x = u_0, u_1, \ldots, u_r = y\}$ and edge set $E(P) = \{u_0u_1, \ldots, u_{r-1}u_r\}$. This path is usually denoted by $P : u_0u_1 \ldots u_r$ and r is the length of P, denoted by $l(P)$. Two xy-paths P and Q are said to be internally disjoint if $V(P) \cap V(Q) = \{x, y\}$. A cycle in G of length r is a path $C : u_0u_1 \ldots u_r$ such that $u_0 = u_r$. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G, and if G contains no cycles, then $g(G) = \infty$. The set of vertices adjacent to $v \in V(G)$ is denoted by $N_G(v)$. The *degree* of v is $d_G(v) = |N_G(v)|$, whereas $\delta(G) = \min_{v \in V(G)} d_G(v)$ is the minimum degree of G.

The distance between two vertices $x, y \in V(G)$, denoted by $d_G(x, y)$, is the length of a shortest xy-path. If there is no xy-path in G, it is said that $d_G(x, y) = \infty$. The diameter of G is defined as $Diam(G) = \max\{d_G(x, y) : x, y \in V(G)\}$. A graph G is connected if for any two distinct vertices $x, y \in V(G)$ there is an xy-path. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose deletion from G produces a disconnected or a trivial graph. There is an important research on this topic (see, e.g., [3]). From Menger's Theorem, Whitney [9] proved in 1932 that a graph G is r-connected, that is, $\kappa(G) \geq r$, if and only if every pair of vertices in $V(G)$ is connected by r internally disjoint paths. In [9] the author also shows that $\kappa(G) \leq \delta(G)$. A graph G is maximally connected if the previous bound is attained, that is, if $\kappa(G) = \delta(G)$.

Given two distinct vertices x, y in a connected graph G, the xy-Menger number with respect to a positive integer ℓ is the maximum number of internally disjoint xy-paths in G whose lengths are at most ℓ. It is denoted by $\zeta_\ell(x, y)$. The Menger number of G with respect to ℓ is defined as $\zeta_\ell(G) = \min\{\zeta_\ell(x, y) : x, y \in V(G)\}$. This parameter was introduced in [5]. Clearly, if $\ell < Diam(G)$, then $\zeta_\ell(G) = 0$ and also, for every integer $\ell \geq |V(G)| - 1$, the Menger number $\zeta_\ell(G) = \kappa(G)$.

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The determination of $\zeta_\ell(G)$ is an open and interesting problem when $Diam(G) \leq \ell \leq |V(G)| - 2$. Observe that $\zeta_\ell(G)$ is an increasing function on ℓ and that $\zeta_\ell(G) \leq \kappa(G)$ for every positive integer ℓ.

For an information system modeled by a graph G, the Menger number can be an important measure of the communication efficiency and fault tolerance. For instance, in a parallel computing system, the efficiency can be analyzed in terms of the number of disjoint routes of information which are able to connect two points in a short period of time. In a real-time system, the information delay must be limited since any message obtained beyond the bound may be worthless. A natural question is to compute or estimate how many routes ensure the transmission of information in an effective time.

Ma, Xu, and Zhu [6] found a lower bound on the *Menger number* of the Cartesian product of two connected graphs G_1 and G_2. Namely, they prove that $\zeta_{\ell_1+\ell_2}(G_1 \Box G_2) \geq \zeta_{\ell_1}(G_1) + \zeta_{\ell_2}(G_2)$. This bound is an equality when G_1 and G_2 are paths and, therefore, $G_1 \Box G_2$ is a grid network.

In this work we study the *Menger number* of the strong product of two connected graphs. The *strong product* $G_1 \boxtimes G_2$ of two connected graphs G_1 and G_2 is defined on the Cartesian product of the vertex sets of the generators, so that two distinct vertices (x_1, x_2) and (y_1, y_2) of $G_1 \boxtimes G_2$ are adjacent if $x_1 = y_1$ and $x_2y_2 \in E(G_2)$, or $x_1y_1 \in E(G_1)$ and $x_2 = y_2$, or $x_1y_1 \in E(G_1)$ and $x_2y_2 \in E(G_2)$. From this definition, it follows that the strong product of two connected graphs is commutative.

It is well known that the product of graphs is an important research topic in Graph Theory (see, e.g. [1,4,7,8,10]). A fundamental principle for network design is extendibility. That is to say, the possibility of building larger versions of a network preserving certain desirable properties. For designing large-scale interconnection networks, the strong product is a useful method to obtain large graphs from smaller ones whose invariants can be easily calculated.

In this paper, we prove that the Menger number $\zeta_\ell(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2)$, for any two connected graphs with at least three vertices. Moreover, if both G_1 and G_2 have also girth at least 5, then we prove that $\zeta_{\ell+2}(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$. These two lower bounds are best possible in a double sense. On the one hand, we provide examples that show that the hypothesis cannot be relaxed. And on the other hand, we give examples of graphs G_1 and G_2 for which both these lower bounds are sharp.

2. Main results

Given two connected graphs G_1 and G_2, in this paper we focus on $\zeta_\ell(G_1 \boxtimes G_2)$, the Menger number of the strong product $G_1 \boxtimes G_2$ with respect to an integer ℓ. First of all, let us notice that $\zeta_\ell(G_1 \boxtimes G_2) = 0$ for integers $\ell < Diam(G_1 \boxtimes G_2) = \max\{Diam(G_1), Diam(G_2)\}$, hence, from now on we assume that $\ell \geq \max\{Diam(G_1), Diam(G_2)\}$.

To estimate the Menger number $\zeta_\ell(G_1 \boxtimes G_2)$, we must find a lower bound on the number of internally disjoint paths of length at most ℓ that join any two arbitrary vertices in $V(G_1 \boxtimes G_2)$. The proof is constructive and in the following lemmas we provide these paths. To do that, for distinct vertices $x_1, y_1 \in V(G_1)$, we consider $\zeta_1 = \zeta_\ell(x_1, y_1)$ internally disjoint x_1y_1-paths $P_1, \ldots, P_{\zeta_1}$ in G_1 of length at most ℓ. Similarly, for distinct vertices $x_2, y_2 \in V(G_2)$, we consider $\zeta_2 = \zeta_\ell(x_2, y_2)$ internally disjoint x_2y_2-paths $Q_1, \ldots, Q_{\zeta_2}$ in G_2 of length at most ℓ. Without loss of generality we assume that $l(P_1) = \min\{l(P_i) : i \in \{1, \ldots, \zeta_1\}\}$ and that $l(Q_1) = \min\{l(Q_j) : j \in \{1, \ldots, \zeta_2\}\}$. Also, for any x_2y_2-path Q_j in G_2 of length at least 2, we denote by $(Q_j)'$ the new path obtained from Q_j by removing its endvertices.

Observe that for every $v \in V(G_2)$, the subgraph of $G_1 \boxtimes G_2$ induced by the set $\{(x_1, v) : x_1 \in V(G_1)\}$ is isomorphic to G_1. For this reason, this subgraph will be denoted by G_1^v. Analogously, for each $u \in V(G_1)$, the set $G_2^u = \{(u, x_2) : x_2 \in V(G_2)\}$ induces a subgraph isomorphic to G_2. Thus, each x_2y_2-path Q_j in G_2 induces an $(u, x_2)(u, y_2)$-path in G_2^u, which will be denoted by Q_j^u.

The first result provides a lower bound on the Menger number between two distinct vertices $(x_1, x_2), (y_1, y_2)$ in $V(G_1 \boxtimes G_2)$ such that either $x_1 = y_1$ or $x_2 = y_2$.

Lemma 2.1. *Let G_1 and G_2 be two connected graphs with at least three vertices. Let $x_i, y_i \in V(G_i)$ be two distinct vertices, $i = 1, 2$. For every integer $\ell \geq \max\{Diam(G_1), Diam(G_2)\}$ the following assertions hold:*

(i) *There exist at least $(\delta(G_1) + 1)\zeta_\ell(G_2)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$. Furthermore, if G_1 has girth at least 5, then there exist at least $\delta(G_1)$ additional internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most $\ell + 2$.*

(ii) *There exist at least $\zeta_\ell(G_1)(\delta(G_2) + 1)$ internally disjoint $(x_1, x_2)(y_1, x_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$. Moreover, if G_2 has girth at least 5, then there exist at least $\delta(G_2)$ additional internally disjoint $(x_1, x_2)(y_1, x_2)$-paths of length at most $\ell + 2$.*

Proof. By the commutativity of the strong product of two graphs, it suffices to prove (i). Denote by $\zeta_2 = \zeta_\ell(G_2)$. Let us consider any vertex $x_1 \in V(G_1)$ and two distinct vertices $x_2, y_2 \in V(G_2)$. Then there are at least ζ_2 internally disjoint x_2y_2-paths, $Q_1, \ldots, Q_{\zeta_2}$, in G_2 of length at most ℓ.

Now, we introduce some general constructions of $(x_1, x_2)(x_1, y_2)$-paths in $G_1 \boxtimes G_2$. Let $u \in N_{G_1}(x_1)$ and $j \in \{1, \ldots, \zeta_2\}$. If $l(Q_j) \geq 2$, then vertices (x_1, x_2) and (x_1, y_2) are adjacent to the first and to the last internal vertex of Q_j^u, respectively. Hence, it makes sense to consider the path $R_{u,j} : (x_1, x_2)(Q_j^u)'(x_1, y_2)$ in $G_1 \boxtimes G_2$. Notice that $l(R_{u,j}) \leq \ell$. Also, when there exists a vertex $w_u \in N_{G_1}(u) \setminus \{x_1\}$, we can consider the $(x_1, x_2)(x_1, y_2)$-path $R_{w_u} : (x_1, x_2)(u, x_2)(Q_1^{w_u})'(u, y_2)(x_1, y_2)$ of length at most $\ell + 2$.

Observe that vertices (x_1, x_2) and (x_1, y_2) belong to the same copy $G_2^{x_1}$ of $G_1 \boxtimes G_2$. Therefore, $Q_1^{x_1}, \dots, Q_{\zeta_2}^{x_1}$ are ζ_2 internally disjoint $(x_1, x_2)(x_1, y_2)$-paths in $G_1 \boxtimes G_2$ of length at most ℓ. To construct the remaining paths, we distinguish whether x_2y_2 belongs to $E(G_2)$ or not.

First, assume that $x_2y_2 \in E(G_2)$, that is, $l(Q_1) = 1$. Let $u \in N_{G_1}(x_1)$. The paths $\widetilde{R}_u : (x_1, x_2)(u, x_2)(x_1, y_2)$ and $\widehat{R}_u : (x_1, x_2)(u, y_2)(x_1, y_2)$ are contained in $G_1 \boxtimes G_2$ and they have length $2 \le \ell$. Moreover, since G_2 is a simple graph, for every $j \in \{2, \dots, \zeta_2\}$, the path Q_j has length at least 2 and there exists the path $R_{u,j}$. Hence, $Q_1^{x_1}, \dots, Q_{\zeta_2}^{x_1}, \widetilde{R}_u, \widehat{R}_u, R_{u,2}, \dots, R_{u,\zeta_2}$, for every $u \in N_{G_1}(x_1)$ are at least $\zeta_2 + 2\delta(G_1) + \delta(G_1)(\zeta_2 - 1) = (\delta(G_1) + 1)\zeta_2 + \delta(G_1)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$.

Second, assume that $x_2y_2 \notin E(G_2)$. For $j \in \{1, \dots, \zeta_2\}$ and $u \in N_{G_1}(x_1)$, we consider the path $R_{u,j}$. Thus, we have $(d_{G_1}(x_1) + 1)\zeta_2$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most ℓ. If there exists a vertex $u \in N_{G_1}(x_1)$ such that $d_{G_1}(u) = 1$, notice that $d_{G_1}(x_1) \ge 2$ and then $(d_{G_1}(x_1) + 1)\zeta_2 \ge 3\zeta_2 \ge 2\zeta_2 + 1 = (\delta(G_1) + 1)\zeta_2 + \delta(G_1)$. Otherwise, there exists a vertex $w_u \in N_{G_1}(u) \setminus \{x_1\}$ for every $u \in N_{G_1}(x_1)$. We assume that $g(G_1) \ge 5$, then $w_u \ne w_v$ for all $u, v \in N_{G_1}(x_1)$ with $u \ne v$. Hence, the paths R_{w_u}, $u \in N_{G_1}(x_1)$, are at least $\delta(G_1)$ internally disjoint $(x_1, x_2)(x_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$. $\square$

Now in the following two lemmas we study the number of internally disjoint paths between two vertices in $V(G_1 \boxtimes G_2)$ which come from two different vertices in G_1 and from another two different ones in G_2. Using paths of length at most ℓ in the generator graphs G_1 and G_2, we construct paths in $G_1 \boxtimes G_2$ whose lengths are also at most ℓ.

Lemma 2.2. *Let G_1 and G_2 be two connected graphs with at least three vertices and $\ell \ge \max\{Diam(G_1), Diam(G_2)\}$ be an integer. For every two distinct vertices $x_1, y_1 \in V(G_1)$ and every two distinct vertices $x_2, y_2 \in V(G_2)$, there exist at least $\zeta_\ell(G_1)\,\zeta_\ell(G_2)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$ of length at most ℓ.*

Proof. Denote by $\zeta_1 = \zeta_\ell(G_1)$ and $\zeta_2 = \zeta_\ell(G_2)$. Let $P_1, \dots, P_{\zeta_1}$ be ζ_1 internally disjoint x_1y_1-paths of length at most ℓ in G_1 and $Q_1, \dots, Q_{\zeta_2}$ be ζ_2 internally disjoint x_2y_2-paths of length at most ℓ in G_2. Let us assume that $P_i : u_0^i u_1^i \dots u_{r_i}^i$, for $i \in \{1, \dots, \zeta_1\}$, and that $Q_j : v_0^j v_1^j \dots v_{s_j}^j$, for $j \in \{1, \dots, \zeta_2\}$, where $(u_0^i, v_0^j) = (x_1, x_2)$ and $(u_{r_i}^i, v_{s_j}^j) = (y_1, y_2)$. For each $i \in \{1, \dots, \zeta_1\}$ and each $j \in \{1, \dots, \zeta_2\}$, associated to the x_1y_1-path P_i in G_1 and to the x_2y_2-path Q_j in G_2, we consider the $(x_1, x_2)(y_1, y_2)$-path $R_{i,j}$ in $G_1 \boxtimes G_2$ as follows:

(i) If $r_i < s_j$ then

$$R_{i,j} : \begin{cases} (u_0^i, v_0^j)(u_1^i, v_1^j) \dots (u_1^i, v_{s_j}^j), & \text{if } r_i = 1 \\ (u_0^i, v_0^j) \dots (u_{r_i-1}^i, v_{r_i-1}^j) \dots (u_{r_i-1}^i, v_{s_j-1}^j)(u_{r_i}^i, v_{s_j}^j), & \text{if } r_i \ge 2. \end{cases}$$

(ii) If $r_i \ge s_j$ then

$$R_{i,j} : \begin{cases} (u_0^i, v_0^j)(u_1^i, v_1^j) \dots (u_{r_i}^i, v_1^j), & \text{if } s_j = 1 \\ (u_0^i, v_0^j) \dots (u_{s_j-1}^i, v_{s_j-1}^j) \dots (u_{r_i-1}^i, v_{s_j-1}^j)(u_{r_i}^i, v_{s_j}^j), & \text{if } s_j \ge 2. \end{cases}$$

The length of the path $R_{i,j}$ is $l(R_{i,j}) = \max\{r_i, s_j\} \le \ell$. Since each path R_{ij} is associated to specific paths P_i and Q_j, they are internally disjoint in $G_1 \boxtimes G_2$ and the proof is complete. $\square$

Using paths of length at most ℓ in the generator graphs G_1 and G_2, we have just constructed $\zeta_\ell(G_1)\zeta_\ell(G_2)$ internally disjoint paths in $G_1 \boxtimes G_2$ of length at most ℓ which join two given vertices in $G_1 \boxtimes G_2$. But if we allow the length of the paths in $G_1 \boxtimes G_2$ to be at most $\ell + 2$, it is possible to construct $\zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$ such paths.

Lemma 2.3. *Let G_1 and G_2 be two connected graphs with at least three vertices and girth at least 5. Let $\ell \ge \max\{Diam(G_1), Diam(G_2)\}$ be an integer. For every two distinct vertices $x_1, y_1 \in V(G_1)$ and every two distinct vertices $x_2, y_2 \in V(G_2)$ there exist at least $\zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$.*

Proof. Let us denote by $\zeta_1 = \zeta_\ell(G_1)$ and $\zeta_2 = \zeta_\ell(G_2)$. Let $P_1, \dots, P_{\zeta_1}$ and $Q_1, \dots, Q_{\zeta_2}$ be internally disjoint paths defined as in the proof of Lemma 2.2, that is, $P_i : u_0^i u_1^i \dots u_{r_i}^i$ and $Q_j : v_0^j v_1^j \dots v_{s_j}^j$, where $(x_1, x_2) = (u_0^i, v_0^j)$ and $(y_1, y_2) = (u_{r_i}^i, v_{s_j}^j)$, for $i \in \{1, \dots, \zeta_1\}$ and $j \in \{1, \dots, \zeta_2\}$. Next, we provide $\zeta_1\,\zeta_2 + \zeta_1 + \zeta_2$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$ of length at most $\ell + 2$.

(I) First, by considering the x_1y_1-path P_1 in G_1 and the x_2y_2-path Q_1 in G_2, we construct three pairwise internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$ of length at most $\ell + 2$. These paths are denoted by $R'_{1,1}$, $\widetilde{R}_{1,1}$ and R^* and their construction is done according to the length of the paths P_1 and Q_1, that is, depending on r_1 and s_1.

(a) If $r_1 = 1$ and $s_1 = 1$, that is, if $P_1 : x_1y_1$ and $Q_1 : x_2y_2$, then

$$R'_{1,1} : (x_1, x_2)(x_1, y_2)(y_1, y_2),$$
$$\widetilde{R}_{1,1} : (x_1, x_2)(y_1, x_2)(y_1, y_2) \quad \text{and}$$
$$R^* : (x_1, x_2)(y_1, y_2)$$

are such paths. Their lengths are $l(R'_{1,1}) = l(\widetilde{R}_{1,1}) = 2$ and $l(R^*) = 1$.

(b) If $r_1 = 1$ and $s_1 \geq 2$, then

$$R'_{1,1} : (u_0^1, v_0^1) \ldots (u_0^1, v_{s_1-1}^1)(u_1^1, v_{s_1}^1),$$
$$\widetilde{R}_{1,1} : (u_0^1, v_0^1)(u_1^1, v_1^1) \ldots (u_1^1, v_{s_1}^1).$$

Notice that $l(R'_{1,1}) = l(\widetilde{R}_{1,1}) = s_1 \leq \ell$. In this case, it is impossible to construct in $G_1 \boxtimes G_2$ one more path induced only by P_1 and Q_1. We solve this problem in two different ways depending on the value ζ_1.

Assume that $\zeta_1 = 1$. Since $x_1y_1 \in E(G_1)$ and G_1 has at least three vertices, there exists a vertex $u \in V(G_1)$ such that either $ux_1 \in E(G_1)$ or $uy_1 \in E(G_1)$. Without loss of generality, we consider that $ux_1 \in E(G_1)$ and hence the first and the last internal vertex of the path Q_1^u are adjacent in $G_1 \boxtimes G_2$ to (x_1, x_2) and (x_1, y_2), respectively. Thus, we obtain the $(x_1, x_2)(y_1, y_2)$-path

$$R^* : (x_1, x_2)(Q_1^u)'(x_1, y_2)(y_1, y_2),$$

which has length $1 + s_1 - 2 + 1 + 1 \leq \ell + 1$.

If $\zeta_1 \geq 2$, since $g(G_1) \geq 5$ and $r_1 = 1$, the path P_2 exists and has length $r_2 \geq 4$. Recall that $u_0^1 = u_0^2 = x_1$, $u_1^1 = u_{r_2}^2 = y_1$, $v_0^1 = x_2$ and $v_{s_1}^1 = y_2$. We consider the path $R^* : (u_0^1, v_0^1)(u_1^1, v_0^1)R\,(u_0^1, v_{s_1}^1)(u_1^1, v_{s_1}^1)$, where

$$R : \begin{cases} (u_{r_2-1}^2, v_0^1)(u_{r_2-2}^2, v_1^1) \ldots (u_2^2, v_1^1)(u_1^2, v_2^1), & \text{if } s_1 = 2 \\ (u_{r_2-1}^2, v_0^1) \ldots (u_{r_2-s_1}^2, v_{s_1-1}^1) \ldots (u_1^2, v_{s_1-1}^1), & \text{if } r_2 > s_1 \text{ and } s_1 \neq 2 \\ (u_{r_2-1}^2, v_1^1) \ldots (u_{r_2-1}^2, v_{s_1-r_2+1}^1) \ldots (u_1^2, v_{s_1-1}^1), & \text{if } r_2 \leq s_1 \text{ and } s_1 \neq 2. \end{cases}$$

The design of this path R^* must be combined with the ones of $R_{2,1}$ and $\widehat{R}_{2,1}$ described below. That is the reason why it becomes necessary to distinguish several cases to construct these three internally disjoint paths associated to the paths P_1, P_2 in G_1 and Q_1 in G_2. Notice that $l(R^*) = \max\{s_1, r_2\} + 2 \leq \ell + 2$.

(c) The case $r_1 \geq 2$ and $s_1 = 1$ is similar to the previous one due to the commutativity of the strong product of graphs.

(d) If $r_1 \geq 2$ and $s_1 \geq 2$, then

$$R'_{1,1} : \begin{cases} (u_0^1, v_0^1) \ldots (u_0^1, v_{s_1-r_1+1}^1) \ldots (u_{r_1-1}^1, v_{s_1}^1)(u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 \leq s_1 \\ (u_0^1, v_0^1)(u_0^1, v_1^1) \ldots (u_{s_1-1}^1, v_{s_1}^1) \ldots (u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 > s_1, \end{cases}$$

$$\widetilde{R}_{1,1} : \begin{cases} (u_0^1, v_0^1)(u_1^1, v_0^1) \ldots (u_{r_1}^1, v_{r_1-1}^1) \ldots (u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 \leq s_1 \\ (u_0^1, v_0^1) \ldots (u_{r_1-s_1+1}^1, v_0^1) \ldots (u_{r_1}^1, v_{s_1-1}^1)(u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 > s_1, \end{cases}$$

and

$$R^* : \begin{cases} (u_0^1, v_0^1) \ldots (u_{r_1-1}^1, v_{r_1-1}^1) \ldots (u_{r_1-1}^1, v_{s_1-1}^1)(u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 \leq s_1 \\ (u_0^1, v_0^1) \ldots (u_{s_1-1}^1, v_{s_1-1}^1) \ldots (u_{r_1-1}^1, v_{s_1-1}^1)(u_{r_1}^1, v_{s_1}^1), & \text{if } r_1 > s_1. \end{cases}$$

In this case $l(R'_{1,1}) = l(\widetilde{R}_{1,1}) = \max\{r_1, s_1\} + 1 \leq \ell + 1$, whereas $l(R^*) \leq \ell$. These three paths prove constructively the desired result when $\zeta_1 = \zeta_2 = 1$.

(II) If $\zeta_1 \geq 2$, then associated to the x_2y_2-path Q_1 in G_2 and to each x_1y_1-path P_i in G_1, $i \in \{2, \ldots, \zeta_1\}$, we construct two $(x_1, x_2)(y_1, y_2)$-paths $R_{i,1}$ and $\widehat{R}_{i,1}$ of length at most $\ell + 2$ in $G_1 \boxtimes G_2$ as follows.

If $s_1 = 1$, then

$$R_{i,1} : (u_0^i, v_0^1) \ldots (u_{r_i-1}^i, v_0^1)(u_{r_i}^i, v_1^1),$$
$$\widehat{R}_{i,1} : (u_0^i, v_0^1)(u_1^i, v_1^1) \ldots (u_{r_i}^i, v_1^1).$$

As we have previously mentioned, the difficulty to construct the paths $R_{i,1}$ and $\widehat{R}_{i,1}$ takes root in the fact that they must be internally disjoint with the path R^* considered in (I).

If $s_1 = 2$, then

$$R_{i,1} : (u_0^i, v_0^1) \ldots (u_{r_i-2}^i, v_0^1)(u_{r_i-1}^i, v_1^1)(u_{r_i}^i, v_2^1),$$
$$\widehat{R}_{i,1} : (u_0^i, v_0^1)(u_1^i, v_1^1)(u_2^i, v_2^1) \ldots (u_{r_i}^i, v_2^1).$$

If $r_i = 3$ and $s_1 \geq 3$, then

$$R_{i,1} : (u_0^i, v_0^1)(u_1^i, v_1^1) \ldots (u_1^i, v_{s_1-1}^1)(u_2^i, v_{s_1}^1)(u_3^i, v_{s_1}^1),$$
$$\widehat{R}_{i,1} : (u_0^i, v_0^1)(u_1^i, v_0^1)(u_2^i, v_1^1) \ldots (u_2^i, v_{s_1-1}^1)(u_3^i, v_{s_1}^1).$$

If $r_i > s_1 \geq 3$, then

$$R_{i,1} : \begin{cases} (u_0^i, v_0^1)(u_1^i, v_1^1) \ldots (u_{r_i-s_1}^i, v_1^1) \ldots (u_{r_i-1}^i, v_{s_1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } s_1 \text{ is odd} \\ (u_0^i, v_0^1)(u_1^i, v_1^1) \ldots (u_{r_i-s_1+1}^i, v_1^1) \ldots (u_{r_i-2}^i, v_{s_1-2}^1)(u_{r_i-2}^i, v_{s_1-1}^1)(u_{r_i-1}^i, v_{s_1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } s_1 \text{ is even,} \end{cases}$$

$$\widehat{R}_{i,1} : \begin{cases} (u_0^i, v_0^1) \ldots (u_{r_i-s_1+1}^i, v_0^1) \ldots (u_{r_i-1}^i, v_{s_1-2}^1)(u_{r_i-1}^i, v_{s_1-1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } s_1 \text{ is odd} \\ (u_0^i, v_0^1) \ldots (u_{r_i-s_1+2}^i, v_0^1) \ldots (u_{r_i-1}^i, v_{s_1-3}^1) \ldots (u_{r_i-1}^i, v_{s_1-1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } s_1 \text{ is even.} \end{cases}$$

If $s_1 \geq r_i > 3$, then

$$R_{i,1}: \begin{cases} (u_0^i, v_0^1)(u_1^1, v_1^1) \dots (u_1^i, v_{s_1-r_i+3}^1) \dots (u_{r_i-2}^i, v_{s_1}^1) \dots (u_{r_i}^i, v_{s_1}^1), & \text{if } r_i \text{ is odd} \\ (u_0^i, v_0^1)(u_1^1, v_1^1) \dots (u_1^i, v_{s_1-r_i+2}^1) \dots (u_{r_i-1}^i, v_{s_1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } r_i \text{ is even,} \end{cases}$$

$$\widehat{R}_{i,1}: \begin{cases} (u_0^i, v_0^1)(u_1^i, v_0^1)(u_2^i, v_1^1) \dots (u_2^i, v_{s_1-r_i+2}^1) \dots (u_{r_i}^i, v_{s_1}^1), & \text{if } r_i \text{ is odd} \\ (u_0^i, v_0^1)(u_1^i, v_0^1)(u_2^i, v_1^1) \dots (u_2^i, v_{s_1-r_i+1}^1) \dots (u_{r_i-1}^i, v_{s_1-2}^1)(u_{r_i-1}^i, v_{s_1-1}^1)(u_{r_i}^i, v_{s_1}^1), & \text{if } r_i \text{ is even.} \end{cases}$$

The length of the paths $R_{i,1}$ and $\widehat{R}_{i,1}$ is at most $\max\{r_i, s_1\} + 2 \leq \ell + 2$. Notice that they are internally disjoint with all the paths described in (I).

If $\zeta_2 = 1$ and $\zeta_1 \geq 2$, then (I) and (II) provide $3 + 2(\zeta_1 - 1)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$, as it is desired.

(III) If $\zeta_2 \geq 2$, then the commutativity of the strong product of graphs leads us to deduce the existence of $2(\zeta_2 - 1)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths $R_{1,j}$ and $\widehat{R}_{1,j}$, for $j \in \{2, \dots, \zeta_2\}$ in $G_1 \boxtimes G_2$, constructed in an analogous way as in (II). They are associated to the x_1y_1-path P_1 in G_1 and to the x_2y_2-paths $Q_2, \dots, Q_{\zeta_2}$ in G_2.

If $\zeta_1 = 1$ and $\zeta_2 \geq 2$, then (I) and (III) provide $3 + 2(\zeta_2 - 1) = 2\zeta_2 + 1$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$ and the proof is finished.

(IV) If $\zeta_1 \geq 2$ and $\zeta_2 \geq 2$ then, for $i \in \{2, \dots, \zeta_1\}$ and $j \in \{2, \dots, \zeta_2\}$, associated to each x_1y_1-path P_i in G_1 and to each x_2y_2-path Q_j in G_2, we consider the path

$$R_{i,j}: \begin{cases} (u_0^i, v_0^j) \dots (u_{r_i-1}^i, v_{r_i-1}^j) \dots (u_{r_i-1}^i, v_{s_j-1}^j)(u_{r_i}^i, v_{s_j}^j), & \text{if } r_i < s_j \\ (u_0^i, v_0^j) \dots (u_{s_j-1}^i, v_{s_j-1}^j) \dots (u_{r_i-1}^i, v_{s_j-1}^j)(u_{r_i}^i, v_{s_j}^j), & \text{if } r_i \geq s_j. \end{cases}$$

It is easy to see that $l(R_{ij}) = \max\{r_i, s_j\} \leq \ell$ and that these $(\zeta_1 - 1)(\zeta_2 - 1)$ paths R_{ij} are internally disjoint with all the previous paths because they are associated to different paths in the generator graphs G_1 and G_2.

If $\zeta_1 \geq 2$ and $\zeta_2 \geq 2$, (I) to (IV) provide $3 + 2(\zeta_2 - 1) + 2(\zeta_1 - 1) + (\zeta_1 - 1)(\zeta_2 - 1) = \zeta_1\zeta_2 + \zeta_1 + \zeta_2$ pairwise internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$ of length at most $\ell + 2$. □

Making use of these previous lemmas, we provide next two lower bounds on the Menger number of the strong product of two connected graphs.

Theorem 2.1. *Let G_1 and G_2 be two connected graphs with at least three vertices and $\ell \geq \max\{Diam(G_1), Diam(G_2)\}$ be an integer. The following assertions hold:*

(i) $\zeta_\ell(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2)$.
(ii) $\zeta_{\ell+2}(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$ *if* $g(G_i) \geq 5$ *for* $i = 1, 2$.

Proof. Let us consider vertices x_1, y_1 in $V(G_1)$ and x_2, y_2 in $V(G_2)$.

(i) If $x_1 = y_1$ and $x_2 \neq y_2$ (resp. if $x_1 \neq y_1$ and $x_2 = y_2$), then, by Lemma 2.1, there exist at least $(\delta(G_1) + 1)\zeta_\ell(G_2) > \zeta_\ell(G_1)\zeta_\ell(G_2)$ (resp. $\zeta_\ell(G_1)(\delta(G_2) + 1) > \zeta_\ell(G_1)\zeta_\ell(G_2)$) internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$. If $x_1 \neq y_1$ and $x_2 \neq y_2$ then, by Lemma 2.2, there exist at least $\zeta_\ell(G_1)\zeta_\ell(G_2)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most ℓ in $G_1 \boxtimes G_2$. Therefore, $\zeta_\ell(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2)$.
(ii) Assume also that G_1 and G_2 have girth at least 5. If $x_1 = y_1$ and $x_2 \neq y_2$, then, by Lemma 2.1, there exist at least $(\delta(G_1) + 1)\zeta_\ell(G_2) + \delta(G_1) \geq \zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_2) + \zeta_\ell(G_1)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$. The same conclusion is obtained when $x_1 \neq y_1$ and $x_2 = y_2$, due to Lemma 2.1. If $x_1 \neq y_1$ and $x_2 \neq y_2$ then, by Lemma 2.3, there exist at least $\zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$ internally disjoint $(x_1, x_2)(y_1, y_2)$-paths of length at most $\ell + 2$ in $G_1 \boxtimes G_2$. Hence, $\zeta_{\ell+2}(G_1 \boxtimes G_2) \geq \zeta_\ell(G_1)\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)$. □

Theorem 2.1(i) provides a tight bound. In fact, the equality $\zeta_\ell(G_1 \boxtimes G_2) = \zeta_\ell(G_1)\zeta_\ell(G_2)$ holds, for instance, when G_1 and G_2 are both isomorphic to the path $P_{\ell+1}$ of length ℓ, or when G_1 and G_2 are isomorphic to the cycle $C_{2\ell+1}$ of length $2\ell + 1$ or when $G_1 = P_{\ell+1}$ and $G_2 = C_{2\ell+1}$ (see Fig. 1).

Also, Theorem 2.1(ii) is best possible in the sense that the hypothesis cannot be relaxed. On the one hand, the bound in Theorem 2.1(ii) may not be attained when at least one of the generator graphs has two vertices. For example, $\zeta_\ell(P_2 \boxtimes P_3) \leq \kappa(P_2 \boxtimes P_3) = 2 < \zeta_\ell(P_2)\zeta_\ell(P_3) + \zeta_\ell(P_2) + \zeta_\ell(P_3)$ for $\ell \geq 2$. On the other hand, the same bound may fail when the hypothesis of girth at least five is not fulfilled. For example, let G_1 be the graph formed by two cycles of length 5 which share a common vertex z, and let G_2 be a cycle of length 4. We consider an integer $\ell \geq \max\{Diam(G_1), Diam(G_2)\} = 4$. Clearly $\zeta_\ell(G_1) = 1$, because z is a cut vertex of G_1, and $\zeta_\ell(G_2) = 2$. Let us consider two distinct vertices $x_1, y_1 \in V(G_1) \setminus \{z\}$ such that any x_1y_1-path in G_1 passes through z. For any two vertices $x_2, y_2 \in V(G_2)$, it is impossible to construct five internally disjoint $(x_1, x_2)(y_1, y_2)$-paths in $G_1 \boxtimes G_2$, because each of these paths must contain a vertex of the subgraph G_2^z. But this graph has only four vertices because it is isomorphic to G_2, that is, to the cycle of length 4.

As a consequence of Theorem 2.1 we obtain the following result.

Corollary 2.1. *Let G_1 and G_2 be two maximally connected graphs with at least three vertices and girth at least 5. If ℓ is an integer such that $\zeta_\ell(G_1) = \kappa(G_1)$ and $\zeta_\ell(G_2) = \kappa(G_2)$, then $\zeta_{\ell+2}(G_1 \boxtimes G_2) = \delta(G_1 \boxtimes G_2)$.*

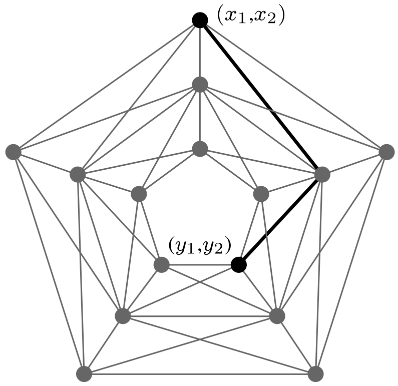

Fig. 1. Unique path of length 2 in $P_3 \boxtimes C_5$ joining vertices (x_1, x_2) and (y_1, y_2).

Proof. Taking into account Theorem 2.1, the maximal connectivity of graphs G_1 and G_2 and the fact that $\delta(G_1 \boxtimes G_2) = \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2)$, it follows that

$$\begin{aligned}\zeta_{\ell+2}(G_1 \boxtimes G_2) &\geq \zeta_\ell(G_1)\,\zeta_\ell(G_2) + \zeta_\ell(G_1) + \zeta_\ell(G_2)\\ &= \kappa(G_1)\,\kappa(G_2) + \kappa(G_1) + \kappa(G_2)\\ &= \delta(G_1)\,\delta(G_2) + \delta(G_1) + \delta(G_2)\\ &= \delta(G_1 \boxtimes G_2).\end{aligned}$$

Since the other inequality is trivial, the desired result is proved. □

As $\zeta_\ell(G) \leq \kappa(G)$ for every graph G and there exists an integer $\ell \leq |V(G)| - 1$ for which the previous inequality is in fact an equality, from Corollary 2.1 it follows the following corollary whose proof is straightforward.

Corollary 2.2. *Let G_1 and G_2 be two maximally connected graphs with at least three vertices and girth at least 5. Then $G_1 \boxtimes G_2$ is maximally connected.*

Applying Corollary 2.1 and considering that for integers $n \geq m$ the Menger number $\zeta_{n-1}(C_n) = \zeta_{n-1}(C_m) = 2$, we determine $\zeta_{n+1}(C_n \boxtimes C_m)$.

Corollary 2.3. *For integers $n \geq m \geq 5$, let C_n and C_m be two cycles of lengths n and m respectively. Then $\zeta_{n+1}(C_n \boxtimes C_m) = 8$.*

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