Counting partial Latin rectangles

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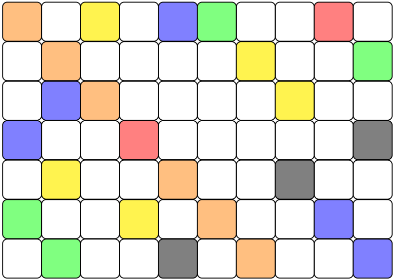
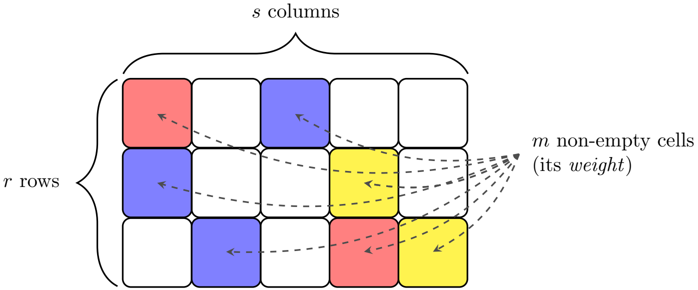
joint work with Raúl Falcón (University of Seville, Spain).

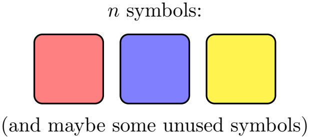
July 7, 2015



Partial Latin rectangles

Here's what I mean by a partial Latin rectangle in this talk:





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set of clashes

set of generalized ordered PLRs with clashes in V (and maybe more)

Example set of clashes V :

$e_1, e_2,$ same cell

$e_1, e_3,$ same cell

$e_3, e_4,$ same cell

$e_1, e_3,$ same symbol and row

$e_1, e_4,$ same symbol and row

$e_2, e_3,$ same symbol and column

Example set of clashes V :

$e_1, e_2,$ same cell

$e_1, e_3,$ same cell

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$e_1, e_3,$ same symbol and row

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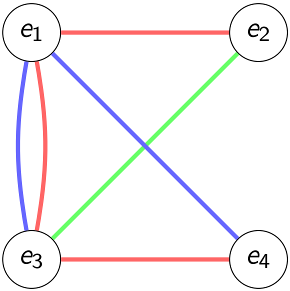
$e_2, e_3,$ same symbol and column



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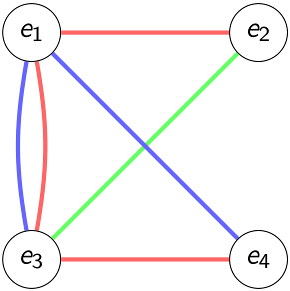

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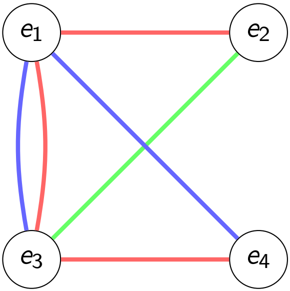

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If we "color" e_1 with (r_1, c_1, s_1) and e_2 with (r_2, c_2, s_2), then we want $r_1 = r_2$ and $c_1 = c_2$ to match the edge coloring.

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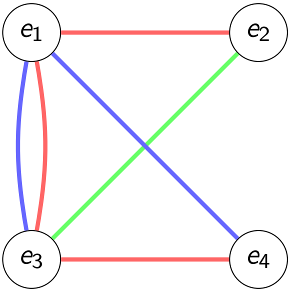

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And so on for the other edges.

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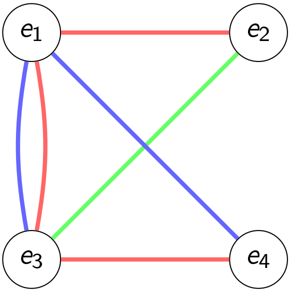

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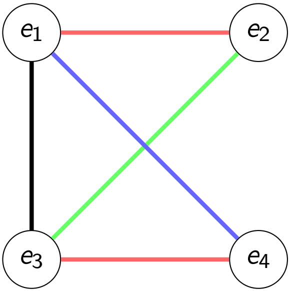

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And so on for the other edges.

Parallel edges imply $(r_i, c_i, s_i) = (r_j, c_j, s_j)$, regardless of the colors of the edges. So we replace them with a single black edge.

If we rephrase in terms of these colorings...

For all $m, r, s, n \geq 1$, we have

$$m!\, \#\mathrm{PLR}(r, s, n; m) =$$

$$(rsn)^m + \sum_{v \geq 2} \sum_{e \geq 1} (-1)^e \binom{m}{v} (rsn)^{m-v+1} \sum_{G \in \Gamma_{e,v}} \frac{v!}{|\mathrm{Aut}(G)|} P(G)$$

where

$$P(G) = P(G; r, s, n) = \sum_{\delta} (-2)^{\mathbf{b}(\delta)} r^{c(H_3)-1} s^{c(H_2)-1} n^{c(H_1)-1}$$

where the sum is over all (red, blue, green, black) edge colorings δ of G.

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What's important here:

- For arbitrary simple graphs G, there is a graph polynomial $P(G)$.
- If we compute $P(G)$ and $|\mathrm{Aut}(G)|$ for small graphs, we find $\#\mathrm{PLR}(r,s,n;m)$ for small m.

We compute these polynomials and automorphism group sizes:

G	v	e	$c(G)$	$\lvert\mathrm{Aut}(G)\rvert$	$P(G) = P(G; r, s, n)$
	2	1	1	2	$\overline{100} - 2$
	3	2	1	2	$P($$)^2$
	3	3	1	6	$\overline{200} - 2$
	4	2	2	8	$\overline{111}\, P($$)^2$
	4	3	1	6	$P($$)^3$
	4	3	1	2	$P($$)^3$
	4	1	1	2	$P($$)P($$)$
	4	4	1	8	$\overline{300} + 6\,\overline{110} - 12\,\overline{100} + 16$
	4	5	1	4	$\overline{300} + 2\,\overline{110} - 4\,\overline{100} + 4$
	4	6	1	24	$\overline{300} - 2$
	5	3	2	4	$\overline{111}\, P($$)^3$
	5	4	2	12	$\overline{111}\, P($$)P($$)$
	6	3	3	48	$\overline{222}\, P($$)^3$

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Here, we use this shorthand:

$$\overline{210} = r^2 s + r^2 n + s^2 r + s^2 n + n^2 r + n^2 s, \text{ and}$$

$$2\,\overline{100} = 2(r + s + t).$$

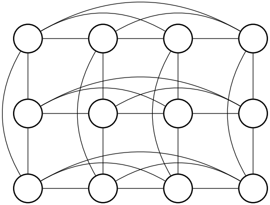
The asymptotic number of partial Latin rectangles of fixed weight...

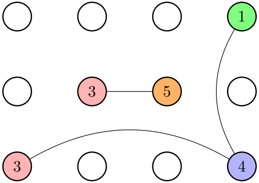
For fixed m, we have

$$\mathbf{m!\,\#\mathrm{PLR}(r,s,n;m)} = (rsn)^m + \binom{m}{2}(rsn)^{m-1}(2-\overline{100}) + \binom{m}{3}(rsn)^{m-2}(14 - 12\,\overline{100} + 6\,\overline{110} + 2\,\overline{200}) + \binom{m}{4}(rsn)^{m-3}(198 - 228\,\overline{100} + 198\,\overline{110} - 84\,\overline{111} + 72\,\overline{200} - 36\,\overline{210} - 12\,\overline{211} + 6\,\overline{221} - 6\,\overline{300} + 3\,\overline{311}) + \binom{m}{5}(rsn)^{m-4}(-6360\,\overline{100} + 7440\,\overline{110} - 6080\,\overline{111} + 2880\,\overline{200} - 2520\,\overline{210} + 820\,\overline{211} + 480\,\overline{220} + 360\,\overline{221} - 180\,\overline{222} - 480\,\overline{300} + 240\,\overline{310} + 160\,\overline{311} - 80\,\overline{321} + 24\,\overline{400} - 20\,\overline{411}) + \binom{m}{6}(rsn)^{m-5}(-13170\,\overline{211} + 17340\,\overline{221} - 15990\,\overline{222} + 7580\,\overline{311} - 7050\,\overline{321} + 3300\,\overline{322} + 1520\,\overline{331} + 180\,\overline{332} - 90\,\overline{333} - 1740\,\overline{411} + 870\,\overline{421} + 90\,\overline{422} - 45\,\overline{432} + 130\,\overline{511} - 15\,\overline{522}) + \binom{m}{7}(rsn)^{m-6}(-10920\,\overline{322} + 15540\,\overline{332} - 15120\,\overline{333} + 7350\,\overline{422} - 7140\,\overline{432} + 3570\,\overline{433} + 1680\,\overline{442} - 2100\,\overline{522} + 1050\,\overline{532} + 210\,\overline{622}) + \binom{m}{8}(rsn)^{m-7}(-3360\,\overline{433} + 5040\,\overline{443} - 5040\,\overline{444} + 2520\,\overline{533} - 2520\,\overline{543} + 1260\,\overline{544} + 630\,\overline{553} - 840\,\overline{633} + 420\,\overline{643} + 105\,\overline{733}) + \textbf{some polynomial of degree } \mathbf{\leq 3m-12}.$$

Method 2: Chromatic Polynomials

Any partial Latin rectangle $\mathrm{PLR}(r, s, n; m)$ can be interpreted as a proper n-coloring of an m-vertex induced subgraph of the $r \times s$ rook's graph.





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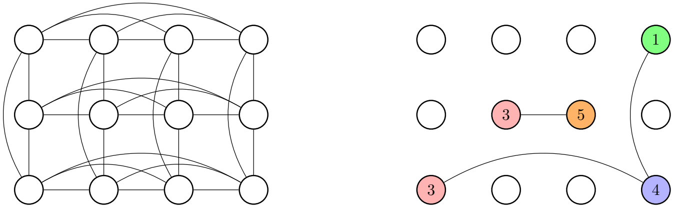

If Π denotes the chromatic polynomial, we thus have

$$\#\mathrm{PLR}(r, s, n; m) = \sum_{M} \Pi(M; n)$$

over all m-vertex induced subgraphs M of the $r \times s$ rook's graph.

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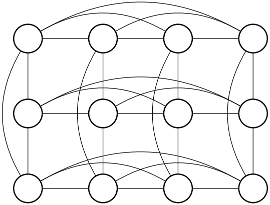

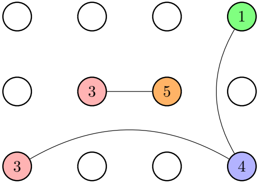

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over all m-vertex induced subgraphs M of the $r \times s$ rook's graph. Or, equivalently, $(0, 1)$-matrices with m ones.

We can permute the rows and columns of a $(0,1)$-matrix with m ones into a canonical form:

$$\begin{array}{|c|c|c|c|c|}
\hline
K_1 & \emptyset & \cdots & \emptyset & \emptyset \\
\hline
\emptyset & K_2 & & \emptyset & \emptyset \\
\hline
\vdots & & \ddots & & \vdots \\
\hline
\emptyset & \emptyset & & K_k & \emptyset \\
\hline
\emptyset & \emptyset & \cdots & \emptyset & \emptyset \\
\hline
\end{array}.$$

The blocks $K_1, K_2, \ldots, K_k$ are in some kind of canonical form under row/column permutations.

If we sum over such canonical forms, for fixed m, we get:

$$m!\,\#\mathrm{PLR}(r,s,n;m) =$$

$$\sum_{k\geq 0}\ \sum_{\substack{(K_1,K_2,\ldots,K_k)\\ m \text{ ones}}}\ \sum_{\text{good } (t_i)_{i=1}^k} [r]_{e_{\mathrm{row}}}[s]_{e_{\mathrm{col}}} \frac{\prod_{i=1}^k \Pi(K_i;n)}{\left(\prod_{i=1}^k |\mathrm{Aut}(G_{K_i})|\right)\left(\prod_{i=1}^\ell k_i!\right)}$$

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- $[r]_{e_{\text{row}}} = r(r-1)\cdots(r-e_{\text{row}}+1)$ and $[s]_{e_{\text{col}}} = s(s-1)\cdots(s-e_{\text{col}}+1)$; e_{row} and e_{col} denote the number of empty rows and columns

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where...

- the t_i's keep track of which matrices are transposed (saves computation),
- $[r]_{e_{\text{row}}} = r(r-1)\cdots(r-e_{\text{row}}+1)$ and $[s]_{e_{\text{col}}} = s(s-1)\cdots(s-e_{\text{col}}+1)$; e_{row} and e_{col} denote the number of empty rows and columns
- k_i, for $i\in\{1,2,\ldots,\ell\}$, be the number of copies of the i-th distinct matrix (given ℓ distinct matrices).

So we compute...

block K	$\lvert \mathrm{Aut}(G_K) \rvert$	$\Pi(K;n)$
$\begin{matrix}1\end{matrix}$	1	n
$\begin{matrix}1 & 1\end{matrix}$	2	$n^2 - n$
$\begin{matrix}1 & 1 & 1\end{matrix}$	6	$n^3 - 3n^2 + 2n$
$\begin{matrix}1 & 1 \\ 1 & 0\end{matrix}$	1	$n^3 - 2n^2 + n$
$\begin{matrix}1 & 1 & 1 & 1\end{matrix}$	24	$n^4 - 6n^3 + 11n^2 - 6n$
$\begin{matrix}1 & 1 & 1 \\ 1 & 0 & 0\end{matrix}$	2	$n^4 - 4n^3 + 5n^2 - 2n$
$\begin{matrix}1 & 1 & 0 \\ 1 & 0 & 1\end{matrix}$	2	$n^4 - 3n^3 + 3n^2 - n$
$\begin{matrix}1 & 1 \\ 1 & 1\end{matrix}$	4	$n^4 - 4n^3 + 6n^2 - 3n$
$\begin{matrix}1 & 1 & 1 & 1 & 1\end{matrix}$	120	$n^5 - 10n^4 + 35n^3 - 50n^2 + 24n$
$\begin{matrix}1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0\end{matrix}$	6	$n^5 - 7n^4 + 17n^3 - 17n^2 + 6n$
$\begin{matrix}1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1\end{matrix}$	2	$n^5 - 5n^4 + 9n^3 - 7n^2 + 2n$
$\begin{matrix}1 & 1 & 1 \\ 1 & 1 & 0\end{matrix}$	2	$n^5 - 6n^4 + 14n^3 - 15n^2 + 6n$
$\begin{matrix}1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0\end{matrix}$	4	$n^5 - 6n^4 + 13n^3 - 12n^2 + 4n$
$\begin{matrix}1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{matrix}$	2	$n^5 - 5n^4 + 9n^3 - 7n^2 + 2n$
$\begin{matrix}1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0\end{matrix}$	2	$n^5 - 5n^4 + 9n^3 - 7n^2 + 2n$
$\begin{matrix}1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0\end{matrix}$	1	$n^5 - 4n^4 + 6n^3 - 4n^2 + n$

...and so on.

And we get **exact formulas** for the number of small-weight partial Latin rectangles:

$$1!\,\#\mathrm{PLR}(r,s,n;1)=\overline{111}.$$

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$$3!\,\#\mathrm{PLR}(r,s,n;3)=$$
$$\overline{333}-3\,\overline{322}+6\,\overline{222}+2\,\overline{311}+6\,\overline{221}-12\,\overline{211}+14\,\overline{111}.$$

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$$\overline{444}-6\,\overline{433}+12\,\overline{333}+11\,\overline{422}+30\,\overline{332}-60\,\overline{322}-6\,\overline{411}-$$
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In this way, we managed to compute the exact formulas for up to weight $m = 14$.

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For the partial Latin rectangle $P = (p_{ij})$ we have $p_{ij} = k$ whenever $x_{ijk} = 1$, and p_{ij} is undefined otherwise.

Thus (from algebraic geometry)

$$\#\mathrm{PLR}(r, s, n) = \dim_{\mathrm{GF}(2)}\bigl(\mathrm{GF}(2)[\mathbf{x}]/I_{r,s,n}\bigr)$$

and

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There are algorithms in algebraic geometry to compute this Hilbert function. In this way, we compute $\#\mathrm{PLR}(r, s, n; m)$ whenever $r, s, n \leq 6$.

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We then use Burnside's Lemma, and sum over possible symmetries to give the number of equivalence classes.

This has been used to find the number of isomorphism and isotopism classes of partial Latin rectangles in small cases.

	# isomorphism classes					
	n					
m	1	2	3	4	5	6
0	1	1	1	1	1	1
1	1	4	5	5	5	5
2		10	50	84	93	94
3		4	221	1120	2112	2548
4		1	525	10128	43955	85234
5			651	60092	674957	2508483
6			415	239302	7679384	59110661
7			136	639098	65404265	1103309385
8			20	1148454	422142208	16466869051
9			5	1374447	2080853035	198621450446
10				1082019	7867483199	1953036511736
11				548440	22843744418	15756857221135
12				176137	50867669444	104784604156741
13				35473	86544642569	576125696499417
14				4696	111836743580	2623564948795633
15				403	108882205792	9901507463165937
16				35	79051125332	30959687376379661
17					42275685836	80100291981771263
18					16420711804	171118574787473668
19					4563456676	300957676311237853
20					894429087	434125855232450974
21					122238972	511227919780309083
22					11569016	488771341028032846
23					759296	376957644290919036
24					33736	232788472371575258
25					1411	114149339445885218
26						44033009520708974
27						13227534274721732
28						3061826358557444
29						540473537486248
30						72090555296085
31						7217657260917
32						540810639064
33						30364554576
34						1285684592
35						40649375
36						1130531
Total	2	20	2029	5319934	534759300182	2815323435872410905

	# isotopism classes					
	n					
m	1	2	3	4	5	6
0	1	1	1	1	1	1
1	1	1	1	1	1	1
2		4	4	4	4	4
3		1	11	11	11	11
4		1	18	52	52	52
5			23	139	221	221
6			15	507	1158	1396
7			6	1161	6310	9130
8			1	2136	33293	72145
9			1	2429	150964	583339
10				2004	554285	4627607
11				975	1594532	33362634
12				364	3539461	210409407
13				72	6017824	1129335392
14				18	7772366	5091624997
15				2	7568187	19140028219
16				2	5493206	59761963636
17					2939617	154544375137
18					1141472	330108625102
19					317980	580559388329
20					62319	837440466326
21					8676	986167409118
22					823	942850011453
23					69	727157075193
24					6	449054224783
25					2	220195944263
26						84941236104
27						25516234965
28						5906586539
29						1042616896
30						139114631
31						13928529
32						1048656
33						59130
34						2846
35						109
36						22
Total	2	8	81	9878	37202839	5431010366322

	# main classes				
	n				
m	1	2	3	4	5
0	1	1	1	1	1
1	1	1	1	1	1
2		2	2	2	2
3		1	5	5	5
4		1	8	18	18
5			9	39	59
6			7	121	256
7			4	253	1224
8			1	442	5997
9			1	495	26188
10				420	94479
11				218	269456
12				96	595649
13				25	1010706
14				8	1304319
15				2	1270356
16				2	923128
17					495565
18					193531
19					54746
20					11052
21					1693
22					192
23					26
24					4
25					2
Total	2	6	39	2148	6239377

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