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Distribution of low-dimensional Malcev algebras over finite fields into isomorphism and isotopism classes

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Abstract

This paper addresses the distribution into isomorphism and isotopism classes of the set $\mathcal{M}_{n,p}$ of n-dimensional Malcev algebras over the finite field $\mathbb{F}_p$, with p prime. This distribution is explicitly determined for dimension 2 and for the sets $\mathcal{M}_{3,p}$, with $p \leq 7$.

Key words: Malcev algebra, finite field, classification
MSC 2000: 17D10, 12E20

1 Introduction

A particular type of non-associative algebras consists of *Moufang-Lie algebras*, which were introduced by Malcev [2] as the tangent algebras of analytic Moufang loops. Later, in 1961, these algebras were called *Malcev algebras* by Sagle [3]. A remarkable problem related to them is that of their distribution into isomorphism classes.

In the paper we consider the set $\mathcal{M}_{n,p}$ of n-dimensional Malcev algebras over the finite field $\mathbb{F}_p$, with p prime. We study the distribution of such algebras, not only into isomorphism classes, which is the usual criterion, but also into isotopism classes. It is a novel contribution on this subject.

The structure of the paper is as follows. In Section 2 we expose some preliminary concepts and results on Malcev algebras. In Section 3 we show the distribution of n-dimensional Malcev algebras in $\mathcal{M}_{n,p}$. As main results, we prove that every pair of two-dimensional non-Abelian Malcev algebras are isomorphic and hence, isotopic. Further, we prove that Malcev algebras in the sets $\mathcal{M}_{3,2}$, $\mathcal{M}_{3,3}$, $\mathcal{M}_{3,5}$ and $\mathcal{M}_{3,7}$ can respectively be distributed into seven, nine, 11 and 13 isomorphism classes. In all the cases, the number of isotopism classes is four.

Note that, for reasons of length, the proofs of the results obtained are omitted in this extended abstract. They are shown in the full paper.

2 Preliminaries

In this section we expose some basic concepts and results on Malcev algebras used throughout the paper. We refer to the article of Sagle [3] for more details about this topic.

A *Malcev algebra* $\mathfrak{m}$ over a field F is a F-vector space endowed with a bilinear product such that

$$u^2 = 0, \text{ for all } u \in \mathfrak{m}, \tag{1}$$

$$uv = -vu, \text{ for all } u, v \in \mathfrak{m}, \tag{2}$$

$$uv \cdot uw = (uv \cdot w)u + (vw \cdot u)u + (wu \cdot u)v, \text{ for all } u, v, w \in \mathfrak{m}. \tag{3}$$

The last condition is equivalent to the *Malcev identity*

$$M(u, v, w) = J(u, v, w)u - J(u, v, uw) = 0, \text{ for all } u, v, w \in \mathfrak{m}, \tag{4}$$

where J is the *Jacobian* defined as $J(u, v, w) = (uv)w + (vw)u + (wu)v$, for all $u, v, w \in \mathfrak{m}$. If $J(u, v, w) = 0$ for all $u, v, w \in \mathfrak{m}$, then this is a Lie algebra.

Given a basis $\{e_1, ..., e_n\}$ of the Malcev algebra $\mathfrak{m}$, this is characterized by its *structure constants* with respect to that basis, that is, by the coefficients $c_{ij}^k \in F$ of the products

$$e_i e_j = \sum_{k=1}^{n} c_{ij}^k e_k \qquad \text{for all} \qquad 1 \leq i < j \leq n. \tag{5}$$

The *centralizer* of a subset $\mathfrak{s}$ of the Malcev algebra $\mathfrak{m}$ is the set $\mathrm{Cen}_{\mathfrak{m}}(\mathfrak{s}) = \{u \in \mathfrak{m} \mid uv = 0, \text{ for all } v \in \mathfrak{s}\}$. An *ideal* of $\mathfrak{m}$ is any vector subspace $\mathfrak{s} \subseteq \mathfrak{m}$ such that $\mathfrak{s}\mathfrak{m} = \{uv \colon u \in \mathfrak{s}, v \in \mathfrak{m}\} \subseteq \mathfrak{s}$. It is called *Abelian* if $\mathfrak{s}\mathfrak{m} = \{0\}$. The *center* of $\mathfrak{m}$ is the ideal $Z(\mathfrak{m}) = \mathrm{Cen}_{\mathfrak{m}}(\mathfrak{m})$. In particular, if $\dim Z(\mathfrak{m}) = n$, then the Malcev algebra $\mathfrak{m}$ is Abelian. The next result follows straightforward from the definition of Malcev algebra.

Lemma 2.1 *Let $\mathfrak{m}$ be an n-dimensional algebra endowed with a bilinear product that satisfies both conditions (1) and (2). If $\dim Z(\mathfrak{m}) \geq n - 2$, then $\mathfrak{m}$ is a Malcev algebra.* □

Two n-dimensional Malcev algebras $\mathfrak{m}$ and $\mathfrak{m}'$ are said to be *isotopic* if there exist three regular linear transformations f, g and h between them such that

$$f(u)g(v) = h(uv) \qquad \text{for all} \qquad u, v \in \mathfrak{m}. \tag{6}$$

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The triple (f, g, h) is said to be an *isotopism* between $\mathfrak{m}$ and $\mathfrak{m}'$. It is denoted as $\mathfrak{m} \simeq \mathfrak{m}'$. If $f = g = h$, then f is an *isomorphism* between $\mathfrak{m}$ and $\mathfrak{m}'$ and these Malcev algebras are said to be *isomorphic*, which is denoted as $\mathfrak{m} \cong \mathfrak{m}'$. Given a basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{m}$ and a basis $\{e'_1, \ldots, e'_n\}$ of $\mathfrak{m}'$, each regular linear application f between these Malcev algebras is uniquely related to a square matrix $M_f = (f_{ij})$ such that $f(e_i) = \sum_{j=1}^{n} f_{ij} e'_j$, for all $i \in \{1, \ldots, n\}$. The next result holds.

Lemma 2.2 *It is verified that*

1. *Isotopisms of Malcev algebras preserve the dimension of their centers.*
2. *The n-dimensional Abelian Malcev algebra is not isotopic to any other Malcev algebra.*
3. *Given a positive integer $n > 2$, every n-dimensional Malcev algebra of basis $\{e_1, \ldots, e_n\}$ whose center has dimension $n - 2$ is isomorphic to one of the following two non-isomorphic Malcev algebras:*
   - *(a) That one determined by the product $e_1 e_2 = e_1$.*
   - *(b) That one determined by the product $e_1 e_2 = e_3$.* □

In the next section we see that the set $\mathcal{M}_{n,p}$ of n-dimensional Malcev algebras over the finite field $\mathbb{F}_p$ and their isomorphism classes can be identified with a pair of algebraic sets defined by zero-dimensional radical ideals. From now on, we denote by $\mathbb{F}_p[\underline{x}]$ the ring of polynomials in the set of variables $\underline{x} = \{x_1, \ldots, x_n\}$ over the finite field $\mathbb{F}_p$. Given an ideal I in $\mathbb{F}_p[\underline{x}]$, the *algebraic set* defined by I is the set $\mathcal{V}(I) = \{\underline{a} \in \mathbb{F}_p^n \colon f(\underline{a}) = 0 \text{ for all } f \in I\}$. If the ideal I is zero-dimensional, then the number of points of the algebraic set $\mathcal{V}(I)$ is less than the Krull dimension of the quotient ring $\mathbb{F}_p[\underline{x}]/I$. The equality holds when the ideal is radical, in which case the dimension coincides with the number of standard monomials of the ideal I.

3 Algebraic sets related to $\mathcal{M}_{n,p}$

In the first place, let us consider the case $n = 2$. Every two-dimensional algebra over a field F, not necessarily finite, with basis $\{e_1, e_2\}$ and endowed with a bilinear product satisfying conditions (1) and (2) is uniquely determined by a pair of structure constants $a, b \in F$ such that $e_1 e_2 = a e_1 + b e_2$. From Lemma 2.1 this is a Malcev algebra, which we denote as $\mathfrak{m}_{ab}$. The next results hold.

Lemma 3.1 *Let $\mathfrak{m}_{ab}$ and $\mathfrak{m}_{a'b'}$ be two isomorphic two-dimensional Malcev algebras over a field F. If $M_f = (f_{ij})$ is the square matrix related to an isomorphism f between both algebras, then it is verified that*

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1. $af_{11} + bf_{12} = a' \det(M_f)$.

2. $af_{21} + bf_{22} = b' \det(M_f)$.

Proposition 3.2 *Every pair of two-dimensional non-Abelian Malcev algebras over a field F are isomorphic.*

We focus now on the case $n > 2$. Firstly, we analyze the problem of determining when an n-dimensional algebra over the finite field $\mathbb{F}_p$, with p prime, is a Malcev algebra. To this end, we consider the set of $(n^3 - n^2)/2$ variables $\underline{c} = \{c_{ij}^k \colon i, j, k \in \{1, \ldots, n\}, i < j\}$ and the n-dimensional algebra $\mathfrak{m}$ of basis $\{e_1, \ldots, e_n\}$ over the polynomial ring $\mathbb{F}_p[\underline{c}]$ endowed with a bilinear product satisfying conditions (1) and (2) and whose structure constants coincide with the variables c_{ij}^k. Given $u, v, w \in \mathfrak{m}$ and $l \in \{1, \ldots, n\}$, let p_{uvwl} be the polynomial in $F[\underline{c}]$ that constitutes the coefficient of e_l in the Malcev identity $M(u, v, w) = 0$. The next result holds.

Theorem 3.3 *The set $\mathcal{M}_{n,p}$ can be identified with the algebraic set defined by the zero-dimensional radical ideal $I_{n,p} = \langle\, \mathrm{p}_{uvwl} \colon u, v, w \in \mathfrak{m}, l \in \{1, \ldots, n\}\,\rangle + \langle\, (c_{ij}^k)^p - c_{ij}^k \colon 1 \leq i < j \leq n, 1 \leq k \leq n\}\,\rangle \subset \mathbb{F}_p[\underline{c}]$. Besides, $|\mathcal{M}_{n,p}| = \dim_{\mathbb{F}_p}(\mathbb{F}_p[\underline{c}])/I_{n,p})$.* □

p	$\lvert\mathcal{M}_{3,p}\rvert$	$\lvert\mathcal{M}_{3,p}/\cong\rvert$	$\lvert\mathcal{M}_{3,p}/\simeq\rvert$
2	120	7	4
3	1,431	9	4
5	31,125	11	4
7	234,955	13	4

Table 1: Distribution of Malcev algebras of $\mathcal{M}_{3,p}$, for all prime $p \leq 7$.

We have implemented Theorem 3.3 in the open computer algebra system for polynomial computations SINGULAR [1] in order to determine in Table 1 the number of Malcev algebras of $\mathcal{M}_{3,p}$ with respect to a given basis $\{e_1, e_2, e_3\}$, for all prime $p \leq 7$. The distribution of these algebras into isomorphism classes, which is explicitly exposed in Theorem 3.5, has also been determined by using this polynomial method. In this respect, given two Malcev algebras $\mathfrak{m}$ and $\mathfrak{m}'$ in $\mathcal{M}_{n,p}$ in $\mathbb{F}_p$, we need to know when a regular linear transformation f between these algebras is an isomorphism. To this end, we consider the set of n^2 variables $\underline{f} = \{f_{11}, \ldots, f_{nn}\}$, which we consider to be the entries of the corresponding matrix M_f of

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our isomorphism f. Let p_{ijl} be the polynomial in the polynomial ring $\mathbb{F}_p[\underline{f}]$ that constitutes the coefficient of e_l in the identity $f(e_ie_j) = f(e_i)f(e_j)$ of Condition (6), for all $i, j, l \in \{1, \ldots, n\}$. The next result holds.

Theorem 3.4 *The set of isomorphisms between two Malcev algebras* $\mathfrak{m}, \mathfrak{m}' \in \mathcal{M}_{n,p}$ *can be identified with the algebraic set defined by the zero-dimensional radical ideal* $I_{\mathfrak{m},\mathfrak{m}'} = \langle\, \mathrm{p}_{ijl} \colon i, j, l \in \{1, \ldots, n\} \,\rangle + \langle\, f_{ij}^p - f_{ij} \colon i, j \in \{1, \ldots, n\} \,\rangle + \langle\, \det(M_f)^{p-1} - 1 \,\rangle \subset \mathbb{F}_p[\underline{f}]$. *The cardinality of this set is* $\dim_{\mathbb{F}_p}(\mathbb{F}_p[\underline{f}])/I_{\mathfrak{m},\mathfrak{m}'})$.

□

Let us denote by $\mathfrak{m}_{u,v,w}$ the Malcev algebra of basis $\{e_1, e_2, e_3\}$ in $\mathcal{M}_{3,p}$ such that $e_1e_2 = u$, $e_1e_3 = v$ and $e_2e_3 = w$. We expose in the next two results the explicit distribution of $\mathcal{M}_{3,2}$, $\mathcal{M}_{3,3}$ and $\mathcal{M}_{3,5}$ into isomorphism and isotopism classes.

Theorem 3.5 *There exist*

1. *seven isomorphism classes in* $\mathcal{M}_{3,2}$*:*

$$\mathfrak{m}_{0,0,0}, \mathfrak{m}_{0,0,e_1}, \mathfrak{m}_{0,0,e_3}, \mathfrak{m}_{0,e_2,e_1}, \mathfrak{m}_{0,e_2,e_1+e_2}, \mathfrak{m}_{0,e_1,e_2}, \mathfrak{m}_{e_3,e_2,e_1}.$$

2. *nine isomorphism classes in* $\mathcal{M}_{3,3}$*:*

$$\mathfrak{m}_{0,0,0}, \mathfrak{m}_{0,0,e_1}, \mathfrak{m}_{0,0,e_3}, \mathfrak{m}_{0,e_2,e_1}, \mathfrak{m}_{0,e_2,e_1+e_2}, \mathfrak{m}_{0,e_2,2e_1}, \mathfrak{m}_{0,e_2,2e_1+e_2}, \mathfrak{m}_{0,e_1,e_2}, \mathfrak{m}_{e_3,e_2,e_1}.$$

3. 11 *isomorphism classes in* $\mathcal{M}_{3,5}$*:*

$$\mathfrak{m}_{0,0,0}, \mathfrak{m}_{0,0,e_1}, \mathfrak{m}_{0,0,e_3}, \mathfrak{m}_{0,e_2,e_1}, \mathfrak{m}_{0,e_2,e_1+e_2}, \mathfrak{m}_{0,e_2,e_1+2e_2}, \mathfrak{m}_{0,e_2,2e_1},$$

$$\mathfrak{m}_{0,e_2,2e_1+e_2}, \mathfrak{m}_{0,e_2,2e_1+2e_2}, \mathfrak{m}_{0,e_1,e_2}, \mathfrak{m}_{e_3,e_2,e_1}.$$

4. 13 *isomorphism classes in* $\mathcal{M}_{3,7}$*:*

$$\mathfrak{m}_{0,0,0}, \mathfrak{m}_{0,0,e_1}, \mathfrak{m}_{0,0,e_3}, \mathfrak{m}_{0,e_2,e_1}, \mathfrak{m}_{0,e_2,e_1+e_2}, \mathfrak{m}_{0,e_2,e_1+2e_2}, \mathfrak{m}_{0,e_2,e_1+3e_2},$$

$$\mathfrak{m}_{0,e_2,3e_1}, \mathfrak{m}_{0,e_2,3e_1+e_2}, \mathfrak{m}_{0,e_2,3e_1+2e_2}, \mathfrak{m}_{0,e_2,3e_1+3e_2}, \mathfrak{m}_{0,e_1,e_2}, \mathfrak{m}_{e_3,e_2,e_1}.$$

□

Theorem 3.6 *There exist four isotopism classes in* $\mathcal{M}_{3,p}$*, for all prime* $p \leq 7$*:*

$$\mathfrak{m}_{0,0,0}, \mathfrak{m}_{e_1,0,0}, \mathfrak{m}_{e_3,e_2,0}, \mathfrak{m}_{e_3,e_2,e_1}.$$

□

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4 Further studies

Dealing with the set $\mathcal{M}_{n,p}$ of n-dimensional Malcev algebras over the finite field $\mathbb{F}_p$, we have enumerated and classified into isomorphism and isotopism classes the elements of the set $\mathcal{M}_{2,p}$, for any prime p, and those of the sets $\mathcal{M}_{3,p}$, with $p \leq 7$. Further studies in the subject are being analyzed to deal with the case $p > 3$ and also to characterize the distinct properties of the algebraic sets related to the *skeleton* of a Malcev algebra, which is a new concept that we wish to introduce as a tool and that we are now analyzing.

We are also trying to associate a combinatorial structure with every Malcev algebra, in the same way as in [4]. It would allow us to use properties of these structures to make easier the study of the algebras.

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