Concurrence designs based on partial Latin rectangles autotopisms.

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Introduction.

- Incidence structures.
- Partial Latin rectangles.

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- An incidence structure is a triple $\mathcal{D} = (\mathcal{V}, \mathcal{B}, I)$, where $\mathcal{V}$ is a set of v *points*, $\mathcal{B}$ is a set of b *blocks* and $I \subseteq \mathcal{V} \times \mathcal{B}$ is an incidence relation.

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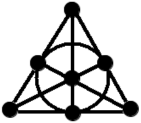
- $\mathcal{D}$ is **k-uniform** if every block contains exactly k points and it is **r-regular** if every point is exactly on r blocks.
- A 1-(v, k, r) *design* is an incidence structure of v points which is k-uniform and r-regular $\rightarrow$ $b \cdot k = v \cdot r$.



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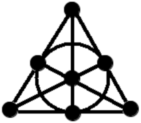
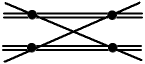


- Two blocks are **equivalent** if they contain the same set of points. The **multiplicity** $mult(x)$ of a block x is the size of its equivalence class.



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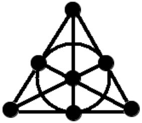

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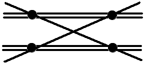

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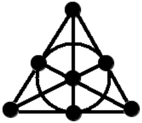

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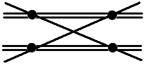

- The design is simple if all its blocks are distinct. Otherwise, it has multiple blocks.
- If all the blocks have the same multiplicity, then the design can be simplified by identifying equivalent blocks: $\mathcal{D} \rightarrow \overline{\mathcal{D}}$.

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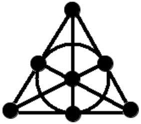

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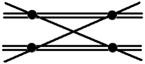

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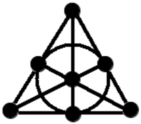

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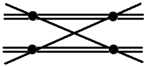

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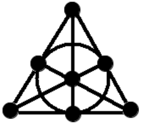

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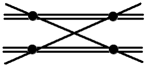

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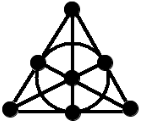
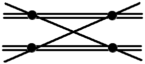
- Two points are *i^{th} associates* if their concurrence is λ_i.
- A *m*-concurrence design is a 1-design with m distinct concurrences $\lambda_1 \ldots, \lambda_m$ among its points, for which there exist m values $n_1, \ldots, n_m$ such that every point has exactly n_i i^{th} associates, for each $i \in [m]$.



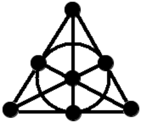
$n_1 = 6$

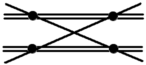


$n_1 = n_2 = n_3 = 1$

Incidence structures.

- An m-concurrence design is a partially balanced incomplete block design (PBIBD) if, for any two k^{th}-associated points P and Q, there exist p_{ij}^k points which are i^{th}-associated to P and j^{th}-associated to Q, where p_{ij}^k only depends on i, j and k.





$$p_{11}^1 = 6 \qquad p_{ij}^k = \begin{cases} 1, & \text{if } i \neq j \neq k \neq i, \\ 0, & \text{otherwise.} \end{cases}$$

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Partial Latin rectangles.

- $\mathcal{PLR}_{r,s,n} = \{r \times s$ partial Latin rectangles based on $[n] = \{1, 2, ..., n\}\}$.

  $r \times s$ arrays in which each cell is either empty or contains one symbol of $[n]$, s.t. each symbol occurs at most once in each row and in each column.

1		3	
	2	4	
			5

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$$P = (p_{ij}) \leftrightarrow x_{ijk} = \begin{cases} 1, & \text{if } p_{ij} = k, \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{r,s,n} \equiv \begin{cases} x_{ijk} \cdot (x_{ijk} - 1) = 0, \forall i \in [r], j \in [s], k \in [n], \\ x_{ijk} \cdot x_{ijl} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [n] \setminus [k], \\ x_{ijk} \cdot x_{ilk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [s] \setminus [j], \\ x_{ijk} \cdot x_{ljk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [r] \setminus [i]. \end{cases}$$

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$$\mathcal{PLR}_{r,s,n:m} \to \sum_{i \in [r], j \in [s], k \in [n]} x_{ijk} = m.$$

Partial Latin rectangles.

		$\lvert\mathcal{PLR}_{r,s,n}\rvert$							
		n							
r	s	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	9
	2		7	13	21	31	43	57	73
	3			34	73	136	229	358	529
	4				209	501	1,045	1,961	3,393
	5					1,546	4,051	9,276	19,081
	6						13,327	37,633	93,289
	7							130,922	394,353
	8								1,441,729
2	2		35	121	325	731	1,447	2,605	4,361
	3			781	3,601	12,781	37,273	93,661	209,761
	4				28,353	162,661	720,181	2,599,185	7,985,761
	5					1,502,171	10,291,951	54,730,201	236,605,001
	6						108,694,843	864,744,637	5,376,213,193
	7							10,256,288,925	92,842,518,721
	8								1,219,832,671,361
3	3			11,776	116,425	805,366	4,193,269	17,464,756	60,983,761
	4				2,423,521	33,199,561	317,651,473	2,263,521,961	12,703,477,825
	5					890,442,316	15,916,515,301	199,463,431,546	1,854,072,020,881
	6						526,905,708,889	11,785,736,969,413	*
4	4				127,545,137	4,146,833,121	87,136,329,169	1,258,840,124,753	*
	5					313,185,347,701	*	*	*

*Excessive cost of computation for a computer system i7-2600, 3.4 GHz.

Max. time of computation: 4,180 seconds ($\mathcal{PLR}_{2,9,13}$).

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		$\lvert \mathcal{PLR}_{r,s,n} \rvert$				
		n				
r	s	9	10	11	12	13
1	1	10	11	12	13	14
	2	91	111	133	157	183
	3	748	1,021	1,354	1,753	2,224
	4	5,509	8,501	12,585	18,001	25,013
	5	36,046	63,591	106,096	169,021	259,026
	6	207,775	424,051	805,597	1,442,173	2,456,299
	7	1047,376	2,501,801	5,470,158	11,109,337	21,204,548
	8	4,596,553	12,975,561	32,989,969	76,751,233	165,625,929
	9	17,572,114	58,941,091	175,721,140	472,630,861	1,163,391,958
	10		234,662,231	824,073,141	258,128,454	7,307,593,151
	11			3,405,357,682	12,470,162,233	40,864,292,184
	12				53,334,454,417	202,976,401,213
	13					896,324,308,634
2	2	6,985	10,411	15,137	21,325	29,251
	3	28,941	815,161	1,458,733	2,482,801	4,050,541
	4	21,582,613	52,585,221	117,667,441	245,278,945	481,597,221
	5	864,742,231	2,756,029,891	7,846,852,421	20,336,594,221	48,689,098,771
	6	27,175,825,171	115,690,051,951	426,999,864,193	1,398,636,508,477	4,141,988,637,463
	7	661,377,377,305	3,836,955,565,101	18,712,512,041,917	78,819,926,380,945	293,220,109,353,081
	8	12,372,136,371,721	99,423,049,782,601	652,303,240,153,313	3,595,671,023,722,081	17,076,864,830,330,761
	9	178,156,152,706,483	2,000,246,352,476,311	17,908,872,286,407,301	131,297,226,011,020,765	808,986,548,443,056,751
	10		31,296,831,902,738,931	385,203,526,838,449,441	*	*
	11			*	*	*
3	3	184,952,170	500,317,981	1,231,810,504	2,803,520,281	5,970,344,446
	4	58,737,345,481	231,769,858,321	802,139,572,873	2,487,656,927,521	7,030,865,002,825
	5	13,451,823,665,776	*	*	*	*

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  - Types ($r, s, n \leq 5$ [Falcón, 2013]):
    Number of entries per row and column and number of occurrences of each symbol. [Keedwell, 1994; Bean et al., 2002].

1		3	4	6
2		5		4
		4	5	1
		2		3

Type: $((4,3,3,2),(2,0,4,2,4),(2,2,2,3,2,1))$.

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- Consider the set of **symmetries (autotopisms)** of $\mathcal{PLR}_{r,s,n}$.

Symmetries of a partial Latin rectangle.

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- S_m: Symmetric group on $[m]$.
- $S_r \times S_s \times S_n$: Set of **isotopisms** of $\mathcal{PLR}_{r,s,n}$.

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Given $P = (p_{ij}) \in \mathcal{PLR}_{r,s,n}$:

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- S_m: Symmetric group on $[m]$.
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Given $P = (p_{ij}) \in \mathcal{PLR}_{r,s,n}$:

- Orthogonal representation: $O(P)=\{(i,j,p_{ij}) \mid i \in [r], j \in [s], p_{ij} \in [n]\}$.

Symmetries of a partial Latin rectangle.

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$$O(P^{\Theta}) = \{(\alpha(i), \beta(j), \gamma(p_{ij})) \mid (i,j,p_{ij}) \in O(P)\}.$$

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- Autotopism group: $\mathfrak{A}_n(P) = \mathfrak{I}_n(P, P)$.

- $\mathcal{PLR}_\Theta = \{P \in \mathcal{PLR}_{r,s,n} \mid \Theta \in \mathfrak{A}_n(P)\}$.

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- Autotopism group: $\mathfrak{A}_n(P) = \mathfrak{I}_n(P,P)$.
- $\mathcal{PLR}_\Theta = \{P \in \mathcal{PLR}_{r,s,n} \mid \Theta \in \mathfrak{A}_n(P)\}$.
- $\mathcal{PLR}_{\Theta:m} = \{P \in \mathcal{PLR}_{r,s,n:m} \mid \Theta \in \mathfrak{A}_n(P)\}$.

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- $\mathcal{PLR}_{\Theta:m} = \{P \in \mathcal{PLR}_{r,s,n:m} \mid \Theta \in \mathfrak{A}_n(P)\}$.

$$|\mathfrak{A}_n(P)| = |\mathfrak{A}_n(Q)|, \forall Q \in \mathfrak{I}_n(P).$$

$$|\mathfrak{I}_{n,P}| = \frac{r! \cdot s! \cdot n!}{|\mathfrak{A}_n(P)|}.$$

Symmetries of a partial Latin rectangle.

$P = (p_{ij}), Q = (q_{ij}) \in \mathcal{PLR}_{r,s,n}$.

POLYNOMIAL METHOD: $\mathfrak{I}_n(P, Q)$.

$$\Theta = (\alpha, \beta, \gamma) \leftrightarrow (a_{ij}, b_{ij}, c_{ij}) \text{ such that } d_{ij} = \begin{cases} 1, & \text{if } \delta(i) = j \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{n,P,Q} \equiv \begin{cases} a_{ij} \cdot (a_{ij} - 1) = 0, \forall i, j \in [r], \\ b_{ij} \cdot (b_{ij} - 1) = 0, \forall i, j \in [s], \\ c_{ij} \cdot (c_{ij} - 1) = 0, \forall i, j \in [n], \\ \sum_{i \in [r]} a_{ij} = 1, \forall j \in [r], \\ \sum_{j \in [r]} a_{ij} = 1, \forall i \in [r], \\ \sum_{i \in [s]} b_{ij} = 1, \forall j \in [s], \\ \sum_{j \in [s]} b_{ij} = 1, \forall i \in [s], \\ \sum_{i \in [n]} c_{ij} = 1, \forall j \in [n], \\ \sum_{j \in [n]} c_{ij} = 1, \forall i \in [n], \\ a_{ik} \cdot b_{jl} \cdot (c_{p_{ij} q_{kl}} - 1) = 0, \forall i, k \in [r], j, l \in [s], \text{ such that } p_{ij}, q_{kl} \in [n], \\ a_{ik} \cdot b_{jl} = 0, \forall i, k \in [r], j, l \in [s], \text{ such that } p_{ij} = \emptyset \text{ or } q_{kl} = \emptyset. \end{cases}$$

Symmetries of a partial Latin rectangle.

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$$\mathfrak{I}_n(P, Q) = \mathcal{V}(I_{n,P,Q})$$

$$|\mathfrak{I}_n(P, Q)| = \dim_{\mathbb{Q}}(\mathbb{Q}[a_{11}, \ldots, c_{nn}]/I_{n,P,Q})$$

Symmetries of a partial Latin rectangle.

$$P \equiv \begin{array}{|c|c|c|c|}\hline 1 & & 3 & \\ \hline & 2 & 4 & \\ \hline & & & 5 \\ \hline \end{array} \in \mathcal{PLR}_{3,4,5}.$$

$$\mathfrak{A}_5(P) = \begin{cases} \Theta_1 = \mathrm{Id}_{3,4,5} = ((1)(2)(3), (1)(2)(3)(4), (1)(2)(3)(4)(5)), \\ \Theta_2 = ((12)(3), (12)(3)(4), (12)(34)(5)). \end{cases}$$

$$|\mathfrak{I}_{5,P}| = \frac{3! \cdot 4! \cdot 5!}{2} = 8,640.$$

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How can we obtain all the 8, 460 partial Latin rectangles?

Symmetries of a partial Latin rectangle.

$P = (p_{ij}) \in \mathcal{PLR}_{r,s,n}$.

POLYNOMIAL METHOD: $\mathfrak{I}_{n,P}$.

$$
I_{n,P} \equiv \begin{cases}
x_{ijk} \cdot (x_{ijk} - 1) = 0, \forall i \in [r], j \in [s], k \in [n], \\
x_{ijk} \cdot x_{ijl} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [n] \setminus [k], \\
x_{ijk} \cdot x_{ilk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [s] \setminus [j], \\
x_{ijk} \cdot x_{ljk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [r] \setminus [i], \\
a_{ij} \cdot (a_{ij} - 1) = 0, \forall i, j \in [r], \\
b_{ij} \cdot (b_{ij} - 1) = 0, \forall i, j \in [s], \\
c_{ij} \cdot (c_{ij} - 1) = 0, \forall i, j \in [n], \\
\sum_{i \in [r]} a_{ij} = 1, \forall j \in [r], \\
\sum_{j \in [r]} a_{ij} = 1, \forall i \in [r], \\
\sum_{i \in [s]} b_{ij} = 1, \forall j \in [s], \\
\sum_{j \in [s]} b_{ij} = 1, \forall i \in [s], \\
\sum_{i \in [n]} c_{ij} = 1, \forall j \in [n], \\
\sum_{j \in [n]} c_{ij} = 1, \forall i \in [n], \\
a_{ik} \cdot b_{jl} \cdot c_{p_{ij} m} \cdot (x_{klm} - 1) = 0, \forall i, k \in [r], j, l \in [s], p_{ij}, m \in [n], \\
a_{ik} \cdot b_{jl} \cdot (x_{klm} - 1) = 0, \forall i, k \in [r], j, l \in [s], m \in [n], \text{ such that } p_{ij} = \emptyset.
\end{cases}
$$

Symmetries of a partial Latin rectangle.

$P = (p_{ij}) \in \mathcal{PLR}_{r,s,n}$.

POLYNOMIAL METHOD: $\mathfrak{I}_{n,P}$.

$$I_{n,P} \equiv \begin{cases} x_{ijk} \cdot (x_{ijk} - 1) = 0, \forall i \in [r], j \in [s], k \in [n], \\ x_{ijk} \cdot x_{ijl} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [n] \setminus [k], \\ x_{ijk} \cdot x_{ilk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [s] \setminus [j], \\ x_{ijk} \cdot x_{ljk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [r] \setminus [i], \\ a_{ij} \cdot (a_{ij} - 1) = 0, \forall i, j \in [r], \\ b_{ij} \cdot (b_{ij} - 1) = 0, \forall i, j \in [s], \\ c_{ij} \cdot (c_{ij} - 1) = 0, \forall i, j \in [n], \\ \sum_{i \in [r]} a_{ij} = 1, \forall j \in [r], \\ \sum_{j \in [r]} a_{ij} = 1, \forall i \in [r], \\ \sum_{i \in [s]} b_{ij} = 1, \forall j \in [s], \\ \sum_{j \in [s]} b_{ij} = 1, \forall i \in [s], \\ \sum_{i \in [n]} c_{ij} = 1, \forall j \in [n], \\ \sum_{j \in [n]} c_{ij} = 1, \forall i \in [n], \\ a_{ik} \cdot b_{jl} \cdot c_{p_{ij} m} \cdot (x_{klm} - 1) = 0, \forall i, k \in [r], j, l \in [s], p_{ij}, m \in [n], \\ a_{ik} \cdot b_{jl} \cdot (x_{klm} - 1) = 0, \forall i, k \in [r], j, l \in [s], m \in [n], \text{ such that } p_{ij} = \emptyset. \end{cases}$$

$\mathfrak{I}_{n,P} = \mathcal{V}(I_P)$ $\quad$ $|\mathfrak{I}_{n,P}| = \dim_{\mathbb{Q}}(\mathbb{Q}[x_{111}, \ldots, c_{nn}]/I_P)$

Symmetries of a partial Latin rectangle.

$P = (p_{ij}) \in \mathcal{PLR}_{r,s,n}$.

POLYNOMIAL METHOD: $\mathfrak{I}_{n,P}$.

(But Gröbner bases are extremely sensitive to the number of variables!!).

$$I_{n,P} \equiv \begin{cases} x_{ijk} \cdot (x_{ijk} - 1) = 0, \forall i \in [r], j \in [s], k \in [n], \\ x_{ijk} \cdot x_{ijl} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [n] \setminus [k], \\ x_{ijk} \cdot x_{ilk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [s] \setminus [j], \\ x_{ijk} \cdot x_{ljk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [r] \setminus [i], \\ a_{ij} \cdot (a_{ij} - 1) = 0, \forall i, j \in [r], \\ b_{ij} \cdot (b_{ij} - 1) = 0, \forall i, j \in [s], \\ c_{ij} \cdot (c_{ij} - 1) = 0, \forall i, j \in [n], \\ \sum_{i \in [r]} a_{ij} = 1, \forall j \in [r], \\ \sum_{j \in [r]} a_{ij} = 1, \forall i \in [r], \\ \sum_{i \in [s]} b_{ij} = 1, \forall j \in [s], \\ \sum_{j \in [s]} b_{ij} = 1, \forall i \in [s], \\ \sum_{i \in [n]} c_{ij} = 1, \forall j \in [n], \\ \sum_{j \in [n]} c_{ij} = 1, \forall i \in [n], \\ a_{ik} \cdot b_{jl} \cdot c_{p_{ij}m} \cdot (x_{klm} - 1) = 0, \forall i, k \in [r], j, l \in [s], p_{ij}, m \in [n], \\ a_{ik} \cdot b_{jl} \cdot (x_{klm} - 1) = 0, \forall i, k \in [r], j, l \in [s], m \in [n], \text{ such that } p_{ij} = \emptyset. \end{cases}$$

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Symmetries of a partial Latin rectangle.

$P = (p_{ij}) \in \mathcal{PLR}_{r,s,n}$.

In order to reduce the number variables, we can consider the **symmetries** of P, i.e., its autotopism group $\mathfrak{A}_n(P)$. It is due to the fact that autotopisms decompose P into blocks.

$$P \equiv \begin{array}{|c|c|c|c|} \hline 1 & & 3 & \\ \hline & 2 & 4 & \\ \hline & & & 5 \\ \hline \end{array} \in \mathcal{PLR}_{3,4,5}.$$

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$P \equiv$

1		3	
	2	4	
			5

$\in \mathcal{PLR}_{3,4,5}$.

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	2	4	
			5

$\in \mathcal{PLR}_{3,4,5}$.

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$$\boxed{\Theta = (\alpha,\beta,\gamma) \to x_{ijk} = x_{\alpha(i)\beta(j)\gamma(k)}}$$

Symmetries of a partial Latin rectangle.

$$\Theta = (\alpha, \beta, \gamma) \in S_r \times S_s \times S_n.$$

POLYNOMIAL METHOD: $\mathcal{PLR}_\Theta$

$$I_\Theta \equiv \begin{cases} x_{ijk} \cdot (x_{ijk} - 1) = 0, \forall i \in [r], j \in [s], k \in [n], \\ x_{ijk} = x_{\alpha(i)\beta(j)\gamma(k)}, \\ x_{ijk} \cdot x_{ijl} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [n] \setminus [k], \\ x_{ijk} \cdot x_{ilk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [s] \setminus [j], \\ x_{ijk} \cdot x_{ljk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [r] \setminus [i]. \end{cases}$$

Symmetries of a partial Latin rectangle.

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$\boxed{\mathcal{PLR}_\Theta = \mathcal{V}(I_\Theta)}$ $\boxed{|\mathcal{PLR}_\Theta| = \dim_{\mathbb{Q}}(\mathbb{Q}[x_{111}, \ldots, x_{rsn}]/I_\Theta).}$

Symmetries of a partial Latin rectangle.

$\Theta = (\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$.

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$$I_\Theta \equiv \begin{cases} x_{ijk} \cdot (x_{ijk} - 1) = 0, \forall i \in [r], j \in [s], k \in [n], \\ x_{ijk} = x_{\alpha(i)\beta(j)\gamma(k)}, \\ x_{ijk} \cdot x_{ijl} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [n] \setminus [k], \\ x_{ijk} \cdot x_{ilk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [s] \setminus [j], \\ x_{ijk} \cdot x_{ljk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [r] \setminus [i]. \end{cases}$$

$$\boxed{\mathcal{PLR}_\Theta = \mathcal{V}(I_\Theta)} \qquad \boxed{|\mathcal{PLR}_\Theta| = \dim_{\mathbb{Q}}(\mathbb{Q}[x_{111}, \ldots, x_{rsn}]/I_\Theta).}$$

If $\Theta = \mathrm{Id}_{r,s,n} = (\mathrm{Id}_r, \mathrm{Id}_s, \mathrm{Id}_n)$, then $I_\Theta = I_{r,s,n}$ and $\mathcal{PLR}_\Theta = \mathcal{PLR}_{r,s,n}$.

Symmetries of a partial Latin rectangle.

$\Theta = (\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$.

POLYNOMIAL METHOD: $\mathcal{PLR}_\Theta$

$$I_\Theta \equiv \begin{cases} x_{ijk} \cdot (x_{ijk} - 1) = 0, \forall i \in [r], j \in [s], k \in [n], \\ x_{ijk} = x_{\alpha(i)\beta(j)\gamma(k)}, \\ x_{ijk} \cdot x_{ijl} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [n] \setminus [k], \\ x_{ijk} \cdot x_{ilk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [s] \setminus [j], \\ x_{ijk} \cdot x_{ljk} = 0, \forall i \in [r], j \in [s], k \in [n], l \in [r] \setminus [i]. \end{cases}$$

$\mathcal{PLR}_\Theta = \mathcal{V}(I_\Theta)$

$|\mathcal{PLR}_\Theta| = \dim_{\mathbb{Q}}(\mathbb{Q}[x_{111}, \ldots, x_{rsn}]/I_\Theta)$.

If $\Theta = \mathrm{Id}_{r,s,n} = (\mathrm{Id}_r, \mathrm{Id}_s, \mathrm{Id}_n)$, then $I_\Theta = I_{r,s,n}$ and $\mathcal{PLR}_\Theta = \mathcal{PLR}_{r,s,n}$.

The number of variables which can be eliminated only depends on the **cycle structure** of Θ.

Symmetries of a partial Latin rectangle.

$$P \equiv \begin{array}{|c|c|c|c|}\hline 1 & & 3 & \\ \hline & 2 & 4 & \\ \hline & & & 5 \\ \hline \end{array} \in \mathcal{PLR}_{3,4,5}.$$

$$\mathfrak{A}_5(P) = \begin{cases} \Theta_1 = \mathrm{Id}_{3,4,5} = ((1)(2)(3), (1)(2)(3)(4), (1)(2)(3)(4)(5)), \\ \Theta_2 = ((12)(3), (12)(3)(4), (12)(34)(5)) \end{cases}.$$

- Cycle structure of $\Theta = (\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$: $z_\Theta = (z_\alpha, z_\beta, z_\gamma)$, where:

  Cycle structure of π: $z_\pi = k^{\lambda_k^\pi} \dots 1^{\lambda_1^\pi}$, being λ_i^π the number of cycles of length i in the decomposition of π as a product of disjoint cycles.

$$z_{\Theta_1} = (1^3, 1^4, 1^5), \qquad z_{\Theta_2} = (21, 21^2, 2^21).$$

- $\mathcal{CS}_n = \{$Cycle structures of $S_n\}$.

The incidence structure $(\mathcal{PLR}_{z:m}, S_z)$.

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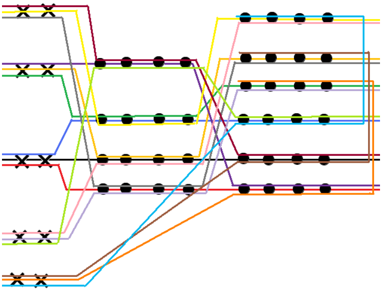

$z = (2, 2, 2^2 1)$
$m = 2$

$|\mathcal{PLR}_{z:m}| = 50$
$|\mathcal{S}_z| = 15$
$\Delta_m(z) = 10 = 2_P + 8_Q$

Two isotopism classes

$P \equiv$

1	
	1

$\rightarrow |\mathfrak{I}_5(P)| = 10$

$Q \equiv$

1	
	2

$\rightarrow |\mathfrak{I}_5(Q)| = 40$

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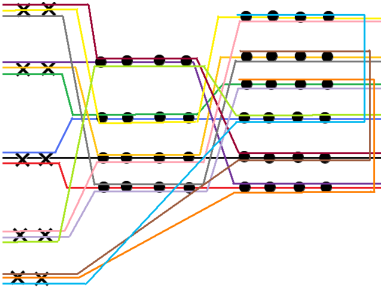

Which are the properties of such incidence structures?

- Multiplicity.
- Regularity.
- Parameters.

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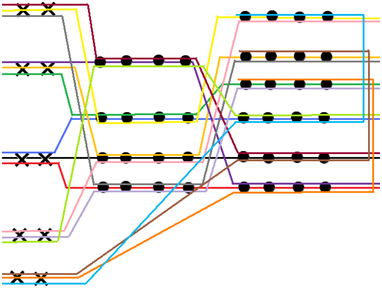

Which is the minimum number of blocks which are necessary to determine all the points of the incidence structure?

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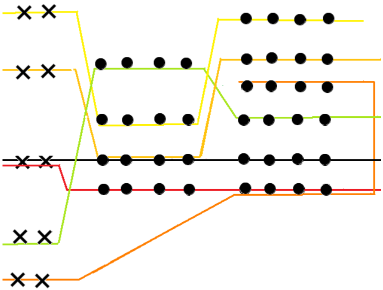

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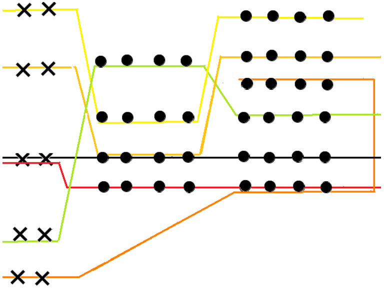

Which is the cost of computation?

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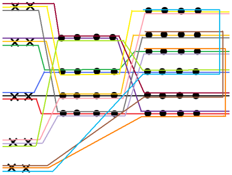

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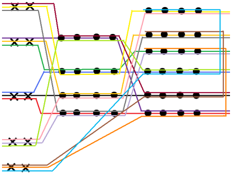

Lemma

All the blocks of $(\mathcal{PLR}_{z:m}, S_z)$ have the **same multiplicity**.

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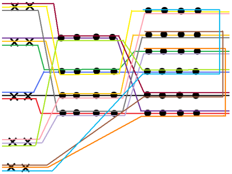

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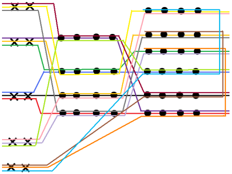

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$\Theta \in S_z \rightarrow$ If $|\mathfrak{A}_z(P)| = |\mathfrak{A}_z(Q)|$, for all $P, Q \in \mathcal{PLR}_{\Theta:m}$, then $(\mathcal{PLR}_{z:m}, S_z)$ is regular.

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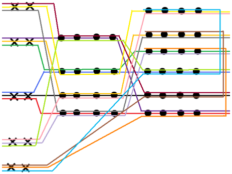

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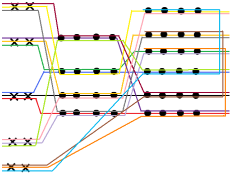

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The incidence structure $(\mathcal{PLR}_{z:m}, S_z)$.

$$z = (2^2, 2^2, 1^4) \in \mathcal{CS}_4 \times \mathcal{CS}_4 \times \mathcal{CS}_4.$$

$$\Theta = ((13)(24), (13)(24), \mathrm{Id}_4) \in S_{z_1}.$$

$$P \equiv \begin{array}{|c|c|c|c|}\hline 1&2&4&3\\\hline 3&1&2&4\\\hline 4&3&1&2\\\hline 2&4&3&1\\\hline\end{array} \not\sim Q \equiv \begin{array}{|c|c|c|c|}\hline 1&2&4&3\\\hline 2&1&3&4\\\hline 4&3&1&2\\\hline 3&4&2&1\\\hline\end{array} \in \mathcal{PLR}_{\Theta:16}.$$

$$\mathfrak{A}_z(P) = \{\Theta\}. \quad \mathfrak{A}_z(Q) = \begin{cases} \Theta, \\ ((12)(34), (12)(34), \mathrm{Id}_4), \\ ((14)(23), (14)(23), \mathrm{Id}_4). \end{cases}$$

$$|\mathfrak{A}_z(P)| = 1. \quad |\mathfrak{A}_z(Q)| = 3.$$

$$\Downarrow$$

$(\mathcal{PLR}_{z:16}, S_z)$ is not regular.

$$|\mathcal{PLR}_{z:16}| = 576 = 432_P + 144_Q, \quad |S_z| = 9, \quad \Delta_{16}(z) = 96 = 48_P + 48_Q.$$

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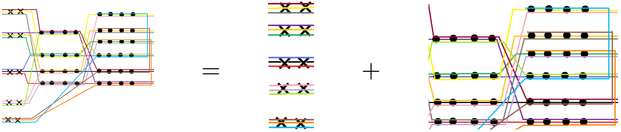

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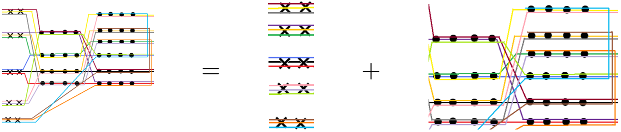

Proposition

The pair $(\mathfrak{I}_{n,P}, S_z)$ is a 1-$(|\mathfrak{I}_{n,P}|, \Delta_P(z), |\mathfrak{A}_z(P)|)$ design, with the incidence relation inherited from $(\mathcal{PLR}_{z:m}, S_z)$, such that:

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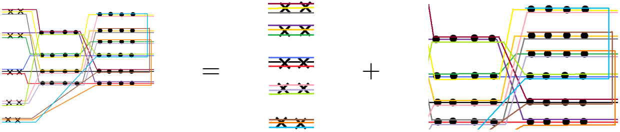

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$Q \in \mathfrak{I}_{n,P} \to$ The number of points which are *concurrent* with Q on exactly λ blocks does not depend on the choice of Q.
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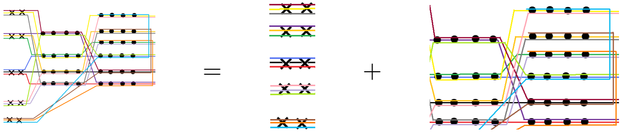

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The 1-design $(\mathfrak{I}_{n,P}, S_z)$ and its dual are m-concurrence designs.

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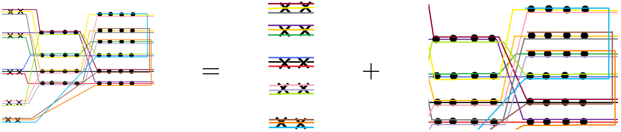

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- $mult(\mathfrak{I}_{n,P}) = \max_{\lambda \in \Lambda}\{\lambda\} + 1$.

The 1-design $(\mathfrak{I}_{n,P}, S_z)$.

r	s	n	m	P	z1	z2	z3	InP	Sz	DPz	AzP	mult	NNt	NtN	Spectrum C	PBIBD
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1
		2	1	1	1	1	11	2	1	2	1	1	1^2	2	1	1
		3	1	1	1	1	21	3	3	1	1	1	1	1	0	2
							111	3	1	3	1	1	1^3	3	1^2	1
		4	1	1	1	1	31	4	8	1	2	2	2	1^2	0	2
							1111	4	1	4	1	1	1^4	4	1^3	1
		5	1	1	1	1	41	5	30	1	6	6	6	1^6	0	2
							221	5	15	1	3	3	3	1^3	0	2
							11111	5	1	5	1	1	1^5	5	1^4	1
	2	2	1	10	1	11	11	4	1	4	1	1	1^4	4	1^3	1
			2	12	1	2	2	2	1	2	1	1	1^2	2	1	1
						11	11	2	1	2	1	1	1^2	2	1	1
		3	1	10	1	11	21	6	3	2	1	1	1^2	2	1^3	2
							111	6	1	6	1	1	1^6	6	1^5	1
			2	12	1	2	21	6	3	2	1	1	1^2	2	1^3	2
						11	111	6	1	6	1	1	1^6	6	1^5	1
		4	1	10	1	11	31	8	8	2	2	2	2^2	2^2	2^4	2
							1111	8	1	8	1	1	1^8	8	1^7	1
			2	12	1	2	22	12	3	4	1	1	1^4	4	1^9	2
						11	1111	12	1	12	1	1	1^(12)	12	1^(11)	1
		5	1	10	1	11	41	10	30	2	6	6	6^2	2^6	6^5	2
							221	10	15	2	3	3	3^2	2^3	3^5	2
							11111	10	1	10	1	1	1^(10)	10	1^9	1
			2	12	1	2	32	20	20	2	2	2	2^2	2^2	2^(10)	2
							221	20	15	4	3	1	3^21^6	42^4	1^5,2.5^4,3^(10	3
						11	11111	20	1	20	1	1	1^(20)	20	1^(19)	1
	3	3	1	100	1	21	21	9	9	1	1	1	1	1	0	2
							111	9	3	3	1	1	1^3	3	1^6	2
						111	111	9	1	9	1	1	1^9	9	1^8	1
			2	120	1	21	21	18	9	2	1	1	1^2	2	1^9	2
						111	111	18	1	18	1	1	1^{18}	18	1^{17}	1
			3	123	1	3	3	6	4	3	2	2	2^3	3^2	2^4	2
						21	21	6	9	2	3	1	31^3	21^4	(3/2)^4 3^1	3
						111	111	6	1	6	1	1	1^6	6	1^5	1
		4	1	100	1	21	31	12	24	1	2	2	2	1^2	0	2
							211	12	18	2	3	1	31^3	21^4	2^9	3
							1111	12	3	4	1	1	1^4	4	1^9	2
						111	31	12	8	3	2	2	2^3	3^2	2^8	2
							211	12	6	6	3	1	3^31^9	63^4	3^82^3	2
							1111	12	1	12	1	1	1^{12}	12	1^{11}	1
			2	120	1	21	22	36	9	4	1	1	1^4	4	1^{27}	2
							211	36	18	2	1	1	1^2	2	1^{18}	2
						111	211	36	6	6	1	1	1^6	6	1^{30}	2
							1111	36	1	36	1	1	1^{36}	36	1^{35}	1
			3	123	1	3	31	24	16	3	2	2	2^3	3^2	2^{16}	2
						21	211	24	18	4	3	1	31^9	41^8	3^{10}2^9(3/2)	0
						111	1111	24	1	24	1	1	1^{24}	{24}	1^{23}	1
		5	1	100	1	21	41	15	90	1	6	6	6	1^6	0	2
							311	15	60	2	8	2	82^4	2^21^{12}	{12}^5	3
							221	15	45	1	3	3	3	1^3	0	2
							2111	15	30	3	6	1	63^4	32^61^3	5^{12}	3
							11111	15	[illegible]	5	1	1	1^5	5	[illegible]	[illegible]

THE 1-DESIGN $(\mathfrak{I}_{n,P}, S_z)$.

In general, $(\mathfrak{I}_{n,P}, S_z)$ is not a PBIBD:

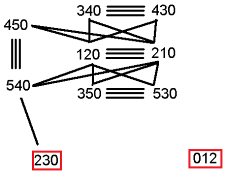
$$z = (1, 21, 2^2 1) \in \mathcal{CS}_1 \times \mathcal{CS}_3 \times \mathcal{CS}_5.$$

$$P \equiv \boxed{1}\boxed{2}\boxed{}$$

$$\begin{cases} |\mathfrak{I}_{n,P}| = 60, \\ |S_z| = 45, \\ \Delta_P(z) = 4, \\ |\mathfrak{A}_z(P)| = 3, \\ \mathrm{mult}(\mathfrak{I}_{n,P}) = 2, \\ \mathrm{mult}(S_z) = 1, \\ \text{3 connected components.} \end{cases} \qquad \begin{cases} \Lambda = \{0, 1, 3\}, \\ n_1 = 52, \quad n_2 = 6, \quad n_3 = 1. \end{cases}$$

$$\Theta = (\mathrm{Id}, (12)(3), (12)(34)(5)) \qquad \Theta = (\mathrm{Id}, (12)(3), (12)(35)(4))$$

$$\Theta = (\mathrm{Id}, (12)(3), (12)(45)(3))$$



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The 1-design $(\mathfrak{I}_{n,P}, S_z)$.

In general, $(\mathfrak{I}_{n,P}, S_z)$ is not a PBIBD:

$$z = (1, 21, 2^2 1) \in \mathcal{CS}_1 \times \mathcal{CS}_3 \times \mathcal{CS}_5.$$

$$P \equiv \boxed{1 \mid 2 \mid \ }$$

$$\begin{cases} |\mathfrak{I}_{n,P}| = 60, \\ |S_z| = 45, \\ \Delta_P(z) = 4, \\ |\mathfrak{A}_z(P)| = 3, \\ \mathrm{mult}(\mathfrak{I}_{n,P}) = 2, \\ \mathrm{mult}(S_z) = 1, \\ \text{3 connected components.} \end{cases} \qquad \begin{cases} \Lambda = \{0, 1, 3\}, \\ n_1 = 52, \quad n_2 = 6, \quad n_3 = 1. \end{cases}$$

$$\Theta = (\mathrm{Id}, (12)(3), (12)(34)(5)) \qquad \Theta = (\mathrm{Id}, (12)(3), (12)(35)(4))$$

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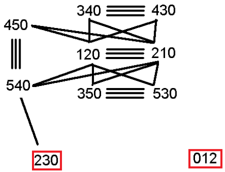

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When a *m*-concurrence design related to a PLR is a PBIBD?

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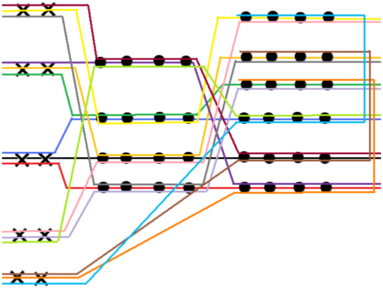
Thank you!!

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