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DETERMINANTS OF LATIN SQUARES OF A GIVEN PATTERN

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Abstract

Cycle structures of autotopisms of Latin squares determine all possible patterns of this kind of design. Moreover, given any isotopism, the number of Latin squares containing it in their autotopism group only depends on the cycle structure of this isotopism. This number has been studied in [2] for Latin squares of order up to 7, by following the classification given in [3]. Specifically, regarding each symbol of a Latin square as a variable, any Latin square can be seen as the vector space associated with the solution of an algebraic system of polynomial equations, which can be solved using Gröbner bases, by following the ideals implemented by Bayer [1] to solve the problem of n-colouring a graph. However, computations for orders higher than 7 have been shown to be very difficult without using some other combinatorial tools. In this sense, we will see in this paper the possibility of studying the determinants of those Latin squares related to a given cycle structure. Specifically, since the determinant of a Latin square can be seen as a polynomial of degree n in n variables, it will determine a new polynomial equation that can be included into the previous system. Moreover, since determinants of Latin squares of order up to 7 determine their isotopic classes [5], we will study the set of isotopic classes of Latin squares of these orders related to each cycle structure.

Introduction and notation

- A ***Latin square*** L of order n is an $n \times n$ array with elements chosen from a set of n distinct symbols (in this paper, it will be the set $[n] = \{1, 2, ..., n\}$) such that each symbol occurs precisely once in each row and each column. The set of Latin squares of order n is denoted by $LS(n)$. Given $L = (l_{i,j}) \in LS(n)$, the ***orthogonal array representation*** of L is the set of n^2 triples $\{(i, j, l_{i,j}) \mid i, j \in [n]\}$.

- The permutation group on $[n]$ is denoted by S_n. Every permutation $\delta \in S_n$ can be uniquely written as a composition of $\mathbf{n}_\delta$ pairwise disjoint cycles, $\delta = C_1^\delta \circ C_2^\delta \circ ... \circ C_{\mathbf{n}_\delta}^\delta$, where for all $i \in [\mathbf{n}_\delta]$, one has $C_i^\delta = \left(c_{i,1}^\delta\ c_{i,2}^\delta\ ...\ c_{i,\lambda_i^\delta}^\delta\right)$, with $c_{i,1}^\delta = \min_j\{c_{i,j}^\delta\}$. The ***cycle structure of*** δ is the sequence $\mathbf{l}_\delta = (\mathbf{l}_1^\delta, \mathbf{l}_2^\delta, ..., \mathbf{l}_n^\delta)$, where $\mathbf{l}_i^\delta$ is the number of cycles of length i in δ, for all $i \in \{1, 2, ..., n\}$. Thus, $\mathbf{l}_1^\delta$ is the cardinal of the set of ***fixed points*** of δ, $Fix(\delta) = \{i \in [n] \mid \delta(i) = i\}$. Given $\sigma \in S_3$, one defines the *conjugate Latin square* $L^\sigma \in LS(n)$ of L, such that if $T = (i, j, l_{i,j}) \in L$, then $(\pi_{\sigma(1)}(T), \pi_{\sigma(2)}(T), \pi_{\sigma(3)}(T)) \in L^\sigma$, where π_i gives the i^{th} coordinate of T, for all $i \in [3]$. In this way, each Latin square L has six conjugate Latin squares associated with it: $L^{Id} = L$, $L^{(12)} = L^t$, $L^{(13)}$, $L^{(23)}$, $L^{(123)}$ and $L^{(132)}$.

- Given $L = (l_{i,j}) \in LS(n)$, it is defined its ***associated matrix*** X_L, which is obtained by replacing each element $l_{i,j}$ by the variable $x_{l_{i,j}}$. The ***determinant*** $\det(L)$ of L is the homogeneous polynomial of degree n in n variables $\det(X_L)$. Two polynomials p_1 and p_2 in $\{x_1, x_2, ..., x_n\}$ are said to be ***similar***, and it is denoted by $p_1 \sim p_2$, if there exists a permutation $\sigma \in S_n$ such that $p_1(x_1, x_2, ..., x_n) = \pm p_2(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})$. Thus, it is verified that isotopic and transposed Latin squares have similar determinants.

- An ***isotopism*** of a Latin square $L = (l_{i,j}) \in LS(n)$ is a triple $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_\mathbf{n} = S_n \times S_n \times S_n$. In this way, α, β and γ are permutations of rows, columns and symbols of L, respectively. The resulting square $\mathbf{L}^\Theta = \{(\alpha(i), \beta(j), \gamma(l_{i,j})) \mid i, j \in [n]\}$ is also a Latin square and it is said to be ***isotopic*** to L. To be isotopic is an equivalence relation and thus, the set of Latin squares being isotopic to a given one is its ***isotopic class***. Given two isotopic Latin squares $L, L' \in LS(n)$ and $\sigma \in S_3$, it is verified that L^σ and L'^σ are also isotopic. Two Latin squares L_1 and L_2 are said to be of the same *type* if L_1 is isotopic to L_2 or L_2^t. It is also an equivalence relation. The number of isotopic classes and types of the set $LS(n)$ is known for all $n \leq 10$ [7]:

n	Isotopic classes	Types
2	1	1
3	1	1
4	2	2
5	2	2
6	22	17
7	564	324
8	1676267	842227
9	115618721533	57810418543
10	208904371354363006	104452188344901572

- The ***cycle structure*** of Θ is the triple $\mathbf{l}_\Theta = (\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$. The set of cycle structures of $LS(n)$ is denoted by $CS(n)$. An isotopism which maps L to itself is an ***autotopism***. The possible cycle structures of the set of non-trivial autotopisms of Latin squares of order up to 11 were obtained in [3]. $\mathfrak{A}_n$ is the set of all possible autotopisms of order n and $\mathfrak{A}_n(\mathbf{l}_\Theta)$ denotes the set of autotopisms having the same cycle structure that a given autotopism Θ. The stabilizer subgroup of L in $\mathfrak{A}_n$ is its ***autotopism group*** $\mathfrak{A}(L)$. Given $L \in LS(n)$, $\Theta \in \mathfrak{A}(L)$ and $\sigma \in S_3$, it is verified that $\Theta^\sigma = (\pi_{\sigma(1)}(\Theta), \pi_{\sigma(2)}(\Theta), \pi_{\sigma(3)}(\Theta)) \in \mathfrak{A}(L^\sigma)$, where π_i gives the i^{th} component of Θ, for all $i \in [3]$. Given $\Theta \in \mathfrak{A}_n$, the set of all Latin squares L such that $\Theta \in \mathfrak{A}(L)$ is denoted by $LS(\Theta)$ and the cardinality of $LS(\Theta)$ is denoted by $\Delta(\Theta)$. Specifically, if Θ_1 and Θ_2 are two autotopisms with the same cycle structure, then $\Delta(\Theta_1) = \Delta(\Theta_2)$.

- Gröbner bases were used in [2] to describe an algorithm that allows one to obtain the number $\Delta(\Theta)$ in a computational way. This algorithm was implemented in SINGULAR [6] to get the number of Latin squares of order ≤ 7 related to any autotopism of a given cycle structure [4]. However, in order to improve the time of computation, it is convenient to combine Gröbner bases with some combinatorial tools. In this sense, the study of the determinants of those Latin squares related to a given cycle structure can be very useful, because they can be seen as polynomials which can be included into the previous Gröbner bases.

Determinants of Latin squares following a given pattern

Given an autotopism $\Theta \in \mathfrak{A}_n$, let us consider the set:

$$\det(\Theta) = \{\det(L) \mid L \in LS(\Theta)\}.$$

We will say that a set $\mathcal{B}$ of polynomials in $\{x_1, x_2, ..., x_n\}$ is a ***basis*** of $\det(\Theta)$ if:

i) Two different polynomials of $\mathcal{B}$ are not similar.

ii) Given $p \in \mathcal{B}$, there exists $L \in LS(\Theta)$ such that $p \sim \det(L)$.

iii) Given $L \in LS(\Theta)$, there exists $p \in \mathcal{B}$ such that $\det(L) \sim p$.

Lemma 1. *Given $\Theta \in \mathfrak{A}_n$, let $\mathcal{B}_1, \mathcal{B}_2$ be two different basis of* $\det(\Theta)$. *Both basis have the same number of elements.* □

The number of elements of any basis of $\det(\Theta)$ will be called its ***order*** and it will be denoted by $\sharp\det(\Theta)$. Since we are interested in those autotopisms $\Theta \in \mathcal{A}_n$ such that $\Delta(\Theta) > 0$, it must be $\sharp\det(\Theta) \geq 1$. Moreover, it is the number of different classes in the quotient set $\det(\Theta)/\sim$.

Proposition 1. *Let $(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma) \in CS(n)$ and let us consider $\Theta_1, \Theta_2 \in \mathfrak{A}_n(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$. There exists a $1-1$ correspondence $f : \det(\Theta_1) \to \det(\Theta_2)$, such that $f(p) \sim p$, for all $p \in \det(\Theta_1)$. As a consequence, a set $\mathcal{B}$ of polynomials in $\{x_1, x_2, ..., x_n\}$ is a basis of $\det(\Theta_1)$ if and only if it is a basis of $\det(\Theta_2)$.* □

Let $(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma) \in CS(n)$ and $\Theta \in \mathfrak{A}_n((\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma))$ and let us consider a basis $\mathcal{B} = \{p_1, p_2, ..., p_m\}$ of $\det(\Theta)$ such that $\mathcal{B} \subseteq \det(\Theta)$. Let $L_1, L_2, ..., L_m \in LS(\Theta)$ be such that $\det(X_{L_i}) = p_i$, for all $i \in [m]$. The set $\{X_{L_i} \mid i \in [m]\}$ will be called a ***set of patterns of*** $(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$.

Theorem 1. *The set of determinants $\mathcal{B}$ of a given set of patterns $\mathcal{P} = \{X_{L_1}, X_{L_2}, ..., X_{L_m}\}$ of a cycle structure $(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma) \in CS(n)$ is a basis of $\det(\Theta)$, for all $\Theta \in \mathfrak{A}_n(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$. Moreover, the Latin squares of $LS(\Theta)$ belong to at least m different isotopic classes of $LS(n)$, where each pattern of $\mathcal{P}$ corresponds to a different isotopic class of $LS(n)$.* □

Proposition 2. *Let $\{L_1, L_2, ..., L_m\} \subseteq LS(n)$ be the set of Latin squares whose associated matrices constitute the set of patterns $\mathcal{P}$ of the cycle structure $\mathbf{l}_\Theta$ of an autotopism $\Theta \in \mathfrak{A}_n$. It is verified that $\mathcal{P}^t = \{X_{L_1^t}, X_{L_2^t}, ..., X_{L_m^t}\}$ is a set of patterns of the cycle structure $\mathbf{l}_{\Theta^{(12)}}$.* □

Patterns of autotopisms of small Latin squares

Finally, we give the classification of the sets of patterns and determinants corresponding to the cycle structures of all autotopisms of Latin squares of order $2 \leq n \leq 6$. To do it, we have used the classification of all possible cycle structures given in [3] and the following result:

Proposition 3. *Given $n \leq 7$ and $\Theta \in \mathfrak{A}_n$, it is verified that* $\sharp\det(\Theta)$ *is smaller than the number of different types of Latin squares of order n.* □

By computing the distinct factorizations of the determinants of the Latin squares related to each cycle structure, we have obtained the following table:

n	$\mathbf{l}_\alpha = \mathbf{l}_\beta$	$\mathbf{l}_\gamma$	$\Delta(\Theta)$	Set of patterns	$\sharp\det(\Theta)$	Classes of determinants
2	(0,1)	(2,0)	2	$\mathcal{P}_2$	1	D_2
3	(0,0,1)	(0,0,1)	3	$\mathcal{P}_3$	1	D_3
		(3,0,0)	6	$\mathcal{P}_3$	1	D_3
	(1,1,0)	(1,1,0)	4	$\mathcal{P}_3$	1	D_3
4	(0,0,0,1)	(0,2,0,0)	8	$\mathcal{P}_{4,a}$	1	$D_{4,1}$
		(2,1,0,0)	8	$\mathcal{P}_{4,b}$	1	$D_{4,2}$
		(4,0,0,0)	24	$\mathcal{P}_{4,a}$	1	$D_{4,1}$
	(0,2,0,0)	(0,2,0,0)	32	$\mathcal{P}_{4,b}$	1	$D_{4,2}$
		(2,1,0,0)	32	$\mathcal{P}_{4,a}$	1	$D_{4,1}$
		(4,0,0,0)	96	$\mathcal{P}_{4,c}$	2	$D_{4,1}, D_{4,2}$
	(1,0,1,0)	(1,0,1,0)	9	$\mathcal{P}_{4,b}$	1	$D_{4,2}$
	(2,1,0,0)	(2,1,0,0)	16	$\mathcal{P}_{4,c}$	2	$D_{4,1}, D_{4,2}$
5	(0,0,0,0,1)	(0,0,0,0,1)	15	$\mathcal{P}_{5,a}$	1	$D_{5,1}$
		(5,0,0,0,0)	120	$\mathcal{P}_{5,a}$	1	$D_{5,1}$
	(1,0,0,1,0)	(1,0,0,1,0)	32	$\mathcal{P}_{5,a}$	1	$D_{5,1}$
	(1,2,0,0,0)	(1,2,0,0,0)	256	$\mathcal{P}_{5,a}$	1	$D_{5,1}$
	(2,0,1,0,0)	(2,0,1,0,0)	144	$\mathcal{P}_{5,b}$	1	$D_{5,2}$
6	(0,0,0,0,0,1)	(0,0,2,0,0,0)	72	$\mathcal{P}_{6,a}$	2	$D_{6,1}, D_{6,2}$
		(1,1,1,0,0,0)	72	$\mathcal{P}_{6,b}$	2	$D_{6,3}, D_{6,4}$
		(2,2,0,0,0,0)	144	$\mathcal{P}_{6,c}$	2	$D_{6,4}, D_{6,5}$
		(3,0,1,0,0,0)	144	$\mathcal{P}_{6,d}$	2	$D_{6,5}, D_{6,6}$
		(4,1,0,0,0,0)	288	$\mathcal{P}_{6,e}$	1	$D_{6,4}$
		(6,0,0,0,0,0)	720	$\mathcal{P}_{6,f}$	1	$D_{6,1}$
	(0,0,2,0,0,0)	(0,0,2,0,0,0)	648	$\mathcal{P}_{6,g}$	2	$D_{6,1}, D_{6,2}$
		(3,0,1,0,0,0)	2592	$\mathcal{P}_{6,h}$	4	$D_{6,3}, D_{6,4}, D_{6,5}, D_{6,6}$
		(6,0,0,0,0,0)	25920	$\mathcal{P}_{6,i}$	2	$D_{6,4}, D_{6,7}$
	(1,0,0,0,1,0)	(1,0,0,0,1,0)	75	$\mathcal{P}_{6,j}$	2	$D_{6,3}, D_{6,8}$
	(0,3,0,0,0,0)	(2,2,0,0,0,0)	36864	$\mathcal{P}_{6,k}$	6	$D_{6,1}, D_{6,2}, D_{6,4}, D_{6,5}, D_{6,6}, D_{6,9}$
		(4,1,0,0,0,0)	110592	$\mathcal{P}_{6,l}$	5	$D_{6,3}, D_{6,4}, D_{6,10}, D_{6,11}, D_{6,12}$
	(2,0,0,1,0,0)	(2,0,0,1,0,0)	768	$\mathcal{P}_{6,m}$	5	$D_{6,3}, D_{6,8}, D_{6,10}, D_{6,13}, D_{6,14}$
	(2,2,0,0,0,0)	(2,2,0,0,0,0)	20480	$\mathcal{P}_{6,n}$	5	$D_{6,3}, D_{6,6}, D_{6,10}, D_{6,15}, D_{6,16}$
	(3,0,1,0,0,0)	(3,0,1,0,0,0)	2592	$\mathcal{P}_{6,o}$	3	$D_{6,4}, D_{6,5}, D_{6,17}$

The sets of patterns of the previous table are indicated below:

$$\mathcal{P}_2 = \left\{\begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}\right\}. \quad \mathcal{P}_3 = \left\{\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \\ x_3 & x_1 & x_2 \end{pmatrix}\right\}. \quad \mathcal{P}_{4,a} = \left\{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_4 & x_2 & x_1 \\ x_4 & x_3 & x_1 & x_2 \end{pmatrix}\right\},$$

$$\mathcal{P}_{4,b} = \left\{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}\right\}, \quad \mathcal{P}_{4,c} = \mathcal{P}_{4,a} \cup \mathcal{P}_{4,b}. \quad \mathcal{P}_{5,a} = \left\{\begin{pmatrix} x_1 & x_4 & x_2 & x_5 & x_3 \\ x_4 & x_2 & x_5 & x_3 & x_1 \\ x_2 & x_5 & x_3 & x_1 & x_4 \\ x_5 & x_3 & x_1 & x_4 & x_2 \\ x_3 & x_1 & x_4 & x_2 & x_5 \end{pmatrix}\right\}, \quad \mathcal{P}_{5,b} = \left\{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_2 & x_1 & x_4 & x_5 & x_3 \\ x_3 & x_4 & x_5 & x_1 & x_2 \\ x_4 & x_5 & x_2 & x_3 & x_1 \\ x_5 & x_3 & x_1 & x_2 & x_4 \end{pmatrix}\right\}.$$

$$\mathcal{P}_{6,a} = \left\{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_5 & x_6 & x_1 \\ x_3 & x_4 & x_5 & x_6 & x_1 & x_2 \\ x_4 & x_5 & x_6 & x_1 & x_2 & x_3 \\ x_5 & x_6 & x_1 & x_2 & x_3 & x_4 \\ x_6 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}, \begin{pmatrix} x_1 & x_5 & x_3 & x_4 & x_2 & x_6 \\ x_2 & x_3 & x_1 & x_5 & x_6 & x_4 \\ x_6 & x_4 & x_5 & x_3 & x_1 & x_2 \\ x_4 & x_2 & x_6 & x_1 & x_5 & x_3 \\ x_5 & x_6 & x_4 & x_2 & x_3 & x_1 \\ x_3 & x_1 & x_2 & x_6 & x_4 & x_5 \end{pmatrix}\right\}. \quad \mathcal{P}_{6,b} = \left\{\begin{pmatrix} x_1 & x_4 & x_3 & x_2 & x_6 & x_5 \\ x_6 & x_5 & x_2 & x_3 & x_4 & x_1 \\ x_5 & x_1 & x_6 & x_4 & x_3 & x_2 \\ x_4 & x_6 & x_5 & x_1 & x_2 & x_3 \\ x_3 & x_2 & x_1 & x_6 & x_5 & x_4 \\ x_2 & x_3 & x_4 & x_5 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_4 & x_5 & x_3 & x_6 & x_2 \\ x_4 & x_5 & x_2 & x_6 & x_3 & x_1 \\ x_5 & x_2 & x_6 & x_4 & x_1 & x_3 \\ x_3 & x_6 & x_4 & x_1 & x_2 & x_5 \\ x_6 & x_3 & x_1 & x_2 & x_5 & x_4 \\ x_2 & x_1 & x_3 & x_5 & x_4 & x_6 \end{pmatrix}\right\}.$$

$$\mathcal{P}_{6,c} = \left\{\begin{pmatrix} x_1 & x_3 & x_6 & x_5 & x_4 & x_2 \\ x_5 & x_6 & x_3 & x_1 & x_2 & x_4 \\ x_4 & x_2 & x_1 & x_3 & x_6 & x_5 \\ x_2 & x_4 & x_5 & x_6 & x_3 & x_1 \\ x_6 & x_5 & x_4 & x_2 & x_1 & x_3 \\ x_3 & x_1 & x_2 & x_4 & x_5 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_3 & x_4 & x_5 & x_6 & x_2 \\ x_5 & x_6 & x_3 & x_4 & x_2 & x_1 \\ x_6 & x_2 & x_1 & x_3 & x_4 & x_5 \\ x_2 & x_1 & x_5 & x_6 & x_3 & x_4 \\ x_4 & x_5 & x_6 & x_2 & x_1 & x_3 \\ x_3 & x_4 & x_2 & x_1 & x_5 & x_6 \end{pmatrix}\right\}. \quad \mathcal{P}_{6,d} = \left\{\begin{pmatrix} x_1 & x_3 & x_5 & x_4 & x_6 & x_2 \\ x_2 & x_5 & x_3 & x_6 & x_4 & x_1 \\ x_5 & x_2 & x_6 & x_3 & x_1 & x_4 \\ x_4 & x_6 & x_2 & x_1 & x_3 & x_5 \\ x_6 & x_4 & x_1 & x_2 & x_5 & x_3 \\ x_3 & x_1 & x_4 & x_5 & x_2 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_3 & x_4 & x_2 & x_6 & x_5 \\ x_6 & x_5 & x_3 & x_4 & x_2 & x_1 \\ x_5 & x_1 & x_6 & x_3 & x_4 & x_2 \\ x_2 & x_6 & x_5 & x_1 & x_3 & x_4 \\ x_4 & x_2 & x_1 & x_6 & x_5 & x_3 \\ x_3 & x_4 & x_2 & x_5 & x_1 & x_6 \end{pmatrix}\right\}.$$

$$\mathcal{P}_{6,e} = \left\{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_6 & x_3 & x_2 & x_1 & x_4 & x_5 \\ x_5 & x_6 & x_1 & x_2 & x_3 & x_4 \\ x_4 & x_5 & x_6 & x_3 & x_2 & x_1 \\ x_3 & x_4 & x_5 & x_6 & x_1 & x_2 \\ x_2 & x_1 & x_4 & x_5 & x_6 & x_3 \end{pmatrix}\right\}. \quad \mathcal{P}_{6,f} = \left\{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_5 & x_6 & x_1 \\ x_3 & x_4 & x_5 & x_6 & x_1 & x_2 \\ x_4 & x_5 & x_6 & x_1 & x_2 & x_3 \\ x_5 & x_6 & x_1 & x_2 & x_3 & x_4 \\ x_6 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}\right\}. \quad \mathcal{P}_{6,g} = \left\{\begin{pmatrix} x_1 & x_2 & x_4 & x_3 & x_6 & x_5 \\ x_2 & x_5 & x_1 & x_6 & x_4 & x_3 \\ x_4 & x_1 & x_6 & x_5 & x_3 & x_2 \\ x_3 & x_6 & x_5 & x_1 & x_2 & x_4 \\ x_6 & x_4 & x_3 & x_2 & x_5 & x_1 \\ x_5 & x_3 & x_2 & x_4 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_5 & x_4 & x_3 & x_6 & x_2 \\ x_2 & x_6 & x_5 & x_4 & x_1 & x_3 \\ x_4 & x_2 & x_1 & x_6 & x_3 & x_5 \\ x_6 & x_1 & x_3 & x_5 & x_2 & x_4 \\ x_3 & x_4 & x_6 & x_2 & x_5 & x_1 \\ x_5 & x_3 & x_2 & x_1 & x_4 & x_6 \end{pmatrix}\right\}.$$

$$\mathcal{P}_{6,h} = \left\{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_6 & x_5 \\ x_2 & x_6 & x_5 & x_1 & x_3 & x_4 \\ x_4 & x_5 & x_1 & x_6 & x_2 & x_3 \\ x_3 & x_1 & x_6 & x_5 & x_4 & x_2 \\ x_6 & x_4 & x_2 & x_3 & x_5 & x_1 \\ x_5 & x_3 & x_4 & x_2 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_6 & x_5 \\ x_2 & x_1 & x_6 & x_5 & x_3 & x_4 \\ x_4 & x_6 & x_5 & x_1 & x_2 & x_3 \\ x_3 & x_5 & x_1 & x_6 & x_4 & x_2 \\ x_6 & x_4 & x_2 & x_3 & x_5 & x_1 \\ x_5 & x_3 & x_4 & x_2 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_3 & x_4 & x_2 & x_5 \\ x_5 & x_1 & x_6 & x_2 & x_3 & x_4 \\ x_4 & x_2 & x_5 & x_1 & x_6 & x_3 \\ x_3 & x_5 & x_2 & x_6 & x_4 & x_1 \\ x_6 & x_4 & x_1 & x_3 & x_5 & x_2 \\ x_2 & x_3 & x_4 & x_5 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_5 & x_2 & x_3 & x_4 \\ x_5 & x_4 & x_6 & x_3 & x_1 & x_2 \\ x_2 & x_3 & x_4 & x_1 & x_6 & x_5 \\ x_6 & x_5 & x_3 & x_4 & x_2 & x_1 \\ x_4 & x_2 & x_1 & x_6 & x_5 & x_3 \\ x_3 & x_1 & x_2 & x_5 & x_4 & x_6 \end{pmatrix}\right\}.$$

$$\mathcal{P}_{6,i} = \left\{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_1 & x_5 & x_6 & x_3 & x_4 \\ x_4 & x_6 & x_1 & x_5 & x_2 & x_3 \\ x_3 & x_5 & x_6 & x_1 & x_4 & x_2 \\ x_6 & x_4 & x_2 & x_3 & x_1 & x_5 \\ x_5 & x_3 & x_4 & x_2 & x_6 & x_1 \end{pmatrix}, \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_1 & x_6 & x_5 & x_3 & x_4 \\ x_4 & x_5 & x_1 & x_6 & x_2 & x_3 \\ x_3 & x_6 & x_5 & x_1 & x_4 & x_2 \\ x_6 & x_4 & x_2 & x_3 & x_1 & x_5 \\ x_5 & x_3 & x_4 & x_2 & x_6 & x_1 \end{pmatrix}\right\}. \quad \mathcal{P}_{6,j} = \left\{\begin{pmatrix} x_1 & x_6 & x_2 & x_5 & x_4 & x_3 \\ x_5 & x_2 & x_6 & x_1 & x_3 & x_4 \\ x_4 & x_1 & x_3 & x_2 & x_6 & x_5 \\ x_6 & x_3 & x_5 & x_4 & x_2 & x_1 \\ x_3 & x_4 & x_1 & x_6 & x_5 & x_2 \\ x_2 & x_5 & x_4 & x_3 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_5 & x_2 & x_3 & x_6 & x_4 \\ x_4 & x_2 & x_5 & x_6 & x_1 & x_3 \\ x_5 & x_6 & x_3 & x_2 & x_4 & x_1 \\ x_3 & x_1 & x_6 & x_4 & x_2 & x_5 \\ x_6 & x_3 & x_4 & x_1 & x_5 & x_2 \\ x_2 & x_4 & x_1 & x_5 & x_3 & x_6 \end{pmatrix}\right\}.$$

$$\mathcal{P}_{6,k} = \left\{\begin{pmatrix} x_1 & x_6 & x_5 & x_4 & x_2 & x_3 \\ x_6 & x_2 & x_3 & x_1 & x_4 & x_5 \\ x_4 & x_3 & x_1 & x_5 & x_6 & x_2 \\ x_5 & x_1 & x_2 & x_6 & x_3 & x_4 \\ x_2 & x_4 & x_6 & x_3 & x_5 & x_1 \\ x_3 & x_5 & x_4 & x_2 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_5 & x_4 & x_2 & x_3 \\ x_6 & x_4 & x_2 & x_1 & x_3 & x_5 \\ x_4 & x_1 & x_3 & x_6 & x_5 & x_2 \\ x_5 & x_2 & x_1 & x_3 & x_6 & x_4 \\ x_2 & x_3 & x_6 & x_5 & x_4 & x_1 \\ x_3 & x_5 & x_4 & x_2 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_5 & x_4 & x_2 & x_3 \\ x_6 & x_4 & x_1 & x_5 & x_3 & x_2 \\ x_4 & x_2 & x_3 & x_1 & x_6 & x_5 \\ x_2 & x_1 & x_6 & x_3 & x_5 & x_4 \\ x_5 & x_3 & x_2 & x_6 & x_4 & x_1 \\ x_3 & x_5 & x_4 & x_2 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_5 & x_4 & x_2 & x_3 \\ x_6 & x_4 & x_3 & x_1 & x_5 & x_2 \\ x_4 & x_1 & x_2 & x_6 & x_3 & x_5 \\ x_2 & x_3 & x_1 & x_5 & x_6 & x_4 \\ x_5 & x_2 & x_6 & x_3 & x_4 & x_1 \\ x_3 & x_5 & x_4 & x_2 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_5 & x_4 & x_2 & x_3 \\ x_6 & x_3 & x_2 & x_1 & x_4 & x_5 \\ x_4 & x_1 & x_3 & x_6 & x_5 & x_2 \\ x_5 & x_2 & x_1 & x_3 & x_6 & x_4 \\ x_2 & x_4 & x_6 & x_5 & x_3 & x_1 \\ x_3 & x_5 & x_4 & x_2 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_3 & x_4 & x_5 & x_3 \\ x_6 & x_4 & x_2 & x_1 & x_3 & x_5 \\ x_4 & x_1 & x_5 & x_6 & x_2 & x_3 \\ x_3 & x_5 & x_1 & x_2 & x_6 & x_4 \\ x_2 & x_3 & x_6 & x_5 & x_4 & x_1 \\ x_5 & x_2 & x_4 & x_3 & x_1 & x_6 \end{pmatrix}\right\}.$$

$$\mathcal{P}_{6,l} = \left\{\begin{pmatrix} x_1 & x_6 & x_2 & x_3 & x_4 & x_5 \\ x_6 & x_5 & x_1 & x_4 & x_3 & x_2 \\ x_3 & x_2 & x_5 & x_1 & x_6 & x_4 \\ x_4 & x_1 & x_6 & x_5 & x_2 & x_3 \\ x_2 & x_3 & x_4 & x_6 & x_5 & x_1 \\ x_5 & x_4 & x_3 & x_2 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_2 & x_3 & x_4 & x_5 \\ x_6 & x_3 & x_1 & x_4 & x_5 & x_2 \\ x_3 & x_2 & x_5 & x_1 & x_6 & x_4 \\ x_4 & x_1 & x_6 & x_5 & x_2 & x_3 \\ x_2 & x_5 & x_4 & x_6 & x_3 & x_1 \\ x_5 & x_4 & x_3 & x_2 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_2 & x_3 & x_4 & x_5 \\ x_6 & x_5 & x_1 & x_4 & x_2 & x_3 \\ x_2 & x_3 & x_5 & x_1 & x_6 & x_4 \\ x_1 & x_4 & x_6 & x_5 & x_3 & x_2 \\ x_3 & x_2 & x_4 & x_6 & x_5 & x_1 \\ x_5 & x_4 & x_3 & x_2 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_2 & x_3 & x_4 & x_5 \\ x_6 & x_2 & x_1 & x_4 & x_5 & x_3 \\ x_2 & x_3 & x_5 & x_1 & x_6 & x_4 \\ x_4 & x_1 & x_6 & x_5 & x_3 & x_2 \\ x_3 & x_5 & x_4 & x_6 & x_2 & x_1 \\ x_5 & x_4 & x_3 & x_2 & x_1 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_2 & x_3 & x_4 & x_5 \\ x_6 & x_2 & x_1 & x_4 & x_5 & x_3 \\ x_4 & x_3 & x_5 & x_1 & x_6 & x_2 \\ x_2 & x_1 & x_6 & x_5 & x_3 & x_4 \\ x_3 & x_5 & x_4 & x_6 & x_2 & x_1 \\ x_5 & x_4 & x_3 & x_2 & x_1 & x_6 \end{pmatrix}\right\}.$$

$$\mathcal{P}_{6,m} = \left\{\begin{pmatrix} x_1 & x_6 & x_5 & x_2 & x_4 & x_3 \\ x_5 & x_2 & x_3 & x_6 & x_1 & x_4 \\ x_4 & x_3 & x_2 & x_5 & x_6 & x_1 \\ x_3 & x_1 & x_6 & x_4 & x_2 & x_5 \\ x_6 & x_4 & x_1 & x_3 & x_5 & x_2 \\ x_2 & x_5 & x_4 & x_1 & x_3 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_5 & x_6 & x_2 & x_4 & x_3 \\ x_5 & x_2 & x_3 & x_6 & x_1 & x_4 \\ x_4 & x_3 & x_2 & x_5 & x_6 & x_1 \\ x_3 & x_6 & x_1 & x_4 & x_2 & x_5 \\ x_6 & x_1 & x_4 & x_3 & x_5 & x_2 \\ x_2 & x_4 & x_5 & x_1 & x_3 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_6 & x_4 & x_2 & x_3 & x_5 \\ x_5 & x_2 & x_3 & x_6 & x_1 & x_4 \\ x_4 & x_3 & x_2 & x_5 & x_6 & x_1 \\ x_6 & x_1 & x_5 & x_4 & x_2 & x_3 \\ x_3 & x_4 & x_6 & x_1 & x_5 & x_2 \\ x_2 & x_5 & x_1 & x_3 & x_4 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_4 & x_5 & x_2 & x_3 & x_5 \\ x_5 & x_2 & x_3 & x_6 & x_1 & x_4 \\ x_4 & x_3 & x_2 & x_5 & x_6 & x_1 \\ x_6 & x_5 & x_1 & x_4 & x_2 & x_3 \\ x_3 & x_6 & x_4 & x_1 & x_5 & x_2 \\ x_2 & x_1 & x_5 & x_3 & x_4 & x_6 \end{pmatrix}, \begin{pmatrix} x_1 & x_5 & x_6 & x_2 & x_3 & x_4 \\ x_6 & x_2 & x_3 & x_1 & x_4 & x_5 \\ x_4 & x_3 & x_2 & x_5 & x_6 & x_1 \\ x_5 & x_6 & x_1 & x_4 & x_2 & x_3 \\ x_3 & x_1 & x_4 & x_6 & x_5 & x_2 \\ x_2 & x_4 & x_5 & x_3 & x_1 & x_6 \end{pmatrix}\right\}.$$

$$\mathcal{P}_{6,n} = \left\{\begin{pmatrix} x_3 & x_2 & x_1 & x_5 & x_6 & x_4 \\ x_5 & x_3 & x_2 & x_6 & x_4 & x_1 \\ x_1 & x_5 & x_3 & x_4 & x_2 & x_6 \\ x_2 & x_6 & x_4 & x_3 & x_1 & x_5 \\ x_6 & x_4 & x_5 & x_1 & x_3 & x_2 \\ x_4 & x_1 & x_6 & x_2 & x_5 & x_3 \end{pmatrix}, \begin{pmatrix} x_3 & x_2 & x_1 & x_5 & x_6 & x_4 \\ x_5 & x_3 & x_2 & x_1 & x_4 & x_6 \\ x_6 & x_5 & x_3 & x_4 & x_2 & x_1 \\ x_2 & x_6 & x_4 & x_3 & x_1 & x_5 \\ x_1 & x_4 & x_5 & x_6 & x_3 & x_2 \\ x_4 & x_1 & x_6 & x_2 & x_5 & x_3 \end{pmatrix}, \begin{pmatrix} x_3 & x_2 & x_1 & x_5 & x_6 & x_4 \\ x_1 & x_3 & x_2 & x_6 & x_4 & x_5 \\ x_6 & x_5 & x_3 & x_4 & x_2 & x_1 \\ x_5 & x_6 & x_4 & x_3 & x_1 & x_2 \\ x_2 & x_4 & x_5 & x_1 & x_3 & x_6 \\ x_4 & x_1 & x_6 & x_2 & x_5 & x_3 \end{pmatrix}, \begin{pmatrix} x_3 & x_2 & x_1 & x_5 & x_6 & x_4 \\ x_5 & x_3 & x_2 & x_1 & x_4 & x_6 \\ x_2 & x_6 & x_3 & x_4 & x_1 & x_5 \\ x_6 & x_5 & x_4 & x_3 & x_2 & x_1 \\ x_1 & x_4 & x_5 & x_6 & x_3 & x_2 \\ x_4 & x_1 & x_6 & x_2 & x_5 & x_3 \end{pmatrix}, \begin{pmatrix} x_3 & x_4 & x_1 & x_5 & x_2 & x_6 \\ x_4 & x_3 & x_2 & x_6 & x_5 & x_1 \\ x_5 & x_6 & x_3 & x_4 & x_1 & x_2 \\ x_2 & x_1 & x_4 & x_3 & x_6 & x_5 \\ x_6 & x_2 & x_5 & x_1 & x_3 & x_4 \\ x_1 & x_5 & x_6 & x_2 & x_4 & x_3 \end{pmatrix}\right\}.$$

$$\mathcal{P}_{6,o} = \left\{\begin{pmatrix} x_2 & x_1 & x_6 & x_5 & x_3 & x_4 \\ x_6 & x_2 & x_4 & x_3 & x_1 & x_5 \\ x_5 & x_3 & x_2 & x_4 & x_6 & x_1 \\ x_1 & x_4 & x_3 & x_2 & x_5 & x_6 \\ x_4 & x_5 & x_1 & x_6 & x_2 & x_3 \\ x_3 & x_6 & x_5 & x_1 & x_4 & x_2 \end{pmatrix}, \begin{pmatrix} x_2 & x_1 & x_5 & x_6 & x_3 & x_4 \\ x_6 & x_2 & x_4 & x_3 & x_1 & x_5 \\ x_5 & x_3 & x_2 & x_4 & x_6 & x_1 \\ x_1 & x_4 & x_3 & x_2 & x_5 & x_6 \\ x_4 & x_5 & x_6 & x_1 & x_2 & x_3 \\ x_3 & x_6 & x_1 & x_5 & x_4 & x_2 \end{pmatrix}, \begin{pmatrix} x_2 & x_1 & x_6 & x_5 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_3 & x_5 & x_6 \\ x_6 & x_3 & x_2 & x_4 & x_1 & x_5 \\ x_5 & x_4 & x_3 & x_2 & x_6 & x_1 \\ x_4 & x_5 & x_1 & x_6 & x_2 & x_3 \\ x_3 & x_6 & x_5 & x_1 & x_4 & x_2 \end{pmatrix}\right\}.$$

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