Partial Latin Squares with no Non-trivial Autotopisms and Six Autoparatopisms

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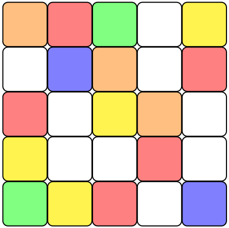
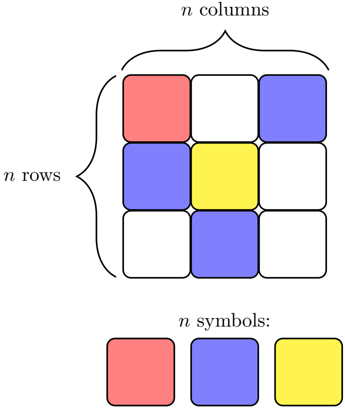
June 16, 2015



Partial Latin squares

No symbol repeats in any row, or any column.



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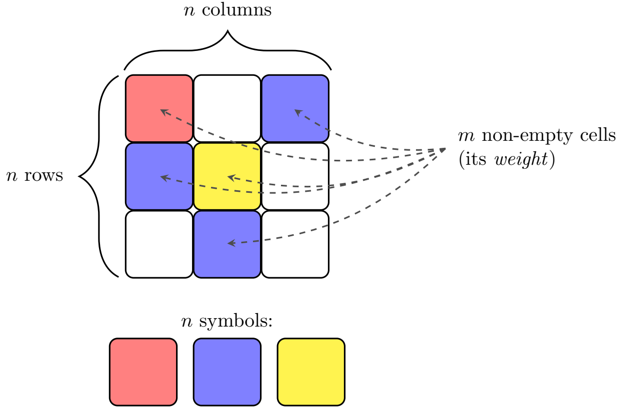

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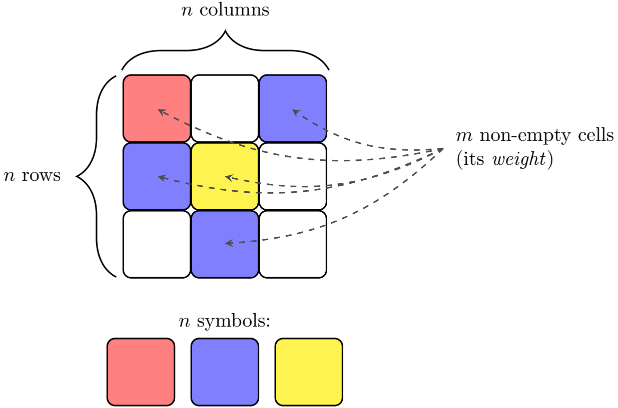

Every row, column, and symbol is "used".

Isotopisms and autotopisms

Starting with a partial Latin square, if we permute the rows, permute the columns, and permute the symbols we obtain another partial Latin square.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \cdot & \cdot \\ 2 & 3 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 5 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 6 & 5 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 6 & 5 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 6 \end{bmatrix} \xrightarrow[\text{swap last two columns}]{\text{swap first two rows}} \begin{bmatrix} 2 & 3 & 1 & \cdot & \cdot & \cdot \\ 1 & 2 & 3 & 4 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 5 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 6 & \cdot & 5 \\ \cdot & \cdot & \cdot & \cdot & 5 & 6 \\ \cdot & \cdot & \cdot & \cdot & 6 & \cdot \end{bmatrix}$$

These operations are called *isotopisms*.

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These operations are called *isotopisms*.

If we end up with original partial Latin square, it's called an *autotopism*.

$$\begin{bmatrix} 1 & \cdot \\ \cdot & 2 \end{bmatrix} \xrightarrow[]{\substack{\text{swap the two rows} \\ \text{swap the two columns} \\ \text{swap the two symbols}}} \begin{bmatrix} 1 & \cdot \\ \cdot & 2 \end{bmatrix}$$

Paratopisms and autoparatopisms

Partial Latin squares are equivalent to their entries, (row, column, symbol) triples:

$$\begin{bmatrix} 1 & 2 \\ 2 & \cdot \end{bmatrix} \quad \longleftrightarrow \quad \begin{matrix} (1,1,1) \\ (1,2,2) \\ (2,1,2) \end{matrix}$$

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We have another operation which preserves "partial Latin square"-ness: we uniformly permute the coordinates of the entries.

$$\begin{matrix} (1,1,1) \\ (1,2,2) \\ (2,1,2) \end{matrix} \quad \xrightarrow{\text{swap first two coordinates}} \quad \begin{matrix} (1,1,1) \\ (2,1,2) \\ (1,2,2) \end{matrix}$$

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Combining this operation with isotopism, gives us *paratopisms*

$$(\alpha, \beta, \gamma; \delta) \in (S_n \times S_n \times S_n) \rtimes S_3.$$

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$$(\alpha, \beta, \gamma; \delta) \in (S_n \times S_n \times S_n) \rtimes S_3.$$

A paratopism that maps a partial Latin square to itself is called an *autoparatopism*. (isotopism, autotopism $\iff \delta = \mathrm{id}$)

Why six autoparatopisms?

If a partial Latin square has two distinct autoparatopisms agreeing at the fourth coordinate

$$(\alpha, \beta, \gamma; \delta) \quad \text{and} \quad (\overline{\alpha}, \overline{\beta}, \overline{\gamma}; \delta)$$

then $(\alpha, \beta, \gamma; \delta)(\overline{\alpha}, \overline{\beta}, \overline{\gamma}; \delta)^{-1}$ is a non-trivial autotopism with a trivial fourth component.

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Therefore, if a partial Latin square has no non-trivial autotopisms...

...it can only have 6 autoparatopisms (6 distinct δ's).

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Question: When does there exist a partial Latin square of order n with weight m with no non-trivial autotopisms and 6 autoparatopisms?

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Question: When does there exist a partial Latin square of order *n* with weight *m* with no non-trivial autotopisms and 6 autoparatopisms?

This is work in progress, and today I'll present one construction.

The first step...

Claim: These partial Latin squares...

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & \cdot \\ 3 & \cdot & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & \cdot \\ 3 & 4 & \cdot & \cdot \\ 4 & \cdot & \cdot & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & \cdot \\ 3 & 4 & 5 & \cdot & \cdot \\ 4 & 5 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & \cdot \\ 3 & 4 & 5 & 6 & \cdot & \cdot \\ 4 & 5 & 6 & \cdot & \cdot & \cdot \\ 5 & 6 & \cdot & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

- ...have no non-trivial autotopisms, and...

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- ...have no non-trivial autotopisms, and...
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To prove , we observe that most rows and columns contain a unique number of entries. [+ a bit of tidying up.]

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- ...have no non-trivial autotopisms, and...
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To prove , we observe that most rows and columns contain a unique number of entries. [+ a bit of tidying up.]
To prove , we just identify two generators of the autoparatopism group (the easy one is the matrix transpose).

The second step...

Claim: These partial Latin squares also...

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & \cdot \\ 3 & 4 & \cdot & 6 & 7 & \cdot & \cdot \\ 4 & 5 & 6 & 7 & \cdot & \cdot & \cdot \\ 5 & 6 & 7 & \cdot & \cdot & \cdot & \cdot \\ 6 & 7 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdot \\ 3 & 4 & \cdot & \cdot & 7 & 8 & \cdot & \cdot \\ 4 & 5 & \cdot & 7 & 8 & \cdot & \cdot & \cdot \\ 5 & 6 & 7 & 8 & \cdot & \cdot & \cdot & \cdot \\ 6 & 7 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 7 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdot \\ 3 & 4 & \cdot & \cdot & \cdot & 8 & 9 & \cdot & \cdot \\ 4 & 5 & \cdot & \cdot & 8 & 9 & \cdot & \cdot & \cdot \\ 5 & 6 & \cdot & 8 & 9 & \cdot & \cdot & \cdot & \cdot \\ 6 & 7 & 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 7 & 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

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We delete all entries except those in the first two rows, the first two columns, or with the last two symbols.

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To prove [dolphin icon], we observe (a) the first two rows/columns map to themselves (unique no. entries),

The second step...

Claim: These partial Latin squares also...

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & \cdot \\ 3 & 4 & \cdot & 6 & 7 & \cdot & \cdot \\ 4 & 5 & 6 & 7 & \cdot & \cdot & \cdot \\ 5 & 6 & 7 & \cdot & \cdot & \cdot & \cdot \\ 6 & 7 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdot \\ 3 & 4 & \cdot & \cdot & 7 & 8 & \cdot & \cdot \\ 4 & 5 & \cdot & 7 & 8 & \cdot & \cdot & \cdot \\ 5 & 6 & 7 & 8 & \cdot & \cdot & \cdot & \cdot \\ 6 & 7 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 7 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdot \\ 3 & 4 & \cdot & \cdot & \cdot & 8 & 9 & \cdot & \cdot \\ 4 & 5 & \cdot & \cdot & 8 & 9 & \cdot & \cdot & \cdot \\ 5 & 6 & \cdot & 8 & 9 & \cdot & \cdot & \cdot & \cdot \\ 6 & 7 & 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 7 & 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

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We delete all entries except those in the first two rows, the first two columns, or with the last two symbols.

To prove (first claim), we observe (a) the first two rows/columns map to themselves (unique no. entries), implying autotopisms have the form $(\alpha, \alpha, \alpha; \mathrm{id})$ where α fixes 1 and 2, and

The second step...

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We delete all entries except those in the first two rows, the first two columns, or with the last two symbols.

To prove , we observe (a) the first two rows/columns map to themselves (unique no. entries), implying autotopisms have the form $(\alpha, \alpha, \alpha; \mathrm{id})$ where α fixes 1 and 2, and (b) the entry $(2, i, i+1)$ exists for $i \in \{2, 3, \ldots, n-1\}$,

The second step...

Claim: These partial Latin squares also...

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & \cdot \\ 3 & 4 & \cdot & 6 & 7 & \cdot & \cdot \\ 4 & 5 & 6 & 7 & \cdot & \cdot & \cdot \\ 5 & 6 & 7 & \cdot & \cdot & \cdot & \cdot \\ 6 & 7 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdot \\ 3 & 4 & \cdot & \cdot & 7 & 8 & \cdot & \cdot \\ 4 & 5 & \cdot & 7 & 8 & \cdot & \cdot & \cdot \\ 5 & 6 & 7 & 8 & \cdot & \cdot & \cdot & \cdot \\ 6 & 7 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 7 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdot \\ 3 & 4 & \cdot & \cdot & \cdot & 8 & 9 & \cdot & \cdot \\ 4 & 5 & \cdot & \cdot & 8 & 9 & \cdot & \cdot & \cdot \\ 5 & 6 & \cdot & 8 & 9 & \cdot & \cdot & \cdot & \cdot \\ 6 & 7 & 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 7 & 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

- ...have no non-trivial autotopisms, and...
- ...have six autoparatopisms.

We delete all entries except those in the first two rows, the first two columns, or with the last two symbols.

To prove this, we observe (a) the first two rows/columns map to themselves (unique no. entries), implying autotopisms have the form $(\alpha, \alpha, \alpha; \mathrm{id})$ where α fixes 1 and 2, and (b) the entry $(2, i, i+1)$ exists for $i \in \{2, 3, \ldots, n-1\}$, so we get the induction "i fixed $\implies$ $i+1$ fixed".

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$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 3 & 4 & 5 & 6 & 7 & \cdot \\
3 & 4 & \cdot & 6 & 7 & \cdot & \cdot \\
4 & 5 & 6 & 7 & \cdot & \cdot & \cdot \\
5 & 6 & 7 & \cdot & \cdot & \cdot & \cdot \\
6 & 7 & \cdot & \cdot & \cdot & \cdot & \cdot \\
7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdot \\
3 & 4 & \cdot & \cdot & 7 & 8 & \cdot & \cdot \\
4 & 5 & \cdot & 7 & 8 & \cdot & \cdot & \cdot \\
5 & 6 & 7 & 8 & \cdot & \cdot & \cdot & \cdot \\
6 & 7 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot \\
7 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdot \\
3 & 4 & \cdot & \cdot & \cdot & 8 & 9 & \cdot & \cdot \\
4 & 5 & \cdot & \cdot & 8 & 9 & \cdot & \cdot & \cdot \\
5 & 6 & \cdot & 8 & 9 & \cdot & \cdot & \cdot & \cdot \\
6 & 7 & 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot \\
7 & 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
$$

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We deleted entries that belong to orbits under the autoparatopism group $\mathrm{apar}(\cdots)$.

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So, if we only delete entries belonging to some of those orbits:

$$\begin{aligned}\text{apar(delete all indicated orbits)} &\geq \mathrm{apar}\text{(delete some orbits)} \\ &\geq \text{apar(delete no orbits)}\end{aligned}$$

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$\geq$ apar(delete no orbits)

And we just showed the these two have size six.

So, we get a bunch of partial Latin squares of order *n* with no non-trivial autotopisms and six autoparatopisms.

Deleting orbits...

The orbits we can delete have sizes 1, 3, or 6.

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \cdot \\
3 & 4 & 5 & \boxed{6} & 7 & 8 & 9 & \boxed{10} & 11 & 12 & 13 & \cdot & \cdot \\
4 & 5 & \boxed{6} & \langle 7 \rangle & 8 & 9 & \langle 10 \rangle & \boxed{11} & 12 & 13 & \cdot & \cdot & \cdot \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \cdot & \cdot & \cdot & \cdot \\
6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \cdot & \cdot & \cdot & \cdot & \cdot \\
7 & 8 & 9 & \langle 10 \rangle & 11 & 12 & 13 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
8 & 9 & \boxed{10} & \boxed{11} & 12 & 13 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
9 & 10 & 11 & 12 & 13 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
10 & 11 & 12 & 13 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
11 & 12 & 13 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
12 & 13 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
13 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \triangle 1
\end{bmatrix}
$$

We can delete some selections of these orbits, giving, for a given n, a range of weights m for which there exists a partial Latin square of order n weight m with no non-trivial autotopisms and six autoparatopisms.

Where we are now...

This is where we are now. We still need to fill in the gaps:

- We can do $m \equiv 0, 1 \pmod 3$ in this way; but what about $m \equiv 2 \pmod 3$?

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- We can do $m \equiv 0, 1 \pmod 3$ in this way; but what about $m \equiv 2 \pmod 3$?
- These partial Latin squares have at most $\binom{n+1}{2} + 1$ entries; can we give such constructions with more entries?

Thank you