Gröbner bases and cocyclic Hadamard matrices

Álvarez, Armario, Falcón, Frau and Gudiel

University of Seville

5th Workshop on Real and Complex Hadamard Matrices and Applications

Outline

Hadamard cocyclic ideals

Hadamard ideals - Kotsireas, Koukouvinos, Seberry (2006)

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G is a group of order $4t$, a *cocycle* ψ over G is a mapping $\psi : G \times G \to \langle -1 \rangle$ satisfying $\psi(1,1) = \psi(g,1) = \psi(1,g) = 1,\ g \in G$ and the cocycle equation:

$$\psi(g_i, g_j)\ \psi(g_i g_j, g_k)\psi(g_i, g_j g_k)\ \psi(g_j, g_k) = 1, \quad g_i, g_j, g_k \in G. \qquad (1)$$

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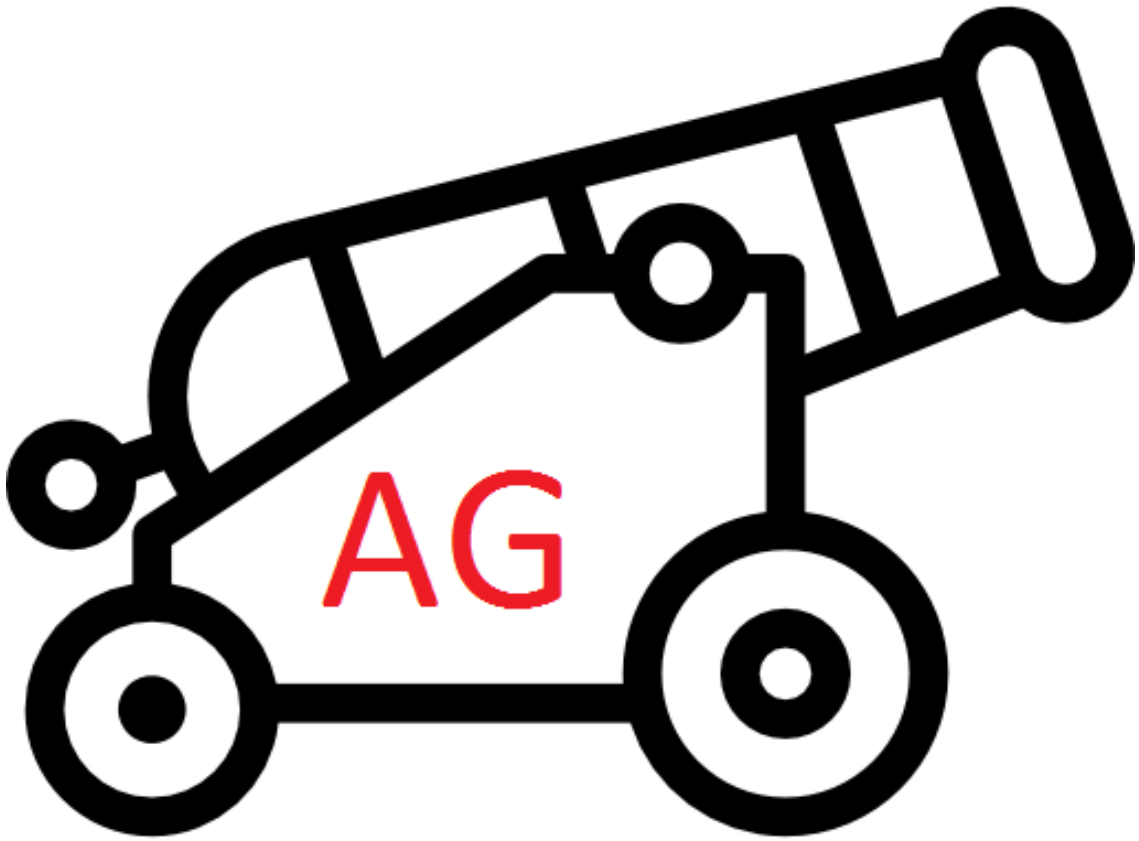
Cons

- $\{$ Cocyclic Hadamard Matrices $\} \subset \{$ Hadamard Matrices $\}$

The idea

Use the Algebraic Geometry artillery (namely Gröbner basis techniques) to determine both the cardinality and the elements of the set $\mathcal{H}_G$ of cocyclic Hadamard matrices over a multiplicative finite group G of $4t$ elements.



First (naive) approach

$\mathbb{Q}[X_G]$ be the polynomial ring over $\{X_G\} = \{x_{i,j} : \ g_i, g_j \in G\}$

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Theorem

The set $\mathcal{H}_G$ can be identified with the set of zeros of the zero-dimensional ideal $I_G = I_G^1 + I_G^2 + I_G^3 + I_G^4 \subset \mathbb{Q}[X_G]$, where

$$\begin{cases} I_G^1 = \langle x_{i,j}^2 - 1 : \ i,j \in G \rangle \\ I_G^2 = \langle x_{i,j}x_{ij,k} - x_{j,k}x_{i,jk} : \ i,j,k \in G \rangle \\ I_G^3 = \langle x_{1,j} - 1, x_{i,1} - 1 : \ i,j \in G \setminus \{1\} \rangle \\ I_G^4 = \langle \sum_{j \in G} x_{i,j} : \ i \in G \setminus \{1\} \rangle \end{cases}$$

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Set of polynomials defining I_G:

$O(t^3)$ polynomials of degree up to 2 over $O(t^2)$ variables

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Open computer algebra system for polynomial computations SINGULAR
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http://personales.us.es/raufalgan/LS/hadamard.lib

$$\begin{cases} G = 1 \Rightarrow \mathbb{Z}_t \times \mathbb{Z}_2^2, G = 2 \Rightarrow D_{4t} \\ opt = 1 \Rightarrow \sharp\mathcal{H}_G, opt = 2 \Rightarrow \mathcal{H}_G \end{cases}$$

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$$t \leq 3$$

Basis of normalized cocycles

Fixed a representative cocycle ρ, and a basis for normalized cocycles $\mathbf{B}$.
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(Álvarez et al.)[2008] The matrix M_ψ is Hadamard if and only if the vector of coordinates $(x_1, \ldots, x_k)_{\mathbf{B}}$ of ψ with regards to $\mathbf{B}$ satisfies the following system of $4t - 1$ equations and k unknowns

$$\begin{cases} (m^1_{2,1})^{x_1} \ldots (m^k_{2,1})^{x_k} + \ldots + (m^1_{2,4t})^{x_1} \ldots (m^k_{2,4t})^{x_k} & = & 0 \\ & \vdots & \\ (m^1_{4t,1})^{x_1} \ldots (m^k_{4t,1})^{x_k} + \ldots + (m^1_{4t,4t})^{x_1} \cdots (m^k_{4t,4t})^{x_k} & = & 0 \end{cases} \tag{2}$$

□

Theorem

The set $\mathcal{H}^{\rho}_{G}$ can be identified with the set of zeros of the following zero-dimensional ideal of $\mathbb{Q}[X]$.

$$J_G := \langle x_i^2 - x_i : \ i \in \{1, \ldots, k-m\} \rangle + \langle \sum_{h=1}^{4t} s_{l,h}(X) : \ l \in \{1, \ldots 4t-1\} \rangle.$$

$s(l,h)$ is defined in terms of paths and intersections, and $deg(s_{l,h}) \leq 2$.

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$$t \leq 7, \ D_{4t}$$

Fine tuning: $\mathbb{Z}_t \times \mathbb{Z}_2^2$

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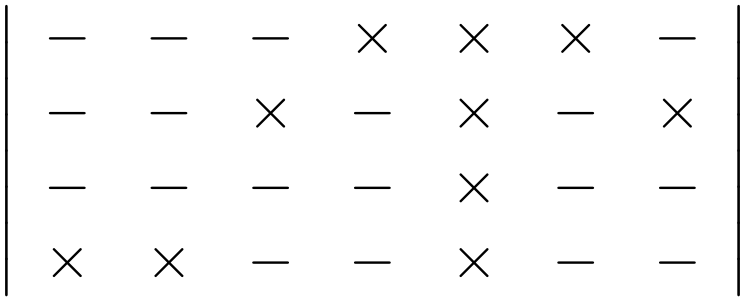
Diagrammatic properties:



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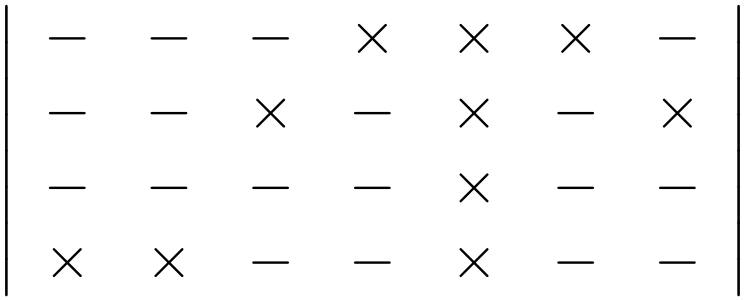

$\texttt{CocAH}(t, col, dist, H)$

- col: parity of columns
- dist: sum of each row
- H: fixed values of some coordinates

Fine tuning: $\mathbb{Z}_t \times \mathbb{Z}_2^2$, example

$t = 31$

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col: $0, 4, 2, 4, 2, 2, 2, 2, 0, 2, 2, 0, 4, 2, 2$

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$$10F6F96990660F6666F06609969F6F0$$

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$\texttt{CocDH}(t, col, dist, H)$

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$$\begin{cases} \text{dist: number of new intersections in each row [2,t]} \\ \text{H: fixed values of some coordinates} \end{cases}$$

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dist $1, 1, 2, 2, 3, 3, 2, 3, 0, 1, 0, 4, 2, 2, 3, 2, 3, 2, 2, 0, 0, 3, 1, 1, 2, 1, 3, 2, 2, 3$

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$H = \underline{5}, \underline{6}, \underline{17}, \underline{36}, \underline{48}, 63, 64, \underline{84}, \underline{95}, 115, \underline{117}$

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$$21172424E984E2E3FC6B06D5527CA70$$

Future work

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Improve the quality of the helping information

to get better results !

Thank you and farewell !!!