Classification of 3-seminets with at most 5 points.

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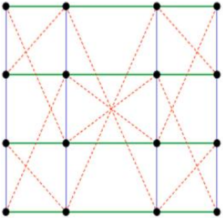
Nets.

$\mathfrak{P} \equiv$ Finite set of points.
$\mathfrak{L} \equiv$ Subsets (*lines*) of $\mathfrak{P}$ s.t. $\exists$ partition of $\mathfrak{L}$ into $k \geq 3$ *parallel* classes:

$$L_1, \ldots, L_k.$$



$(\mathfrak{P}, L_1, \ldots, L_k)$ is a ***k*-net** [Bruck, 1963] if:

- Any two lines of different classes intersect in **exactly one** point.
- Every point belongs to **exactly one** line of each class.

Nets.

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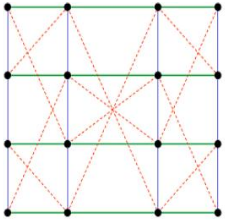

$\equiv$

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

- All parallel classes have the same number of lines.
- Every line contains the same number of points (**order**).
- Two k-nets are *isomorphic* if there exists a bijection between their sets of points which preserves collinearity in each parallel class.
- Any 3-net of order n is uniquely identified with a *Latin square* of the same order.

Classification of Latin Squares.

- Orthogonal representation: $O(P) = \{(\text{row}, \ \text{column}, \ \text{symbol})\}$.
- Classification:
  1. Isotopism: Permutations of rows, columns and symbols.
  2. Parastrophism:
     $\pi \in S_3 \to O(P^\pi) = \{(l_{\pi(1)}, l_{\pi(2)}, l_{\pi(3)}) \mid (l_1, l_2, l_3) \in O(P)\}$.
  3. Paratopism (main classes): Composition of isotopism and parastrophism.
- Isotopic LS $\equiv$ Isomorphic 3-nets.
- Paratopic LS $\equiv$ Isomorphic 3-nets after relabeling parallel classes.

n	$\lvert LS_n \rvert$	IC	MC
1	1	1	1
2	2	1	1
3	12	1	1
4	576	2	2
5	161280	2	2
6	812851200	22	12
7	61479419904000	564	147
8	108776032459082956800	1676267	283657
9	5524751496156892842531225600	115618721533	19270853541
10	9982437658213039871725064756920320000	208904371354363006	34817397894749939
11	776966836171770144107444346734230682311065600000	12216177315369229261482540	2036029552582883134196099

[McKay and Wanless, 2005; Hulpke et al., 2011]

Seminets.

$\mathfrak{P} \equiv$ Finite set of points.
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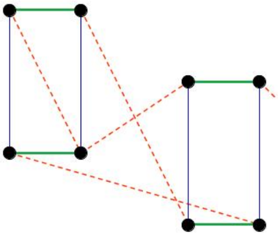


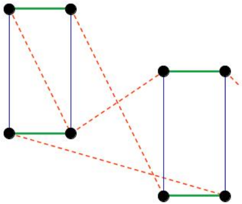
$(\mathfrak{P}, L_1, \ldots, L_k)$ is a ***k*-seminet** [Ušan, 1977] if:

- Any two lines of different classes intersect in **at most one** point.
- Every point belongs to **exactly one** line of each class.

SEMINETS.

$\mathfrak{P} \equiv$ Finite set of points.
$\mathfrak{L} \equiv$ Subsets (*lines*) of $\mathfrak{P}$ s.t. $\exists$ partition of $\mathfrak{L}$ into $k \geq 3$ *parallel* classes:

$$L_1, \ldots, L_k.$$



$$\equiv \begin{array}{cccc} 1 & 2 & \cdot & \cdot \\ \cdot & \cdot & 1 & 4 \\ 3 & 1 & \cdot & \cdot \\ \cdot & \cdot & 2 & 3 \end{array}$$

- Parallel classes can have different number of lines: r, s and n.
- Lines can contain different number of points.
- It can contain **skew** lines: Non-parallel lines without common points.
- Any 3-seminet with parallel classes of r, s and n lines can be uniquely identified with an $r \times s$ **partial Latin rectangle** based on n symbols.

Partial Latin rectangles.

- An $r \times s$ partial Latin rectangle based on a set of n symbols is an $r \times s$ array in which each cell is either empty or contains one element chosen from a set of symbols, $[n] = \{1, 2, ..., n\}$, s.t. each symbol occurs at most once in each row and in each column.

- The number of filled cells is its size. Their positions determine the shape:

$$P = \begin{pmatrix} 1 & \cdot & 3 & 4 & 6 \\ 2 & \cdot & 5 & \cdot & 4 \\ \cdot & \cdot & 4 & 5 & 1 \\ \cdot & \cdot & 2 & \cdot & 3 \end{pmatrix} \in \mathcal{PLR}^{6}_{4\times 5,12} \to \mathrm{Sh}(P) = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

- $r = s = n \to$ Partial Latin square.
- Size $r \cdot s \to$ Latin rectangle (square if $r = s$).
- **Applications**: Algebra (quasigroups), Experimental Designs, Cryptography.

Partial Latin rectangles.

$|PLR^n_{r\times s}|$ upper bounded for $r = s = n$ [Ghandehari, 2005].
$|IC|$ and $|MC|$ lower bounded for $r = s = n \leq 6$ [Adams, 2003].

Order n	Size m	$\lvert PLS_{n,m}\rvert$
1	1	1
2	1	8
	2	16
	3	8
	4	2
3	1	27
	2	270
	3	1,278
	4	3,078
	5	3,834
	6	2,412
	7	756
	8	108
	9	12
4	1	64
	2	1,728
	3	25,920
	4	239,760
	5	1,437,696
	6	5,728,896
	7	15,326,208
	8	27,534,816
	9	32,971,008
	10	25,941,504
	11	13,153,536
	12	4,215,744
	13	847,872
	14	110,592
	15	9,216
	16	576

[Falcón, 2012]

Some exact values have recently been obtained by applying Gröbner bases in an equivalent planar assignment problem:

$$\begin{cases} \sum_{k\in[n]} x_{ijk} \leq 1, \forall i,j \in [n], \\ \sum_{j\in[n]} x_{ijk} \leq 1, \forall i,k \in [n], \\ \sum_{i\in[n]} x_{ijk} \leq 1, \forall j,k \in [n], \\ \sum_{i,j,k\in[n]} x_{ijk} = m, \\ x_{ijk} \in \{0,1\}, \forall i,j,k \in [n], \end{cases}$$

Partial Latin rectangles.

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Type and structure of a PLR.

$$\sum_{i,j,k\in[n]} x_{ijk} = m \to \begin{cases} \sum_{j,k\in[n]} x_{ijk} = T_1(P,i), \leftarrow \text{ Rows.} \\ \sum_{i,k\in[n]} x_{ijk} = T_2(P,j), \leftarrow \text{ Columns.} \\ \sum_{i,j\in[n]} x_{ijk} = T_3(P,k), \leftarrow \text{ Symbols.} \end{cases}$$

$$P = \begin{pmatrix} 1 & \cdot & 3 & 4 & 6 \\ 2 & \cdot & 5 & \cdot & 4 \\ \cdot & \cdot & 4 & 5 & 1 \\ \cdot & \cdot & 2 & \cdot & 3 \end{pmatrix} \in \mathcal{PLR}^6_{4\times 5,12} \to \mathrm{Sh}(P) = \left(\begin{array}{c|ccccc} & 2 & 0 & 4 & 2 & 4 \\ \hline 4 & 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

Row type: $T_1(P) = (4,3,3,2)$

Column type: $T_2(P) = (2,0,4,2,4)$.

Symbol type: $T_3(P) = (2,2,2,3,2,1)$.

[Keedwell, 1994; Bean, 2002]

Type and structure of a PLR.

- $\mathcal{T}_{l,w} = \{(t_1, \dots, t_l)$ of *weight* $\sum_{i \in [l]} t_i = w$, s.t. $t_i \in \mathbb{N}\}$.
- Structure of $T \in \mathcal{T}_{l,w}$: $\mathrm{st}(T) = w^{\lambda_w^T} \dots 1^{\lambda_1^T}$, where λ_i^T is the number of occurrences of i in T.
- $\mathcal{Z}_{l,w}$: Set of possible structures of length l and weight w.

$$P = \begin{pmatrix} 1 & \cdot & 3 & 4 & 6 \\ 2 & \cdot & 5 & \cdot & 4 \\ \cdot & \cdot & 4 & 5 & 1 \\ \cdot & \cdot & 2 & \cdot & 3 \end{pmatrix} \in \mathcal{PLR}^6_{4\times 5,12} \to \mathrm{Sh}(P) = \left(\begin{array}{c|ccccc} & 2 & 0 & 4 & 2 & 4 \\ \hline 4 & 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

Row type: $T_1(P) = (4,3,3,2) \to \mathrm{st}(T_1(P)) = 43^2 2$.

Column type: $T_2(P) = (2,0,4,2,4) \to \mathrm{st}(T_2(P)) = 4^2 2^2$.

Symbol type: $T_3(P) = (2,2,2,3,2,1) \to \mathrm{st}(T_3(P)) = 32^4 1$.

Type and structure of a PLR.

- Given $R \in \mathcal{T}_{r,w}$, $C \in \mathcal{T}_{s,w}$ and $S \in \mathcal{T}_{n,w}$:

$$\mathcal{PLR}^n_{(R,C)} = \{P \in \mathcal{PLR}^n_{r\times s} \colon\ T_1(P) = R \text{ and } T_2(P) = C\}.$$

$$\mathcal{PLR}_{(R,C,S)} = \{P \in \mathcal{PLR}^n_{r\times s} \colon\ T_1(P) = R,\, T_2(P) = C \text{ and } T_3(P) = S\}.$$

Lemma

$|\mathcal{PLR}^n_{(R,C)}|$ and $|\mathcal{PLR}_{(R,C,S)}|$ only depend on the structures of R, C, S.

Type and structure of a PLR.

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Lemma

$|\mathcal{PLR}^n_{(R,C)}|$ and $|\mathcal{PLR}_{(R,C,S)}|$ only depend on the structures of R, C, S.

- Given $z_1 \in \mathcal{Z}_{l_1,w}$, $z_2 \in \mathcal{Z}_{l_2,w}$ and $z_3 \in \mathcal{Z}_{l_3,w}$:

$$\Delta^n_{r\times s}(z_1, z_2) = |\mathcal{PLR}^n_{(R,C)}|, \forall R \in \mathcal{T}_{r,w}, C \in \mathcal{T}_{s,w}, \text{ s.t. } \mathrm{st}(R) = z_1, \mathrm{st}(C) = z_2.$$
$$\Delta^n_{r\times s}(z_1, z_2, z_3) = |\mathcal{PLR}^n_{(R,C,S)}|, \forall R \in \mathcal{T}_{r,w}, C \in \mathcal{T}_{s,w}, S \in \mathcal{T}_{n,w}, \text{ s.t. } \mathrm{st}(R) = z_1, \mathrm{st}(C) = z_2, \mathrm{st}(S) = z_3.$$

Type and structure of a PLR.

Proposition

$$|\mathcal{PLR}^n_{r\times s}| = \sum_{(l,l')\in[r]\times[s]} \sum_{w\in[l\cdot s]} \sum_{(z,z')\in \mathcal{Z}_{l,w}\times\mathcal{Z}_{l',w}} \frac{r!}{(r-l)!\cdot\prod_{i\in[w]}\lambda_i!}\cdot\Delta^n_{r\times s}(z,z')$$

Where:

$$\Delta^n_{l\times s}(z,z') = \sum_{l''\in[n]} \sum_{z''\in\mathcal{Z}_{l'',w}} \Delta^n_{r\times s}(z,z',z'').$$

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Problem

How to obtain $\Delta^n_{r\times s}(z,z')$ and $\Delta^n_{r\times s}(z,z',z'')$?

Type and structure of a PLR.

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$$|\mathcal{PLR}^n_{(R,C)}| \to \mathfrak{A}(R,C)$$

- $\mathfrak{A}(R,C)$: $(0,1)$-matrices having R and C as row and column sum vectors.
- $\mathcal{PLR}^n_M$: Set of PLR of n symbols having $M \in \mathfrak{A}(R,C)$ as shape.

$$|\mathcal{PLR}^n_{(R,C)}| = \sum_{M\in\mathfrak{A}_{(R,C)}} |\mathcal{PLR}^n_M|.$$

Equivalent problems:

- n-edge-colouring a bipartite graph of incidence matrix $\mathrm{Sh}(P)$ (Existence problem is NP-complete even for $n=3$ [Holyer, 1981]).
- 1-*color tomography problem* [Kuba, 1999]: Reconstructing a binary matrix starting from its row and column sums.

$\Delta^n_{r\times s}(z,z')$.

Gale-Ryser theorem [Gale, Ryser, 1957]: $\mathfrak{A}(R,C) \neq \emptyset \Leftrightarrow C \preceq R^*$.
$R = (3,5,2,2) \rightarrow R^* = (4,4,2,1,1) \succeq (3,3,3,2,1)$. *(Dominance order)*.

Formulas and algorithms:

- Monomial symmetric functions [Sukhatme, 1938; David, 1951 ($\leq$ 12)].
- Character of the symmetric group [Snapper, 1971].
- Lower bound [Wei, 1982].
- Recurrence formulas [Wang, 1988; Wang and Zhang, 1998; Pérez Salvador, 2002].
- General formulas [Dias, 2002].
- Asymptotic methods [Barvinok, 2010].
- Combinatorial methods [Brualdi, 1980; Brualdi, 2006; Fonseca, 2009].
- Simulation methods (social networks, ecology) [Snijders, 1991; Rao, 1996; Chen, 2005; Bezakova, 2007; Blanchet, 2009].

$\Delta^n_{r\times s}(z,z')$.

Algebraic approach: Gröbner bases of boolean ideals for counting problems [Bayer, 1982; Alon, 1995; Bernasconi, 1997].

THEOREM

$R = (\mathrm{r}_1,\ldots,\mathrm{r}_r) \in \mathcal{T}_{r,w}$ and $C = (\mathrm{c}_1,\ldots,\mathrm{c}_s) \in \mathcal{T}_{s,w}$ s.t. $C \preceq R^*$.

$\mathfrak{A}(R,C) = V(I)$, where:

$$I = \langle\, (\sum_{j\in[s]} x_{ij} - \mathrm{r}_i) :\ i \in [r]\,\rangle + \langle\, (\sum_{i\in[r]} x_{ij} - \mathrm{c}_j) :\ j \in [s]\,\rangle$$

$$+\langle\, x_{ij}\cdot(1-x_{ij}) :\ i\in[r], j\in[s]\,\rangle \subseteq \mathbb{Q}[x_{11},\ldots,x_{rs}].$$

Moreover, $|\mathfrak{A}(R,C)| = \dim_{\mathbb{Q}}(\mathbb{Q}[x_{11},\ldots,x_{rs}]/I)$.

Type and structure of a PLR.

Proposition

$$|\mathcal{PLR}^n_{r\times s}| = \sum_{(l,l')\in[r]\times[s]} \sum_{w\in[l\cdot s]} \sum_{(z,z')\in\mathcal{Z}_{l,w}\times\mathcal{Z}_{l',w}} \frac{r!}{(r-l)!\cdot\prod_{i\in[w]}\lambda_i!} \cdot \Delta^n_{r\times s}(z,z')$$

Where:

$$\Delta^n_{l\times s}(z,z') = \sum_{l''\in[n]} \sum_{z''\in\mathcal{Z}_{l'',w}} \Delta^n_{r\times s}(z,z',z'').$$

Problem

How to obtain $\Delta^n_{r\times s}(z,z')$ and $\Delta^n_{r\times s}(z,z',z'')$?

$\Delta^n_{r\times s}(z,z',z'')$.

Theorem

$R=(\mathrm{r}_1,\dots,\mathrm{r}_r)\in\mathcal{T}_{r,w}$, $C=(\mathrm{c}_1,\dots,\mathrm{c}_s)\in\mathcal{T}_{s,w}$ and $S=(\mathrm{s}_1,\dots,\mathrm{s}_n)\in\mathcal{T}_{n,w}$.

$\mathcal{PLR}_{(R,C,S)}=V(I)$, where:

$$I=\langle\,(\sum_{i\in[r]}x_{ijk})\cdot(1-\sum_{i\in[r]}x_{ijk}):\ j\in[s],k\in[n]\,\rangle+\langle\,\mathrm{r}_i-\sum_{j\in[s],k\in[n]}x_{ijk}:\ i\in[r]\,\rangle+$$

$$\langle\,(\sum_{j\in[s]}x_{ijk})\cdot(1-\sum_{j\in[s]}x_{ijk}):\ i\in[r],k\in[n]\,\rangle+\langle\,\mathrm{c}_j-\sum_{i\in[r],k\in[n]}x_{ijk}:\ j\in[s]\,\rangle+$$

$$\langle\,(\sum_{k\in[n]}x_{ijk})\cdot(1-\sum_{k\in[n]}x_{ijk}):\ i\in[r],j\in[s]\,\rangle+\langle\,\mathrm{s}_k-\sum_{i\in[r],j\in[s]}x_{ijk}:\ k\in[n]\,\rangle+$$

$$\langle\,x_{ijk}\cdot(1-x_{ijk}):\ i\in[r],j\in[s],k\in[n]\,\rangle\subseteq\mathbb{Q}[x_{111},\dots,x_{rsn}].$$

Moreover, $|\mathcal{PLR}_{(R,C,S)}|=\dim_{\mathbb{Q}}(\mathbb{Q}[x_{111},\dots,x_{rsn}]/I)$.

$\Delta^n_{r\times s}(z, z', z'')$.

Theorem

$R = (\mathrm{r}_1, \ldots, \mathrm{r}_r) \in \mathcal{T}_{r,w}$, $C = (\mathrm{c}_1, \ldots, \mathrm{c}_s) \in \mathcal{T}_{s,w}$ and $S = (\mathrm{s}_1, \ldots, \mathrm{s}_n) \in \mathcal{T}_{n,w}$.
Two PLR $P = (p_{rc}), Q = (q_{rc}) \in \mathcal{PLR}_{(R,C,S)}$ are isotopic if the following system has solution:

$$\begin{cases} \sum_{j\in[r]} x_{ij} = 1, \forall i \in [r], \\ \sum_{j\in[s]} y_{ij} = 1, \forall i \in [s], \\ \sum_{j\in[n]} z_{ij} = 1, \forall i \in [n], \\ \sum_{i\in[r]} x_{ij} = 1, \forall j \in [r], \\ \sum_{i\in[s]} y_{ij} = 1, \forall j \in [s], \\ \sum_{i\in[n]} z_{ij} = 1, \forall j \in [n], \\ x_{ik} \cdot y_{jl} \cdot (z_{p_{ij} q_{kl}} - 1) = 0, \forall i, j \in [r] \text{ and } k, l \in [s] \text{ s.t. } p_{ij}, q_{kl} \in [n], \\ x_{ik} \cdot y_{jl} = 0, \forall i, j \in [r] \text{ and } k, l \in [s] \text{ s.t. } q_{kl} = \emptyset, \end{cases}$$

$\Delta^n_{r\times s}(z, z', z'')$.

z_1	z_2	z_3	$\Delta(Z)$	$\Delta_I(Z)$	$\Delta_P(Z)$
1	1	1	1	1	1
2	1^2	1^2	2	1	1
1^2	1^2	1^2	4	1	1
3	1^3	1^3	6	1	1
21	21	21	1	1	1
		1^3	6	1	1
	1^3	1^3	18	1	1
1^3	1^3	1^3	36	1	1
4	1^4	1^4	24	1	1
31	21^2	21^2	4	1	1
		1^4	24	1	1
	1^4	1^4	96	1	1
2^2	2^2	2^2	2	1	1
		21^2	4	1	1
		1^4	24	1	1
	21^2	21^2	12	2	2
		1^4	48	1	1
	1^4	1^4	144	1	1
21^2	21^2	21^2	40	5	3
		1^4	120	2	2
	1^4	1^4	288	1	1
1^4	1^4	1^4	576	1	1
5	1^5	1^5	120	1	1
41	21^3	21^3	18	1	1
		1^5	120	1	1
	1^5	1^5	600	1	1

z_1	z_2	z_3	$\Delta(Z)$	$\Delta_I(Z)$	$\Delta_P(Z)$
32	2^21	2^21	6	2	2
		21^3	24	2	2
		1^5	120	1	1
	21^3	21^3	90	3	3
		1^5	360	1	1
	1^5	1^5	1,200	1	1
31^2	31^2	2^21	4	1	1
		21^3	24	1	1
		1^5	120	1	1
	2^21	2^21	12	2	2
		21^3	60	3	3
		1^5	240	1	1
	21^3	21^3	252	5	4
		1^5	840	2	2
	1^5	1^5	2,400	1	1
2^21	2^21	2^21	58	8	4
		21^3	180	8	6
		1^5	600	2	2
	21^3	21^3	504	8	6
		1^5	1440	2	2
	1^5	1^5	3,600	1	1
21^3	21^3	21^3	1,296	8	4
		1^5	3,240	2	2
	1^5	1^5	7,200	1	1
1^5	1^5	1^5	14,400	1	1

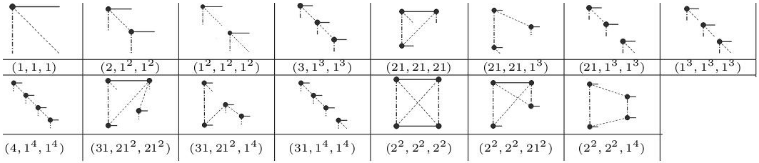
(In both tables, $Z = (z_1, z_2, z_3)$.)

3-SEMINETS.

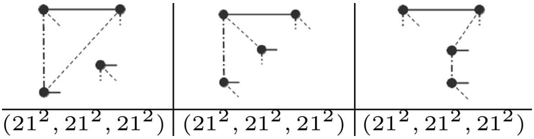
THEOREM

The number of isomorphism classes of 3-seminets with one, two, three, four and five points are 1, 4, 11, 52 and 220, respectively. That of paratopism classes are 1, 2, 5, 18 and 59, respectively. □

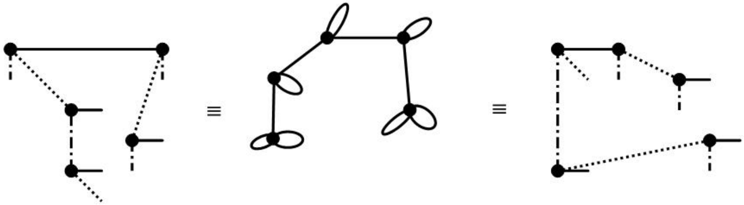
34 of the 85 3-seminets are uniquely determined by the PLR's structures.



The rest are not uniquely determined:



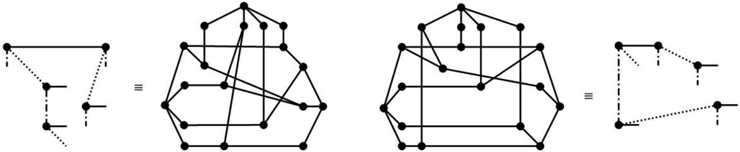
Seminets-graphs.



Seminet-graph (G_1):

- Vertices $\equiv$ points.
- Connected vertices $\equiv$ collinear points.
- Lines containing only one point are identified with loops.
- It can be related to more than one paratopism class.

Seminets-graphs.



Seminet-graph (G_2):

- Vertices $\equiv$ Points, lines and parallel classes.

$$\{u_1, \dots, u_n\} \cup \{v_{1,1}, \dots, v_{1,l_1}, v_{2,1}, \dots, v_{3,l_3}\} \cup \{w_1, w_2, w_3\}.$$

- Each vertex w_i is connected to all the vertices $v_{i,j}$, for all $i \in [3]$ and $j \in [l_i]$.
- each vertex $v_{i,j}$ is connected to those vertices u_k such that the line related to the former contains the point associated to the latter.
- Uniquely related to a paratopism class.
- If G_2 is not acyclic, then its girth is 6.

Seminets-graphs.

We have considered:

- $\mathfrak{l} \equiv$ Number of vertices of $G_1(S)$ contained in at least one loop.
- $\mathfrak{a} \equiv$ Number of articulation points of $G_1(S)$.
- $\mathfrak{t} \equiv$ Number of transversal of $G_1(S)$.
- $\mathfrak{c} \equiv$ Clustering coefficient of $G_1(S)$.
- $\mathfrak{st}_1 \equiv$ Number of spanning trees in $G_1(S)$.
- $\mathfrak{st}_2 \equiv$ Number of spanning trees in $G_2(S)$.

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Classification of 3-seminets with at most 5 points.

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Thank you for your attention!!