The compressed shape of a partial Latin rectangle.

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Basic notions.

- An $r \times s$ partial Latin rectangle based on a set of n symbols is an $r \times s$ array in which each cell is either empty or contains one element chosen from a set of symbols, $[n] = \{1, 2, ..., n\}$, s.t. each symbol occurs at most once in each row and in each column.

- The number of filled cells is its size. Their positions determine the shape:

$$P = \begin{pmatrix} 1 & \cdot & 3 & 4 & 6 \\ 2 & \cdot & 5 & \cdot & 4 \\ \cdot & \cdot & 4 & 5 & 1 \\ \cdot & \cdot & 2 & \cdot & 3 \end{pmatrix} \in \mathcal{PLR}^6_{4\times 5,12} \to \mathrm{Sh}(P) = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

- $r = s = n \to$ Partial Latin square.
- Size $r \cdot s \to$ Latin rectangle (square if $r = s$).
- **Applications**: Algebra (quasigroups), Experimental Designs, Cryptography.

Classification of Latin Rectangles.

- Orthogonal representation: $O(P) = \{(\text{row}, \text{ column}, \text{ symbol})\}$.
- Classification:
  1. Isotopism: Permutations of rows, columns and symbols.
  2. Parastrophism (main classes):
     $\pi \in S_3 \to O(P^\pi) = \{(l_{\pi(1)}, l_{\pi(2)}, l_{\pi(3)}) \mid (l_1, l_2, l_3) \in O(P)\}$.

n	$\lvert LS_n \rvert$	IC	MC
1	1	1	1
2	2	1	1
3	12	1	1
4	576	2	2
5	161280	2	2
6	812851200	22	12
7	61479419904000	564	147
8	108776032459082956800	1676267	283657
9	5524751496156892842531225600	115618721533	19270853541
10	9982437658213039871725064756920320000	208904371354363006	34817397894749939
11	776966836171770144107444346734230682311065600000	12216177315369229261482540	2036029552582883134196099

[Hulpke et al., 2011]

What about partial Latin rectangles?

$|PLR^n_{r\times s}|$ upper bounded for $r = s = n$ [Ghandehari, 2005].
$|IC|$ and $|MC|$ lower bounded for $r = s = n \leq 6$ [Adams, 2003].

Order n	Size m	$\lvert PLS_{n,m}\rvert$
1	1	1
2	1	8
	2	16
	3	8
	4	2
3	1	27
	2	270
	3	1,278
	4	3,078
	5	3,834
	6	2,412
	7	756
	8	108
	9	12
4	1	64
	2	1,728
	3	25,920
	4	239,760
	5	1,437,696
	6	5,728,896
	7	15,326,208
	8	27,534,816
	9	32,971,008
	10	25,941,504
	11	13,153,536
	12	4,215,744
	13	847,872
	14	110,592
	15	9,216
	16	576

[Falcón, 2012]

Some exact values have recently been obtained by applying Gröbner bases in an equivalent planar assignment problem:

$$\begin{cases} \sum_{k\in[n]} x_{ijk} \leq 1, \forall i,j \in [n], \\ \sum_{j\in[n]} x_{ijk} \leq 1, \forall i,k \in [n], \\ \sum_{i\in[n]} x_{ijk} \leq 1, \forall j,k \in [n], \\ \sum_{i,j,k\in[n]} x_{ijk} = m, \\ x_{ijk} \in \{0,1\}, \forall i,j,k \in [n], \end{cases}$$

What about partial Latin rectangles?

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Type and structure of a PLR.

$$\sum_{i,j,k\in[n]} x_{ijk} = m \to \begin{cases} \sum_{j,k\in[n]} x_{ijk} = T_1(P,i), \leftarrow \text{ Rows.} \\ \sum_{i,k\in[n]} x_{ijk} = T_2(P,j), \leftarrow \text{ Columns.} \\ \sum_{i,j\in[n]} x_{ijk} = T_3(P,k), \leftarrow \text{ Symbols.} \end{cases}$$

$$P = \begin{pmatrix} 1 & \cdot & 3 & 4 & 6 \\ 2 & \cdot & 5 & \cdot & 4 \\ \cdot & \cdot & 4 & 5 & 1 \\ \cdot & \cdot & 2 & \cdot & 3 \end{pmatrix} \in \mathcal{PLR}^6_{4\times 5,12} \to \mathrm{Sh}(P) = \left(\begin{array}{c|ccccc} & 2 & 0 & 4 & 2 & 4 \\ \hline 4 & 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

Row type: $T_1(P) = (4,3,3,2)$

Column type: $T_2(P) = (2,0,4,2,4)$.

Symbol type: $T_3(P) = (2,2,2,3,2,1)$.

[Keedwell, 1994; Bean, 2002]

Type and structure of a PLR.

- $\mathcal{T}_{l,w} = \{(t_1, \ldots, t_l)$ of *weight* $\sum_{i\in[l]} t_i = w$, s.t. $t_i \in \mathbb{N}\}$.
- Structure of $T \in \mathcal{T}_{l,w}$: $\mathrm{st}(T) = w^{\lambda_w^T} \ldots 1^{\lambda_1^T}$, where λ_i^T is the number of occurrences of i in T.
- $\mathcal{Z}_{l,w}$: Set of possible structures of length l and weight w.

$$P = \begin{pmatrix} 1 & \cdot & 3 & 4 & 6 \\ 2 & \cdot & 5 & \cdot & 4 \\ \cdot & \cdot & 4 & 5 & 1 \\ \cdot & \cdot & 2 & \cdot & 3 \end{pmatrix} \in \mathcal{PLR}^6_{4\times 5,12} \to \mathrm{Sh}(P) = \left(\begin{array}{c|ccccc} & 2 & 0 & 4 & 2 & 4 \\ \hline 4 & 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

Row type: $T_1(P) = (4,3,3,2) \to \mathrm{st}(T_1(P)) = 43^22.$

Column type: $T_2(P) = (2,0,4,2,4) \to \mathrm{st}(T_2(P)) = 4^22^2.$

Symbol type: $T_3(P) = (2,2,2,3,2,1) \to \mathrm{st}(T_3(P)) = 32^41.$

Type and structure of a PLR.

Some examples:

a) $F(n; n-k, 1^k)$-squares [Hedayat, 1970] $\rightarrow (k^n, k^n, n^k)$.

b) Cycles [Donovan, 2002] $\rightarrow (2^2, 2^2, \cdot)$.

c) k-plexes [Wanless, 2002] $\rightarrow (k^n, k^n, k^n)$.

d) Intercalates [Colbourn, 2007] $\rightarrow (2^2, 2^2, 2^2)$.

e) The problem of completing partial Latin squares has also dealt with several structures:

- $(k^l, l^k, \cdot)$ [Ryser, 1951].
- $((n-k)^n, (n-k)^n, (n-k)^n)$, $k = 1, 2$ [Andersen and Hilton, 1997].
- $(n^2 2^{n-2}, n^2 2^{n-2})$ [Adams et al., 2008].

Type and structure of a PLR.

- Given $R \in \mathcal{T}_{r,w}$, $C \in \mathcal{T}_{s,w}$ and $S \in \mathcal{T}_{n,w}$:

$$\mathcal{PLR}^n_{(R,C)} = \{P \in \mathcal{PLR}^n_{r\times s} \colon\ T_1(P) = R \text{ and } T_2(P) = C\}.$$

$$\mathcal{PLR}_{(R,C,S)} = \{P \in \mathcal{PLR}^n_{r\times s} \colon\ T_1(P) = R,\ T_2(P) = C \text{ and } T_3(P) = S\}.$$

Lemma

$|\mathcal{PLR}^n_{(R,C)}|$ and $|\mathcal{PLR}_{(R,C,S)}|$ only depend on the structures of R, C, S.

Type and structure of a PLR.

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Lemma

$|\mathcal{PLR}^n_{(R,C)}|$ and $|\mathcal{PLR}_{(R,C,S)}|$ only depend on the structures of R, C, S.

- Given $z_1 \in \mathcal{Z}_{l_1,w}$, $z_2 \in \mathcal{Z}_{l_2,w}$ and $z_3 \in \mathcal{Z}_{l_3,w}$:

$$\Delta^n_{r\times s}(z_1, z_2) = |\mathcal{PLR}^n_{(R,C)}|, \forall R \in \mathcal{T}_{r,w}, C \in \mathcal{T}_{s,w}, \text{ s.t. } \mathrm{st}(R) = z_1, \mathrm{st}(C) = z_2.$$

$$\Delta^n_{r\times s}(z_1, z_2, z_3) = |\mathcal{PLR}^n_{(R,C,S)}|, \forall R \in \mathcal{T}_{r,w}, C \in \mathcal{T}_{s,w}, S \in \mathcal{T}_{n,w}, \text{ s.t. } \mathrm{st}(R) = z_1, \mathrm{st}(C) = z_2, \mathrm{st}(S) = z_3.$$

Type and structure of a PLR.

Proposition

$$|\mathcal{PLR}^n_{r\times s}| = \sum_{(l,l')\in[r]\times[s]} \sum_{w\in[l\cdot s]} \sum_{(z,z')\in\mathcal{Z}_{l,w}\times\mathcal{Z}_{l',w}} \frac{r!}{(r-l)!\cdot\prod_{i\in[w]}\lambda_i!} \cdot \Delta^n_{r\times s}(z,z')$$

Where:

$$\Delta^n_{l\times s}(z,z') = \sum_{l''\in[n]} \sum_{z''\in\mathcal{Z}_{l'',w}} \Delta^n_{r\times s}(z,z',z'').$$

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Problem

How to obtain $\Delta^n_{r\times s}(z,z')$ and $\Delta^n_{r\times s}(z,z',z'')$?

Type and structure of a PLR.

Proposition

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Where:

$$\Delta^n_{l\times s}(z,z') = \sum_{l''\in[n]} \sum_{z''\in\mathcal{Z}_{l'',w}} \Delta^n_{r\times s}(z,z',z'').$$

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THE SET $\mathfrak{A}(R, C)$.

$$P = \begin{pmatrix} 1 & \cdot & 3 & 4 & 6 \\ 2 & \cdot & 5 & \cdot & 4 \\ \cdot & \cdot & 4 & 5 & 1 \\ \cdot & \cdot & 2 & \cdot & 3 \end{pmatrix} \in \mathcal{PLR}^{6}_{4\times 5,12} \rightarrow \mathrm{Sh}(P) = \left(\begin{array}{c|ccccc} & 2 & 0 & 4 & 2 & 4 \\ \hline 4 & 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$|\mathcal{PLR}^{n}_{(R,C)}| \rightarrow \mathfrak{A}(R, C)$$

- $\mathfrak{A}(R, C)$: $(0, 1)$-matrices having R and C as row and column sum vectors.
- $\mathcal{PLR}^{n}_{M}$: Set of PLR of n symbols having $M \in \mathfrak{A}(R, C)$ as shape.

$$|\mathcal{PLR}^{n}_{(R,C)}| = \sum_{M \in \mathfrak{A}_{(R,C)}} |\mathcal{PLR}^{n}_{M}|.$$

Equivalent problems:

- n-edge-colouring a bipartite graph of incidence matrix $\mathrm{Sh}(P)$ (Existence problem is NP-complete even for $n = 3$ [Holyer, 1981]).
- 1-*color tomography problem* [Kuba, 1999]: Reconstructing a binary matrix starting from its row and column sums.

The set $\mathfrak{A}(R, C)$.

Gale-Ryser theorem [Gale, Ryser, 1957]: $\mathfrak{A}(R, C) \neq \emptyset \Leftrightarrow C \preceq R^*$.
$R = (3, 5, 2, 2) \rightarrow R^* = (4, 4, 2, 1, 1) \succeq (3, 3, 3, 2, 1)$. *(Dominance order)*.

Formulas and algorithms:

- Monomial symmetric functions [Sukhatme, 1938; David, 1951 ($\leq$ 12)].
- Character of the symmetric group [Snapper, 1971].
- Lower bound [Wei, 1982].
- Recurrence formulas [Wang, 1988; Wang and Zhang, 1998; Pérez Salvador, 2002].
- General formulas [Dias, 2002].
- Asymptotic methods [Barvinok, 2010].
- Combinatorial methods [Brualdi, 1980; Brualdi, 2006; Fonseca, 2009].
- Simulation methods (social networks, ecology) [Snijders, 1991; Rao, 1996; Chen, 2005; Bezakova, 2007; Blanchet, 2009].

The set $\mathfrak{A}(R, C)$.

Algebraic approach: Gröbner bases of boolean ideals for counting problems [Bayer, 1982; Alon, 1995; Bernasconi, 1997].

Theorem

$R = (\mathrm{r}_1, \ldots, \mathrm{r}_r) \in \mathcal{T}_{r,w}$ and $C = (\mathrm{c}_1, \ldots, \mathrm{c}_s) \in \mathcal{T}_{s,w}$ s.t. $C \preceq R^*$.

$\mathfrak{A}(R, C) = V(I)$, where:

$$I = \langle\, (\sum_{j \in [s]} x_{ij} - \mathrm{r}_i) \colon\ i \in [r]\,\rangle + \langle\, (\sum_{i \in [r]} x_{ij} - \mathrm{c}_j) \colon\ j \in [s]\,\rangle$$
$$+ \langle\, x_{ij} \cdot (1 - x_{ij}) :\ i \in [r], j \in [s]\,\rangle \subseteq \mathbb{Q}[x_{11}, \ldots, x_{rs}].$$

Moreover, $|\mathfrak{A}(R, C)| = \dim_{\mathbb{Q}}(\mathbb{Q}[x_{11}, \ldots, x_{rs}]/I)$.

THE SET $\mathfrak{A}(R, C)$.

z_1	z_2	$\lvert\mathfrak{A}(R,C)\rvert$
1	1	1
1^2	1^2	2
21	1^3	3
1^3	1^3	6
31	1^4	4
2^2	1^4	6
21^2	21^2	5
	1^4	12
1^4	1^4	24
41	1^5	5
32	1^5	10
31^2	21^3	7
	1^5	20
2^21	2^21	5
	21^3	36
	1^5	30
21^3	21^3	27
	1^5	60
1^5	1^5	120
41^2	21^4	9
321	2^21^2	8
	21^4	22
31^3	31^3	10
	2^21^2	18
	21^4	48
2^3	2^3	6
	2^21^2	15
	21^4	36
2^21^2	2^21^2	34
	21^4	78
21^4	21^4	168
421	2^21^3	11

z_1	z_2	$\lvert\mathfrak{A}(R,C)\rvert$
421^2	321^3	18
	2^31^2	39
41^4	41^4	17
	321^3	46
	2^31^2	84
3^22	2^4	12
	2^31^2	31
3^21^2	3^21^2	4
	32^21	12
	321^3	30
	2^4	28
	2^31^2	68
32^21	32^21	24
	321^3	58
	2^4	48
	2^31^2	117
321^3	321^3	141
	2^4	108
	2^31^2	258
2^4	2^4	90
	2^31^2	204
2^31^2	2^31^2	453
4^21	2^41	9
432	2^41	22
431^2	3^21^3	6
	32^21^2	19
	2^41	48
42^21	3^21^3	17
	32^21^2	36
	2^41	78
421^3	421^3	28
	3^21^3	42

z_1	z_2	$\lvert\mathfrak{A}(R,C)\rvert$
4^22	2^5	20
4^21^2	3^221^2	4
	32^31	16
	2^5	45
43^2	2^5	30
4321	3^221^2	21
	32^31	49
	2^5	170
431^3	431^3	9
	42^21^2	25
	3^221^2	48
	32^31	112
	2^5	240
42^3	3^221^2	42
	32^31	87
	2^5	180
42^21^2	42^21^2	54
	3^221^2	109
	32^31	198
	2^5	390
3^31	3^31	10
	3^22^2	18
	3^221^2	42
	32^31	87
	2^5	180
3^22^2	3^22^2	34
	3^221^2	80
	32^31	156
	2^5	310
3^221^2	3^221^2	186
	32^31	358
	2^5	680

z_1	z_2	$\lvert\mathfrak{A}(R,C)\rvert$
4321^2	42^31	76
	3^31^2	69
	3^22^21	148
	32^4	306
42^31	42^31	147
	3^31^2	138
	3^22^21	273
	32^4	555
3^32	3^32	27
	3^31^2	64
	3^22^21	120
	32^4	528
3^31^2	3^31^2	146
	3^22^21	276
	32^4	528
3^22^21	3^22^21	506
	32^4	934
32^4	32^4	1656
4^231	3^321	18
	3^22^3	39
4^22^2	3^321	30
	3^22^3	68
4^221^2	4^221^2	4
	43^21^2	10
	432^21	31
	42^4	76
	3^321	66
	3^22^3	153
43^22	3^321	58
	3^22^3	117
43^21^2	43^21^2	29
	432^21	60

The set $\mathfrak{A}(R, C)/\sim$.

$$M = \left(\begin{array}{c|ccccc} & 2 & 0 & 4 & 2 & 4 \\ \hline 4 & 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \sim N = \left(\begin{array}{c|ccccc} & 2 & 0 & 4 & 2 & 4 \\ \hline 4 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

- Two $(0,1)$-matrices of $\mathfrak{A}(R, C)$ are equivalent ($\sim$) if there are equal up to permutation of rows and columns.

- $M \sim N \Rightarrow |\mathcal{PLR}^n_M| = |\mathcal{PLR}^n_N|$.

$$\mu_M = \sharp\{N \in \mathfrak{A}(R, C) \mid N \sim M\}.$$

$$|\mathcal{PLR}^n_{(R,C)}| = \sum_{M \in \mathfrak{A}_{(R,C)}} |\mathcal{PLR}^n_M| = \sum_{M \in \mathfrak{A}_{(R,C)}/\sim} \mu_M \cdot |\mathcal{PLR}^n_M|.$$

The set $\mathfrak{A}(R, C)/\sim$.

$$M = \left(\begin{array}{c|ccccc} & 2 & 0 & 4 & 2 & 4 \\ \hline 4 & 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \sim N = \left(\begin{array}{c|ccccc} & 2 & 0 & 4 & 2 & 4 \\ \hline 4 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

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$$|\mathcal{PLR}^n_{(R,C)}| = \sum_{M \in \mathfrak{A}_{(R,C)}} |\mathcal{PLR}^n_M| = \sum_{M \in \mathfrak{A}_{(R,C)}/\sim} \mu_M \cdot |\mathcal{PLR}^n_M|.$$

The set $\mathfrak{A}(R, C)/\sim$.

$$\mu_M = \sharp\{N \in \mathfrak{A}(R, C) \mid N \sim M\} = \frac{|\mathrm{Perm}(R, C)|}{|\mathrm{Perm}(M, M)|} = \frac{\prod_{i \in [s]} \lambda_i^R! \cdot \lambda_i^C!}{|\mathrm{Perm}(M, M)|}.$$

- $\mathrm{Perm}(R, C) = \{(\alpha, \beta) \in S_r \times S_s$ s.t. $R^\alpha = R$ and $C^\beta = C\}$.
- $\mathrm{Perm}(M, N) = \{(\alpha, \beta) \in S_r \times S_s \mid M^{(\alpha, \beta)} = N\}$.

THE SET $\mathfrak{A}(R,C)/\sim$.

$$\mu_M = \sharp\{N \in \mathfrak{A}(R,C) \mid N \sim M\} = \frac{|\mathrm{Perm}(R,C)|}{|\mathrm{Perm}(M,M)|} = \frac{\prod_{i\in[s]} \lambda_i^R! \cdot \lambda_i^C!}{|\mathrm{Perm}(M,M)|}.$$

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- $\mathrm{Perm}(M,N) = \{(\alpha,\beta) \in S_r \times S_s \mid M^{(\alpha,\beta)} = N\}$.

THEOREM

$R \in \mathcal{T}_{r,w}, C \in \mathcal{T}_{s,w}$ s.t. $C \preceq R^*$. $M = (m_{ij}), N = (n_{ij}) \in \mathfrak{A}(R,C)$.

$\mathrm{Perm}(M,N) = V(I)$, where:

$$I = \langle\, (1 - \sum_{j\in[s]} x_{ij}) \colon\ i \in [r]\,\rangle + \langle\, (1 - \sum_{i\in[r]} x_{ij}) \colon\ j \in [s]\,\rangle + \langle\, (1 - \sum_{j\in[s]} y_{ij}) \colon\ i \in [r]\,\rangle +$$

$$\langle\, (1 - \sum_{i\in[r]} y_{ij}) \colon\ j \in [s]\,\rangle + \langle\, x_{ij} \cdot y_{kl} \colon\ i,j \in [r], k,l \in [s] \text{ and } m_{i,k} \neq n_{j,l}\,\rangle +$$

$$\langle\, x_{ij} \cdot (1 - x_{ij}) \,:\ i,j \in [r]\,\rangle + \langle\, y_{ij} \cdot (1 - y_{ij}) \,:\ i,j \in [s]\,\rangle \subseteq \mathbb{Q}[x_{11}, \ldots, x_{kk}, y_{11}, \ldots, y_{nn}].$$

Moreover, $|\mathrm{Perm}(M,N)| = \dim_{\mathbb{Q}}(\mathbb{Q}[x_{11}, \ldots, x_{kk}, y_{11}, \ldots, y_{nn}]/I)$.

The set $\mathfrak{A}(R, C)/\sim$.

μ_M only depends on non-zero rows and columns with at least one 0.

Compressed shape: $M \in \mathfrak{A}(R, C) \to \widetilde{M} \in \mathfrak{A}(\widetilde{R}, \widetilde{C})$, s. t. $\mu_{\widetilde{M}} = \mu_M$.

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \widetilde{M} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- In case of eliminating all the cells of M: $\widetilde{M} = (1)$ and $\mu_M = 1$.
- $\widetilde{R}$ and $\widetilde{C}$, only depend on the R and C.
- The process is always reversible:

$$\left(\begin{array}{c|cccc} R\backslash C & 3 & 1 & 1 & 1 \\ \hline 3 & . & . & . & . \\ 2 & . & . & . & . \\ 1 & . & . & . & . \\ 0 & . & . & . & . \end{array}\right) \to \left(\begin{array}{c|cccc} R\backslash C & 3 & 1 & 1 & 1 \\ \hline 3 & . & . & . & . \\ 2 & . & . & . & . \\ 1 & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to$$

$$\to \left(\begin{array}{c|cccc} R\backslash C & 3 & 1 & 1 & 1 \\ \hline 3 & 1 & . & . & . \\ 2 & 1 & . & . & . \\ 1 & 1 & . & . & . \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{c|cccc} R\backslash C & 3 & 1 & 1 & 1 \\ \hline 3 & 1 & . & . & . \\ 2 & 1 & . & . & . \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We focus on structures of R and C such that $\widetilde{M} = M, \forall M \in \mathfrak{A}(R, C)$.

Algorithm: Computing $\mathfrak{A}(R, C)/\sim$.

Input: $R = (\mathrm{r}_1, \ldots, \mathrm{r}_r) \in \mathcal{T}_{r,w}$ and $C = (\mathrm{c}_1, \ldots, \mathrm{c}_s) \in \mathcal{T}_{s,w}$ s.t. $C \preceq R^*$.
Output: $\mathfrak{A}(R, C)/\sim$.

$S = \mathfrak{A}(R, C)$, $S' = \emptyset$, $S'' = \emptyset$, $p_{RC} = |\mathrm{Perm}(R, C)| = \prod_{i \in [s]} \lambda_i^R! \cdot \lambda_i^C!$.

While $S \neq \emptyset$
    Take $M \in S$.
    $S = S \setminus \{M\}$.
    If $\nexists N \in S'$ s.t. $M \sim N$ $(\equiv \mathrm{Perm}(M, N) = \emptyset, \forall N \in S')$ **then**
        $\mu_M = p_{RC}/|\mathrm{Perm}(M, M)|$.
        $\rho_M = 1$.
        $S' = S' \cup \{M\}$.
    else
        $\rho_N = \rho_N + 1$.
        If $\rho_N = \mu_N$ **then**
            $S' = S' \setminus \{N\}$.
            $S'' = S'' \cup \{N\}$.
Return $\{(M, \mu_M) \mid M \in S''\}$.

The set $\widetilde{\mathrm{Sh}}(z_1, z_2)$ of compressed shapes, up to permutation of rows and columns.

z_1	z_2	$\lvert\widetilde{\mathrm{Sh}}(z_1, z_2)\rvert$
1	1	1
1^2	1^2	2
21	1^3	3
1^3	1^3	6
31	1^4	4
2^2	1^4	6
21^2	21^2	$4, 1$
	1^4	12
1^4	1^4	24
41	1^5	5
32	1^5	10
31^2	21^3	$6, 1$
	1^5	20
2^21	2^21	$4, 1$
	21^3	6^2
	1^5	30
21^3	21^3	$18, 9$
	1^5	60
1^5	1^5	120
41^2	21^4	$8, 1$
321	2^21^2	$4, 2^2$
	21^4	$12, 6, 4$
31^3	31^3	$9, 1$
	2^21^2	$12, 6$
	21^4	$36, 12$
2^3	2^3	6
	2^21^2	$12, 3$
	21^4	36
2^21^2	2^21^2	$16, 8, 4^2, 2$

z_1	z_2	$\lvert\widetilde{\mathrm{Sh}}(z_1, z_2)\rvert$
421^2	321^3	$6^2, 3, 2, 1$
	2^31^2	$12^3, 3$
41^4	41^4	$16, 1$
	321^3	$36, 6, 4$
	2^31^2	$48, 36$
3^22	2^4	12
	2^31^2	$12^2, 6, 1$
3^21^2	3^21^2	4
	32^21	$8, 2^2$
	321^3	$12^2, 6$
	2^4	$24, 4$
	2^31^2	$24^2, 12, 6, 2$
32^21	32^21	$4^4, 2^4$
	321^3	$12^2, 6^5, 3, 1$
	2^4	24^2
	2^31^2	$24^2, 12^5, 6, 3$
321^3	321^3	$36, 18^3, 9^5, 3^2$
	2^4	$72, 36$
	2^31^2	$72, 36^3, 18^4, 6$
2^4	2^4	$72, 18$
	2^31^2	$144, 36, 24$
2^31^2	2^31^2	$144^2, 36^4, 12, 9$
4^21	2^41	$8, 1$
432	2^41	$12, 6, 4$
431^2	3^21^3	6
	32^21^2	$8, 4^2, 2, 1$
	2^41	$24, 12, 8, 4$
42^21	3^21^3	$12, 3, 2$
	32^21^2	$8^2, 4^4, 2^2$

z_1	z_2	$\lvert\widetilde{\mathrm{Sh}}(z_1, z_2)\rvert$
4^22	2^5	20
4^21^2	3^221^2	4
	32^31	$12, 2^2$
	2^5	$40, 5$
43^2	2^5	30
4321	3^221^2	$4^3, 2^4, 1$
	32^31	$6^4, 3^8, 1$
	2^5	$60, 30, 20$
431^3	431^3	9
	42^21^2	$12, 6^2, 2, 1$
	3^221^2	$12^3, 6^2$
	32^31	$36, 18^2, 9^4, 3, 1$
	2^5	$180, 60$
42^3	3^221^2	$24, 6^3$
	32^31	$36, 18^2, 9, 6$
	2^5	180
42^21^2	42^21^2	$16, 8^3, 4^3, 2$
	3^221^2	$16^3, 8^4, 4^4, 2, 1$
	32^31	$24^4, 12^6, 6^5$
	2^5	$240, 120, 30$
3^31	3^31	$9, 1$
	3^22^2	$12, 6$
	3^221^2	$24, 6^3$
	32^31	$36, 18^2, 9, 6$
	2^5	180
3^22^2	3^22^2	$16, 8, 4^2, 2$
	3^221^2	$16^3, 8^2, 4^3, 2^2$
	32^31	$24^4, 12^4, 6^2$
	2^5	$120^2, 60, 10$

z_1	z_2	$\lvert\widetilde{\mathrm{Sh}}(z_1, z_2)\rvert$
4321^2	42^31	$12^3, 6^4, 3^5, 1$
	3^31^2	$24, 12^3, 6, 3$
	3^22^21	$8^{12}, 4^{10}, 2^6$
	32^4	$48^2, 24^7, 12^2, 8, 6, 4$
42^31	42^31	$36, 18^5, 9, 6^2$
	3^31^2	$72, 18^3, 12$
	3^22^21	$24^7, 12^7, 6^3, 3$
	32^4	$144, 72^4, 36^2, 24, 18, 9$
3^32	3^32	$18, 9$
	3^31^2	$36, 18, 9, 1$
	3^22^21	$24^3, 12^2, 6^4$
	32^4	$144, 72^4, 36^2, 24$
3^31^2	3^31^2	$72, 36, 18^2, 2$
	3^22^21	$48^3, 24^3, 12^4, 6^2$
	32^4	$144^3, 36^2, 24$
3^22^21	3^22^21	$16^{24}, 8^{12}, 4^6, 2$
	32^4	$96^5, 48^7, 24^4, 12, 6, 4$
32^4	32^4	$576, 288^2, 144^3, 72$
4^231	3^321	$6^2, 3, 2, 1$
	3^22^3	$12^3, 3$
4^22^2	3^321	$12^2, 6$
	3^22^3	$24^2, 12, 6, 2$
4^221^2	4^221^2	4
	43^21^2	$8, 2$
	432^21	$8^2, 4^2, 2^3, 1$
	42^4	$48, 12, 8^2$
	3^321	$24, 12^2, 6^3$
	3^22^3	$48, 24^2, 12^4, 4, 3, 2$
43^22	3^321	$12^2, 6^5, 3, 1$

The set $\widetilde{\mathrm{Sh}}(z_1, z_2)$ of compressed shapes, up to permutation of rows and columns.

The exponents indicate the number of compressed shapes which take each value.

z_1	z_2	$\lvert\widetilde{\mathrm{Sh}}(z_1, z_2)\rvert$
$4^2 1^2$	$32^3 1$	$12,\ 2^2$

Three equivalence classes in $\mathfrak{A}(R, C)/\sim$. One of them corresponds to 12 shapes of $\mathcal{PLR}_{(R,C)}$ and each of the others to 2 shapes.

$$\widetilde{\mathrm{Sh}}(4^2 1^2, 32^3 1) = \left\{ \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

The set $\mathcal{PLR}_{(R,C,S),M}$.

$$\mathcal{PLR}_{(R,C,S),M} = \mathcal{PLR}_{(R,C,S)} \cap \mathcal{PLR}_M.$$

Theorem

$R = (\mathrm{r}_1, \ldots, \mathrm{r}_r) \in \mathcal{T}_{r,w}$, $C = (\mathrm{c}_1, \ldots, \mathrm{c}_s) \in \mathcal{T}_{s,w}$, $S = (\mathrm{s}_1, \ldots, \mathrm{s}_n) \in \mathcal{T}_{n,w}$ and $M = (m_{ij}) \in \mathfrak{A}(R, C)$.

$\mathcal{PLR}_{(R,C,S),M} = V(I)$, where:

$$I = \langle \sum_{k\in[n]} x_{ijk} - 1 : \ i \in [r], j \in [s] \text{ s.t. } m_{i,j} = 1 \rangle +$$
$$+\langle (\sum_{j\in[s]} x_{ijk}) \cdot (1 - \sum_{j\in[s]} x_{ijk}) : \ i \in [r], k \in [n] \rangle +$$
$$+\langle (\sum_{i\in[r]} x_{ijk}) \cdot (1 - \sum_{j\in[s]} x_{ijk}) : \ j \in [s], k \in [n] \rangle +$$
$$+\langle x_{ijk} \cdot (1 - x_{ijk}) : i \in [r], j \in [s], k \in [n] \rangle \subseteq \mathbb{Q}[x_{111}, \ldots, x_{rsn}].$$

Moreover, $|\mathcal{PLR}_{(R,C,S),M}| = \dim_{\mathbb{Q}}(\mathbb{Q}[x_{111}, \ldots, x_{rsn}]/I)$.

The number of PLR.

s	$\lvert\mathcal{PLS}_{n,s}\rvert$, n = 1	2	3	4	5
1	1	8	27	64	125
2		16	270	1,728	7,000
3		8	1,278	25,920	233,000
4		2	3,078	239,760	5,159,000
5			3,834	1,437,696	80,602,200
6			2,412	5,728,896	920,160,000
7			756	15,326,208	7,845,192,000
8			108	27,534,816	50,648,616,000
9			12	32,971,008	249,687,408,000
10				25,941,504	944,069,668,800
11				13,153,536	2,741,210,616,000
12				4,215,744	6,104,014,872,000
13				847,872	
14				110,592	
15				9,216	
16				576	
17					
18					
19					547,608,096,000
20					107,330,054,400
21					14,590,224,000
22					1,388,160,000
23					91,008,000
24					4,032,000
25					161,280
$\lvert\mathcal{PLS}_n\rvert$	1	34	11,775	127,545,136	

The number of PLR.

	$\lvert \mathcal{IC}_{n,s} \rvert$				
	n				
s	1	2	3	4	5
1	1	4	5	5	5
2		10	50	88	93
3		4	221	1,120	2,112
4		1	525	10,172	43,955
5			651	60,092	674,957
6			415	239,510	18,326,384
7			136	639,098	22,627,758
8			20	1,148,898	422,222,624
9			5	1,374,447	2,080,853,035
10				1,082,435	7,867,483,199
11				548,440	22,843,744,418
12				176,313	50,867,237,444
13				35,473	
14				4,728	
15				403	
16				39	
17					
18					
19					4,563,456,676
20					894,429,087
21					122,238,972
22					11,569,024
23					759,296
24					33,736
25					1,411
$\lvert \mathcal{IC}_n \rvert$	1	19	2,028	5,321,261	

The number of PLR.

	$\lvert\mathfrak{I}\mathfrak{C}_{n,s}\rvert$				
	n				
s	1	2	3	4	5
1	1	1	1	1	1
2		4	4	4	4
3		1	11	11	11
4		1	18	52	52
5			23	139	221
6			15	507	1,158
7			6	1,161	6,310
8			1	2,136	33,293
9			1	2,429	150,964
10				2,004	554,285
11				975	1,594,532
12				364	3,539,431
13				72	
14				18	
15				2	
16				2	
17					
18					
19					317,980
20					62,319
21					8,676
22					823
23					69
24					6
25					2
$\lvert\mathfrak{I}\mathfrak{C}_n\rvert$	1	7	80	9,877	

The number of PLR.

	$\lvert MC_{n,s} \rvert$				
	n				
s	1	2	3	4	5
1	1	1	1	1	1
2		2	2	2	2
3		1	5	5	5
4		1	8	18	18
5			9	39	59
6			7	121	256
7			4	253	1,224
8			1	442	5,997
9			1	495	26,188
10				420	94,479
11				218	269,456
12				96	595,641
13				25	
14				8	
15				2	
16				2	
17					
18					
19					54,746
20					11,052
21					1,693
22					192
23					26
24					4
25					2
$\lvert MC_n \rvert$	1	5	38	2,147	

References.

- R. M. Falcón, The set of autotopisms of partial Latin squares, Discrete Mathematics (2011), doi: 10.1016/j.disc.2011.11.013 (in press).
- A. Hulpke, P. Kaski and P. R. J. Östergård, The number of Latin squares of order 11, Math. Comp. 80 (2011), no. 274, 1197–1219.
- B. D. McKay and I. M. Wanless, On the number of Latin squares, Ann. Comb. 9 (2005), no. 3, 335–344.
- D. S. Stones, The many formulae for the number of Latin rectangles, Electron. J. Combin. 17 (2010), no. 1, Article 1, 46 pp.

The compressed shape of a partial Latin rectangle.

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Thank you for your attention!!